

Intermediate

Alg-ebra

Intermediate Algebra

SENIOR CONTRIBUTING AUTHOR

LYNN MARECEK, SANTA ANA COLLEGE



OpenStax
Rice University
6100 Main Street MS-375
Houston, Texas 77005

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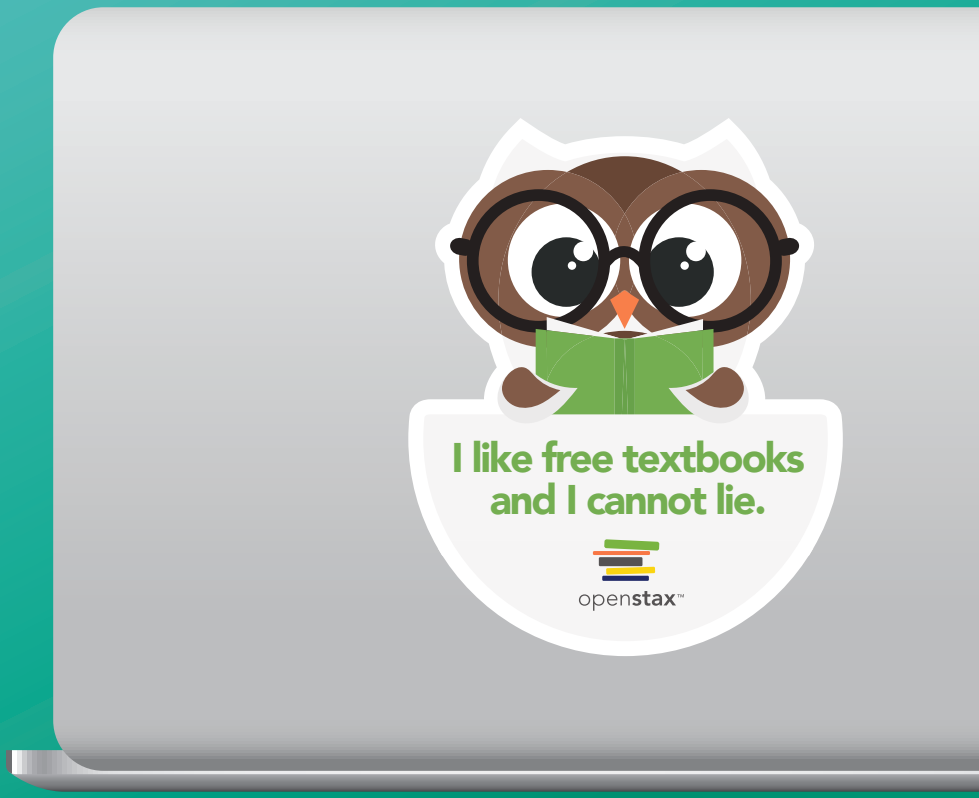
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Table of Contents

Preface 1

1 Foundations 5

1.1 Use the Language of Algebra 5
1.2 Integers 24
1.3 Fractions 41
1.4 Decimals 55
1.5 Properties of Real Numbers 72

2 Solving Linear Equations 97

2.1 Use a General Strategy to Solve Linear Equations 97
2.2 Use a Problem Solving Strategy 114
2.3 Solve a Formula for a Specific Variable 132
2.4 Solve Mixture and Uniform Motion Applications 149
2.5 Solve Linear Inequalities 168
2.6 Solve Compound Inequalities 187
2.7 Solve Absolute Value Inequalities 198

3 Graphs and Functions 225

3.1 Graph Linear Equations in Two Variables 225
3.2 Slope of a Line 254
3.3 Find the Equation of a Line 279
3.4 Graph Linear Inequalities in Two Variables 295
3.5 Relations and Functions 314
3.6 Graphs of Functions 329

4 Systems of Linear Equations 367

4.1 Solve Systems of Linear Equations with Two Variables 367
4.2 Solve Applications with Systems of Equations 389
4.3 Solve Mixture Applications with Systems of Equations 407
4.4 Solve Systems of Equations with Three Variables 420
4.5 Solve Systems of Equations Using Matrices 433
4.6 Solve Systems of Equations Using Determinants 446
4.7 Graphing Systems of Linear Inequalities 460

5 Polynomials and Polynomial Functions 487

5.1 Add and Subtract Polynomials 487
5.2 Properties of Exponents and Scientific Notation 501
5.3 Multiply Polynomials 524
5.4 Dividing Polynomials 540

6 Factoring 565

6.1 Greatest Common Factor and Factor by Grouping 565
6.2 Factor Trinomials 574
6.3 Factor Special Products 592
6.4 General Strategy for Factoring Polynomials 605
6.5 Polynomial Equations 615

7 Rational Expressions and Functions 639

7.1 Multiply and Divide Rational Expressions 639
7.2 Add and Subtract Rational Expressions 655
7.3 Simplify Complex Rational Expressions 670
7.4 Solve Rational Equations 682
7.5 Solve Applications with Rational Equations 697
7.6 Solve Rational Inequalities 721

8 **Roots and Radicals 743**

8.1 Simplify Expressions with Roots 743
8.2 Simplify Radical Expressions 757
8.3 Simplify Rational Exponents 774
8.4 Add, Subtract, and Multiply Radical Expressions 789
8.5 Divide Radical Expressions 800
8.6 Solve Radical Equations 812
8.7 Use Radicals in Functions 826
8.8 Use the Complex Number System 834

9 **Quadratic Equations and Functions 859**

9.1 Solve Quadratic Equations Using the Square Root Property 859
9.2 Solve Quadratic Equations by Completing the Square 872
9.3 Solve Quadratic Equations Using the Quadratic Formula 887
9.4 Solve Quadratic Equations in Quadratic Form 900
9.5 Solve Applications of Quadratic Equations 908
9.6 Graph Quadratic Functions Using Properties 921
9.7 Graph Quadratic Functions Using Transformations 947
9.8 Solve Quadratic Inequalities 968

10 **Exponential and Logarithmic Functions 989**

10.1 Finding Composite and Inverse Functions 989
10.2 Evaluate and Graph Exponential Functions 1006
10.3 Evaluate and Graph Logarithmic Functions 1021
10.4 Use the Properties of Logarithms 1036
10.5 Solve Exponential and Logarithmic Equations 1047

11 **Conics 1069**

11.1 Distance and Midpoint Formulas; Circles 1069
11.2 Parabolas 1084
11.3 Ellipses 1105
11.4 Hyperbolas 1122
11.5 Solve Systems of Nonlinear Equations 1136

12 **Sequences, Series and Binomial Theorem 1163**

12.1 Sequences 1163
12.2 Arithmetic Sequences 1176
12.3 Geometric Sequences and Series 1186
12.4 Binomial Theorem 1202

Index 1347

PREFACE

Welcome to *Intermediate Algebra,* an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

About OpenStax

OpenStax is a nonprofit based at Rice University, and it's our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 25 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

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Format

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About *Intermediate Algebra*

Intermediate Algebra is designed to meet the scope and sequence requirements of a one-semester Intermediate algebra course. The book's organization makes it easy to adapt to a variety of course syllabi. The text expands on the fundamental concepts of algebra while addressing the needs of students with diverse backgrounds and learning styles. Each topic builds upon previously developed material to demonstrate the cohesiveness and structure of mathematics.

Coverage and Scope

Intermediate Algebra continues the philosophies and pedagogical features of *Prealgebra* and *Elementary Algebra*, by Lynn Marecek and MaryAnne Anthony-Smith. By introducing the concepts and vocabulary of algebra in a nurturing, non-threatening environment while also addressing the needs of students with diverse backgrounds and learning styles, the book helps students gain confidence in their ability to succeed in the course and become successful college students.

The material is presented as a sequence of small, and clear steps to conceptual understanding. The order of topics was carefully planned to emphasize the logical progression throughout the course and to facilitate a thorough understanding of each concept. As new ideas are presented, they are explicitly related to previous topics.

Chapter 1: Foundations
Chapter 1 reviews arithmetic operations with whole numbers, integers, fractions, decimals and real numbers, to give the student a solid base that will support their study of algebra.

Chapter 2: Solving Linear Equations and Inequalities
In Chapter 2, students learn to solve linear equations using the Properties of Equality and a general strategy. They use a problem-solving strategy to solve number, percent, mixture and uniform motion applications. Solving a formula for a specific variable, and also solving both linear and compound inequalities is presented.

Chapter 3: Graphs and Functions
Chapter 3 covers the rectangular coordinate system where students learn to plot graph linear equations in two variables, graph with intercepts, understand slope of a line, use the slope-intercept form of an equation of a line, find the equation of a line, and create graphs of linear inequalities. The chapter also introduces relations and

functions as well as graphing of functions.

Chapter 4: Systems of Linear Equations
Chapter 4 covers solving systems of equations by graphing, substitution, and elimination; solving applications with systems of equations, solving mixture applications with systems of equations, and graphing systems of linear inequalities. Systems of equations are also solved using matrices and determinants.

Chapter 5: Polynomials and Polynomial Functions
In Chapter 5, students learn how to add and subtract polynomials, use multiplication properties of exponents, multiply polynomials, use special products, divide monomials and polynomials, and understand integer exponents and scientific notation.

Chapter 6: Factoring
In Chapter 6, students learn the process of factoring expressions and see how factoring is used to solve quadratic equations.

Chapter 7: Rational Expressions and Functions
In Chapter 7, students work with rational expressions, solve rational equations and use them to solve problems in a variety of applications, and solve rational inequalities.

Chapter 8: Roots and Radical
In Chapter 8, students simplify radical expressions, rational exponents, perform operations on radical expressions, and solve radical equations. Radical functions and the complex number system are introduced

Chapter 9: Quadratic Equations
In Chapter 9, students use various methods to solve quadratic equations and equations in quadratic form and learn how to use them in applications. Students will graph quadratic functions using their properties and by transformations.

Chapter 10: Exponential and Logarithmic Functions
In Chapter 10, students find composite and inverse functions, evaluate, graph, and solve both exponential and logarithmic functions.

Chapter 11: Conics
In Chapter 11, the properties and graphs of circles, parabolas, ellipses and hyperbolas are presented. Students also solve applications using the conics and solve systems of nonlinear equations.

Chapter 12: Sequences, Series and the Binomial Theorem
In Chapter 12, students are introduced to sequences, arithmetic sequences, geometric sequences and series and the binomial theorem.

All chapters are broken down into multiple sections, the titles of which can be viewed in the **Table of Contents**.

Key Features and Boxes

Examples Each learning objective is supported by one or more worked examples, which demonstrate the problem-solving approaches that students must master. Typically, we include multiple examples for each learning objective to model different approaches to the same type of problem, or to introduce similar problems of increasing complexity.

All examples follow a simple two- or three-part format. First, we pose a problem or question. Next, we demonstrate the solution, spelling out the steps along the way. Finally (for select examples), we show students how to check the solution. Most examples are written in a two-column format, with explanation on the left and math on the right to mimic the way that instructors "talk through" examples as they write on the board in class.

Be Prepared! Each section, beginning with Section 2.1, starts with a few "Be Prepared!" exercises so that students can determine if they have mastered the prerequisite skills for the section. Reference is made to specific Examples from previous sections so students who need further review can easily find explanations. Answers to these exercises can be found in the supplemental resources that accompany this title.

Try It



Try it The Try It feature includes a pair of exercises that immediately follow an Example, providing the student with an immediate opportunity to solve a similar problem with an easy reference to the example. In the Web View version of the text, students can click an Answer link directly below the question to check their understanding. In the PDF, answers to the Try It exercises are located in the Answer Key.

How To



How To Examples use a three column format to demonstrate how to solve an example with a certain procedure. The first column states the formal step, the second column is in words as the teacher would explain the process, and then the third column is the actual math. A How To procedure box follows each of these How To examples and summarizes the series of steps from the example. These procedure boxes provide an easy reference for students.

Media

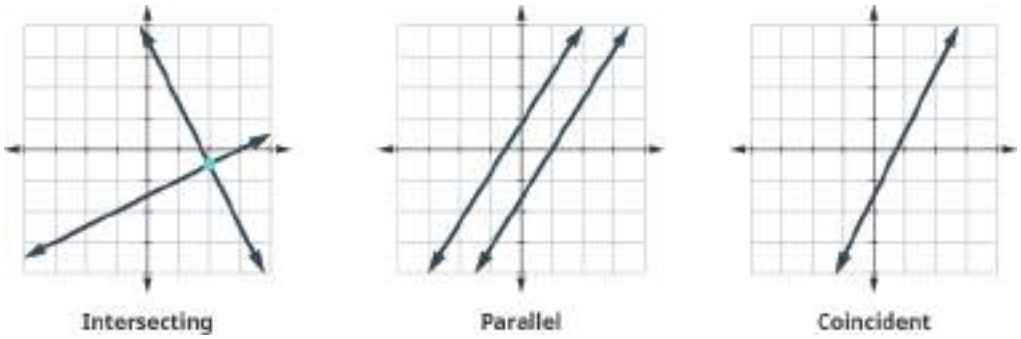
Media The "Media" icon appears at the conclusion of each section, just prior to the Self Check. This icon marks a list of links to online video tutorials that reinforce the concepts and skills introduced in the section.

Disclaimer: While we have selected tutorials that closely align to our learning objectives, we did not produce these tutorials, nor were they specifically produced or tailored to accompany *Intermediate Algebra*.

Self Check The Self Check includes the learning objectives for the section so that students can self-assess their mastery and make concrete plans to improve.

Art Program

Intermediate Algebra contains many figures and illustrations. Art throughout the text adheres to a clear, understated style, drawing the eye to the most important information in each figure while minimizing visual distractions.



Section Exercises and Chapter Review

Section Exercises Each section of every chapter concludes with a well-rounded set of exercises that can be assigned as homework or used selectively for guided practice. Exercise sets are named *Practice Makes Perfect* to encourage completion of homework assignments.

Exercises correlate to the learning objectives. This facilitates assignment of personalized study plans based on individual student needs.

Exercises are carefully sequenced to promote building of skills.

Values for constants and coefficients were chosen to practice and reinforce arithmetic facts.

Even and odd-numbered exercises are paired.

Exercises parallel and extend the text examples and use the same instructions as the examples to help students easily recognize the connection.

Applications are drawn from many everyday experiences, as well as those traditionally found in college math texts.

Everyday Math highlights practical situations using the concepts from that particular section

Writing Exercises are included in every exercise set to encourage conceptual understanding, critical thinking, and literacy.

Chapter review Each chapter concludes with a review of the most important takeaways, as well as additional practice problems that students can use to prepare for exams.

Key Terms provide a formal definition for each bold-faced term in the chapter.

Key Concepts summarize the most important ideas introduced in each section, linking back to the relevant Example(s) in case students need to review.

Chapter Review Exercises include practice problems that recall the most important concepts from each section.

Practice Test includes additional problems assessing the most important learning objectives from the chapter.

Answer Key includes the answers to all Try It exercises and every other exercise from the Section Exercises, Chapter Review Exercises, and Practice Test.

Additional Resources

Student and Instructor Resources

We've compiled additional resources for both students and instructors, including Getting Started Guides, manipulative mathematics worksheets, an answer key to the Be Prepared Exercises, and an answer guide to the section review exercises. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

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About the Authors

Senior Contributing Author

Lynn Marecek, Santa Ana College

Lynn Marecek has been teaching mathematics at Santa Ana College for many years has focused her career on meeting the needs of developmental math students. At Santa Ana College, she has been awarded the Distinguished Faculty Award, Innovation Award, and the Curriculum Development Award four times. She is a Coordinator of the Freshman Experience Program, the Department Facilitator for Redesign, and a member of the Student Success and Equity Committee, and the Basic Skills Initiative Task Force.

She is the coauthor with MaryAnne Anthony-Smith of *Strategies for Success: Study Skills for the College Math Student*, *Prealgebra* published by OpenStax and *Elementary Algebra* published by OpenStax.

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1 FOUNDATIONS

Figure 1.1 This hand may change someone's life. Amazingly, it was created using a special kind of printer known as a 3D printer. (credit: U.S. Food and Drug Administration/Wikimedia Commons)

Chapter Outline

1.1 Use the Language of Algebra
1.2 Integers
1.3 Fractions
1.4 Decimals
1.5 Properties of Real Numbers

Introduction

For years, doctors and engineers have worked to make artificial limbs, such as this hand for people who need them. This particular product is different, however, because it was developed using a 3D printer. As a result, it can be printed much like you print words on a sheet of paper. This makes producing the limb less expensive and faster than conventional methods.

Biomedical engineers are working to develop organs that may one day save lives. Scientists at NASA are designing ways to use 3D printers to build on the moon or Mars. Already, animals are benefitting from 3D-printed parts, including a tortoise shell and a dog leg. Builders have even constructed entire buildings using a 3D printer.

The technology and use of 3D printers depends on the ability to understand the language of algebra. Engineers must be able to translate observations and needs in the natural world to complex mathematical commands that can provide directions to a printer. In this chapter, you will review the language of algebra and take your first steps toward working with algebraic concepts.

1.1 Use the Language of Algebra

Learning Objectives

By the end of this section, you will be able to:

- Find factors, prime factorizations, and least common multiples
- Use variables and algebraic symbols
- Simplify expressions using the order of operations
- Evaluate an expression
- Identify and combine like terms
- Translate an English phrase to an algebraic expression

Be Prepared!

This chapter is intended to be a brief review of concepts that will be needed in an Intermediate Algebra course. A more thorough introduction to the topics covered in this chapter can be found in the *Elementary Algebra* chapter,

Foundations.

Find Factors, Prime Factorizations, and Least Common Multiples

The numbers 2, 4, 6, 8, 10, 12 are called multiples of 2. A **multiple** of 2 can be written as the product of a counting number and 2.

Multiples of 2:

2,	4,	6,	8,	10,	12, ...
$2 \cdot 1$	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 5$	$2 \cdot 6$

Similarly, a multiple of 3 would be the product of a counting number and 3.

Multiples of 3:

3,	6,	9,	12,	15,	18, ...
$3 \cdot 1$	$3 \cdot 2$	$3 \cdot 3$	$3 \cdot 4$	$3 \cdot 5$	$3 \cdot 6$

We could find the multiples of any number by continuing this process.

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108

Multiple of a Number

A number is a **multiple** of n if it is the product of a counting number and n.

Another way to say that 15 is a multiple of 3 is to say that 15 is **divisible** by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

Divisible by a Number

If a number m is a multiple of *n*, then *m* is **divisible** by *n*.

If we were to look for patterns in the multiples of the numbers 2 through 9, we would discover the following divisibility tests:

Divisibility Tests

A number is divisible by:

2 if the last digit is 0, 2, 4, 6, or 8.

3 if the sum of the digits is divisible by 3.

5 if the last digit is 5 or 0.

6 if it is divisible by both 2 and 3.

10 if it ends with 0.

EXAMPLE 1.1

Is 5,625 divisible by ⓐ 2? ⓑ 3? ⓒ 5 or 10? ⓓ 6?

Solution

ⓐ

Is 5,625 divisible by 2?

Does it end in 0, 2, 4, 6 or 8?	No. 5,625 is not divisible by 2.

ⓑ

Is 5,625 divisible by 3?

What is the sum of the digits?	$5+6+2+5=18$
Is the sum divisible by 3?	Yes. 5,625 is divisible by 3.

ⓒ

Is 5,625 divisible by 5 or 10?

What is the last digit? It is 5.	5,625 is divisible by 5 but not by 10.

ⓓ

Is 5,625 divisible by 6?

Is it divisible by both 2 and 3?	No, 5,625 is not divisible by 2, so 5,625 is not divisible by 6.

TRY IT : : 1.1 Is 4,962 divisible by ⓐ 2? ⓑ 3? ⓒ 5? ⓓ 6? ⓔ 10?

TRY IT : : 1.2 Is 3,765 divisible by ⓐ 2? ⓑ 3? ⓒ 5? ⓓ 6? ⓔ 10?

In mathematics, there are often several ways to talk about the same ideas. So far, we've seen that if *m* is a multiple of *n*, we can say that *m* is divisible by *n*. For example, since 72 is a multiple of 8, we say 72 is divisible by 8. Since 72 is a multiple of 9, we say 72 is divisible by 9. We can express this still another way.

Since $8 \cdot 9 = 72$, we say that 8 and 9 are **factors** of 72. When we write $72 = 8 \cdot 9$, we say we have factored 72.



Other ways to factor 72 are $1 \cdot 72$, $2 \cdot 36$, $3 \cdot 24$, $4 \cdot 18$, and $6 \cdot 12$. The number 72 has many factors: $1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36,$ and 72.

Factors

If $a \cdot b = m$, then *a* and *b* are **factors** of *m*.

Some numbers, such as 72, have many factors. Other numbers have only two factors. A **prime number** is a counting number greater than 1 whose only factors are 1 and itself.

Prime number and Composite number

A **prime number** is a counting number greater than 1 whose only factors are 1 and the number itself.

A **composite number** is a counting number that is not prime. A composite number has factors other than 1 and the number itself.

The counting numbers from 2 to 20 are listed in the table with their factors. Make sure to agree with the "prime" or "composite" label for each!

Number	Factors	Prime or Composite?
2	1,2	Prime
3	1,3	Prime
4	1,2,4	Composite
5	1,5	Prime
6	1,2,3,6	Composite
7	1,7	Prime
8	1,2,4,8	Composite
9	1,3,9	Composite
10	1,2,5,10	Composite
11	1,11	Prime

Number	Factors	Prime or Composite?
12	1,2,3,4,6,12	Composite
13	1,13	Prime
14	1,2,7,14	Composite
15	1,3,5,15	Composite
16	1,2,4,8,16	Composite
17	1,17	Prime
18	1,2,3,6,9,18	Composite
19	1,19	Prime
20	1,2,4,5,10,20	Composite

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. Notice that the only even prime number is 2.

A composite number can be written as a unique product of primes. This is called the **prime factorization** of the number. Finding the prime factorization of a composite number will be useful in many topics in this course.

Prime Factorization

The **prime factorization** of a number is the product of prime numbers that equals the number.

To find the prime factorization of a composite number, find any two factors of the number and use them to create two branches. If a factor is prime, that branch is complete. Circle that prime. Otherwise it is easy to lose track of the prime numbers.

If the factor is not prime, find two factors of the number and continue the process. Once all the branches have circled primes at the end, the factorization is complete. The composite number can now be written as a product of prime numbers.

EXAMPLE 1.2 HOW TO FIND THE PRIME FACTORIZATION OF A COMPOSITE NUMBER

Factor 48.

Solution

Step 1. Find two factors whose product is the given number. Use these numbers to create two branches.	$48 = 2 \cdot 24$	
Step 2. If a factor is prime, that branch is complete. Circle the prime.	2 is prime. Circle the prime.	
Step 3. If a factor is not prime, write it as the product of two factors and continue the process.	24 is not prime. Break it into 2 more factors.	
	4 and 6 are not prime. Break them each into two factors.	
	2 and 3 are prime, so circle them.	

Step 4. Write the composite number as the product of all the circled primes.		$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

We say $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer.

If we first factored 48 in a different way, for example as $6 \cdot 8$, the result would still be the same. Finish the prime factorization and verify this for yourself.

TRY IT : : 1.3 Find the prime factorization of 80.

TRY IT : : 1.4 Find the prime factorization of 60.



HOW TO : : FIND THE PRIME FACTORIZATION OF A COMPOSITE NUMBER.

Step 1. Find two factors whose product is the given number, and use these numbers to create two branches.

Step 2. If a factor is prime, that branch is complete. Circle the prime, like a leaf on the tree.

Step 3. If a factor is not prime, write it as the product of two factors and continue the process.

Step 4. Write the composite number as the product of all the circled primes.

One of the reasons we look at primes is to use these techniques to find the **least common multiple** of two numbers. This will be useful when we add and subtract fractions with different denominators.

Least Common Multiple

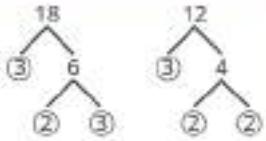
The **least common multiple (LCM)** of two numbers is the smallest number that is a multiple of both numbers.

To find the least common multiple of two numbers we will use the Prime Factors Method. Let's find the LCM of 12 and 18 using their prime factors.

EXAMPLE 1.3 HOW TO FIND THE LEAST COMMON MULTIPLE USING THE PRIME FACTORS METHOD

Find the least common multiple (LCM) of 12 and 18 using the prime factors method.

Solution

Step 1. Write each number as a product of primes.		
Step 2. List the primes of each number. Match primes vertically when possible.	List the primes of 12. List the primes of 18. Line up with the primes of 12 when possible. If not create a new column.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$
Step 3. Bring down the number from each column.		$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 3$

Step 4. Multiply the factors.		LCM = 36

Notice that the prime factors of 12 $(2 \cdot 2 \cdot 3)$ and the prime factors of 18 $(2 \cdot 3 \cdot 3)$ are included in the LCM $(2 \cdot 2 \cdot 3 \cdot 3)$. So 36 is the least common multiple of 12 and 18.

By matching up the common primes, each common prime factor is used only once. This way you are sure that 36 is the *least* common multiple.

TRY IT : : 1.5 Find the LCM of 9 and 12 using the Prime Factors Method.

TRY IT : : 1.6 Find the LCM of 18 and 24 using the Prime Factors Method.

HOW TO : : FIND THE LEAST COMMON MULTIPLE USING THE PRIME FACTORS METHOD.

Step 1. Write each number as a product of primes.
Step 2. List the primes of each number. Match primes vertically when possible.
Step 3. Bring down the columns.
Step 4. Multiply the factors.

Use Variables and Algebraic Symbols

In algebra, we use a letter of the alphabet to represent a number whose value may change. We call this a **variable** and letters commonly used for variables are x, y, a, b, c.

Variable

A **variable** is a letter that represents a number whose value may change.

A number whose value always remains the same is called a **constant**.

Constant

A **constant** is a number whose value always stays the same.

To write algebraically, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using. There are four basic arithmetic operations: addition, subtraction, multiplication, and division. We'll list the symbols used to indicate these operations below.

Operation Symbols

Operation	Notation	Say:	The result is...
Addition	$a + b$	a plus b	the sum of a and b
Subtraction	$a - b$	a minus b	the difference of a and b
Multiplication	$a \cdot b$, ab, $(a)(b)$, $(a)b$, $a(b)$	a times b	the product of a and b
Division	$a \div b$, a/b, $\frac{a}{b}$, $b\overline{)a}$	a divided by b	the quotient of a and b; a is called the dividend, and b is called the divisor

When two quantities have the same value, we say they are equal and connect them with an equal sign.

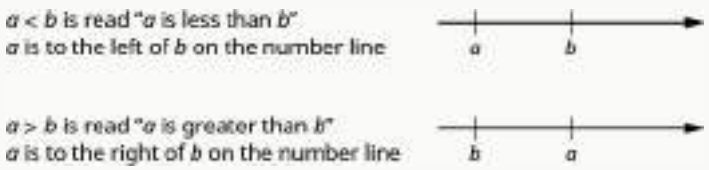
Equality Symbol

$a = b$ is read "a is equal to b."

The symbol "=" is called the equal sign.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols "<" and ">".

Inequality



The expressions $a < b$ or $a > b$ can be read from left to right or right to left, though in English we usually read from left to right. In general,

$a < b$ is equivalent to $b > a$. For example, $7 < 11$ is equivalent to $11 > 7$.

$a > b$ is equivalent to $b < a$. For example, $17 > 4$ is equivalent to $4 < 17$.

Inequality Symbols

Inequality Symbols	Words
$a \neq b$	a is *not equal to b.*
$a < b$	a is *less than b.*
$a \leq b$	a is *less than or equal to b.*
$a > b$	a is *greater than b.*
$a \geq b$	a is *greater than or equal to b.*

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English. They help identify an **expression**, which can be made up of number, a variable, or a combination of numbers and variables using operation symbols. We will introduce three types of grouping symbols now.

Grouping Symbols

Parentheses	()
Brackets	[]
Braces	{ }

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$$8(14 - 8) \qquad 21 - 3[2 + 4(9 - 8)] \qquad 24 \div \{13 - 2[1(6 - 5) + 4]\}$$

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. A sentence has a subject and a verb. In algebra, we have *expressions* and *equations*.

Expression

An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols.

Expression	Words	English Phrase
$3+5$	3 plus 5	the sum of three and fi e
$n-1$	n minus one	the diffe ence of n and one
$6\cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the English phrases do not form a complete sentence because the phrase does not have a verb.

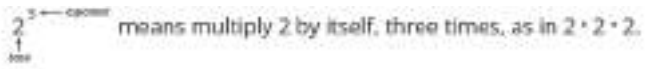
An **equation** is two expressions linked by an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

Equation

An **equation** is two expressions connected by an equal sign.

Equation	English Sentence
$3+5=8$	The sum of three and fi e is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fi ty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

Suppose we need to multiply 2 nine times. We could write this as $2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2$. This is tedious and it can be hard to keep track of all those 2s, so we use exponents. We write $2\cdot 2\cdot 2$ as 2^3 and $2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2$ as 2^9. In expressions such as 2^3, the 2 is called the *base* and the 3 is called the *exponent*. The exponent tells us how many times we need to multiply the base.



We read 2^3 as "two to the third power" or "two cubed."

Exponential Notation

We say 2^3 is in *exponential notation* and $2\cdot 2\cdot 2$ is in *expanded notation*.

a^n means multiply a by itself, n times.



The expression a^n is read a to the n^{th} power.

While we read a^n as "a to the n^{th} power", we usually read:

a^2 "a squared"

a^3 "a cubed"

We'll see later why a^2 and a^3 have special names.

Table 1.1 shows how we read some expressions with exponents.

Expression	In Words	
7^2	7 to the second power or	7 squared
5^3	5 to the third power or	5 cubed
9^4	9 to the fourth power	
12^5	12 to the fifth power	

Table 1.1

Simplify Expressions Using the Order of Operations

To **simplify an expression** means to do all the math possible. For example, to simplify $4 \cdot 2 + 1$ we would first multiply $4 \cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$$\begin{array}{c} 4 \cdot 2 + 1 \\ 8 + 1 \\ 9 \end{array}$$

By not using an equal sign when you simplify an expression, you may avoid confusing expressions with equations.

Simplify an Expression

To **simplify an expression**, do all operations in the expression.

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the **order of operations**. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression $4 + 3 \cdot 7$. Some students simplify this getting 49, by adding $4 + 3$ and then multiplying that result by 7. Others get 25, by multiplying $3 \cdot 7$ first and then adding 4.

The same expression should give the same result. So mathematicians established some guidelines that are called the order of operations.



HOW TO : : USE THE ORDER OF OPERATIONS.

Step 1. Parentheses and Other Grouping Symbols
- Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

Step 2. Exponents
- Simplify all expressions with exponents.

Step 3. Multiplication and Division
- Perform all multiplication and division in order from left to right. These operations have equal priority.

Step 4. Addition and Subtraction
- Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, "How will I remember the order?" Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase "Please Excuse My Dear Aunt Sally".

Parentheses	**P**lease
Exponents	**E**xcuse
Multiplication **D**ivision	**M**y **D**ear
Addition **S**ubtraction	**A**unt **S**ally

It's good that "**M**y **D**ear" goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, "**A**unt **S**ally" goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

EXAMPLE 1.4

Simplify: $18 \div 6 + 4(5 - 2)$.

Solution

	$18 \div 6 + 4(5 - 2)$
Parentheses? Yes, subtract first.	$18 \div 6 + 4(3)$
Exponents? No.	
Multiplication or division? Yes.	
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication of division? No.	
Any addition or subtraction? Yes.	
Add.	15

TRY IT : : 1.7 Simplify: $30 \div 5 + 10(3 - 2)$.

TRY IT : : 1.8 Simplify: $70 \div 10 + 4(6 - 2)$.

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

EXAMPLE 1.5

Simplify: $5 + 2^3 + 3[6 - 3(4 - 2)]$.

Solution

	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Are there any parentheses (or other grouping symbols)? Yes.	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Focus on the parentheses that are inside the brackets. Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes. Simplify exponents.	$5 + 8 + 3[0]$
Is there any multiplication or division? Yes.	

Multiply.	$5+8+0$
Is there any addition of subtraction? Yes.	
Add.	$13+0$
Add.	13

TRY IT : : 1.9 Simplify: $9+5^3-[4(9+3)]$.

TRY IT : : 1.10 Simplify: $7^2-2[4(5+1)]$.

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

Evaluate an Expression

To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

EXAMPLE 1.6

Evaluate when $x=4$: ⓐ x^2 ⓑ 3^x ⓒ $2x^2+3x+8$.

Solution

ⓐ

	x^2
Replace x with 4.	4^2
Use definition of exponent.	$4\cdot 4$
Simplify.	16

ⓑ

	3^x
Replace x with 4.	3^4
Use definition of exponent.	$3\cdot 3\cdot 3\cdot 3$
Simplify.	81

ⓒ

	$2x^2 + 3x + 8$
Replace x with 4.	$2(4)^2 + 3(4) + 8$
Follow the order of operations.	$2(16) + 3(4) + 8$
	$32 + 12 + 8$
	52

TRY IT : : 1.11 Evaluate when $x = 3$, ⓐ x^2 ⓑ 4^x ⓒ $3x^2 + 4x + 1$.

TRY IT : : 1.12 Evaluate when $x = 6$, ⓐ x^3 ⓑ 2^x ⓒ $6x^2 - 4x - 7$.

Identify and Combine Like Terms

Algebraic expressions are made up of terms. A **term** is a constant, or the product of a constant and one or more variables.

Term

A **term** is a constant or the product of a constant and one or more variables.

Examples of terms are 7, y, $5x^2$, $9a$, and b^5.

The constant that multiplies the variable is called the **coefficient**.

Coefficient

The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \cdot x$.

Some terms share common traits. When two terms are constants or have the same variable and exponent, we say they are **like terms**.

Look at the following 6 terms. Which ones seem to have traits in common?

$$5x \quad 7 \quad n^2 \quad 4 \quad 3x \quad 9n^2$$

We say,

7 and 4 are like terms.

$5x$ and $3x$ are like terms.

n^2 and $9n^2$ are like terms.

Like Terms

Terms that are either constants or have the same variables raised to the same powers are called **like terms.**

If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable.

Simplify.	$4x + 7x + x$
Add the coefficient	$12x$

EXAMPLE 1.7 HOW TO COMBINE LIKE TERMS

Simplify: $2x^2 + 3x + 7 + x^2 + 4x + 5$.

Solution

Step 1. Identify the like terms.	$2x^2+3x+7+x^2+4x+5$ $2x^2+3x+7+x^2+4x+5$ $2x^2+3x+7+x^2+4x+5$
Step 2. Rearrange the expression so the like terms are together.	$2x^2+x^2+3x+4x+7+5$ $2x^2+x^2+3x+4x+7+5$
Step 3. Combine like terms.	$3x^2+7x+12$

TRY IT : : 1.13 Simplify: $3x^2+7x+9+7x^2+9x+8$.

TRY IT : : 1.14 Simplify: $4y^2+5y+2+8y^2+4y+5$.

HOW TO : : COMBINE LIKE TERMS.

Step 1. Identify like terms.
Step 2. Rearrange the expression so like terms are together.
Step 3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Translate an English Phrase to an Algebraic Expression

We listed many operation symbols that are used in algebra. Now, we will use them to translate English phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. Table 1.7 summarizes them.

Operation	Phrase	Expression
Addition	*a* plus *b* the sum of a and *b* *a* increased by *b* *b* more than *a* the total of *a* and *b* *b* added to *a*	$a+b$
Subtraction	*a* minus b the difference of *a* and *b* *a* decreased by *b* *b* less than *a* *b* subtracted from *a*	$a-b$
Multiplication	*a* times *b* the product of a and b twice *a*	$a \cdot b,\ ab,\ a(b),\ (a)(b)$ $2a$
Division	*a* divided by *b* the quotient of *a* and *b* the ratio of *a* and *b* *b* divided into *a*	$a \div b,\ a/b,\ \frac{a}{b},\ b\overline{)a}$

Table 1.7

Look closely at these phrases using the four operations:

the **sum** *of a and b*

the **difference** *of a and b*

the **product** *of a and b*

the **quotient** *of a and b*

Each phrase tells us to operate on two numbers. Look for the words *of* and *and* to find the numbers.

EXAMPLE 1.8

Translate each English phrase into an algebraic expression:

ⓐ the difference of $14x$ and 9

ⓑ the quotient of $8y^2$ and 3

ⓒ twelve more than y

ⓓ seven less than $49x^2$

Solution

ⓐ The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

the *difference of* 14*x and* 9

14*x* minus 9

14*x* – 9

ⓑ The key word is *quotient*, which tells us the operation is division.

the quotient of $8y^2$ and 3

divide $8y^2$ by 3

$8y^2 \div 3$

This can also be written $8y^2/3$ or $\frac{8y^2}{3}$.

ⓒ The key words are *more than.* They tell us the operation is addition. *More than* means "added to."

twelve more than y

twelve added to y

$y + 12$

ⓓ The key words are *less than*. They tell us to subtract. *Less than* means "subtracted from."

seven less than $49x^2$

seven subtracted from $49x^2$

$49x^2 - 7$

TRY IT : : 1.15 Translate the English phrase into an algebraic expression:

ⓐ the difference of $14x^2$ and 13

ⓑ the quotient of $12x$ and 2

ⓒ 13 more than z

ⓓ 18 less than $8x$

TRY IT : : 1.16 Translate the English phrase into an algebraic expression:

ⓐ the sum of $17y^2$ and 19

ⓑ the product of 7 and y

ⓒ Eleven more than x

ⓓ Fourteen less than $11a$

We look carefully at the words to help us distinguish between multiplying a sum and adding a product.

EXAMPLE 1.9

Translate the English phrase into an algebraic expression:

ⓐ eight times the sum of x and y

ⓑ the sum of eight times x and y

Solution

There are two operation words—*times* tells us to multiply and *sum* tells us to add.

ⓐ Because we are multiplying 8 times the sum, we need parentheses around the sum of x and y, $(x + y)$. This forces us to determine the sum first. (Remember the order of operations.)

eight times the sum of x and y

$8(x + y)$

ⓑ To take a sum, we look for the words *of* and *and* to see what is being added. Here we are taking the sum *of* eight times x and y.

the sum of eight times x and y

$8x + y$

TRY IT : : 1.17 Translate the English phrase into an algebraic expression:

ⓐ four times the sum of p and q

ⓑ the sum of four times p and q

TRY IT : : 1.18 Translate the English phrase into an algebraic expression:

ⓐ the difference of two times x and 8

ⓑ two times the difference of x and 8

Later in this course, we'll apply our skills in algebra to solving applications. The first step will be to translate an English phrase to an algebraic expression. We'll see how to do this in the next two examples.

EXAMPLE 1.10

The length of a rectangle is 14 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Solution

Write a phrase about the length of the rectangle.	14 less than the width
Substitute w for "the width."	w
Rewrite *less than* as *subtracted from*.	14 subtracted from w
Translate the phrase into algebra.	$w - 14$

TRY IT : : 1.19

The length of a rectangle is 7 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

TRY IT : : 1.20

The width of a rectangle is 6 less than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

The expressions in the next example will be used in the typical coin mixture problems we will see soon.

EXAMPLE 1.11

June has dimes and quarters in her purse. The number of dimes is seven less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution

Write a phrase about the number of dimes.	seven less than four times the number of quarters
Substitute q for the number of quarters.	7 less than 4 times q
Translate 4 times q.	7 less than $4q$
Translate the phrase into algebra.	$4q - 7$

TRY IT : : 1.21

Geoffrey has dimes and quarters in his pocket. The number of dimes is eight less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

TRY IT : : 1.22

Lauren has dimes and nickels in her purse. The number of dimes is three more than seven times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

1.1 EXERCISES

Practice Makes Perfect

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10.

1. 84

2. 96

3. 896

4. 942

5. 22,335

6. 39,075

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

7. 86

8. 78

9. 455

10. 400

11. 432

12. 627

In the following exercises, find the least common multiple of each pair of numbers using the prime factors method.

13. 8, 12

14. 12, 16

15. 28, 40

16. 84, 90

17. 55, 88

18. 60, 72

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.

19. $2^3 - 12 \div (9 - 5)$

20. $3^2 - 18 \div (11 - 5)$

21. $2 + 8(6 + 1)$

22. $4 + 6(3 + 6)$

23. $20 \div 4 + 6(5 - 1)$

24. $33 \div 3 + 4(7 - 2)$

25. $3(1 + 9 \cdot 6) - 4^2$

26. $5(2 + 8 \cdot 4) - 7^2$

27. $2[1 + 3(10 - 2)]$

28. $5[2 + 4(3 - 2)]$

29. $8 + 2[7 - 2(5 - 3)] - 3^2$

30. $10 + 3[6 - 2(4 - 2)] - 2^4$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

31. When $x = 2$,

ⓐ x^6

ⓑ 4^x

ⓒ $2x^2 + 3x - 7$

32. When $x = 3$,

ⓐ x^5

ⓑ 5^x

ⓒ $3x^2 - 4x - 8$

33. When $x = 4,\ y = 1$

$x^2 + 3xy - 7y^2$

34. When $x = 3,\ y = 2$

$6x^2 + 3xy - 9y^2$

35. When $x = 10,\ y = 7$

$(x - y)^2$

36. When $a = 3,\ b = 8$

$a^2 + b^2$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.

37. $7x + 2 + 3x + 4$

38. $8y + 5 + 2y - 4$

39. $10a + 7 + 5a - 2 + 7a - 4$

40. $7c + 4 + 6c - 3 + 9c - 1$

41. $3x^2 + 12x + 11 + 14x^2 + 8x + 5$

42. $5b^2 + 9b + 10 + 2b^2 + 3b - 4$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the phrases into algebraic expressions.

43.

ⓐ the difference of $5x^2$ and $6xy$

ⓑ the quotient of $6y^2$ and $5x$

ⓒ Twenty-one more than y^2

ⓓ $6x$ less than $81x^2$

44.

ⓐ the difference of $17x^2$ and $5xy$

ⓑ the quotient of $8y^3$ and $3x$

ⓒ Eighteen more than a^2;

ⓓ $11b$ less than $100b^2$

45.

ⓐ the sum of $4ab^2$ and $3a^2b$

ⓑ the product of $4y^2$ and $5x$

ⓒ Fifteen more than m

ⓓ $9x$ less than $121x^2$

46.

ⓐ the sum of $3x^2y$ and $7xy^2$

ⓑ the product of $6xy^2$ and $4z$

ⓒ Twelve more than $3x^2$

ⓓ $7x^2$ less than $63x^3$

47.

ⓐ eight times the difference of y and nine

ⓑ the difference of eight times y and 9

48.

ⓐ seven times the difference of y and one

ⓑ the difference of seven times y and 1

49.

ⓐ five times the sum of $3x$ and y

ⓑ the sum of five times $3x$ and y

50.

ⓐ eleven times the sum of $4x^2$ and $5x$

ⓑ the sum of eleven times $4x^2$ and $5x$

51. Eric has rock and country songs on his playlist. The number of rock songs is 14 more than twice the number of country songs. Let *c* represent the number of country songs. Write an expression for the number of rock songs.

52. The number of women in a Statistics class is 8 more than twice the number of men. Let m represent the number of men. Write an expression for the number of women.

53. Greg has nickels and pennies in his pocket. The number of pennies is seven less than three the number of nickels. Let *n* represent the number of nickels. Write an expression for the number of pennies.

54. Jeannette has $\$5$ and $\$10$ bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Writing Exercises

55. Explain in your own words how to find the prime factorization of a composite number.

56. Why is it important to use the order of operations to simplify an expression?

57. Explain how you identify the like terms in the expression $8a^2 + 4a + 9 - a^2 - 1$.

58. Explain the difference between the phrases "4 times the sum of *x* and *y*" and "the sum of 4 times *x* and *y*".

Self Check

ⓐ *Use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
identify multiples and apply divisibility tests.			
find prime factorizations and least common multiples.			
use variables and algebraic symbols.			
simplify expressions using the order of operations.			
evaluate an expression.			
identify and combine like terms.			
translate English phrases to algebraic expressions.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

1.2 Integers

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with absolute value
- Add and subtract integers
- Multiply and divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate phrases to expressions with integers
- Use integers in applications

Be Prepared!

A more thorough introduction to the topics covered in this section can be found in the *Elementary Algebra* chapter, Foundations.

Simplify Expressions with Absolute Value

A **negative numbers** is a number less than 0. The negative numbers are to the left of zero on the number line. See Figure 1.2.

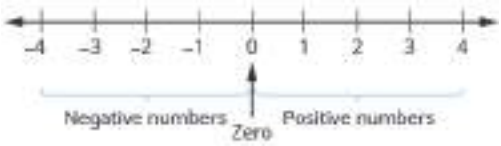


Figure 1.2 The number line shows the location of positive and negative numbers.

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, each one is called the **opposite** of the other. The opposite of 2 is -2, and the opposite of -2 is 2.

Opposite

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

Figure 1.3 illustrates the definition.

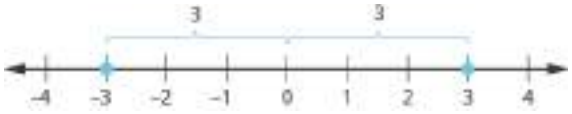


Figure 1.3 The opposite of 3 is -3.

Opposite Notation

$-a$ means the opposite of the number a

The notation $-a$ is read as "the opposite of a."

We saw that numbers such as 3 and -3 are opposites because they are the same distance from 0 on the number line. They are both three units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Absolute Value

The **absolute value** of a number is its distance from 0 on the number line.

The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

For example,

-5 is 5 units away from 0, so $|-5| = 5$.
5 is 5 units away from 0, so $|5| = 5$.

Figure 1.4 illustrates this idea.

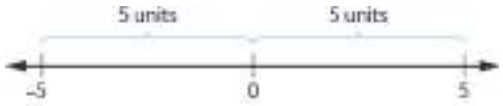


Figure 1.4 The numbers 5 and -5 are 5 units away from 0.

The absolute value of a number is never negative because distance cannot be negative. The only number with absolute value equal to zero is the number zero itself because the distance from 0 to 0 on the number line is zero units.

In the next example, we'll order expressions with absolute values.

EXAMPLE 1.12

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

ⓐ $|-5|__ -|-5|$ ⓑ $8__ -|-8|$ ⓒ $-9__ -|-9|$ ⓓ $-(-16)__|-16|$.

Solution

ⓐ

	$\|-5\| __ -\|-5\|$
Simplify.	$5 __ -5$
Order.	$5 > -5$
	$\|-5\| > -\|-5\|$

ⓑ

	$8 __ -\|-8\|$
Simplify.	$8 __ -8$
Order.	$8 > -8$
	$8 > -\|-8\|$

ⓒ

	$-9 __ -\|-9\|$
Simplify.	$-9 __ -9$
Order.	$-9 = -9$
	$-9 = -\|-9\|$

ⓓ

	$-(-16) __ \|-16\|$
Simplify.	$16 __ 16$
Order.	$16 = 16$
	$-(-16) = \|-16\|$

TRY IT : : 1.23 Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

ⓐ $-9__ -|-9|$ ⓑ $2__ -|-2|$ ⓒ $-8__|-8|$ ⓓ $-(-9)__|-9|$.

TRY IT : : 1.24 Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

ⓐ $7__ - |-7|$ ⓑ $-(-10)__|-10|$ ⓒ $|-4|__ - |-4|$ ⓓ $-1__|-1|$.

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Grouping Symbols

Parentheses	()	Braces	{ }
Brackets	[]	Absolute value	\| \|

In the next example, we simplify the expressions inside absolute value bars first just like we do with parentheses.

EXAMPLE 1.13

Simplify: $24 - |19 - 3(6 - 2)|$.

Solution

$$24 - |19 - 3(6 - 2)|$$

Work inside parentheses fir t:	
subtract 2 from 6.	$24 - \|19 - 3(4)\|$
Multiply 3(4).	$24 - \|19 - 12\|$
Subtract inside the absolute value bars.	$24 - \|7\|$
Take the absolute value.	$24 - 7$
Subtract.	17

TRY IT : : 1.25 Simplify: $19 - |11 - 4(3 - 1)|$.

TRY IT : : 1.26 Simplify: $9 - |8 - 4(7 - 5)|$.

Add and Subtract Integers

So far, we have only used the counting numbers and the whole numbers.

Counting numbers	1, 2, 3 …
Whole numbers	0, 1, 2, 3….

Our work with opposites gives us a way to define the **integers**. The whole numbers and their opposites are called the integers. The integers are the numbers … −3, −2, −1, 0, 1, 2, 3…

Integers

The whole numbers and their opposites are called the **integers**.

The integers are the numbers

$$\ldots -3, -2, -1, 0, 1, 2, 3\ldots,$$

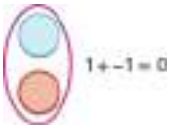
Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more challenging.

We will use two color counters to model addition and subtraction of negatives so that you can visualize the procedures instead of memorizing the rules.

We let one color (blue) represent positive. The other color (red) will represent the negatives.



If we have one positive counter and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero.



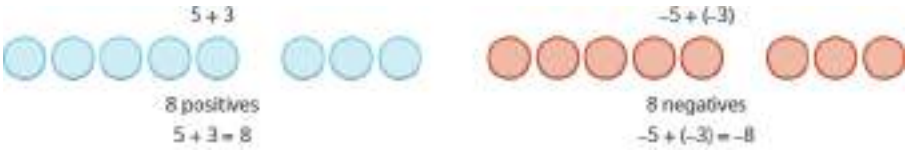
We will use the counters to show how to add:

$$5+3 \qquad -5+(-3) \qquad -5+3 \qquad 5+(-3)$$

The first example, $5+3$, adds 5 positives and 3 positives—both positives.

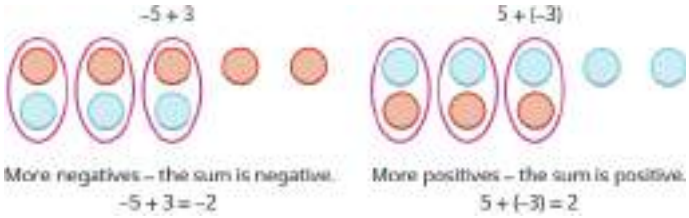
The second example, $-5+(-3)$, adds 5 negatives and 3 negatives—both negatives.

When the signs are the same, the counters are all the same color, and so we add them. In each case we get 8—either 8 positives or 8 negatives.



So what happens when the signs are different? Let's add $-5+3$ and $5+(-3)$.

When we use counters to model addition of positive and negative integers, it is easy to see whether there are more positive or more negative counters. So we know whether the sum will be positive or negative.



EXAMPLE 1.14

Add: ⓐ $-1+(-4)$ ⓑ $-1+5$ ⓒ $1+(-5)$.

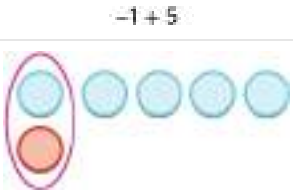
Solution

ⓐ



1 negative plus 4 negatives is 5 negatives −5

ⓑ



There are more positives, so the sum is positive. 4

ⓒ

	1 + (−5) 
There are more negatives, so the sum is negative.	−4

TRY IT : : 1.27 Add: ⓐ $-2+(-4)$ ⓑ $-2+4$ ⓒ $2+(-4)$.

TRY IT : : 1.28 Add: ⓐ $-2+(-5)$ ⓑ $-2+5$ ⓒ $2+(-5)$.

We will continue to use counters to model the subtraction. Perhaps when you were younger, you read "$5-3$" as "5 take away 3." When you use counters, you can think of subtraction the same way!

We will use the counters to show to subtract:

$$5-3 \qquad -5-(-3) \qquad -5-3 \qquad 5-(-3)$$

The first example, $5-3$, we subtract 3 positives from 5 positives and end up with 2 positives.

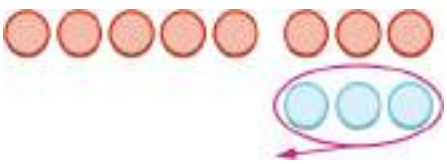
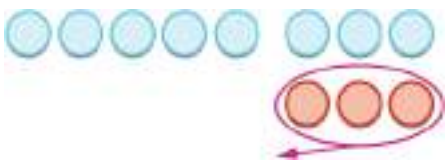
In the second example, $-5-(-3)$, we subtract 3 negatives from 5 negatives and end up with 2 negatives.

Each example used counters of only one color, and the "take away" model of subtraction was easy to apply.



What happens when we have to subtract one positive and one negative number? We'll need to use both blue and red counters as well as some neutral pairs. If we don't have the number of counters needed to take away, we add neutral pairs. Adding a neutral pair does not change the value. It is like changing quarters to nickels—the value is the same, but it looks different.

Let's look at $-5-3$ and $5-(-3)$.

	−5 − 3	5 − (−3)
Model the first number.		
We now add the needed neutral pairs.		
We remove the number of counters modeled by the second number.		

Count what is left.		
	$-5-3=-8$	$5-(-3)=-8$
	-8	8

EXAMPLE 1.15

Subtract: ⓐ $3-1$ ⓑ $-3-(-1)$ ⓒ $-3-1$ ⓓ $3-(-1)$.

Solution

ⓐ



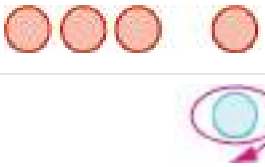
Take 1 positive from 3 positives and get 2 positives.	2

ⓑ



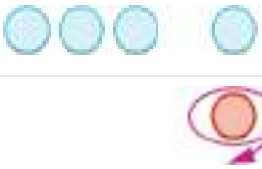
Take 1 positive from 3 negatives and get 2 negatives.	-2

ⓒ



Take 1 positive from the one added neutral pair.

ⓓ



Take 1 negative from the one added neutral pair.

> **TRY IT : :** 1.29 Subtract: ⓐ $6-4$ ⓑ $-6-(-4)$ ⓒ $-6-4$ ⓓ $6-(-4)$.

> **TRY IT : :** 1.30 Subtract: ⓐ $7-4$ ⓑ $-7-(-4)$ ⓒ $-7-4$ ⓓ $7-(-4)$.

Have you noticed that *subtraction of signed numbers can be done by adding the opposite?* In the last example, $-3-1$ is the same as $-3+(-1)$ and $3-(-1)$ is the same as $3+1$. You will often see this idea, the Subtraction Property, written as follows:

Subtraction Property

$$a-b=a+(-b)$$

Subtracting a number is the same as adding its opposite.

EXAMPLE 1.16

Simplify: ⓐ $13-8$ and $13+(-8)$ ⓑ $-17-9$ and $-17+(-9)$ ⓒ $9-(-15)$ and $9+15$ ⓓ $-7-(-4)$ and $-7+4$.

Solution

ⓐ

	$13-8$	and	$13+(-8)$
Subtract.	5		5

ⓑ

	$-17-9$	and	$-17+(-9)$
Subtract.	-26		-26

ⓒ

	$9-(-15)$	and	$9+15$
Subtract.	24		24

ⓓ

	$-7-(-4)$	and	$-7+4$
Subtract.	-3		-3

TRY IT : : 1.31

Simplify: ⓐ $21-13$ and $21+(-13)$ ⓑ $-11-7$ and $-11+(-7)$ ⓒ $6-(-13)$ and $6+13$ ⓓ $-5-(-1)$ and $-5+1$.

TRY IT : : 1.32

Simplify: ⓐ $15-7$ and $15+(-7)$ ⓑ $-14-8$ and $-14+(-8)$ ⓒ $4-(-19)$ and $4+19$ ⓓ $-4-(-7)$ and $-4+7$.

What happens when there are more than three integers? We just use the order of operations as usual.

EXAMPLE 1.17

Simplify: $7-(-4-3)-9$.

Solution

	$7-(-4-3)-9$
Simplify inside the parentheses fir t.	$7-(-7)-9$
Subtract left to right.	$14-9$
Subtract.	5

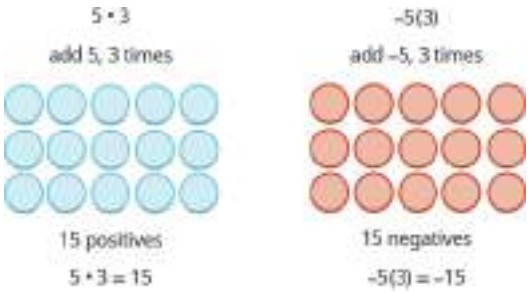
TRY IT : : 1.33 Simplify: $8-(-3-1)-9$.

TRY IT : : 1.34 Simplify: $12 - (-9 - 6) - 14$.

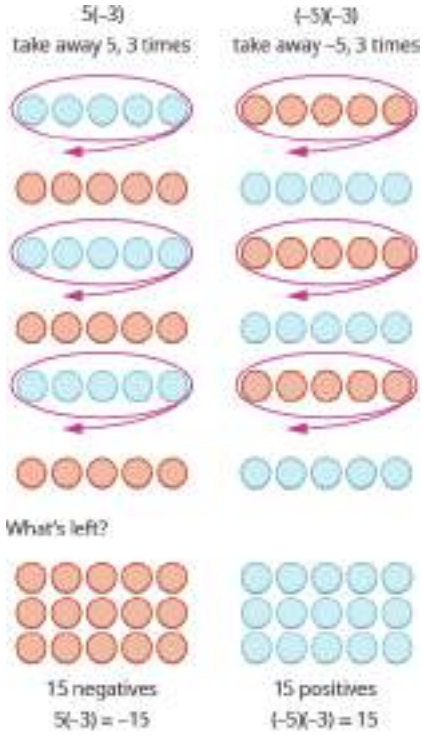
Multiply and Divide Integers

Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show multiplication of integers. Let's look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction. Here, we are using the model just to help us discover the pattern.

We remember that $a \cdot b$ means **add** a, b **times**.



The next two examples are more interesting. What does it mean to multiply 5 by -3? It means subtract 5, 3 times. Looking at subtraction as "taking away", it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs on the workspace.



In summary:

$$5 \cdot 3 = 15 \qquad -5(3) = -15$$
$$5(-3) = -15 \qquad (-5)(-3) = 15$$

Notice that for multiplication of two signed numbers, when the

signs are the **same**, the product is **positive**.
signs are **diffe ent**, the product is **negative**.

What about division? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $15 \cdot 3 = 15$. In words, this expression says that 15 can be divided into 3 groups of 5 each because adding five three times gives 15. If you look at some examples of multiplying integers, you might figure out the rules for dividing integers.

$5 \cdot 3 = 15$ so $15 \div 3 = 5$ $\quad -5(3) = -15$ so $-15 \div 3 = -5$
$(-5)(-3) = 15$ so $15 \div (-3) = -5$ $\quad 5(-3) = -15$ so $-15 \div (-3) = 5$

Division follows the same rules as multiplication with regard to signs.

Multiplication and Division of Signed Numbers

For multiplication and division of two signed numbers:

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

If the signs are the same, the result is positive.

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

If the signs are different, the result is negative.

EXAMPLE 1.18

Multiply or divide: ⓐ $-100 \div (-4)$ ⓑ $7 \cdot 6$ ⓒ $4(-8)$ ⓓ $-27 \div 3$.

Solution

ⓐ

$-100 \div (-4)$

Divide, with signs that are the same the quotient is positive. $\quad 25$

ⓑ

$7 \cdot 6$

Multiply, with same signs. $\quad 42$

ⓒ

$4(-8)$

Multiply, with diffe ent signs. $\quad -32$

ⓓ

$-27 \div 3$

Divide, with diffe ent signs, the quotient is negative. $\quad -9$

TRY IT : : 1.35 Multiply or divide: ⓐ $-115 \div (-5)$ ⓑ $5 \cdot 12$ ⓒ $9(-7)$ ⓓ $-63 \div 7$.

TRY IT : : 1.36 Multiply or divide: ⓐ $-117 \div (-3)$ ⓑ $3 \cdot 13$ ⓒ $7(-4)$ ⓓ $-42 \div 6$.

When we multiply a number by 1, the result is the same number. Each time we multiply a number by -1, we get its opposite!

Multiplication by -1

$$-1a = -a$$

Multiplying a number by -1 gives its opposite.

Simplify Expressions with Integers

What happens when there are more than two numbers in an expression? The order of operations still applies when negatives are included. Remember Please Excuse My Dear Aunt Sally?

Let's try some examples. We'll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

EXAMPLE 1.19

Simplify: ⓐ $(-2)^4$ ⓑ -2^4.

Solution

Notice the difference in parts (a) and (b). In part (a), the exponent means to raise what is in the parentheses, the -2 to the 4th power. In part (b), the exponent means to raise just the 2 to the 4th power and then take the opposite.

ⓐ

	$(-2)^4$
Write in expanded form.	$(-2)(-2)(-2)(-2)$
Multiply.	$4(-2)(-2)$
Multiply.	$-8(-2)$
Multiply.	16

ⓑ

	-2^4
Write in expanded form.	$-(2 \cdot 2 \cdot 2 \cdot 2)$
We are asked to fin the opposite of 2^4.	
Multiply.	$-(4 \cdot 2 \cdot 2)$
Multiply.	$-(8 \cdot 2)$
Multiply.	-16

TRY IT : : 1.37 Simplify: ⓐ $(-3)^4$ ⓑ -3^4.

TRY IT : : 1.38 Simplify: ⓐ $(-7)^2$ ⓑ -7^2.

The last example showed us the difference between $(-2)^4$ and -2^4. This distinction is important to prevent future errors. The next example reminds us to multiply and divide in order left to right.

EXAMPLE 1.20

Simplify: ⓐ $8(-9) \div (-2)^3$ ⓑ $-30 \div 2 + (-3)(-7)$.

Solution

ⓐ

	$8(-9) \div (-2)^3$
Exponents fir t.	$8(-9) \div (-8)$
Multiply.	$-72 \div (-8)$
Divide.	9

ⓑ

	$-30 \div 2 + (-3)(-7)$
Multiply and divide left to right, so divide fir t.	$-15 + (-3)(-7)$
Multiply.	$-15 + 21$
Add.	6

TRY IT : : 1.39 Simplify: ⓐ $12(-9) \div (-3)^3$ ⓑ $-27 \div 3 + (-5)(-6)$.

TRY IT : : 1.40 Simplify: ⓐ $18(-4) \div (-2)^3$ ⓑ $-32 \div 4 + (-2)(-7)$.

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers.

EXAMPLE 1.21

Evaluate $4x^2 - 2xy + 3y^2$ when $x = 2,\ y = -1$.

Solution

	$4x^2 - 2xy + 3y^2$
Substitute $x = 2, y = -1$. Use parentheses to show multiplication.	$4(2)^2 - 2(2)(-1) + 3(-1)^2$
Simplify exponents.	$4 \cdot 4 - (-4) + 3 \cdot 1$
Multiply.	$16 - (-4) + 3$
Subtract.	$20 + 3$
Add.	23

TRY IT : : 1.41 Evaluate: $3x^2 - 2xy + 6y^2$ when $x = 1,\ y = -2$.

TRY IT : : 1.42 Evaluate: $4x^2 - xy + 5y^2$ when $x = -2,\ y = 3$.

Translate Phrases to Expressions with Integers

Our earlier work translating English to algebra also applies to phrases that include both positive and negative numbers.

EXAMPLE 1.22

Translate and simplify: the sum of 8 and -12, increased by 3.

Solution

	the **sum** of 8 and -12 increased by 3
Translate.	$[8+(-12)]+3$
Simplify. Be careful not to confuse the brackets with an absolute value sign.	$(-4)+3$
Add.	-1

TRY IT : : 1.43 Translate and simplify the sum of 9 and -16, increased by 4.

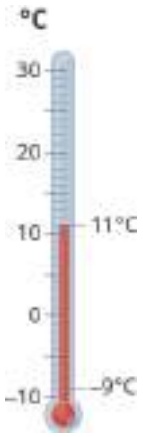
TRY IT : : 1.44 Translate and simplify the sum of -8 and -12, increased by 7.

Use Integers in Applications

We'll outline a plan to solve applications. It's hard to find something if we don't know what we're looking for or what to call it! So when we solve an application, we first need to determine what the problem is asking us to find. Then we'll write a phrase that gives the information to find it. We'll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

EXAMPLE 1.23 HOW TO SOLVE APPLICATION PROBLEMS USING INTEGERS

The temperature in Kendallville, Indiana one morning was 11 degrees. By mid-afternoon, the temperature had dropped to -9 degrees. What was the difference in the morning and afternoon temperatures?



Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	The difference of the morning and afternoon temperatures
Step 3. Write a phrase that gives the information to find it.	the difference *of* 11 *and* −9
Step 4. Translate the phrase to an expression.	11 − (−9)
Step 5. Simplify the expression.	20
Step 6. Answer the question with a complete sentence.	The difference in temperatures was 20 degrees.

TRY IT : : 1.45

The temperature in Anchorage, Alaska one morning was 15 degrees. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference in the morning and afternoon temperatures?

TRY IT : : 1.46

The temperature in Denver was -6 degrees at lunchtime. By sunset the temperature had dropped to -15 degrees. What was the difference in the lunchtime and sunset temperatures?



HOW TO : : USE INTEGERS IN APPLICATIONS.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what we are asked to find.
Step 3. **Write a phrase** that gives the information to find it.
Step 4. **Translate** the phrase to an expression.
Step 5. **Simplify** the expression.
Step 6. **Answer** the question with a complete sentence.

MEDIA : :

Access this online resource for additional instruction and practice with integers.

- **Subtracting Integers with Counters (https://openstax.org/l/25subintegers)**

1.2 EXERCISES

Practice Makes Perfect

Simplify Expressions with Absolute Value

In the following exercises, fill in $<$, $>$, or $=$ for each of the following pairs of numbers.

59.
ⓐ $|-7|___ -|-7|$
ⓑ $6___ -|-6|$
ⓒ $|-11|___ -11$
ⓓ $-(-13)___ -|-13|$

60.
ⓐ $-|-9|___|-9|$
ⓑ $-8___|-8|$
ⓒ $|-1|___ -1$
ⓓ $-(-14)___ -|-14|$

61.
ⓐ $-|2|___ -|-2|$
ⓑ $-12___ -|-12|$
ⓒ $|-3|___ -3$
ⓓ $|-19|___ -(-19)$

62.
ⓐ $-|-4|___ -|4|$
ⓑ $5___ -|-5|$
ⓒ $-|-10|___ -10$
ⓓ $-|-0|___ -(-0)$

In the following exercises, simplify.

63. $|15-7|-|14-6|$

64. $|17-8|-|13-4|$

65. $18-|2(8-3)|$

66. $15-|3(8-5)|$

67. $18-|12-4(4-1)+3|$

68. $27-|19+4(3-1)-7|$

69. $10-3|9-3(3-1)|$

70. $13-2|11-2(5-2)|$

Add and Subtract Integers

In the following exercises, simplify each expression.

71.
ⓐ $-7+(-4)$
ⓑ $-7+4$
ⓒ $7+(-4).$

72.
ⓐ $-5+(-9)$
ⓑ $-5+9$
ⓒ $5+(-9)$

73. $48+(-16)$

74. $34+(-19)$

75. $-14+(-12)+4$

76. $-17+(-18)+6$

77. $19+2(-3+8)$

78. $24+3(-5+9)$

79.
ⓐ $13-7$
ⓑ $-13-(-7)$
ⓒ $-13-7$
ⓓ $13-(-7)$

80.
ⓐ $15 - 8$
ⓑ $-15 - (-8)$
ⓒ $-15 - 8$
ⓓ $15 - (-8)$

81. $-17 - 42$

82. $-58 - (-67)$

83. $-14 - (-27) + 9$

84. $64 + (-17) - 9$

85. ⓐ $44 - 28$ ⓑ $44 + (-28)$

86. ⓐ $35 - 16$ ⓑ $35 + (-16)$

87. ⓐ $27 - (-18)$ ⓑ $27 + 18$

88. ⓐ $46 - (-37)$ ⓑ $46 + 37$

89. $(2 - 7) - (3 - 8)$

90. $(1 - 8) - (2 - 9)$

91. $-(6 - 8) - (2 - 4)$

92. $-(4 - 5) - (7 - 8)$

93. $25 - [10 - (3 - 12)]$

94. $32 - [5 - (15 - 20)]$

Multiply and Divide Integers

In the following exercises, multiply or divide.

95.
ⓐ $-4 \cdot 8$
ⓑ $13(-5)$
ⓒ $-24 \div 6$
ⓓ $-52 \div (-4)$

96.
ⓐ $-3 \cdot 9$
ⓑ $9(-7)$
ⓒ $35 \div (-7)$
ⓓ $-84 \div (-6)$

97.
ⓐ $-28 \div 7$
ⓑ $-180 \div 15$
ⓒ $3(-13)$
ⓓ $-1(-14)$

98.
ⓐ $-36 \div 4$
ⓑ $-192 \div 12$
ⓒ $9(-7)$
ⓓ $-1(-19)$

Simplify and Evaluate Expressions with Integers

In the following exercises, simplify each expression.

99. ⓐ $(-2)^6$ ⓑ -2^6

100. ⓐ $(-3)^5$ ⓑ -3^5

101. $5(-6) + 7(-2) - 3$

102. $8(-4) + 5(-4) - 6$

103. $-3(-5)(6)$

104. $-4(-6)(3)$

105. $(8 - 11)(9 - 12)$

106. $(6 - 11)(8 - 13)$

107. $26 - 3(2 - 7)$

108. $23 - 2(4 - 6)$

109. $65 \div (-5) + (-28) \div (-7)$

110. $52 \div (-4) + (-32) \div (-8)$

111. $9 - 2[3 - 8(-2)]$

112. $11 - 3[7 - 4(-2)]$

113. $8 - |2 - 4(4 - 1) + 3|$

114. $7 - |5 - 3(4 - 1) - 6|$

115. $9 - 3|2(2 - 6) - (3 - 7)|$

116. $5 - 2|2(1 - 4) - (2 - 5)|$

117. $(-3)^2 - 24 \div (8 - 2)$

118. $(-4)^2 - 32 \div (12 - 4)$

In the following exercises, evaluate each expression.

119. $y + (-14)$ when
ⓐ $y = -33$ ⓑ $y = 30$

120. $x + (-21)$ when
ⓐ $x = -27$ ⓑ $x = 44$

121. $(x + y)^2$ when
$x = -3,\ y = 14$

122. $(y + z)^2$ when
$y = -3,\ z = 15$

123. $9a - 2b - 8$ when
$a = -6$ and $b = -3$

124. $7m - 4n - 2$ when
$m = -4$ and $n = -9$

125. $3x^2 - 4xy + 2y^2$ when
$x = -2,\ y = -3$

126. $4x^2 - xy + 3y^2$ when
$x = -3,\ y = -2$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

127. the sum of 3 and -15, increased by 7

128. the sum of -8 and -9, increased by 23

129.
ⓐ the difference of 10 and -18
ⓑ subtract 11 from -25

130.
ⓐ the difference of -5 and -30
ⓑ subtract -6 from -13

131. the quotient of -6 and the sum of a and b

132. the product of -13 and the difference of c and d

Use Integers in Applications

In the following exercises, solve.

133. Temperature On January 15, the high temperature in Anaheim, California, was 84°. That same day, the high temperature in Embarrass, Minnesota, was $-12°$. What was the difference between the temperature in Anaheim and the temperature in Embarrass?

134. Temperature On January 21, the high temperature in Palm Springs, California, was $89°$, and the high temperature in Whitefield, New Hampshire, was $-31°$. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?

135. Football On the first down, the Chargers had the ball on their 25-yard line. On the next three downs, they lost 6 yards, gained 10 yards, and lost 8 yards. What was the yard line at the end of the fourth down?

136. Football On the first down, the Steelers had the ball on their 30-yard line. On the next three downs, they gained 9 yards, lost 14 yards, and lost 2 yards. What was the yard line at the end of the fourth down?

137. Checking Account Mayra has $124 in her checking account. She writes a check for $152. What is the new balance in her checking account?

138. Checking Account Reymonte has a balance of $-\$49$ in his checking account. He deposits $281 to the account. What is the new balance?

Writing Exercises

139. Explain why the sum of -8 and 2 is negative, but the sum of 8 and -2 is positive.

140. Give an example from your life experience of adding two negative numbers.

141. In your own words, state the rules for multiplying and dividing integers.

142. Why is $-4^3 = (-4)^3$?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with absolute value.			
add and subtract integers.			
multiply and divide integers.			
simplify and evaluate expressions with integers.			
translate English phrases to algebraic expressions.			
use integers in applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

1.3 Fractions

Learning Objectives

By the end of this section, you will be able to:

- Simplify fractions
- Multiply and divide fractions
- Add and subtract fractions
- Use the order of operations to simplify fractions
- Evaluate variable expressions with fractions

Be Prepared!

A more thorough introduction to the topics covered in this section can be found in the *Elementary Algebra* chapter, Foundations.

Simplify Fractions

A **fraction** is a way to represent parts of a whole. The fraction $\frac{2}{3}$ represents two of three equal parts. See Figure 1.5. In the fraction $\frac{2}{3}$, the 2 is called the **numerator** and the 3 is called the **denominator**. The line is called the fraction bar.

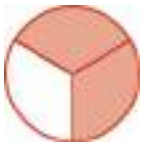


Figure 1.5 In the circle, $\frac{2}{3}$ of the circle is shaded—2 of the 3 equal parts.

Fraction

A **fraction** is written $\frac{a}{b}$, where $b \neq 0$ and

a is the **numerator** and *b* is the **denominator**.

A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

Fractions that have the same value are **equivalent fractions**. The Equivalent Fractions Property allows us to find equivalent fractions and also simplify fractions.

Equivalent Fractions Property

If *a*, *b*, and *c* are numbers where $b \neq 0$, $c \neq 0$,

then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator.

For example,

$\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.

$\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors using the Equivalent Fractions Property.

EXAMPLE 1.24 HOW TO SIMPLIFY A FRACTION

Simplify: $-\frac{315}{770}$.

Solution

Step 1. Rewrite the numerator and denominator to show the common factors. If needed, use a factor tree.	Rewrite 315 and 770 as the product of the primes.	$-\frac{315}{770}$ $-\frac{3\cdot 3\cdot 5\cdot 7}{2\cdot 5\cdot 7\cdot 11}$
Step 2. Simplify using the equivalent fractions property by dividing out common factors.	Mark the common factors 5 and 7. Divide out the common factors	$-\frac{3\cdot 3\cdot \cancel{5}\cdot \cancel{7}}{2\cdot \cancel{5}\cdot \cancel{7}\cdot 11}$ $-\frac{3\cdot 3}{2\cdot 11}$
Step 3. Multiply the remaining factors, if necessary.		$-\frac{9}{22}$

TRY IT : : 1.47 Simplify: $-\frac{69}{120}$.

TRY IT : : 1.48 Simplify: $-\frac{120}{192}$.

We now summarize the steps you should follow to simplify fractions.

HOW TO : : SIMPLIFY A FRACTION.

Step 1. Rewrite the numerator and denominator to show the common factors.
If needed, factor the numerator and denominator into prime numbers first.

Step 2. Simplify using the Equivalent Fractions Property by dividing out common factors.

Step 3. Multiply any remaining factors.

Multiply and Divide Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions.

To multiply fractions, we multiply the numerators and multiply the denominators.

Fraction Multiplication

If *a*, *b*, *c*, and *d* are numbers where $b \neq 0$, and $d \neq 0$, then

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step. In **Example 1.25**, we will multiply negative and a positive, so the product will be negative.

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, *a*, can be written as $\frac{a}{1}$. So, for example, $3=\frac{3}{1}$.

EXAMPLE 1.25

Multiply: $-\frac{12}{5}(-20x)$.

Solution

The first step is to find the sign of the product. Since the signs are the same, the product is positive.

	$-\frac{12}{5}(-20x)$
Determine the sign of the product. The signs are the same, so the product is positive.	$\frac{12}{5}(20x)$
Write $20x$ as a fraction.	$\frac{12}{5}\left(\frac{20x}{1}\right)$
Multiply.	$\frac{12 \cdot 20x}{5 \cdot 1}$
Rewrite 20 to show the common factor 5 and divide it out.	$\frac{12 \cdot 4 \cdot 5 \cdot x}{5 \cdot 1}$
Simplify.	$48x$

TRY IT : : 1.49 Multiply: $\frac{11}{3}(-9a)$.

TRY IT : : 1.50 Multiply: $\frac{13}{7}(-14b)$.

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, we need some vocabulary. The **reciprocal** of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. Since 4 is written in fraction form as $\frac{4}{1}$, the reciprocal of 4 is $\frac{1}{4}$.

To divide fractions, we multiply the first fraction by the reciprocal of the second.

Fraction Division

If *a*, *b*, *c*, and *d* are numbers where $b \neq 0$, $c \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

To divide fractions, we multiply the first fraction by the **reciprocal** of the second.

We need to say $b \neq 0$, $c \neq 0$, and $d \neq 0$, to be sure we don't divide by zero!

EXAMPLE 1.26

Find the quotient: $-\frac{7}{8} \div \left(-\frac{14}{27}\right)$.

Solution

	$-\frac{7}{8}\div\left(-\frac{14}{27}\right)$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{7}{18}\left(-\frac{27}{14}\right)$
Determine the sign of the product, and then multiply.	$\frac{7\cdot 27}{18\cdot 14}$
Rewrite showing common factors.	$\frac{7\cdot 9\cdot 3}{9\cdot 2\cdot 7\cdot 2}$
Remove common factors.	$\frac{3}{2\cdot 2}$
Simplify.	$\frac{3}{4}$

TRY IT : : 1.51 Divide: $-\frac{7}{27}\div\left(-\frac{35}{36}\right)$.

TRY IT : : 1.52 Divide: $-\frac{5}{14}\div\left(-\frac{15}{28}\right)$.

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

Complex Fraction

A **complex fraction** is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:

$$\frac{\frac{6}{7}}{3} \qquad \frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{5}{6}}$$

To simplify a complex fraction, remember that the fraction bar means division. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4}\div\frac{5}{8}$.

EXAMPLE 1.27

Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$.

Solution

$$\frac{\frac{x}{2}}{\frac{xy}{6}}$$

Rewrite as division. $\frac{x}{2} \div \frac{xy}{6}$

Multiply the fir t fraction by the reciprocal of the second. $\frac{x}{2} \cdot \frac{6}{xy}$

Multiply. $\frac{x \cdot 6}{2 \cdot xy}$

Look for common factors. $\frac{x \cdot 3 \cdot 2}{2 \cdot x \cdot y}$

Divide common factors and simplify. $\frac{3}{y}$

TRY IT : : 1.53 Simplify: $\frac{\frac{a}{8}}{\frac{ab}{6}}$.

TRY IT : : 1.54 Simplify: $\frac{\frac{p}{2}}{\frac{pq}{8}}$.

Add and Subtract Fractions

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

Fraction Addition and Subtraction

If *a*, *b*, and *c* are numbers where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \text{ and } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

Least Common Denominator

The **least common denominator** (LCD) of two fractions is the least common multiple (LCM) of their denominators.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

EXAMPLE 1.28 HOW TO ADD OR SUBTRACT FRACTIONS

Add: $\frac{7}{12} + \frac{5}{18}$.

Solution

Step 1. Do they have a common denominator? • No—rewrite each fraction with the LCD (least common denominator).	No. Find the LCD of 12, 18.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $LCD = 2 \cdot 2 \cdot 3 \cdot 3$ $LCD = 36$ LCD is 36.
	We multiply the numerator and denominator of each fraction by the factor needed to get the denominator to be 36. Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!	$\frac{7}{12} + \frac{5}{18}$ $\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$ $\frac{21}{36} + \frac{10}{36}$
Step 2. Add or subtract the fractions.	Add.	$\frac{31}{36}$
Step 3. Simplify, if possible.	Since 31 is prime, its only factors are 1 and 31. Since 31 does not go into 36, the answer is simplified.	

TRY IT : : 1.55 Add: $\frac{7}{12} + \frac{11}{15}$.

TRY IT : : 1.56 Add: $\frac{13}{15} + \frac{17}{20}$.

HOW TO : : ADD OR SUBTRACT FRACTIONS.

Step 1. Do they have a common denominator?
- Yes—go to step 2.
- No—rewrite each fraction with the LCD (least common denominator).
 - Find the LCD.
 - Change each fraction into an equivalent fraction with the LCD as its denominator.

Step 2. Add or subtract the fractions.

Step 3. Simplify, if possible.

We now have all four operations for fractions. Table 1.18 summarizes fraction operations.

Fraction Multiplication	Fraction Division
$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$
Multiply the numerators and multiply the denominators	Multiply the first fraction by the reciprocal of the second.
Fraction Addition	**Fraction Subtraction**
$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
Add the numerators and place the sum over the common denominator.	Subtract the numerators and place the difference over the common denominator.
To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.	

Table 1.18

When starting an exercise, always identify the operation and then recall the methods needed for that operation.

EXAMPLE 1.29

Simplify: ⓐ $\frac{5x}{6} - \frac{3}{10}$ ⓑ $\frac{5x}{6} \cdot \frac{3}{10}$.

Solution

First ask, "What is the operation?" Identifying the operation will determine whether or not we need a common denominator. Remember, we need a common denominator to add or subtract, but not to multiply or divide.

ⓐ

What is the operation? The operation is subtraction.

Do the fractions have a common denominator? No. $\frac{5x}{6} - \frac{3}{10}$

Find the LCD of 6 and 10 The LCD is 30.

$$6 = 2 \cdot 3$$
$$10 = 2 \cdot 5$$
$$\text{LCD} = 2 \cdot 3 \cdot 5$$
$$\text{LCD} = 30$$

Rewrite each fraction as an equivalent fraction with the LCD. $\frac{5x \cdot 5}{6 \cdot 5} - \frac{3 \cdot 3}{10 \cdot 3}$

$\frac{25x}{30} - \frac{9}{30}$

Subtract the numerators and place the difference over the common denominators. $\frac{25x - 9}{30}$

Simplify, if possible. There are no common factors.

The fraction is simplified.

ⓑ

What is the operation? Multiplication.	$\frac{25x}{6} \cdot \frac{3}{10}$
To multiply fractions, multiply the numerators and multiply the denominators.	$\frac{25x \cdot 3}{6 \cdot 10}$
Rewrite, showing common factors.	$\frac{5 \cdot 5x \cdot 3}{2 \cdot 3 \cdot 2 \cdot 5 \cdot 3}$
Remove common factors.	
Simplify.	$\frac{x}{4}$

Notice, we needed an LCD to add $\frac{25x}{6} - \frac{3}{10}$, but not to multiply $\frac{25x}{6} \cdot \frac{3}{10}$.

TRY IT : : 1.57 Simplify: ⓐ $\frac{3a}{4} - \frac{8}{9}$ ⓑ $\frac{3a}{4} \cdot \frac{8}{9}$.

TRY IT : : 1.58 Simplify: ⓐ $\frac{4k}{5} - \frac{1}{6}$ ⓑ $\frac{4k}{5} \cdot \frac{1}{6}$.

Use the Order of Operations to Simplify Fractions

The fraction bar in a fraction acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

HOW TO : : SIMPLIFY AN EXPRESSION WITH A FRACTION BAR.

Step 1. Simplify the expression in the numerator. Simplify the expression in the denominator.

Step 2. Simplify the fraction.

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.

$$\frac{-1}{3} = -\frac{1}{3} \qquad \frac{\text{negative}}{\text{positive}} = \text{negative}$$

$$\frac{1}{-3} = -\frac{1}{3} \qquad \frac{\text{positive}}{\text{negative}} = \text{negative}$$

Placement of Negative Sign in a Fraction

For any positive numbers a and b,

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

EXAMPLE 1.30

Simplify: $\frac{4(-3) + 6(-2)}{-3(2) - 2}$.

Solution

The fraction bar acts like a grouping symbol. So completely simplify the numerator and the denominator separately.

	$\frac{4(-3)+6(-2)}{-3(2)-2}$
Multiply.	$\frac{-12+(-12)}{-6-2}$
Simplify.	$\frac{-24}{-8}$
Divide.	3

TRY IT : : 1.59 Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$.

TRY IT : : 1.60 Simplify: $\frac{7(-1)+9(-3)}{-5(3)-2}$.

Now we'll look at complex fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator as the fraction bar means division.

EXAMPLE 1.31 HOW TO SIMPLIFY COMPLEX FRACTIONS

Simplify: $\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$.

Solution

Step 1. Simplify the numerator.	$\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$ $\frac{\frac{1}{4}}{4+3^2}$
Step 2. Simplify the denominator.	$\frac{\frac{1}{4}}{4+9}$ $\frac{\frac{1}{4}}{13}$
Step 3. Divide the numerator by the denominator. Simplify if possible.	$\frac{1}{4} \div \frac{13}{1}$ $\frac{1}{4} \cdot \frac{1}{13}$ $\frac{1}{52}$

TRY IT : : 1.61 Simplify: $\frac{\left(\frac{1}{3}\right)^2}{2^3+2}$.

TRY IT : : 1.62 Simplify: $\frac{1+4^2}{\left(\frac{1}{4}\right)^2}$.

HOW TO : : SIMPLIFY COMPLEX FRACTIONS.

Step 1. Simplify the numerator.
Step 2. Simplify the denominator.
Step 3. Divide the numerator by the denominator. Simplify if possible.

EXAMPLE 1.32

Simplify: $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$.

Solution

It may help to put parentheses around the numerator and the denominator.

	$\dfrac{\left(\frac{1}{2}+\frac{2}{3}\right)}{\left(\frac{3}{4}-\frac{1}{6}\right)}$
Simplify the numerator (LCD = 6) and simplify the denominator (LCD = 12).	$\dfrac{\left(\frac{3}{6}+\frac{4}{6}\right)}{\left(\frac{9}{12}-\frac{2}{12}\right)}$
Simplify.	$\dfrac{\left(\frac{7}{6}\right)}{\left(\frac{7}{12}\right)}$
Divide the numerator by the denominator.	$\frac{7}{6} \div \frac{7}{12}$
Simplify.	$\frac{7}{6} \cdot \frac{12}{7}$
Divide out common factors.	$\dfrac{\cancel{7} \cdot \cancel{6} \cdot 2}{\cancel{6} \cdot \cancel{7} \cdot 1}$
Simplify.	2

TRY IT : : 1.63 Simplify: $\dfrac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$.

TRY IT : : 1.64 Simplify: $\dfrac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$.

Evaluate Variable Expressions with Fractions

We have evaluated expressions before, but now we can evaluate expressions with fractions. Remember, to evaluate an expression, we substitute the value of the variable into the expression and then simplify.

EXAMPLE 1.33

Evaluate $2x^2y$ when $x=\frac{1}{4}$ and $y=-\frac{2}{3}$.

Solution

Substitute the values into the expression.

	$2x^2y$
Substitute $\frac{1}{4}$ for x and $-\frac{2}{3}$ for y.	$2\left(\frac{1}{4}\right)^2\left(-\frac{2}{3}\right)$
Simplify exponents first.	$2\left(\frac{1}{16}\right)\left(-\frac{2}{3}\right)$
Multiply; divide out the common factors. Notice we write 16 as $2 \cdot 2 \cdot 4$ to make it easy to remove common factors.	$\frac{2 \cdot 1 \cdot 2}{2 \cdot 2 \cdot 4 \cdot 3}$
Simplify.	$-\frac{1}{12}$

TRY IT : : 1.65 Evaluate $3ab^2$ when $a = -\frac{2}{3}$ and $b = -\frac{1}{2}$.

TRY IT : : 1.66 Evaluate $4c^3d$ when $c = -\frac{1}{2}$ and $d = -\frac{4}{3}$.

MEDIA : :

Access this online resource for additional instruction and practice with fractions.

- **Adding Fractions with Unlike Denominators (https://openstax.org/l/25addfractions)**

1.3 EXERCISES

Practice Makes Perfect

Simplify Fractions

In the following exercises, simplify.

143. $-\frac{108}{63}$

144. $-\frac{104}{48}$

145. $\frac{120}{252}$

146. $\frac{182}{294}$

147. $\frac{14x^2}{21y}$

148. $\frac{24a}{32b^2}$

149. $-\frac{210a^2}{110b^2}$

150. $-\frac{30x^2}{105y^2}$

Multiply and Divide Fractions

In the following exercises, perform the indicated operation.

151. $-\frac{3}{4}\left(-\frac{4}{9}\right)$

152. $-\frac{3}{8}\cdot\frac{4}{15}$

153. $\left(-\frac{14}{15}\right)\left(\frac{9}{20}\right)$

154. $\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$

155. $\left(-\frac{63}{84}\right)\left(-\frac{44}{90}\right)$

156. $\left(-\frac{33}{60}\right)\left(-\frac{40}{88}\right)$

157. $\frac{3}{7}\cdot 21n$

158. $\frac{5}{6}\cdot 30m$

159. $\frac{3}{4}\div\frac{x}{11}$

160. $\frac{2}{5}\div\frac{y}{9}$

161. $\frac{5}{18}\div\left(-\frac{15}{24}\right)$

162. $\frac{7}{18}\div\left(-\frac{14}{27}\right)$

163. $\frac{8u}{15}\div\frac{12v}{25}$

164. $\frac{12r}{25}\div\frac{18s}{35}$

165. $\frac{3}{4}\div(-12)$

166. $-15\div\left(-\frac{5}{3}\right)$

In the following exercises, simplify.

167. $\frac{-\frac{8}{21}}{\frac{12}{35}}$

168. $\frac{-\frac{9}{16}}{\frac{33}{40}}$

169. $\frac{-\frac{4}{5}}{2}$

170. $\frac{\frac{5}{3}}{10}$

171. $\frac{\frac{m}{3}}{\frac{n}{2}}$

172. $\frac{-\frac{3}{8}}{-\frac{y}{12}}$

Add and Subtract Fractions

In the following exercises, add or subtract.

173. $\frac{7}{12}+\frac{5}{8}$

174. $\frac{5}{12}+\frac{3}{8}$

175. $\frac{7}{12}-\frac{9}{16}$

176. $\frac{7}{16}-\frac{5}{12}$

177. $-\frac{13}{30}+\frac{25}{42}$

178. $-\frac{23}{30}+\frac{5}{48}$

179. $-\frac{39}{56}-\frac{22}{35}$

180. $-\frac{33}{49}-\frac{18}{35}$

181. $-\frac{2}{3}-\left(-\frac{3}{4}\right)$

182. $-\frac{3}{4}-\left(-\frac{4}{5}\right)$

183. $\frac{x}{3}+\frac{1}{4}$

184. $\frac{x}{5}-\frac{1}{4}$

185.
ⓐ $\frac{2}{3}+\frac{1}{6}$
ⓑ $\frac{2}{3}\div\frac{1}{6}$

186.
ⓐ $-\frac{2}{5}-\frac{1}{8}$
ⓑ $-\frac{2}{5}\cdot\frac{1}{8}$

187.
ⓐ $\frac{5n}{6}\div\frac{8}{15}$
ⓑ $\frac{5n}{6}-\frac{8}{15}$

188.
ⓐ $\frac{3a}{8}\div\frac{7}{12}$
ⓑ $\frac{3a}{8}-\frac{7}{12}$

189.
ⓐ $-\frac{4x}{9}-\frac{5}{6}$
ⓑ $-\frac{4k}{9}\cdot\frac{5}{6}$

190.
ⓐ $-\frac{3y}{8}-\frac{4}{3}$
ⓑ $-\frac{3y}{8}\cdot\frac{4}{3}$

191.
ⓐ $-\frac{5a}{3}+\left(-\frac{10}{6}\right)$
ⓑ $-\frac{5a}{3}\div\left(-\frac{10}{6}\right)$

192.
ⓐ $\frac{2b}{5}+\frac{8}{15}$
ⓑ $\frac{2b}{5}\div\frac{8}{15}$

Use the Order of Operations to Simplify Fractions

In the following exercises, simplify.

193. $\frac{5\cdot 6-3\cdot 4}{4\cdot 5-2\cdot 3}$

194. $\frac{8\cdot 9-7\cdot 6}{5\cdot 6-9\cdot 2}$

195. $\frac{5^2-3^2}{3-5}$

196. $\frac{6^2-4^2}{4-6}$

197. $\frac{7\cdot 4-2(8-5)}{9\cdot 3-3\cdot 5}$

198. $\frac{9\cdot 7-3(12-8)}{8\cdot 7-6\cdot 6}$

199. $\frac{9(8-2)-3(15-7)}{6(7-1)-3(17-9)}$

200. $\frac{8(9-2)-4(14-9)}{7(8-3)-3(16-9)}$

201. $\frac{2^3+4^2}{\left(\frac{2}{3}\right)^2}$

202. $\frac{3^3-3^2}{\left(\frac{3}{4}\right)^2}$

203. $\frac{\left(\frac{3}{5}\right)^2}{\left(\frac{3}{7}\right)^2}$

204. $\frac{\left(\frac{3}{4}\right)^2}{\left(\frac{5}{8}\right)^2}$

205. $\frac{2}{\frac{1}{3}+\frac{1}{5}}$

206. $\frac{5}{\frac{1}{4}+\frac{1}{3}}$

207. $\frac{\frac{7}{8}-\frac{2}{3}}{\frac{1}{2}+\frac{3}{8}}$

208. $\frac{\frac{3}{4}-\frac{3}{5}}{\frac{1}{4}+\frac{2}{5}}$

Mixed Practice

In the following exercises, simplify.

209. $-\frac{3}{8}\div\left(-\frac{3}{10}\right)$

210. $-\frac{3}{12}\div\left(-\frac{5}{9}\right)$

211. $-\frac{3}{8}+\frac{5}{12}$

212. $-\frac{1}{8}+\frac{7}{12}$

213. $-\frac{7}{15}-\frac{y}{4}$

214. $-\frac{3}{8}-\frac{x}{11}$

215. $\frac{11}{12a}\cdot\frac{9a}{16}$

216. $\frac{10y}{13}\cdot\frac{8}{15y}$

217. $\frac{1}{2}+\frac{2}{3}\cdot\frac{5}{12}$

218. $\frac{1}{3}+\frac{2}{5}\cdot\frac{3}{4}$

219. $1-\frac{3}{5}\div\frac{1}{10}$

220. $1-\frac{5}{6}\div\frac{1}{12}$

221. $\frac{3}{8}-\frac{1}{6}+\frac{3}{4}$

222. $\frac{2}{5}+\frac{5}{8}-\frac{3}{4}$

223. $12\left(\frac{9}{20}-\frac{4}{15}\right)$

224. $8\left(\frac{15}{16}-\frac{5}{6}\right)$

225. $\frac{\frac{5}{8}+\frac{1}{6}}{\frac{19}{24}}$

226. $\frac{\frac{1}{6}+\frac{3}{10}}{\frac{14}{30}}$

227. $\left(\frac{5}{9}+\frac{1}{6}\right)\div\left(\frac{2}{3}-\frac{1}{2}\right)$

228. $\left(\frac{3}{4}+\frac{1}{6}\right)\div\left(\frac{5}{8}-\frac{1}{3}\right)$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

229. $\frac{7}{10}-w$ when

ⓐ $w=\frac{1}{2}$ ⓑ $w=-\frac{1}{2}$

230. $\frac{5}{12}-w$ when

ⓐ $w=\frac{1}{4}$ ⓑ $w=-\frac{1}{4}$

231. $2x^2y^3$ when

$x=-\frac{2}{3}$ and $y=-\frac{1}{2}$

232. $8u^2v^3$ when

$u=-\frac{3}{4}$ and $v=-\frac{1}{2}$

233. $\frac{a+b}{a-b}$ when

$a=-3,\ b=8$

234. $\frac{r-s}{r+s}$ when

$r=10,\ s=-5$

Writing Exercises

235. Why do you need a common denominator to add or subtract fractions? Explain.

236. How do you find the LCD of 2 fractions?

237. Explain how you find the reciprocal of a fraction.

238. Explain how you find the reciprocal of a negative number.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify fractions.			
multiply and divide fractions.			
add and subtract fractions.			
use the order of operations to simplify fractions.			
evaluate variable expressions with fractions.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

1.4 Decimals

Learning Objectives

By the end of this section, you will be able to:

- Round decimals
- Add and subtract decimals
- Multiply and divide decimals
- Convert decimals, fractions, and percents
- Simplify expressions with square roots
- Identify integers, rational numbers, irrational numbers, and real numbers
- Locate fractions and decimals on the number line

Be Prepared!

A more thorough introduction to the topics covered in this section can be found in the *Elementary Algebra* chapter, Foundations.

Round Decimals

Decimals are another way of writing fractions whose denominators are powers of ten.

$$0.1 = \frac{1}{10} \quad \text{is “one tenth”}$$

$$0.01 = \frac{1}{100} \quad \text{is “one hundredth”}$$

$$0.001 = \frac{1}{1000} \quad \text{is “one thousandth”}$$

$$0.0001 = \frac{1}{10{,}000} \quad \text{is “one ten-thousandth”}$$

Just as in whole numbers, each digit of a decimal corresponds to the place value based on the powers of ten. Figure 1.6 shows the names of the place values to the left and right of the decimal point.



Figure 1.6

When we work with decimals, it is often necessary to round the number to the nearest required place value. We summarize the steps for rounding a decimal here.

HOW TO : : ROUND DECIMALS.

Step 1. Locate the given place value and mark it with an arrow.

Step 2. Underline the digit to the right of the place value.

Step 3. Is the underlined digit greater than or equal to 5?

- Yes: add 1 to the digit in the given place value.
- No: do not change the digit in the given place value

Step 4. Rewrite the number, deleting all digits to the right of the rounding digit.

EXAMPLE 1.34

Round 18.379 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Solution

Round 18.379.

ⓐ to the nearest hundredth

Locate the hundredths place with an arrow.	
Underline the digit to the right of the given place value.	
Because 9 is greater than or equal to 5, add 1 to the 7.	
Rewrite the number, deleting all digits to the right of the rounding digit.	18.38
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest hundredth is 18.38.

ⓑ to the nearest tenth

Locate the tenths place with an arrow.	
Underline the digit to the right of the given place value.	
Because 7 is greater than or equal to 5, add 1 to the 3.	
Rewrite the number, deleting all digits to the right of the rounding digit.	18.4

Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest tenth is 18.4.

ⓒ to the nearest whole number

Locate the ones place with an arrow.	ones place ↓ 18.379
Underline the digit to the right of the given place value.	ones place ↓ 18.379
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	18.379 delete do not add 1
Rewrite the number, deleting all digits to the right of the rounding digit.	18
	So, 18.379 rounded to the nearest whole number is 18.

TRY IT : : 1.67 Round 6.582 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

TRY IT : : 1.68 Round 15.2175 to the nearest ⓐ thousandth ⓑ hundredth ⓒ tenth.

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

HOW TO : : ADD OR SUBTRACT DECIMALS.

Step 1. Determine the sign of the sum or difference.
Step 2. Write the numbers so the decimal points line up vertically.
Step 3. Use zeros as placeholders, as needed.
Step 4. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.
Step 5. Write the sum or difference with the appropriate sign.

EXAMPLE 1.35

Add or subtract: ⓐ $-23.5 - 41.38$ ⓑ $14.65 - 20$.

Solution

ⓐ

$-23.5 - 41.38$

The diffe ence will be negative. To subtract, we add the numerals. Write the numbers so the decimal points line up vertically.

$$\begin{array}{r} 23.5 \\ +41.38 \end{array}$$

Put 0 as a placeholder after the 5 in 23.5.
Remember, $\frac{5}{10} = \frac{50}{100}$ so $0.5 = 0.50$.

$$\begin{array}{r} 23.50 \\ +41.38 \end{array}$$

Add the numbers as if they were whole numbers. Then place the decimal point in the sum.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline 64.88 \end{array}$$

Write the result with the correct sign.

$-23.5 - 41.38 = -64.88$

ⓑ

$14.65 - 20$

The diffe ence will be negative. To subtract, we subtract 14.65 from 20.

Write the numbers so the decimal points line up vertically.

$$\begin{array}{r} 20 \\ -14.65 \end{array}$$

Remember, 20 is a whole number, so place the decimal point after the 0.

Put in zeros to the right as placeholders.

$$\begin{array}{r} 20.00 \\ -14.65 \end{array}$$

Subtract and place the decimal point in the answer.

$$\begin{array}{r} {}^{1}\cancel{2}{}^{9}\cancel{0}{}^{9}\cancel{0}{}^{10}0 \\ -14.65 \\ \hline 5.35 \end{array}$$

Write the result with the correct sign.

$14.65 - 20 = -5.35$

TRY IT : : 1.69 Add or subtract: ⓐ $-4.8 - 11.69$ ⓑ $9.58 - 10$.

TRY IT : : 1.70 Add or subtract: ⓐ $-5.123 - 18.47$ ⓑ $37.42 - 50$.

Multiply and Divide Decimals

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. We multiply the numbers temporarily ignoring the decimal point and then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product. Finally, we write the product with the appropriate sign.

HOW TO : : MULTIPLY DECIMALS.

Step 1. Determine the sign of the product.

Step 2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.

Step 3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.

Step 4. Write the product with the appropriate sign.

EXAMPLE 1.36

Multiply: $(-3.9)(4.075)$.

Solution

	$(-3.9)(4.075)$
The signs are different. The product will be negative.	The product will be negative.
Write in vertical format, lining up the numbers on the right.	4.075 x 3.9
Multiply.	4.075 x 3.9 36675 12225 158925
Add the number of decimal places in the factors (1 + 3). Place the decimal point 4 places from the right. (−3.9) (4.075) 1 place 3 places	4.075 x 3.9 36675 12225 15.8925 4 places
The signs are the different, so the product is negative.	$(-3.9)(4.075) = -15.8925$

TRY IT : : 1.71 Multiply: $-4.5(6.107)$.

TRY IT : : 1.72 Multiply: $-10.79(8.12)$.

Often, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

HOW TO : : MULTIPLY A DECIMAL BY A POWER OF TEN.

Step 1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.

Step 2. Add zeros at the end of the number as needed.

EXAMPLE 1.37

Multiply: 5.63 by ⓐ 10 ⓑ 100 ⓒ 1000.

Solution

By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

ⓐ

	5.63 (10)
There is 1 zero in 10, so move the decimal point 1 place to the right.	5.63
	5.63

ⓑ

	5.63(100)
There are 2 zeroes in 100, so move the decimal point 2 places to the right.	5.63
	563

ⓒ

	5.63(1,000)
There are 3 zeroes in 1,000, so move the decimal point 3 place to the right.	5.63
A zero must be added to the end.	5,630

TRY IT : : 1.73 Multiply 2.58 by ⓐ 10 ⓑ 100 ⓒ 1000.

TRY IT : : 1.74 Multiply 14.2 by ⓐ 10 ⓑ 100 ⓒ 1000.

Just as with multiplication, division of signed decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed and the sign of the quotient. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{c} \qquad b\overline{)a} \quad \begin{matrix} c \\ \text{quotient} \end{matrix} \quad \begin{matrix} \\ \text{divisor/dividend} \end{matrix}$$

We'll write the steps to take when dividing decimals for easy reference.

HOW TO : : DIVIDE DECIMALS.

Step 1. Determine the sign of the quotient.

Step 2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.

Step 3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.

Step 4. Write the quotient with the appropriate sign.

EXAMPLE 1.38

Divide: $-25.65 \div (-0.06)$.

Solution

Remember, you can "move" the decimals in the divisor and dividend because of the Equivalent Fractions Property.

	$-25.65 \div (-0.06)$
The signs are the same.	The quotient is positive.
Make the divisor a whole number by "moving" the decimal point all the way to the right.	
"Move" the decimal point in the dividend the same number of places.	$0.06\overline{)25.65}$
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	$\begin{array}{r} 427.5 \\ 006.\overline{)2565.0} \\ -24 \\ \hline 16 \\ -12 \\ \hline 45 \\ -42 \\ \hline 30 \\ 30 \end{array}$
Write the quotient with the appropriate sign.	$-25.65 \div (-0.06) = 427.5$

TRY IT : : 1.75 Divide: $-23.492 \div (-0.04)$.

TRY IT : : 1.76 Divide: $-4.11 \div (-0.12)$.

Convert Decimals, Fractions, and Percents

In our work, it is often necessary to change the form of a number. We may have to change fractions to decimals or decimals to percent.

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal 0.03. the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.

$$0.03 = \frac{3}{100}$$

The steps to take to convert a decimal to a fraction are summarized in the procedure box.

HOW TO : : CONVERT A DECIMAL TO A PROPER FRACTION AND A FRACTION TO A DECIMAL.

Step 1. To convert a decimal to a proper fraction, determine the place value of the final digit.

Step 2. Write the fraction.

- numerator—the "numbers" to the right of the decimal point
- denominator—the place value corresponding to the final digit

Step 3. To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

EXAMPLE 1.39

Write: ⓐ 0.374 as a fraction ⓑ $-\frac{5}{8}$ as a decimal.

Solution

ⓐ

	0.374
Determine the place value of the final digit.	0.3 tenths, 7 hundredths, 4 thousandths
Write the fraction for 0.374: The numerator is 374. The denominator is 1,000.	$\frac{374}{1000}$
Simplify the fraction.	$\frac{2 \cdot 187}{2 \cdot 500}$
Divide out the common factors.	$\frac{187}{500}$
	so, $0.374 = \frac{187}{500}$

ⓑ Since a fraction bar means division, we begin by writing the fraction $\frac{5}{8}$ as $8\overline{)5.}$ Now divide.

$$\begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ \underline{48} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

so, $-\frac{5}{8} = -0.625$

TRY IT : : 1.77 Write: ⓐ 0.234 as a fraction ⓑ $-\frac{7}{8}$ as a decimal.

TRY IT : : 1.78 Write: ⓐ 0.024 as a fraction ⓑ $-\frac{3}{8}$ as a decimal.

A **percent** is a ratio whose denominator is 100. Percent means per hundred. We use the percent symbol, %, to show percent. Since a percent is a ratio, it can easily be expressed as a fraction. Percent means per 100, so the denominator of the fraction is 100. We then change the fraction to a decimal by dividing the numerator by the denominator. After doing this many times, you may see the pattern.

To convert a percent number to a decimal number, we move the decimal point two places to the left.

6%	78%	2.7%	135%
0.06	0.78	0.027	1.35

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent. After many conversions, you may recognize the pattern.

To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

0.05	0.83	1.05	0.075	0.3
5%	83%	105%	7.5%	30%



HOW TO : : CONVERT A PERCENT TO A DECIMAL AND A DECIMAL TO A PERCENT.

Step 1. To convert a percent to a decimal, move the decimal point two places to the left after removing the percent sign.

Step 2. To convert a decimal to a percent, move the decimal point two places to the right and then add the percent sign.

EXAMPLE 1.40

Convert each:

ⓐ percent to a decimal: 62%, 135%, and 13.7%.

ⓑ decimal to a percent: 0.51, 1.25, and 0.093.

Solution

ⓐ

	62%	135%	35.7%
Move the decimal point two places to the left.	0.62	1.35	0.357

ⓑ

	0.51	1.25	0.093
Move the decimal point two places to the right.	51%	125%	9.3%

TRY IT : : 1.79

Convert each:

ⓐ percent to a decimal: 9%, 87%, and 3.9%.

ⓑ decimal to a percent: 0.17, 1.75, and 0.0825.

TRY IT : : 1.80

Convert each:

ⓐ percent to a decimal: 3%, 91%, and 8.3%.

ⓑ decimal to a percent: 0.41, 2.25, and 0.0925.

Simplify Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2 and read it "n squared." The result is called the **square of a number** *n*. For example, 8^2 is read "8 squared" and 64 is called the *square* of 8. Similarly, 121 is the square of 11 because 11^2 is 121. It will be helpful to learn to recognize the perfect square numbers.

Square of a number

If $n^2 = m$, then *m* is the **square** of *n*.

What about the squares of negative numbers? We know that when the signs of two numbers are the same, their product is positive. So the square of any negative number is also positive.

$$(-3)^2 = 9 \qquad (-8)^2 = 64 \qquad (-11)^2 = 121 \qquad (-15)^2 = 225$$

Because $10^2 = 100$, we say 100 is the square of 10. We also say that 10 is a *square root* of 100. A number whose square is *m* is called a **square root of a number** *m*.

Square Root of a Number

If $n^2 = m$, then *n* is a **square root** of *m*.

Notice $(-10)^2 = 100$ also, so -10 is also a square root of 100. Therefore, both 10 and -10 are square roots of 100. So, every positive number has two square roots—one positive and one negative. The radical sign, $\sqrt{m}$, denotes the positive square root. The positive square root is called the **principal square root**. When we use the radical sign that always means we want the principal square root.

Square Root Notation

$\sqrt{m}$ is read "the square root of m."



If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

The square root of *m*, $\sqrt{m}$, is the positive number whose square is *m*.

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{100} = 10$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{100} = -10$. We read $-\sqrt{100}$ as "the opposite of the principal square root of 10."

EXAMPLE 1.41

Simplify: ⓐ $\sqrt{25}$ ⓑ $\sqrt{121}$ ⓒ $-\sqrt{144}$.

Solution

ⓐ

	$\sqrt{25}$
Since $5^2 = 25$	5

ⓑ

	$\sqrt{121}$
Since $11^2 = 121$	11

ⓒ

	$-\sqrt{144}$
The negative is in front of the radical sign.	-12

TRY IT : : 1.81 Simplify: ⓐ $\sqrt{36}$ ⓑ $\sqrt{169}$ ⓒ $-\sqrt{225}$.

TRY IT : : 1.82 Simplify: ⓐ $\sqrt{16}$ ⓑ $\sqrt{196}$ ⓒ $-\sqrt{100}$.

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as *counting numbers*, *whole numbers*, and *integers*. What is the difference between these types of numbers? Difference could be confused with subtraction. How about asking how we distinguish between these types of numbers?

Counting numbers	$1, 2, 3, 4, \ldots.$
Whole numbers	$0, 1, 2, 3, 4, \ldots.$
Integers	$\ldots. -3, -2, -1, 0, 1, 2, 3, \ldots.$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A **rational number** is a number that can be written as a ratio of two integers.

In general, any decimal that ends after a number of digits (such as 7.3 or -1.2684) is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction. The decimal for $\frac{1}{3}$ is the number $0.\overline{3}$. The bar over the 3 indicates that the number 3 repeats infinitely. Continuously has an important meaning in calculus. The number(s) under the bar is called the repeating block and it repeats continuously.

Since all integers can be written as a fraction whose denominator is 1, the integers (and so also the counting and whole numbers. are rational numbers.

Every rational number can be written both as a ratio of integers $\frac{p}{q}$, *where p and q are integers and* $q \neq 0$, *and as a decimal that stops or repeats.*

Rational Number

A **rational number** is a number of the form $\frac{p}{q}$, where *p* and *q* are integers and $q \neq 0$.

Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes! The number π (the Greek letter *pi*, pronounced "pie"), which is very important in describing circles, has a decimal form that does not stop or repeat. We use three dots (...) to indicate the decimal does not stop or repeat.

$$\pi = 3.141592654...$$

The square root of a number that is not a perfect square is a decimal that does not stop or repeat.

A numbers whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call this an **irrational number**.

Irrational Number

An **irrational number** is a number that cannot be written as the ratio of two integers.

Its decimal form does not stop and does not repeat.

Let's summarize a method we can use to determine whether a number is rational or irrational.

Rational or Irrational

If the decimal form of a number

- *repeats or stops*, the number is a **rational number**.
- *does not repeat and does not stop*, the number is an **irrational number.**

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of **real numbers**.

Real Number

A **real number** is a number that is either rational or irrational.

Later in this course we will introduce numbers beyond the real numbers. Figure 1.7 illustrates how the number sets we've used so far fit together.

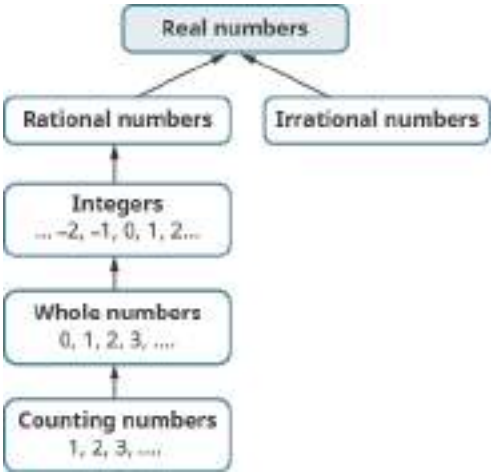


Figure 1.7 This chart shows the number sets that make up the set of real numbers.

Does the term "real numbers" seem strange to you? Are there any numbers that are not "real," and, if so, what could they be? Can we simplify $\sqrt{-25}$? Is there a number whose square is -25?

$$(\)^2 = -25\,?$$

None of the numbers that we have dealt with so far has a square that is -25. Why? Any positive number squared is positive. Any negative number squared is positive. So we say there is no real number equal to $\sqrt{-25}$. The square root of a negative number is not a real number.

EXAMPLE 1.42

Given the numbers $-7,\ \frac{14}{5},\ 8,\ \sqrt{5},\ 5.9,\ -\sqrt{64},$ list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers.

Solution

ⓐ Remember, the whole numbers are $0,\ 1,\ 2,\ 3,\ \ldots,$ so 8 is the only whole number given.

ⓑ The integers are the whole numbers and their opposites (which includes 0). So the whole number 8 is an integer, and -7 is the opposite of a whole number so it is an integer, too. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. So the integers are $-7,\ 8,$ and $-\sqrt{64}$.

ⓒ Since all integers are rational, then $-7,\ 8,$ and $-\sqrt{64}$ are rational. Rational numbers also include fractions and decimals that repeat or stop, so $\frac{14}{5}$ and 5.9 are rational. So the list of rational numbers is $-7,\ \frac{14}{5},\ 8,\ 5.9,$ and $-\sqrt{64}$.

ⓓ Remember that 5 is not a perfect square, so $\sqrt{5}$ is irrational.

ⓔ All the numbers listed are real numbers.

TRY IT : : 1.83

Given the numbers $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49},$ list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers.

TRY IT : : 1.84

Given numbers $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975...,$ list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers.

Locate Fractions and Decimals on the Number Line

We now want to include fractions and decimals on the number line. Let's start with fractions and locate $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5$ and $\frac{8}{3}$ on the number line.

We'll start with the whole numbers 3 and -5 because they are the easiest to plot. See Figure 1.8.

The proper fractions listed are $\frac{1}{5}$ and $-\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between 0 and 1. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We plot $\frac{1}{5}$.

Similarly, $-\frac{4}{5}$ is between 0 and -1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$.

Finally, look at the improper fractions $\frac{7}{4}, \frac{9}{2}, \frac{8}{3}$. Locating these points may be easier if you change each of them to a mixed number.

$$\frac{7}{4} = 1\frac{3}{4} \qquad -\frac{9}{2} = -4\frac{1}{2} \qquad \frac{8}{3} = 2\frac{2}{3}$$

Figure 1.8 shows the number line with all the points plotted.

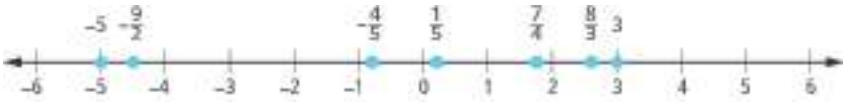


Figure 1.8

EXAMPLE 1.43

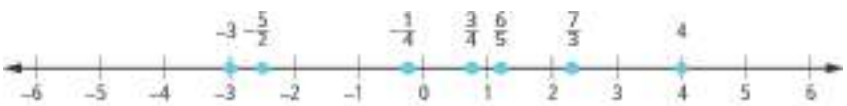
Locate and label the following on a number line: $4, \frac{3}{4}, -\frac{1}{4}, -3, \frac{6}{5}, -\frac{5}{2},$ and $\frac{7}{3}$.

Solution

Locate and plot the integers, $4, -3$.

Locate the proper fraction $\frac{3}{4}$ first. The fraction $\frac{3}{4}$ is between 0 and 1. Divide the distance between 0 and 1 into four equal parts, then we plot $\frac{3}{4}$. Similarly plot $-\frac{1}{4}$.

Now locate the improper fractions $\frac{6}{5}, -\frac{5}{2},$ and $\frac{7}{3}$. It is easier to plot them if we convert them to mixed numbers and then plot them as described above: $\frac{6}{5} = 1\frac{1}{5}, -\frac{5}{2} = -2\frac{1}{2}, \frac{7}{3} = 2\frac{1}{3}$.



TRY IT : : 1.85 Locate and label the following on a number line: $-1, \frac{1}{3}, \frac{6}{5}, -\frac{7}{4}, \frac{9}{2}, 5, -\frac{8}{3}$.

TRY IT : : 1.86 Locate and label the following on a number line: $-2, \frac{2}{3}, \frac{7}{5}, -\frac{7}{4}, \frac{7}{2}, 3, -\frac{7}{3}$.

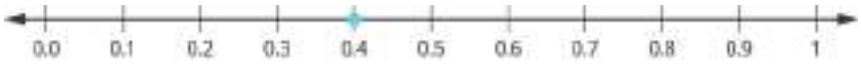
Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

EXAMPLE 1.44

Locate on the number line: ⓐ 0.4 ⓑ -0.74.

Solution

ⓐ The decimal number 0.4 is equivalent to $\frac{4}{10}$, a proper fraction, so 0.4 is located between 0 and 1. On a number line, divide the interval between 0 and 1 into 10 equal parts. Now label the parts 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. We write 0 as 0.0 and 1 as 1.0, so that the numbers are consistently in tenths. Finally, mark 0.4 on the number line.



ⓑ The decimal -0.74 is equivalent to $-\frac{74}{100}$, so it is located between 0 and -1. On a number line, mark off and label the hundredths in the interval between 0 and -1.



TRY IT : : 1.87 Locate on the number line: ⓐ 0.6 ⓑ -0.25.

TRY IT : : 1.88 Locate on the number line: ⓐ 0.9 ⓑ -0.75.

MEDIA : :

Access this online resource for additional instruction and practice with decimals.

- **Arithmetic Basics: Dividing Decimals (https://openstax.org/l/25dividedecim)**

1.4 EXERCISES

Practice Makes Perfect

Round Decimals

In the following exercises, round each number to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

239. 5.781

240. 1.638

241. 0.299

242. 0.697

243. 63.479

244. 84.281

Add and Subtract Decimals

In the following exercises, add or subtract.

245. $-16.53 - 24.38$

246. $-19.47 - 32.58$

247. $-38.69 + 31.47$

248. $-29.83 + 19.76$

249. $72.5 - 100$

250. $86.2 - 100$

251. $91.75 - (-10.462)$

252. $94.69 - (-12.678)$

253. $55.01 - 3.7$

254. $59.08 - 4.6$

255. $2.51 - 7.4$

256. $3.84 - 6.1$

Multiply and Divide Decimals

In the following exercises, multiply.

257. $94.69 - (-12.678)$

258. $(-8.5)(1.69)$

259. $(-5.18)(-65.23)$

260. $(-9.16)(-68.34)$

261. $(0.06)(21.75)$

262. $(0.08)(52.45)$

263. $(9.24)(10)$

264. $(6.531)(10)$

265. $(0.025)(100)$

266. $(0.037)(100)$

267. $(55.2)(1000)$

268. $(99.4)(1000)$

In the following exercises, divide. Round money monetary answers to the nearest cent.

269. $\$117.25 \div 48$

270. $\$109.24 \div 36$

271. $1.44 \div (-0.3)$

272. $-1.15 \div (-0.05)$

273. $5.2 \div 2.5$

274. $14 \div 0.35$

Convert Decimals, Fractions and Percents

In the following exercises, write each decimal as a fraction.

275. 0.04

276. 1.464

277. 0.095

278. -0.375

In the following exercises, convert each fraction to a decimal.

279. $\frac{17}{20}$

280. $\frac{17}{4}$

281. $-\frac{310}{25}$

282. $-\frac{18}{11}$

In the following exercises, convert each percent to a decimal.

283. 71%

284. 150%

285. 39.3%

286. 7.8%

In the following exercises, convert each decimal to a percent.

287. 1.56

288. 3

289. 0.0625

290. 2.254

Simplify Expressions with Square Roots

In the following exercises, simplify.

291. $\sqrt{64}$

292. $\sqrt{169}$

293. $\sqrt{144}$

294. $-\sqrt{4}$

295. $-\sqrt{100}$

296. $-\sqrt{121}$

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercises, list the ⓐ whole numbers, ⓑ integers, ⓒ rational numbers, ⓓ irrational numbers, ⓔ real numbers for each set of numbers.

297. $-8,\ 0,\ 1.95286\ldots,\ \frac{12}{5},\ \sqrt{36},\ 9$

298. $-9,\ -3\frac{4}{9},\ -\sqrt{9},\ 0.4\overline{09},\ \frac{11}{6},\ 7$

299. $-\sqrt{100},\ -7,\ -\frac{8}{3},\ -1,\ 0.77,\ 3\frac{1}{4}$

300. $-6,\ -\frac{5}{2},\ 0,\ 0.\overline{714285},\ 2\frac{1}{5},\ \sqrt{14}$

Locate Fractions and Decimals on the Number Line

In the following exercises, locate the numbers on a number line.

301. $\frac{3}{10},\ \frac{7}{2},\ \frac{11}{6},\ 4$

302. $\frac{7}{10},\ \frac{5}{2},\ \frac{13}{8},\ 3$

303. $\frac{3}{4},\ -\frac{3}{4},\ 1\frac{2}{3},\ -1\frac{2}{3},\ \frac{5}{2},\ -\frac{5}{2}$

304. $\frac{2}{5},\ -\frac{2}{5},\ 1\frac{3}{4},\ -1\frac{3}{4},\ \frac{8}{3},\ -\frac{8}{3}$

305. ⓐ 0.8 ⓑ −1.25

306. ⓐ −0.9 ⓑ −2.75

307. ⓐ −1.6 ⓑ 3.25

308. ⓐ 3.1 ⓑ −3.65

Writing Exercises

309. How does knowing about U.S. money help you learn about decimals?

310. When the Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.

311. In your own words, explain the difference between a rational number and an irrational number.

312. Explain how the sets of numbers (counting, whole, integer, rational, irrationals, reals) are related to each other.

Self Check

ⓐ *Use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
round decimals.			
add and subtract decimals.			
multiply and divide decimals.			
convert decimals, fractions and percents.			
simplify expressions with square roots.			
identify integers, rational numbers, irrational numbers and real numbers.			
locate fractions and decimals on the number line.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

1.5 Properties of Real Numbers

Learning Objectives

By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the properties of identity, inverse, and zero
- Simplify expressions using the Distributive Property

Be Prepared!

A more thorough introduction to the topics covered in this section can be found in the *Elementary Algebra* chapter, Foundations.

Use the Commutative and Associative Properties

The order we add two numbers doesn't affect the result. If we add $8+9$ or $9+8$, the results are the same—they both equal 17. So, $8+9=9+8$. The order in which we add does not matter!

Similarly, when multiplying two numbers, the order does not affect the result. If we multiply $9\cdot 8$ or $8\cdot 9$ the results are the same—they both equal 72. So, $9\cdot 8=8\cdot 9$. The order in which we multiply does not matter!

These examples illustrate the Commutative Property.

Commutative Property

of Addition	If a and b are real numbers, then	$a+b=b+a$.
of Multiplication	If a and b are real numbers, then	$a\cdot b=b\cdot a$.

When adding or multiplying, changing the *order* gives the same result.

The Commutative Property has to do with order. We subtract $9-8$ and $8-9$, and see that $9-8\neq 8-9$. Since changing the order of the subtraction does not give the same result, we know that *subtraction is not commutative*.

Division is not commutative either. Since $12\div 3\neq 3\div 12$, changing the order of the division did not give the same result. The commutative properties apply only to addition and multiplication!

Addition and multiplication *are* commutative.

Subtraction and division are *not* commutative.

When adding three numbers, changing the grouping of the numbers gives the same result. For example, $(7+8)+2=7+(8+2)$, since each side of the equation equals 17.

This is true for multiplication, too. For example, $\left(5\cdot\frac{1}{3}\right)\cdot 3=5\cdot\left(\frac{1}{3}\cdot 3\right)$, since each side of the equation equals 5.

These examples illustrate the Associative Property.

Associative Property

of Addition	If a, b, and c are real numbers, then	$(a+b)+c=a+(b+c)$.
of Multiplication	If a, b, and c are real numbers, then	$(a\cdot b)\cdot c=a\cdot(b\cdot c)$.

When adding or multiplying, changing the *grouping* gives the same result.

The Associative Property has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either.

$$(10-3)-2\neq 10-(3-2) \qquad (24\div 4)\div 2\neq 24\div(4\div 2)$$
$$7-2\neq 10-1 \qquad 6\div 2\neq 24\div 2$$
$$5\neq 9 \qquad 3\neq 12$$

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the Commutative Property of addition to write the like terms together.

EXAMPLE 1.45

Simplify: $18p + 6q + 15p + 5q$.

Solution

	$18p + 6q + 15p + 5q$
Use the Commutative Property of addition to reorder so that like terms are together.	$18p + 15p + 6q + 5q$
Add like terms.	$33p + 11q$

TRY IT : : 1.89 Simplify: $23r + 14s + 9r + 15s$.

TRY IT : : 1.90 Simplify: $37m + 21n + 4m - 15n$.

When we have to simplify algebraic expressions, we can often make the work easier by applying the Commutative Property or Associative Property first.

EXAMPLE 1.46

Simplify: $\left(\frac{5}{13} + \frac{3}{4}\right) + \frac{1}{4}$.

Solution

	$\left(\frac{5}{13} + \frac{3}{4}\right) + \frac{1}{4}$
Notice that the last 2 terms have a common denominator, so change the grouping.	$\frac{5}{13} + \left(\frac{3}{4} + \frac{1}{4}\right)$
Add in parentheses fir t.	$\frac{5}{13} + \left(\frac{4}{4}\right)$
Simplify the fraction.	$\frac{5}{13} + 1$
Add.	$1\frac{5}{13}$
Convert to an improper fraction.	$\frac{18}{13}$

TRY IT : : 1.91 Simplify: $\left(\frac{7}{15} + \frac{5}{8}\right) + \frac{3}{8}$.

TRY IT : : 1.92 Simplify: $\left(\frac{2}{9} + \frac{7}{12}\right) + \frac{5}{12}$.

Use the Properties of Identity, Inverse, and Zero

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the **additive identity**. The **Identity Property of Addition** that states that for any real number a, $a + 0 = a$ and $0 + a = a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the **multiplicative identity**. The **Identity Property of Multiplication** that states that for any real number a, $a \cdot 1 = a$ and $1 \cdot a = a$.

We summarize the Identity Properties here.

Identity Property

of Addition For any real number a: $a+0=a$ $0+a=a$

0 is the **additive identity**

of Multiplication For any real number a: $a\cdot 1=a$ $1\cdot a=a$

1 is the **multiplicative identity**

What number added to 5 gives the additive identity, 0? We know

$$5+(-5)=0$$

The missing number was the opposite of the number!

We call $-a$ the **additive inverse** of a. *The opposite of a number is its additive inverse.* A number and its opposite add to zero, which is the additive identity. This leads to the **Inverse Property of Addition** that states for any real number a, $a+(-a)=0$.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, $\frac{2}{3}$ times what results in 1? We know

$$\frac{2}{3}\cdot\frac{3}{2}=1$$

The missing number was the reciprocal of the number!

We call $\frac{1}{a}$ the **multiplicative inverse** of *a*. *The reciprocal of a number is its multiplicative inverse.* This leads to the **Inverse Property of Multiplication** that states that for any real number a, $a\neq 0$, $a\cdot\frac{1}{a}=1$.

We'll formally state the inverse properties here.

Inverse Property

of Addition For any real number a, $a+(-a)=0$

$-a$ is the **additive inverse** of a

A number and its *opposite* add to zero.

of Multiplication For any real number a, $a\neq 0$, $a\cdot\frac{1}{a}=1$.

$\frac{1}{a}$ is the **multiplicative inverse** of a.

A number and its *reciprocal* multiply to one.

The Identity Property of addition says that when we add 0 to any number, the result is that same number. What happens when we multiply a number by 0? Multiplying by 0 makes the product equal zero.

What about division involving zero? What is $0\div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So, $0\div 3=0$.

We can check division with the related multiplication fact. So we know $0\div 3=0$ because $0\cdot 3=0$.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact:

$$4\div 0=\square \text{ means } \square\cdot 0=4$$

Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4. We conclude that there is no answer to $4\div 0$ and so we say that division by 0 is **undefined**.

We summarize the properties of zero here.

Properties of Zero

Multiplication by Zero: For any real number *a*,

$a\cdot 0=0$ $0\cdot a=0$ The product of any number and 0 is 0.

Division by Zero: For any real number a, $a \neq 0$

$\frac{0}{a} = 0$	Zero divided by any real number, except itself, is zero.
$\frac{a}{0}$ is undefine	Division by zero is undefined

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

EXAMPLE 1.47

Simplify: $-84n + (-73n) + 84n$.

Solution

	$-84n + (-73n) + 84n$
Notice that the first and third terms are opposites; use the Commutative Property of addition to re-order the terms.	$-84n + 84n + (-73n)$
Add left to right.	$0 + (-73n)$
Add.	$-73n$

TRY IT : : 1.93 Simplify: $-27a + (-48a) + 27a$.

TRY IT : : 1.94 Simplify: $39x + (-92x) + (-39x)$.

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

EXAMPLE 1.48

Simplify: $\frac{7}{15} \cdot \frac{8}{23} \cdot \frac{15}{7}$.

Solution

	$\frac{7}{15} \cdot \frac{8}{23} \cdot \frac{15}{7}$
Notice the first and third terms are reciprocals, so use the Commutative Property of multiplication to re-order the factors.	$\frac{7}{15} \cdot \frac{15}{7} \cdot \frac{8}{23}$
Multiply left to right.	$1 \cdot \frac{8}{23}$
Multiply.	$\frac{8}{23}$

TRY IT : : 1.95 Simplify: $\frac{9}{16} \cdot \frac{5}{49} \cdot \frac{16}{9}$.

TRY IT : : 1.96 Simplify: $\frac{6}{17} \cdot \frac{11}{25} \cdot \frac{17}{6}$.

The next example makes us aware of the distinction between dividing 0 by some number or some number being divided by 0.

EXAMPLE 1.49

Simplify: ⓐ $\frac{0}{n+5}$, where $n \neq -5$ ⓑ $\frac{10-3p}{0}$, where $10-3p \neq 0$.

Solution

ⓐ

	$\frac{0}{n+5}$
Zero divided by any real number except itself is 0.	0

ⓑ

	$\frac{10-3p}{0}$
Division by 0 is undefined	undefine

TRY IT : : 1.97 Simplify: ⓐ $\frac{0}{m+7}$, where $m \neq -7$ ⓑ $\frac{18-6c}{0}$, where $18-6c \neq 0$.

TRY IT : : 1.98 Simplify: ⓐ $\frac{0}{d-4}$, where $d \neq 4$ ⓑ $\frac{15-4q}{0}$, where $15-4q \neq 0$.

Simplify Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need $9.25—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times $9 so $27 and 3 times 1 quarter, so 75 cents. In total, they need $27.75. If you think about doing the math in this way, you are using the Distributive Property.

Distributive Property

If a, b, and c are real numbers, then

$$a(b+c) = ab+ac$$
$$(b+c)a = ba+ca$$
$$a(b-c) = ab-ac$$
$$(b-c)a = ba-ca$$

In algebra, we use the Distributive Property to remove parentheses as we simplify expressions.

EXAMPLE 1.50

Simplify: $3(x+4)$.

Solution

	$3(x+4)$
Distribute.	$3 \cdot x + 3 \cdot 4$
Multiply.	$3x+12$

TRY IT : : 1.99 Simplify: $4(x+2)$.

TRY IT : : 1.100 Simplify: $6(x+7)$.

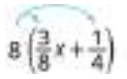
Some students find it helpful to draw in arrows to remind them how to use the Distributive Property. Then the first step in **Example 1.50** would look like this:



EXAMPLE 1.51

Simplify: $8\left(\frac{3}{8}x+\frac{1}{4}\right)$.

Solution



Distribute.	$8\cdot\frac{3}{8}x+8\cdot\frac{1}{4}$
Multiply.	$3x+2$

TRY IT : : 1.101 Simplify: $6\left(\frac{5}{6}y+\frac{1}{2}\right)$.

TRY IT : : 1.102 Simplify: $12\left(\frac{1}{3}n+\frac{3}{4}\right)$.

Using the Distributive Property as shown in the next example will be very useful when we solve money applications in later chapters.

EXAMPLE 1.52

Simplify: $100(0.3+0.25q)$.

Solution



Distribute.	$100(0.3)+100(0.25q)$
Multiply.	$30+25q$

TRY IT : : 1.103 Simplify: $100(0.7+0.15p)$.

TRY IT : : 1.104 Simplify: $100(0.04+0.35d)$.

When we distribute a negative number, we need to be extra careful to get the signs correct!

EXAMPLE 1.53

Simplify: $-11(4-3a)$.

Solution

	$-11(4-3a)$
Distribute.	$-11\cdot 4-(-11)\cdot 3a$
Multiply.	$-44-(-33a)$
Simplify.	$-44+33a$

Notice that you could also write the result as $33a-44$. Do you know why?

TRY IT : : 1.105 Simplify: $-5(2-3a)$.

TRY IT : : 1.106 Simplify: $-7(8-15y)$.

In the next example, we will show how to use the Distributive Property to find the opposite of an expression.

EXAMPLE 1.54

Simplify: $-(y+5)$.

Solution

	$-(y+5)$
Multiplying by -1 results in the opposite.	$-1(y+5)$
Distribute.	$-1\cdot y+(-1)\cdot 5$
Simplify.	$-y+(-5)$
Simplify.	$-y-5$

TRY IT : : 1.107 Simplify: $-(z-11)$.

TRY IT : : 1.108 Simplify: $-(x-4)$.

There will be times when we'll need to use the Distributive Property as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the Distributive Property, which removes the parentheses. The next two examples will illustrate this.

EXAMPLE 1.55

Simplify: $8-2(x+3)$

Solution

We follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract.

	$8-2(x+3)$
Distribute.	$8-2\cdot x-2\cdot 3$
Multiply.	$8-2x-6$
Combine like terms.	$-2x+2$

TRY IT : : 1.109 Simplify: $9-3(x+2)$.

TRY IT : : 1.110 Simplify: $7x-5(x+4)$.

EXAMPLE 1.56

Simplify: $4(x-8)-(x+3)$.

Solution

	$4(x-8)-(x+3)$
Distribute.	$4x-32-x-3$
Combine like terms.	$3x-35$

TRY IT : : 1.111 Simplify: $6(x-9)-(x+12)$.

TRY IT : : 1.112 Simplify: $8(x-1)-(x+5)$.

All the properties of real numbers we have used in this chapter are summarized here.

Commutative Property
When adding or multiplying, changing the *order* gives the same result

of addition If a, b are real numbers, then $a+b = b+a$

of multiplication If a, b are real numbers, then $a \cdot b = b \cdot a$

Associative Property
When adding or multiplying, changing the *grouping* gives the same result.

of addition If $a, b,$ and c are real numbers, then $(a+b)+c = a+(b+c)$

of multiplication If $a, b,$ and c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive Property

If $a, b,$ and c are real numbers, then

$$a(b+c) = ab+ac$$
$$(b+c)a = ba+ca$$
$$a(b-c) = ab-ac$$
$$(b-c)a = ba-ca$$

Identity Property

of addition For any real number a: $a+0=a$

0 is the **additive identity** $0+a=a$

of multiplication For any real number a: $a \cdot 1 = a$

1 is the **multiplicative identity** $1 \cdot a = a$

Inverse Property

of addition For any real number a, $a + (-a) = 0$

$-a$ is the **additive inverse** of a

A number and its *opposite* add to zero.

of multiplication For any real number a, $a \neq 0$ $a \cdot \frac{1}{a} = 1$

$\frac{1}{a}$ is the **multiplicative inverse** of a

A number and its *reciprocal* multiply to one.

Properties of Zero

For any real number a, $a \cdot 0 = 0$

$0 \cdot a = 0$

For any real number a, $a \neq 0$, $\frac{0}{a} = 0$

For any real number a, $\frac{a}{0}$ is undefine

1.5 EXERCISES

Practice Makes Perfect

Use the Commutative and Associative Properties

In the following exercises, simplify.

313. $43m + (-12n) + (-16m) + (-9n)$

314. $-22p + 17q + (-35p) + (-27q)$

315. $\frac{3}{8}g + \frac{1}{12}h + \frac{7}{8}g + \frac{5}{12}h$

316. $\frac{5}{6}a + \frac{3}{10}b + \frac{1}{6}a + \frac{9}{10}b$

317. $6.8p + 9.14q + (-4.37p) + (-0.88q)$

318. $9.6m + 7.22n + (-2.19m) + (-0.65n)$

319. $-24 \cdot 7 \cdot \frac{3}{8}$

320. $-36 \cdot 11 \cdot \frac{4}{9}$

321. $\left(\frac{5}{6} + \frac{8}{15}\right) + \frac{7}{15}$

322. $\left(\frac{11}{12} + \frac{4}{9}\right) + \frac{5}{9}$

323. $17(0.25)(4)$

324. $36(0.2)(5)$

325. $[2.48(12)](0.5)$

326. $[9.731(4)](0.75)$

327. $12\left(\frac{5}{6}p\right)$

328. $20\left(\frac{3}{5}q\right)$

Use the Properties of Identity, Inverse and Zero

In the following exercises, simplify.

329. $19a + 44 - 19a$

330. $27c + 16 - 27c$

331. $\frac{1}{2} + \frac{7}{8} + \left(-\frac{1}{2}\right)$

332. $\frac{2}{5} + \frac{5}{12} + \left(-\frac{2}{5}\right)$

333. $10(0.1d)$

334. $100(0.01p)$

335. $\frac{3}{20} \cdot \frac{49}{11} \cdot \frac{20}{3}$

336. $\frac{13}{18} \cdot \frac{25}{7} \cdot \frac{18}{13}$

337. $\frac{0}{u - 4.99}$, where $u \neq 4.99$

338. $0 \div \left(y - \frac{1}{6}\right)$, where $x \neq \frac{1}{6}$

339. $\frac{32 - 5a}{0}$, where $32 - 5a \neq 0$

340. $\frac{28 - 9b}{0}$, where $28 - 9b \neq 0$

341. $\left(\frac{3}{4} + \frac{9}{10}m\right) \div 0$, where $\frac{3}{4} + \frac{9}{10}m \neq 0$

342. $\left(\frac{5}{16}n - \frac{3}{7}\right) \div 0$, where $\frac{5}{16}n - \frac{3}{7} \neq 0$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

343. $8(4y + 9)$

344. $9(3w + 7)$

345. $6(c - 13)$

346. $7(y-13)$

347. $\frac{1}{4}(3q+12)$

348. $\frac{1}{5}(4m+20)$

349. $9\left(\frac{5}{9}y-\frac{1}{3}\right)$

350. $10\left(\frac{3}{10}x-\frac{2}{5}\right)$

351. $12\left(\frac{1}{4}+\frac{2}{3}r\right)$

352. $12\left(\frac{1}{6}+\frac{3}{4}s\right)$

353. $15\cdot\frac{3}{5}(4d+10)$

354. $18\cdot\frac{5}{6}(15h+24)$

355. $r(s-18)$

356. $u(v-10)$

357. $(y+4)p$

358. $(a+7)x$

359. $-7(4p+1)$

360. $-9(9a+4)$

361. $-3(x-6)$

362. $-4(q-7)$

363. $-(3x-7)$

364. $-(5p-4)$

365. $16-3(y+8)$

366. $18-4(x+2)$

367. $4-11(3c-2)$

368. $9-6(7n-5)$

369. $22-(a+3)$

370. $8-(r-7)$

371. $(5m-3)-(m+7)$

372. $(4y-1)-(y-2)$

373. $9(8x-3)-(-2)$

374. $4(6x-1)-(-8)$

375. $5(2n+9)+12(n-3)$

376. $9(5u+8)+2(u-6)$

377. $14(c-1)-8(c-6)$

378. $11(n-7)-5(n-1)$

379. $6(7y+8)-(30y-15)$

380. $7(3n+9)-(4n-13)$

Writing Exercises

381. In your own words, state the Associative Property of addition.

382. What is the difference between the additive inverse and the multiplicative inverse of a number?

383. Simplify $8\left(x-\frac{1}{4}\right)$ using the Distributive Property and explain each step.

384. Explain how you can multiply $4(\$5.97)$ without paper or calculator by thinking of $\$5.97$ as $6-0.03$ and then using the Distributive Property.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
use the commutative and associative properties.			
use the properties of identity, inverse and zero.			
simplify expressions using the distributive property.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

CHAPTER 1 REVIEW

KEY TERMS

absolute value The absolute value of a number is its distance from 0 on the number line.

additive identity The number 0 is the additive identity because adding 0 to any number does not change its value.

additive inverse The opposite of a number is its additive inverse.

coefficient The coefficient of a term is the constant that multiplies the variable in a term.

complex fraction A fraction in which the numerator or the denominator is a fraction is called a complex fraction.

composite number A composite number is a counting number that is not prime. It has factors other than 1 and the number itself.

constant A constant is a number whose value always stays the same.

denominator In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the denominator *b* is the number of equal parts the whole has been divided into.

divisible by a number If a number *m* is a multiple of *n*, then *m* is divisible by *n*.

equation An equation is two expressions connected by an equal sign.

equivalent fractions Equivalent fractions are fractions that have the same value.

evaluate an expression To evaluate an expression means to find the value of the expression when the variables are replaced by a given number.

expression An expression is a number, a variable, or a combination of numbers and variables using operation symbols.

factors If $a \cdot b = m$, then *a* and *b* are factors of *m*.

fraction A fraction is written $\frac{a}{b}$, where $b \neq 0$, and *a* is the numerator and *b* is the denominator. A fraction represents parts of a whole.

integers The whole numbers and their opposites are called the integers.

irrational number An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

least common denominator The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

least common multiple The least common multiple (LCM) of two numbers is the smallest number that is a multiple of both numbers.

like terms Terms that are either constants or have the same variables raised to the same powers are called like terms.

multiple of a number A number is a multiple of *n* if it is the product of a counting number and *n.*

multiplicative identity The number 1 is the multiplicative identity because multiplying 1 by any number does not change its value.

multiplicative inverse The reciprocal of a number is its multiplicative inverse.

negative numbers Numbers less than 0 are negative numbers.

numerator In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the numerator *a* indicates how many parts are included.

opposite The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

order of operations The order of operations are established guidelines for simplifying an expression.

percent A percent is a ratio whose denominator is 100.

prime factorization The prime factorization of a number is the product of prime numbers that equals the number.

prime number A prime number is a counting number greater than 1 whose only factors are 1 and the number itself.

principal square root The positive square root is called the principal square root.

rational number A rational number is a number of the form $\frac{p}{q}$, where *p* and *q* are integers and $q \neq 0$. Its decimal form stops or repeats.

real number A real number is a number that is either rational or irrational.

reciprocal The reciprocal of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator.

simplify an expression To simplify an expression means to do all the math possible.

square of a number If $n^2 = m$, then m is the square of n.

square root of a number If $n^2 = m$, then n is a square root of m.

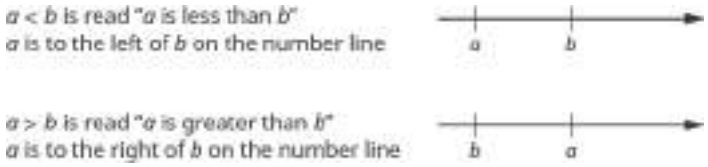
term A term is a constant, or the product of a constant and one or more variables.

variable A variable is a letter that represents a number whose value may change.

KEY CONCEPTS

1.1 Use the Language of Algebra

- **Divisibility Tests**
 A number is divisible by:
 2 if the last digit is 0, 2, 4, 6, or 8.
 3 if the sum of the digits is divisible by 3.
 5 if the last digit is 5 or 0.
 6 if it is divisible by both 2 and 3.
 10 if it ends with 0.
- **How to find the prime factorization of a composite number.**
 Step 1. Find two factors whose product is the given number, and use these numbers to create two branches.
 Step 2. If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
 Step 3. If a factor is not prime, write it as the product of two factors and continue the process.
 Step 4. Write the composite number as the product of all the circled primes.
- **How To Find the least common multiple using the prime factors method.**
 Step 1. Write each number as a product of primes.
 Step 2. List the primes of each number. Match primes vertically when possible.
 Step 3. Bring down the columns.
 Step 4. Multiply the factors.
- **Equality Symbol**
 $a = b$ is read "a is equal to b."
 The symbol "=" is called the equal sign.
- **Inequality**



- **Inequality Symbols**

Inequality Symbols	Words
$a \neq b$	*a* is *not equal to b.*
$a < b$	*a* is *less than b.*
$a \leq b$	*a* is *less than or equal to b.*
$a > b$	*a* is *greater than b.*
$a \geq b$	*a* is *greater than or equal to b.*

Table 1.33

- **Grouping Symbols**
 Parentheses ()
 Brackets []
 Braces { }
- **Exponential Notation**
 a^n means multiply *a* by itself, *n* times.
 The expression a^n is read *a* to the n^{th} power.
- **Simplify an Expression**
 To simplify an expression, do all operations in the expression.
- **How to use the order of operations.**
 Step 1. Parentheses and Other Grouping Symbols
 - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

 Step 2. Exponents
 - Simplify all expressions with exponents.

 Step 3. Multiplication and Division
 - Perform all multiplication and division in order from left to right. These operations have equal priority.

 Step 4. Addition and Subtraction
 - Perform all addition and subtraction in order from left to right. These operations have equal priority.
- **How to combine like terms.**
 Step 1. Identify like terms.
 Step 2. Rearrange the expression so like terms are together.
 Step 3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Operation	Phrase	Expression
Addition	*a* plus *b* the sum of a and *b* *a* increased by *b* *b* more than *a* the total of *a* and *b* *b* added to *a*	$a+b$
Subtraction	*a* minus b the difference of *a* and *b* *a* decreased by *b* *b* less than *a* *b* subtracted from *a*	$a-b$
Multiplication	*a* times *b* the product of a and b twice *a*	$a \cdot b,\ ab,\ a(b),\ (a)(b)$ $2a$
Division	*a* divided by *b* the quotient of *a* and *b* the ratio of *a* and *b* *b* divided into *a*	$a \div b,\ a/b,\ \frac{a}{b},\ b\overline{)a}$

Table 1.34

1.2 Integers

- **Opposite Notation**

 $-a$ means the opposite of the number a

 The notation $-a$ is read as "the opposite of a."

- **Absolute Value**
 The absolute value of a number is its distance from 0 on the number line.
 The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.
 Absolute values are always greater than or equal to zero.

- **Grouping Symbols**

Parentheses	()	Braces	{ }
Brackets	[]	Absolute value	\| \|

- **Subtraction Property**

 $a-b=a+(-b)$

 Subtracting a number is the same as adding its opposite.

- **Multiplication and Division of Signed Numbers**
 For multiplication and division of two signed numbers:

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

If the signs are the same, the result is positive.

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

If the signs are different, the result is negative.

- **Multiplication by -1**
 $-1a = -a$
 Multiplying a number by -1 gives its opposite.
- **How to Use Integers in Applications.**
 Step 1. **Read** the problem. Make sure all the words and ideas are understood
 Step 2. **Identify** what we are asked to find.
 Step 3. **Write a phrase** that gives the information to find it.
 Step 4. **Translate** the phrase to an expression.
 Step 5. **Simplify** the expression.
 Step 6. **Answer** the question with a complete sentence.

1.3 Fractions

- **Equivalent Fractions Property**
 If *a*, *b*, and *c* are numbers where $b \neq 0,\ c \neq 0,$ then
 $$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$
- **How to simplify a fraction.**
 Step 1. Rewrite the numerator and denominator to show the common factors.
 If needed, factor the numerator and denominator into prime numbers first.
 Step 2. Simplify using the Equivalent Fractions Property by dividing out common factors.
 Step 3. Multiply any remaining factors.
- **Fraction Multiplication**
 If *a*, *b*, *c*, and *d* are numbers where $b \neq 0,$ and $d \neq 0,$ then
 $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$
 To multiply fractions, multiply the numerators and multiply the denominators.
- **Fraction Division**
 If *a*, *b*, *c*, and *d* are numbers where $b \neq 0,\ c \neq 0,$ and $d \neq 0,$ then
 $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$
 To divide fractions, we multiply the first fraction by the reciprocal of the second.
- **Fraction Addition and Subtraction**
 If *a*, *b*, and *c* are numbers where $c \neq 0,$ then
 $$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \text{ and } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$
 To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.
- **How to add or subtract fractions.**
 Step 1. Do they have a common denominator?
 - Yes—go to step 2.
 - No—rewrite each fraction with the LCD (least common denominator).
 - Find the LCD.
 - Change each fraction into an equivalent fraction with the LCD as its denominator.

 Step 2. Add or subtract the fractions.

Step 3. Simplify, if possible.

- **How to simplify an expression with a fraction bar.**

 Step 1. Simplify the expression in the numerator. Simplify the expression in the denominator.

 Step 2. Simplify the fraction.

- **Placement of Negative Sign in a Fraction**
 For any positive numbers a and b,
 $$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}.$$

- **How to simplify complex fractions.**

 Step 1. Simplify the numerator.

 Step 2. Simplify the denominator.

 Step 3. Divide the numerator by the denominator. Simplify if possible.

1.4 Decimals

- **How to round decimals.**

 Step 1. Locate the given place value and mark it with an arrow.

 Step 2. Underline the digit to the right of the place value.

 Step 3. Is the underlined digit greater than or equal to 5?

 - Yes: add 1 to the digit in the given place value.
 - No: do not change the digit in the given place value

 Step 4. Rewrite the number, deleting all digits to the right of the rounding digit.

- **How to add or subtract decimals.**

 Step 1. Determine the sign of the sum or difference.

 Step 2. Write the numbers so the decimal points line up vertically.

 Step 3. Use zeros as placeholders, as needed.

 Step 4. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.

 Step 5. Write the sum or difference with the appropriate sign

- **How to multiply decimals.**

 Step 1. Determine the sign of the product.

 Step 2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.

 Step 3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.

 Step 4. Write the product with the appropriate sign.

- **How to multiply a decimal by a power of ten.**

 Step 1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.

 Step 2. Add zeros at the end of the number as needed.

- **How to divide decimals.**

 Step 1. Determine the sign of the quotient.

 Step 2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.

 Step 3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.

 Step 4. Write the quotient with the appropriate sign.

- **How to convert a decimal to a proper fraction and a fraction to a decimal.**

 Step 1. To convert a decimal to a proper fraction, determine the place value of the final digit.

 Step 2. Write the fraction.

 - numerator—the "numbers" to the right of the decimal point
 - denominator—the place value corresponding to the final digit

Step 3. To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

- **How to convert a percent to a decimal and a decimal to a percent.**

 Step 1. To convert a percent to a decimal, move the decimal point two places to the left after removing the percent sign.

 Step 2. To convert a decimal to a percent, move the decimal point two places to the right and then add the percent sign.

- **Square Root Notation**
 $\sqrt{m}$ is read "the square root of *m*."

 If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

 The square root of *m*, $\sqrt{m}$, is the positive number whose square is *m*.

- **Rational or Irrational**
 If the decimal form of a number
 - *repeats or stops*, the number is a rational number.
 - *does not repeat and does not stop*, the number is an irrational number.

- **Real Numbers**

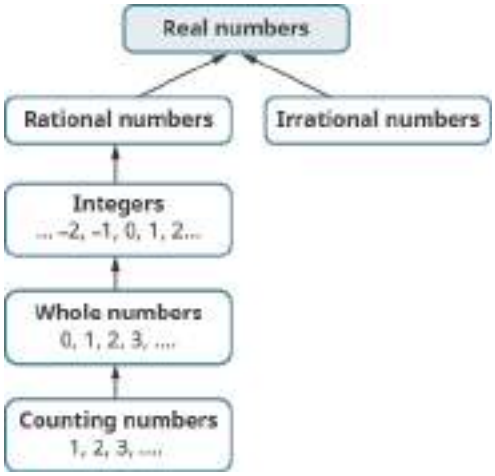


Figure 1.9

1.5 Properties of Real Numbers

Commutative Property
When adding or multiplying, changing the *order* gives the same result

of addition If a, b are real numbers, then $a + b = b + a$

of multiplication If a, b are real numbers, then $a \cdot b = b \cdot a$

Associative Property
When adding or multiplying, changing the *grouping* gives the same result.

of addition If $a, b,$ and c are real numbers, then $(a + b) + c = a + (b + c)$

of multiplication If $a, b,$ and c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive Property

If a, b, and c are real numbers, then

$$a(b+c) = ab+ac$$
$$(b+c)a = ba+ca$$
$$a(b-c) = ab-ac$$
$$(b-c)a = ba-ca$$

Identity Property

of addition For any real number a: $a+0=a$

0 is the **additive identity** $0+a=a$

of multiplication For any real number a: $a\cdot 1=a$

1 is the **multiplicative identity** $1\cdot a=a$

Inverse Property

of addition For any real number a, $a+(-a)=0$

$-a$ is the **additive inverse** of a

A number and its *opposite* add to zero.

of multiplication For any real number a, $a\neq 0$ $a\cdot\frac{1}{a}=1$

$\frac{1}{a}$ is the **multiplicative inverse** of a

A number and its *reciprocal* multiply to one.

Properties of Zero

For any real number a, $a\cdot 0=0$

$0\cdot a=0$

For any real number a, $a\neq 0$, $\frac{0}{a}=0$

For any real number a, $\frac{a}{0}$ is undefine

REVIEW EXERCISES

1.1 Use the Language of Algebra

Identify Multiples and Factors

385. Use the divisibility tests to determine whether 180 is divisible by 2, by 3, by 5, by 6, and by 10.

386. Find the prime factorization of 252.

387. Find the least common multiple of 24 and 40.

In the following exercises, simplify each expression.

388. $24\div 3+4(5-2)$

389. $7+3[6-4(5-4)]-3^2$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

390. When $x = 4$, ⓐ x^3 ⓑ 5^x ⓒ $2x^2 - 5x + 3$

391. $2x^2 - 4xy - 3y^2$ when $x = 3, \quad y = 1$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.

392. $12y + 7 + 2y - 5$

393. $14x^2 - 9x + 11 - 8x^2 + 8x - 6$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the phrases into algebraic expressions.

394.

ⓐ the sum of $4ab^2$ and $7a^3b^2$

ⓑ the product of $6y^2$ and $3y$

ⓒ twelve more than $5x$

ⓓ $5y$ less than $8y^2$

395.

ⓐ eleven times the difference of y and two

ⓑ the difference of eleven times y and two

396. Dushko has nickels and pennies in his pocket. The number of pennies is four less than five the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

1.2 Integers

Simplify Expressions with Absolute Value

In the following exercise, fill in $<$, $>$, or $=$ for each of the following pairs of numbers.

397.

ⓐ $-|7|___ - |-7|$

ⓑ $-8___ - |-8|$

ⓒ $|-13|___ - 13$

ⓓ $|-12|___ - (-12)$

In the following exercises, simplify.

398. $9 - |3(4 - 8)|$

399. $12 - 3|1 - 4(4 - 2)|$

Add and Subtract Integers

In the following exercises, simplify each expression.

400. $-12 + (-8) + 7$

401.

ⓐ $15 - 7$

ⓑ $-15 - (-7)$

ⓒ $-15 - 7$

ⓓ $15 - (-7)$

402. $-11 - (-12) + 5$

403. ⓐ $23 - (-17)$ ⓑ $23 + 17$

404. $-(7 - 11) - (3 - 5)$

Multiply and Divide Integers

In the following exercise, multiply or divide.

405. ⓐ $-27 \div 9$ ⓑ $120 \div (-8)$
ⓒ $4(-14)$ ⓓ $-1(-17)$

Simplify and Evaluate Expressions with Integers

In the following exercises, simplify each expression.

406. ⓐ $(-7)^3$ ⓑ -7^3

407. $(7-11)(6-13)$

408. $63 \div (-9) + (-36) \div (-4)$

409. $6-3|4(1-2)-(7-5)|$

410. $(-2)^4 - 24 \div (13-5)$

For the following exercises, evaluate each expression.

411. $(y+z)^2$ when $y=-4,\ z=7$

412. $3x^2 - 2xy + 4y^2$ when $x=-2,\ y=-3$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

413. the sum of -4 and -9, increased by 23

414. ⓐ the difference of 17 and -8 ⓑ subtract 17 from -25

Use Integers in Applications

In the following exercise, solve.

415. **Temperature** On July 10, the high temperature in Phoenix, Arizona, was 109°, and the high temperature in Juneau, Alaska, was 63°. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?

1.3 Fractions

Simplify Fractions

In the following exercises, simplify.

416. $\frac{204}{228}$

417. $-\frac{270x^3}{198y^2}$

Multiply and Divide Fractions

In the following exercises, perform the indicated operation.

418. $\left(-\frac{14}{15}\right)\left(\frac{10}{21}\right)$

419. $\frac{6x}{25} \div \frac{9y}{20}$

420. $\frac{-\frac{4}{9}}{\frac{8}{21}}$

Add and Subtract Fractions

In the following exercises, perform the indicated operation.

421. $\frac{5}{18} + \frac{7}{12}$

422. $\frac{11}{36} - \frac{15}{48}$

423. ⓐ $\frac{5}{8} + \frac{3}{4}$ ⓑ $\frac{5}{8} \div \frac{3}{4}$

424. ⓐ $-\frac{3y}{10}-\frac{5}{6}$ ⓑ $-\frac{3y}{10}\cdot\frac{5}{6}$

Use the Order of Operations to Simplify Fractions

In the following exercises, simplify.

425. $\frac{4\cdot 3-2\cdot 5}{-6\cdot 3+2\cdot 3}$

426. $\frac{4(7-3)-2(4-9)}{-3(4+2)+7(3-6)}$

427. $\frac{4^3-4^2}{\left(\frac{4}{5}\right)^2}$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

428. $4x^2y^2$ when $x=\frac{2}{3}$ and $y=-\frac{3}{4}$

429. $\frac{a+b}{a-b}$ when $a=-4,\quad b=6$

1.4 Decimals

Round Decimals

430. Round 6.738 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Add and Subtract Decimals

In the following exercises, perform the indicated operation.

431. $-23.67+29.84$

432. $54.3-100$

433. $79.38-(-17.598)$

Multiply and Divide Decimals

In the following exercises, perform the indicated operation.

434. $(-2.8)(3.97)$

435. $(-8.43)(-57.91)$

436. $(53.48)(10)$

437. $(0.563)(100)$

438. $\$118.35\div 2.6$

439. $1.84\div(-0.8)$

Convert Decimals, Fractions and Percents

In the following exercises, write each decimal as a fraction.

440. $\frac{13}{20}$

441. $-\frac{240}{25}$

In the following exercises, convert each fraction to a decimal.

442. $-\frac{5}{8}$

443. $\frac{14}{11}$

In the following exercises, convert each decimal to a percent.

444. 2.43

445. 0.0475

Simplify Expressions with Square Roots

In the following exercises, simplify.

446. $\sqrt{289}$

447. $\sqrt{-121}$

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercise, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers for each set of numbers

448.
$-8,\ 0,\ 1.95286\ldots,\ \frac{12}{5},\ \sqrt{36},\ 9$

Locate Fractions and Decimals on the Number Line

In the following exercises, locate the numbers on a number line.

449. $\frac{3}{4},\ -\frac{3}{4},\ 1\frac{1}{3},\ -1\frac{2}{3},\ \frac{7}{2},\ -\frac{5}{2}$

450. ⓐ 3.2 ⓑ -1.35

1.5 Properties of Real Numbers

Use the Commutative and Associative Properties

In the following exercises, simplify.

451. $\frac{5}{8}x+\frac{5}{12}y+\frac{1}{8}x+\frac{7}{12}y$

452. $-32\cdot 9\cdot\frac{5}{8}$

453. $\left(\frac{11}{15}+\frac{3}{8}\right)+\frac{5}{8}$

Use the Properties of Identity, Inverse and Zero

In the following exercises, simplify.

454. $\frac{4}{7}+\frac{8}{15}+\left(-\frac{4}{7}\right)$

455. $\frac{13}{15}\cdot\frac{9}{17}\cdot\frac{15}{13}$

456. $\frac{0}{x-3},\ x\neq 3$

457. $\frac{5x-7}{0},\ 5x-7\neq 0$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

458. $8(a-4)$

459. $12\left(\frac{2}{3}b+\frac{5}{6}\right)$

460. $18\cdot\frac{5}{6}(2x-5)$

461. $(x-5)p$

462. $-4(y-3)$

463. $12-6(x+3)$

464. $6(3x-4)-(-5)$

465. $5(2y+3)-(4y-1)$

PRACTICE TEST

466. Find the prime factorization of 756.

467. Combine like terms: $5n + 8 + 2n - 1$

468. Evaluate when $x = -2$ and $y = 3$: $\frac{|3x - 4y|}{6}$

469. Translate to an algebraic expression and simplify:

ⓐ eleven less than negative eight

ⓑ the difference of -8 and -3, increased by 5

470. Dushko has nickels and pennies in his pocket. The number of pennies is seven less than four times the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

471. Round 28.1458 to the nearest

ⓐ hundredth ⓑ thousandth

472. Convert

ⓐ $\frac{5}{11}$ to a decimal ⓑ 1.15 to a percent

473. Locate $\frac{3}{5}$, 2.8, and $-\frac{5}{2}$ on a number line.

In the following exercises, simplify each expression.

474. $8 + 3[6 - 3(5 - 2)] - 4^2$

475. $-(4 - 9) - (9 - 5)$

476. $56 \div (-8) + (-27) \div (-3)$

477. $16 - 2|3(1 - 4) - (8 - 5)|$

478. $-5 + 2(-3)^2 - 9$

479. $\frac{180}{204}$

480. $-\frac{7}{18} + \frac{5}{12}$

481. $\frac{4}{5} \div \left(-\frac{12}{25}\right)$

482. $\frac{9 - 3 \cdot 9}{15 - 9}$

483. $\frac{4(-3 + 2(3 - 6))}{3(11 - 3(2 + 3))}$

484. $\frac{5}{13} \cdot 47 \cdot \frac{13}{5}$

485. $\frac{-\frac{5}{9}}{\frac{10}{21}}$

486. $-4.8 + (-6.7)$

487. $34.6 - 100$

488. $-12.04 \cdot (4.2)$

489. $-8 \div 0.05$

490. $\sqrt{-121}$

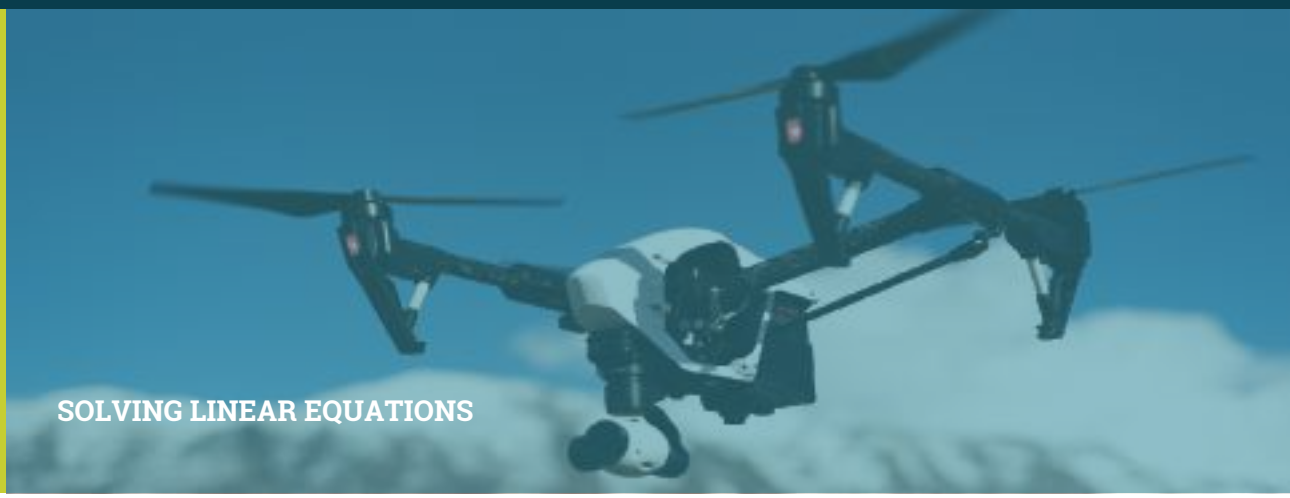
491. $\left(\frac{8}{13} + \frac{5}{7}\right) + \frac{2}{7}$

492. $5x + (-8y) - 6x + 3y$

493. ⓐ $\frac{0}{9}$ ⓑ $\frac{11}{0}$

494. $-3(8x - 5)$

495. $6(3y - 1) - (5y - 3)$



2 SOLVING LINEAR EQUATIONS

Figure 2.1 This drone is flying high in the sky while its pilot remains safely on the ground. (credit: "Unsplash" / Pixabay)

Chapter Outline

2.1 Use a General Strategy to Solve Linear Equations
2.2 Use a Problem Solving Strategy
2.3 Solve a Formula for a Specific Variable
2.4 Solve Mixture and Uniform Motion Applications
2.5 Solve Linear Inequalities
2.6 Solve Compound Inequalities
2.7 Solve Absolute Value Inequalities

Introduction

Imagine being a pilot, but not just any pilot—a drone pilot. Drones, or unmanned aerial vehicles, are devices that can be flown remotely. They contain sensors that can relay information to a command center where the pilot is located. Larger drones can also carry cargo. In the near future, several companies hope to use drones to deliver materials and piloting a drone will become an important career. Law enforcement and the military are using drones rather than send personnel into dangerous situations. Building and piloting a drone requires the ability to program a set of actions, including taking off, turning, and landing. This, in turn, requires the use of linear equations. In this chapter, you will explore linear equations, develop a strategy for solving them, and relate them to real-world situations.

2.1 Use a General Strategy to Solve Linear Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve linear equations using a general strategy
- Classify equations
- Solve equations with fraction or decimal coefficients

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{3}{2}(12x + 20)$.
 If you missed this problem, review Example 1.51.
2. Simplify: $5 - 2(n + 1)$.
 If you missed this problem, review Example 1.55.

3. Find the LCD of $\frac{5}{6}$ and $\frac{1}{4}$.

If you missed this problem, review Example 1.28.

Solve Linear Equations Using a General Strategy

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that makes it a true statement. Any value of the variable that makes the equation true is called a **solution** to the equation. It is the answer to the puzzle!

Solution of an Equation

A **solution** of an equation is a value of a variable that makes a true statement when substituted into the equation.

To determine whether a number is a solution to an equation, we substitute the value for the variable in the equation. If the resulting equation is a true statement, then the number is a solution of the equation.



HOW TO : : DETERMINE WHETHER A NUMBER IS A SOLUTION TO AN EQUATION.

Step 1. Substitute the number for the variable in the equation.

Step 2. Simplify the expressions on both sides of the equation.

Step 3. Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

EXAMPLE 2.1

Determine whether the values are solutions to the equation: $5y + 3 = 10y - 4$.

ⓐ $y = \frac{3}{5}$ ⓑ $y = \frac{7}{5}$

Solution

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

ⓐ

	$5y + 3 = 10y - 4$
Substitute $\frac{3}{5}$ for y.	$5\left(\frac{3}{5}\right) + 3 \stackrel{?}{=} 10\left(\frac{3}{5}\right) - 4$
Multiply.	$3 + 3 \stackrel{?}{=} 6 - 4$
Simplify.	$6 \neq 2$

Since $y = \frac{3}{5}$ does not result in a true equation, $y = \frac{3}{5}$ is not a solution to the equation $5y + 3 = 10y - 4$.

ⓑ

	$5y+3=10y-4$
Substitute $\frac{7}{5}$ for y.	$5\left(\frac{7}{5}\right)+3 \stackrel{?}{=} 10\left(\frac{7}{5}\right)-4$
Multiply.	$7+3 \stackrel{?}{=} 14-4$
Simplify.	$10=10$ ✓

Since $y=\frac{7}{5}$ results in a true equation, $y=\frac{7}{5}$ is a solution to the equation $5y+3=10y-4$.

TRY IT : : 2.1 Determine whether the values are solutions to the equation: $9y+2=6y+3$.

ⓐ $y=\frac{4}{3}$ ⓑ $y=\frac{1}{3}$

TRY IT : : 2.2 Determine whether the values are solutions to the equation: $4x-2=2x+1$.

ⓐ $x=\frac{3}{2}$ ⓑ $x=-\frac{1}{2}$

There are many types of equations that we will learn to solve. In this section we will focus on a **linear equation**.

Linear Equation

A **linear equation** is an equation in one variable that can be written, where a and b are real numbers and $a \neq 0$, as:

$$ax+b=0$$

To solve a linear equation it is a good idea to have an overall strategy that can be used to solve any linear equation. In the next example, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

EXAMPLE 2.2 HOW TO SOLVE A LINEAR EQUATION USING A GENERAL STRATEGY

Solve: $7(n-3)-8=-15$.

Solution

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do—all n's are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n-29+29=-15+29$ $7n=14$
Step 4. Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7}=\frac{14}{7}$ $n=2$

Step 5. Check the solution.	Let $n = 2$ Subtract.	**Check:** $7(n-3)-8=-15$ $7(2-3)-8\stackrel{?}{=}-15$ $7(-1)-8\stackrel{?}{=}-15$ $-7-8\stackrel{?}{=}-15$ $-15=-15$ ✓

TRY IT : : 2.3 Solve: $2(m-4)+3=-1$.

TRY IT : : 2.4 Solve: $5(a-3)+5=-10$.

These steps are summarized in the General Strategy for Solving Linear Equations below.

HOW TO : : SOLVE LINEAR EQUATIONS USING A GENERAL STRATEGY.

Step 1. Simplify each side of the equation as much as possible.
Use the Distributive Property to remove any parentheses.
Combine like terms.

Step 2. Collect all the variable terms on one side of the equation.
Use the Addition or Subtraction Property of Equality.

Step 3. Collect all the constant terms on the other side of the equation.
Use the Addition or Subtraction Property of Equality.

Step 4. Make the coefficient of the variable term equal to 1.
Use the Multiplication or Division Property of Equality.
State the solution to the equation.

Step 5. Check the solution.
Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 2.3

Solve: $\frac{2}{3}(3m-6)=5-m$.

Solution

	$\frac{2}{3}(3m-6)=5-m$
Distribute.	$2m-4=5-m$
Add m to both sides to get the variables only on the left.	$2m+m-4=5-m+m$
Simplify.	$3m-4=5$
Add 4 to both sides to get constants only on the right.	$3m-4+4=5+4$
Simplify.	$3m=9$
Divide both sides by three.	$\frac{3m}{3}=\frac{9}{3}$
Simplify.	$m=3$
Check: $\frac{2}{3}(3m-6)=5-m$	

Let $m = 3$.	$\frac{2}{3}(3 \cdot 3 - 6) \stackrel{?}{=} 5 - 3$
	$\frac{2}{3}(9 - 6) \stackrel{?}{=} 2$
	$\frac{2}{3}(3) \stackrel{?}{=} 2$
	$2 = 2 \checkmark$

TRY IT : : 2.5 Solve: $\frac{1}{3}(6u + 3) = 7 - u$.

TRY IT : : 2.6 Solve: $\frac{2}{3}(9x - 12) = 8 + 2x$.

We can solve equations by getting all the variable terms to either side of the equal sign. By collecting the variable terms on the side where the coefficient of the variable is larger, we avoid working with some negatives. This will be a good strategy when we solve inequalities later in this chapter. It also helps us prevent errors with negatives.

EXAMPLE 2.4

Solve: $4(x - 1) - 2 = 5(2x + 3) + 6$.

Solution

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract $4x$ from each side to get the variables only on the right since $10 > 4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 from each side to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$
Divide both sides by 6.	$\frac{-27}{6}=\frac{6x}{6}$
Simplify.	$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$
Let $x = -\frac{9}{2}$.	$4\left(-\frac{9}{2}-1\right)-2 \stackrel{?}{=} 5\left(2\left(-\frac{9}{2}\right)+3\right)+6$
	$4\left(-\frac{11}{2}\right)-2 \stackrel{?}{=} 5(-9+3)+6$
	$-22-2 \stackrel{?}{=} 5(-6)+6$
	$-24 \stackrel{?}{=} -30+6$
	$-24=-24 \checkmark$

TRY IT : : 2.7 Solve: $6(p-3)-7=5(4p+3)-12.$

TRY IT : : 2.8 Solve: $8(q+1)-5=3(2q-4)-1.$

EXAMPLE 2.5

Solve: $10[3-8(2s-5)]=15(40-5s).$

Solution

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add $160s$ to both sides to get the variables to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$
Subtract 600 from both sides to get the constants to the left.	$430-600=600+85s-600$
Simplify.	$-170=85s$
Divide both sides by 85.	$\frac{-170}{85}=\frac{85s}{85}$
Simplify.	$-2=s$
Check:	$10[3-8(2s-5)]=15(40-5s)$
Let $s=-2$.	$10[3-8(2(-2)-5)]\stackrel{?}{=}15(40-5(-2))$
	$10[3-8(-4-5)]\stackrel{?}{=}15(40+10)$
	$10[3-8(-9)]\stackrel{?}{=}15(50)$
	$10(3+72)\stackrel{?}{=}750$
	$10(75)\stackrel{?}{=}750$
	$750=750\checkmark$

TRY IT : : 2.9 Solve: $6[4-2(7y-1)]=8(13-8y).$

TRY IT : : 2.10 Solve: $12[1-5(4z-1)]=3(24+11z).$

Classify Equations

Whether or not an equation is true depends on the value of the variable. The equation $7x+8=-13$ is true when we replace the variable, *x*, with the value $-3,$ but not true when we replace *x* with any other value. An equation like this is called a **conditional equation**. All the equations we have solved so far are conditional equations.

Conditional Equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a

conditional equation.

Now let's consider the equation $7y + 14 = 7(y + 2)$. Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for *y*.

Solve:

	$7y + 14 = 7(y + 2)$
Distribute.	$7y + 14 = 7y + 14$
Subtract $7y$ to each side to get the y's to one side.	$7y - 7y + 14 = 7y - 7y + 14$
Simplify—the *y*'s are eliminated.	$14 = 14$
	But $14 = 14$ is true.

This means that the equation $7y + 14 = 7(y + 2)$ is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable is called an **identity**.

Identity

An equation that is true for any value of the variable is called an **identity**.
The solution of an identity is all real numbers.

What happens when we solve the equation $-8z = -8z + 9$?

Solve:

	$-8z = -8z + 9$
Add $8z$ to both sides to leave the constant alone on the right.	$-8z + 8z = -8z + 8z + 9$
Simplify—the z's are eliminated.	$0 \neq 9$
	But $0 \neq 9$.

Solving the equation $-8z = -8z + 9$ led to the false statement $0 = 9$. The equation $-8z = -8z + 9$ will not be true for any value of *z*. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **contradiction**.

Contradiction

An equation that is false for all values of the variable is called a **contradiction**.
A contradiction has no solution.

The next few examples will ask us to classify an equation as conditional, an identity, or as a contradiction.

EXAMPLE 2.6

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6(2n - 1) + 3 = 2n - 8 + 5(2n + 1)$.

Solution

	$6(2n-1)+3=2n-8+5(2n+1)$
Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract $12n$ from each side to get the n's to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$
This is a true statement.	The equation is an identity.
	The solution is all real numbers.

TRY IT : : 2.11

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $4+9(3x-7)=-42x-13+23(3x-2)$.

TRY IT : : 2.12

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $8(1-3x)+15(2x+7)=2(x+50)+4(x+3)+1$.

EXAMPLE 2.7

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $8+3(a-4)=0$.

Solution

	$8+3(a-4)=0$
Distribute.	$8+3a-12=0$
Combine like terms.	$3a-4=0$
Add 4 to both sides.	$3a-4+4=0+4$
Simplify.	$3a=4$
Divide.	$\frac{3a}{3}=\frac{4}{3}$
Simplify.	$a=\frac{4}{3}$
The equation is true when $a=\frac{4}{3}$.	This is a conditional equation.
	The solution is $a=\frac{4}{3}$.

TRY IT : : 2.13

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q+3)-5=19$.

TRY IT : : 2.14

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14(k-8)=95$.

EXAMPLE 2.8

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $5m+3(9+3m)=2(7m-11)$.

Solution

	$5m+3(9+3m)=2(7m-11)$
Distribute.	$5m+27+9m=14m-22$
Combine like terms.	$14m+27=14m-22$
Subtract $14m$ from both sides.	$14m+27-14m=14m-22-14m$
Simplify.	$27\neq -22$
But $27\neq -22$.	The equation is a contradiction.
	It has no solution.

TRY IT : : 2.15

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $12c+5(5+3c)=3(9c-4)$.

TRY IT : : 2.16

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $4(7d+18)=13(3d-2)-11d$.

We summarize the methods for classifying equations in the table.

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	True for any value of the variable	All real numbers
Contradiction	False for all values of the variable	No solution

Table 2.9

Solve Equations with Fraction or Decimal Coefficients

We could use the General Strategy to solve the next example. This method would work fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator (LCD) of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called *clearing* the equation of fractions.

To clear an equation of decimals, we think of all the decimals in their fraction form and then find the LCD of those denominators.

EXAMPLE 2.9 HOW TO SOLVE EQUATIONS WITH FRACTION OR DECIMAL COEFFICIENTS

Solve: $\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$.

Solution

Step 1. Find the least common denominator of *all* the fractions and decimals in the equation.	What is the LCD of $\frac{1}{12}$, $\frac{5}{6}$, and $\frac{3}{4}$?	$\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$ LCD = 12
Step 2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.	Multiply both sides of the equation by the LCD, 12. Use the Distributive Property. Simplify—and notice, no more fractions!	$12\left(\frac{1}{12}x+\frac{5}{6}\right)=12\left(\frac{3}{4}\right)$ $12\cdot\frac{1}{12}x+12\cdot\frac{5}{6}=12\cdot\frac{3}{4}$ $x+10=9$
Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the variable term, subtract 10. Simplify.	$x+10-10=9-10$ $x=-1$ **Check:** $\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$ $\frac{1}{12}(-1)+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{10}{12}\stackrel{?}{=}\frac{9}{12}$ $\frac{9}{12}=\frac{9}{12}$ ✓

TRY IT : : 2.17 Solve: $\frac{1}{4}x+\frac{1}{2}=\frac{5}{8}$.

TRY IT : : 2.18 Solve: $\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$.

Notice in the previous example, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve. We then used the General Strategy for Solving Linear Equations.

HOW TO : : SOLVE EQUATIONS WITH FRACTION OR DECIMAL COEFFICIENTS.

Step 1. Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.

Step 2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.

Step 3. Solve using the General Strategy for Solving Linear Equations.

EXAMPLE 2.10

Solve: $5=\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y$.

Solution

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation.	$5=\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y$
The LCD is 12.	
Multiply both sides of the equation by 12.	$12(5)=12\cdot\left(\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y\right)$
Distribute.	$12(5)=12\cdot\frac{1}{2}y+12\cdot\frac{2}{3}y-12\cdot\frac{3}{4}y$
Simplify—notice, no more fractions.	$60=6y+8y-9y$
Combine like terms.	$60=5y$
Divide by five.	$\frac{60}{5}=\frac{5y}{5}$
Simplify.	$12=y$
Check:	$5=\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y$
Let $y=12$.	$5\stackrel{?}{=}\frac{1}{2}(12)+\frac{2}{3}(12)-\frac{3}{4}(12)$
	$5\stackrel{?}{=}6+8-9$
	$5=5\checkmark$

TRY IT : : 2.19 Solve: $7=\frac{1}{2}x+\frac{3}{4}x-\frac{2}{3}x.$

TRY IT : : 2.20 Solve: $-1=\frac{1}{2}u+\frac{1}{4}u-\frac{2}{3}u.$

In the next example, we'll distribute before we clear the fractions.

EXAMPLE 2.11

Solve: $\frac{1}{2}(y-5)=\frac{1}{4}(y-1).$

Solution

	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2}\cdot y-\frac{1}{2}\cdot 5=\frac{1}{4}\cdot y-\frac{1}{4}\cdot 1$
Simplify.	$\frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$
Multiply by the LCD, four.	$4\left(\frac{1}{2}y-\frac{5}{2}\right)=4\left(\frac{1}{4}y-\frac{1}{4}\right)$
Distribute.	$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$

Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$

An alternate way to solve this equation is to clear the fractions without distributing first. If you multiply the factors correctly, this method will be easier.

	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Multiply by the LCD, 4.	$4\cdot\frac{1}{2}(y-5)=4\cdot\frac{1}{4}(y-1)$
Multiply four times the fractions.	$2(y-5)=1(y-1)$
Distribute.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$
Check:	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Let $y=9$.	$\frac{1}{2}(9-5)\stackrel{?}{=}\frac{1}{4}(9-1)$
Finish the check on your own.	

TRY IT : : 2.21 Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$.

TRY IT : : 2.22 Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$.

When you multiply both sides of an equation by the LCD of the fractions, make sure you multiply each term by the LCD—even if it does not contain a fraction.

EXAMPLE 2.12

Solve: $\frac{4q+3}{2}+6=\frac{3q+5}{4}$

Solution

	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Multiply both sides by the LCD, 4.	$4\left(\frac{4q+3}{2}+6\right)=4\left(\frac{3q+5}{4}\right)$
Distribute.	$4\left(\cdot\frac{4q+3}{2}\right)+4\cdot 6=4\cdot\left(\frac{3q+5}{4}\right)$
Simplify.	$2(4q+3)+24=3q+5$
	$8q+6+24=3q+5$

	$8q+30=3q+5$
Collect the variables to the left.	$8q-3q+30=3q-3q+5$
Simplify.	$5q+30=5$
Collect the constants to the right.	$5q+30-30=5-30$
Simplify.	$5q=-25$
Divide both sides by five.	$\frac{5q}{5}=\frac{-25}{5}$
Simplify.	$q=-5$
Check:	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Let $q=-5$.	$\frac{4(-5)+3}{2}+6\stackrel{?}{=}\frac{3(-5)+5}{4}$
Finish the check on your own.	

TRY IT : : 2.23 Solve: $\frac{3r+5}{6}+1=\frac{4r+3}{3}$.

TRY IT : : 2.24 Solve: $\frac{2s+3}{2}+1=\frac{3s+2}{4}$.

Some equations have decimals in them. This kind of equation may occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.7=\frac{7}{10}$ and $0.29=\frac{29}{100}$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the least common denominator.

The next example uses an equation that is typical of the ones we will see in the money applications in a later section. Notice that we will clear all decimals by multiplying by the LCD of their fraction form.

EXAMPLE 2.13

Solve: $0.25x+0.05(x+3)=2.85$.

Solution

Look at the decimals and think of the equivalent fractions:

$$0.25=\frac{25}{100},\qquad 0.05=\frac{5}{100},\qquad 2.85=2\frac{85}{100}.$$

Notice, the LCD is 100. By multiplying by the LCD we will clear the decimals from the equation.

	$0.25x + 0.05(x + 3) = 2.85$
Distribute first.	$0.25x + 0.05x + 0.15 = 2.85$
Combine like terms.	$0.30x + 0.15 = 2.85$
To clear decimals, multiply by 100.	$100(0.30x + 0.15) = 100(2.85)$
Distribute.	$30x + 15 = 285$
Subtract 15 from both sides.	$30x + 15 - 15 = 285 - 15$
Simplify.	$30x = 270$
Divide by 30.	$\frac{30x}{30} = \frac{270}{30}$
Simplify.	$x = 9$
Check it yourself by substituting $x = 9$ into the original equation.	

TRY IT : : 2.25 Solve: $0.25n + 0.05(n + 5) = 2.95.$

TRY IT : : 2.26 Solve: $0.10d + 0.05(d - 5) = 2.15.$

2.1 EXERCISES

Practice Makes Perfect

Solve Equations Using the General Strategy

In the following exercises, determine whether the given values are solutions to the equation.

1. $6y + 10 = 12y$

ⓐ $y = \frac{5}{3}$

ⓑ $y = -\frac{1}{2}$

2. $4x + 9 = 8x$

ⓐ $x = -\frac{7}{8}$

ⓑ $x = \frac{9}{4}$

3. $8u - 1 = 6u$

ⓐ $u = -\frac{1}{2}$

ⓑ $u = \frac{1}{2}$

4. $9v - 2 = 3v$

ⓐ $v = -\frac{1}{3}$

ⓑ $v = \frac{1}{3}$

In the following exercises, solve each linear equation.

5. $15(y - 9) = -60$

6. $-16(3n + 4) = 32$

7. $-(w - 12) = 30$

8. $-(t - 19) = 28$

9. $51 + 5(4 - q) = 56$

10. $-6 + 6(5 - k) = 15$

11. $3(10 - 2x) + 54 = 0$

12. $-2(11 - 7x) + 54 = 4$

13. $\frac{2}{3}(9c - 3) = 22$

14. $\frac{3}{5}(10x - 5) = 27$

15. $\frac{1}{5}(15c + 10) = c + 7$

16. $\frac{1}{4}(20d + 12) = d + 7$

17. $3(4n - 1) - 2 = 8n + 3$

18. $9(2m - 3) - 8 = 4m + 7$

19. $12 + 2(5 - 3y) = -9(y - 1) - 2$

20. $-15 + 4(2 - 5y) = -7(y - 4) + 4$

21. $5 + 6(3s - 5) = -3 + 2(8s - 1)$

22. $-12 + 8(x - 5) = -4 + 3(5x - 2)$

23. $4(p - 4) - (p + 7) = 5(p - 3)$

24. $3(a - 2) - (a + 6) = 4(a - 1)$

25. $4[5 - 8(4c - 3)] = 12(1 - 13c) - 8$

26. $5[9 - 2(6d - 1)] = 11(4 - 10d) - 139$

27. $3[-9 + 8(4h - 3)] = 2(5 - 12h) - 19$

28. $3[-14 + 2(15k - 6)] = 8(3 - 5k) - 24$

29. $5[2(m + 4) + 8(m - 7)] = 2[3(5 + m) - (21 - 3m)]$

30. $10[5(n + 1) + 4(n - 1)] = 11[7(5 + n) - (25 - 3n)]$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

31. $23z + 19 = 3(5z - 9) + 8z + 46$

32. $15y + 32 = 2(10y - 7) - 5y + 46$

33. $18(5j - 1) + 29 = 47$

34. $24(3d - 4) + 100 = 52$

35. $22(3m - 4) = 8(2m + 9)$

36. $30(2n - 1) = 5(10n + 8)$

37. $7v + 42 = 11(3v + 8) - 2(13v - 1)$

38. $18u - 51 = 9(4u + 5) - 6(3u - 10)$

39. $45(3y - 2) = 9(15y - 6)$

40. $60(2x - 1) = 15(8x + 5)$

41. $9(14d + 9) + 4d = 13(10d + 6) + 3$

42. $11(8c + 5) - 8c = 2(40c + 25) + 5$

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation with fraction coefficients.

43. $\frac{1}{4}x - \frac{1}{2} = -\frac{3}{4}$

44. $\frac{3}{4}x - \frac{1}{2} = \frac{1}{4}$

45. $\frac{5}{6}y - \frac{2}{3} = -\frac{3}{2}$

46. $\frac{5}{6}y - \frac{1}{3} = -\frac{7}{6}$

47. $\frac{1}{2}a + \frac{3}{8} = \frac{3}{4}$

48. $\frac{5}{8}b + \frac{1}{2} = -\frac{3}{4}$

49. $2 = \frac{1}{3}x - \frac{1}{2}x + \frac{2}{3}x$

50. $2 = \frac{3}{5}x - \frac{1}{3}x + \frac{2}{5}x$

51. $\frac{1}{3}w + \frac{5}{4} = w - \frac{1}{4}$

52. $\frac{1}{2}a - \frac{1}{4} = \frac{1}{6}a + \frac{1}{12}$

53. $\frac{1}{3}b + \frac{1}{5} = \frac{2}{5}b - \frac{3}{5}$

54. $\frac{1}{3}x + \frac{2}{5} = \frac{1}{5}x - \frac{2}{5}$

55. $\frac{1}{4}(p - 7) = \frac{1}{3}(p + 5)$

56. $\frac{1}{5}(q + 3) = \frac{1}{2}(q - 3)$

57. $\frac{1}{2}(x + 4) = \frac{3}{4}$

58. $\frac{1}{3}(x + 5) = \frac{5}{6}$

59. $\frac{4n + 8}{4} = \frac{n}{3}$

60. $\frac{3p + 6}{3} = \frac{p}{2}$

61. $\frac{3x + 4}{2} + 1 = \frac{5x + 10}{8}$

62. $\frac{10y - 2}{3} + 3 = \frac{10y + 1}{9}$

63. $\frac{7u - 1}{4} - 1 = \frac{4u + 8}{5}$

64. $\frac{3v - 6}{2} + 5 = \frac{11v - 4}{5}$

In the following exercises, solve each equation with decimal coefficients.

65. $0.4x + 0.6 = 0.5x - 1.2$

66. $0.7x + 0.4 = 0.6x + 2.4$

67. $0.9x - 1.25 = 0.75x + 1.75$

68. $1.2x - 0.91 = 0.8x + 2.29$

69. $0.05n + 0.10(n + 8) = 2.15$

70. $0.05n + 0.10(n + 7) = 3.55$

71. $0.10d + 0.25(d + 5) = 4.05$

72. $0.10d + 0.25(d + 7) = 5.25$

Everyday Math

73. Fencing Micah has 74 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, *L*, by solving the equation $2L + 2(L - 2.5) = 74$.

74. Stamps Paula bought $22.82 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was eight less than the number of 49-cent stamps. Solve the equation $0.49s + 0.21(s - 8) = 22.82$ for *s*, to find the number of 49-cent stamps Paula bought.

Writing Exercises

75. Using your own words, list the steps in the general strategy for solving linear equations.

76. Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

77. What is the first step you take when solving the equation $3 - 7(y - 4) = 38$? Why is this your first step?

78. If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?

79. If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?

80. For the equation $0.35x + 2.1 = 3.85$, how do you clear the decimal?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve linear equations using a general strategy.			
classify equations.			
solve equations with fraction or decimal coefficients.			

ⓑ *If most of your checks were:*

...confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

...with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

...no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

2.2 Use a Problem Solving Strategy

Learning Objectives

By the end of this section, you will be able to:

- Use a problem solving strategy for word problems
- Solve number word problems
- Solve percent applications
- Solve simple interest applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Translate "six less than twice x" into an algebraic expression.
 If you missed this problem, review **Example 1.8**.
2. Convert 4.5% to a decimal.
 If you missed this problem, review **Example 1.40**.
3. Convert 0.6 to a percent.
 If you missed this problem, review **Example 1.40**.

Have you ever had any negative experiences in the past with word problems? When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. Realize that your negative experiences with word problems are in your past. To move forward you need to calm your fears and change your negative feelings.

Start with a fresh slate and begin to think positive thoughts. Repeating some of the following statements may be helpful to turn your thoughts positive. Thinking positive thoughts is a first step towards success.

I think I can! I think I can!

While word problems were hard in the past, I think I can try them now.

I am better prepared now—I think I will begin to understand word problems.

I am able to solve equations because I practiced many problems and I got help when I needed it—I can try that with word problems.

It may take time, but I can begin to solve word problems.

You are now well prepared and you are ready to succeed. If you take control and believe you can be successful, you will be able to master word problems.

Use a Problem Solving Strategy for Word Problems

Now that we can solve equations, we are ready to apply our new skills to word problems. We will develop a strategy we can use to solve any word problem successfully.

EXAMPLE 2.14

Normal yearly snowfall at the local ski resort is 12 inches more than twice the amount it received last season. The normal yearly snowfall is 62 inches. What was the snowfall last season at the ski resort?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	What was the snowfall last season?
Step 3. Name what we are looking for and choose a variable to represent it.	Let $s =$ the snowfall last season.
Step 4. Translate. Restate the problem in one sentence with all the important information.	The normal snowfall was twelve more than twice the amount last year
Translate into an equation.	$62 = 2s + 12$

Step 5. Solve the equation.	$62 = 2s + 12$
Subtract 12 from each side.	$62 - 12 = 2s + 12 - 12$
Simplify.	$50 = 2s$
Divide each side by two.	$\frac{50}{2} = \frac{2s}{2}$
Simplify.	$25 = s$
Step 6. Check: First, is our answer reasonable? Yes, having 25 inches of snow seems OK. The problem says the normal snowfall is twelve inches more than twice the number of last season. Twice 25 is 50 and 12 more than that is 62.	
Step 7. Answer the question.	The snowfall last season was 25 inches.

TRY IT : : 2.27

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was three more than twice the number of notebooks. He bought seven textbooks. How many notebooks did he buy?

TRY IT : : 2.28

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

We summarize an effective strategy for problem solving.



HOW TO : : USE A PROBLEM SOLVING STRATEGY FOR WORD PROBLEMS.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what you are looking for.
Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.
Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
Step 5. **Solve** the equation using proper algebra techniques.
Step 6. **Check** the answer in the problem to make sure it makes sense.
Step 7. **Answer** the question with a complete sentence.

Solve Number Word Problems

We will now apply the problem solving strategy to "number word problems." Number word problems give some clues about one or more numbers and we use these clues to write an equation. Number word problems provide good practice for using the Problem Solving Strategy.

EXAMPLE 2.15

The sum of seven times a number and eight is thirty-six. Find the number.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number
Step 3. Name what you are looking for and choose a variable to represent it.	Let n = the number.
Step 4. Translate: Restate the problem as one sentence. Translate into an equation.	The sum of seven times a number and 8 is 36 $7n+8 = 36$
Step 5. Solve the equation. Subtract eight from each side and simplify. Divide each side by seven and simplify.	$7n+8=36$ $7n=28$ $n=4$
Step 6. Check. Is the sum of seven times four plus eight equal to 36?	$7\cdot 4+8 \stackrel{?}{=} 36$ $28+8 \stackrel{?}{=} 36$ $36 = 36$ ✓
Step 7. Answer the question.	The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

TRY IT : : 2.29 The sum of four times a number and two is fourteen. Find the number.

TRY IT : : 2.30 The sum of three times a number and seven is twenty-five. Find the number.

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far, we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

EXAMPLE 2.16

The sum of two numbers is negative fifteen. One number is nine less than the other. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	two numbers
Step 3. Name what you are looking for by choosing a variable to represent the first number. "One number is nine less than the other."	Let $n = 1^{\text{st}}$ number. $n-9 = 2^{\text{nd}}$ number
Step 4. Translate. Write as one sentence. Translate into an equation.	The sum of two numbers is negative fifteen. 1st number + 2nd number is negative fifteen $n + n-9 = -15$

Step 5. Solve the equation. Combine like terms. Add nine to each side and simplify. Simplify.	$n+n-9=-15$ $2n-9=-15$ $2n=-6$ $n=-3$ 1st number $n-9$ 2nd number $-3-9$ -12
Step 6. Check. Is -12 nine less than -3? $-3-9 \stackrel{?}{=} -12$ $-12 = -12$ ✓ Is their sum -15? $-3+(-12) \stackrel{?}{=} -15$ $-15 = -15$ ✓	
Step 7. Answer the question.	The numbers are -3 and -12.

TRY IT : : 2.31

The sum of two numbers is negative twenty-three. One number is seven less than the other. Find the numbers.

TRY IT : : 2.32

The sum of two numbers is negative eighteen. One number is forty more than the other. Find the numbers.

Some number problems involve **consecutive integers**. Consecutive integers are integers that immediately follow each other. Examples of **consecutive integers** are:

$$\begin{aligned} &1, \quad 2, \quad 3, \quad 4 \\ &-10, \quad -9, \quad -8, \quad -7 \\ &150, \quad 151, \quad 152, \quad 153 \end{aligned}$$

Notice that each number is one more than the number preceding it. Therefore, if we define the first integer as n, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

n	1^{st} integer
$n+1$	2^{nd} consecutive integer
$n+2$	3^{rd} consecutive integer etc.

We will use this notation to represent consecutive integers in the next example.

EXAMPLE 2.17

Find three consecutive integers whose sum is -54.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	three consecutive integers
Step 3. Name each of the three numbers	Let $n = 1^{\text{st}}$ integer. $n+1 = 2^{\text{nd}}$ consecutive integer $n+2 = 3^{\text{rd}}$ consecutive integer
Step 4. Translate. Restate as one sentence. Translate into an equation.	The sum of the three integers is -54. $n+n+1+n+2=-54$
Step 5. Solve the equation. Combine like terms. Subtract three from each side. Divide each side by three.	$n+n+1+n+2=-54$ $3n+3=-54$ $3n=-57$ $n=-19$ 1st integer $n+1$ 2nd integer $-19+1$ -18 $n+2$ 3rd integer $-19+2$ -17
Step 6. Check. $-19+(-18)+(-17) = -54$ $-54 = -54$ ✓	
Step 7. Answer the question.	The three consecutive integers are $-17,\ -18,$ and -19.

TRY IT : : 2.33 Find three consecutive integers whose sum is -96.

TRY IT : : 2.34 Find three consecutive integers whose sum is -36.

Now that we have worked with consecutive integers, we will expand our work to include **consecutive even integers** and **consecutive odd integers**. Consecutive even integers are even integers that immediately follow one another. Examples of consecutive even integers are:

$$24,\ 26,\ 28$$
$$-12,\ -10,\ -8$$

Notice each integer is two more than the number preceding it. If we call the first one *n*, then the next one is $n+2$. The one after that would be $n+2+2$ or $n+4$.

n	1^{st} even integer
$n+2$	2^{nd} consecutive even integer
$n+4$	3^{rd} consecutive even integer etc.

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 63, 65, and 67.

$$63,\ 65,\ 67$$
$$n,\ n+2,\ n+4$$

n	1^{st} odd integer
$n+2$	2^{nd} consecutive odd integer
$n+4$	3^{rd} consecutive odd integer etc.

Does it seem strange to have to add two (an even number) to get the next odd number? Do we get an odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you do not have to do anything different. The pattern is still the same—to get to the next odd or the next even integer, add two.

EXAMPLE 2.18

Find three consecutive even integers whose sum is 120.

Solution

Step 1. Read the problem.

Step 2. Identify what you are looking for. three consecutive even integers

Step 3. Name. Let $n =$ 1^{st} even integer.

$n+2 =$ 2^{nd} consecutive even integer

$n+4 =$ 3^{rd} consecutive even integer

Step 4. Translate.

Restate as one sentence. The sum of the three even integers is 120.

Translate into an equation. $n + n + 2 + n + 4 = 120$

Step 5. Solve the equation. $n + n + 2 + n + 4 = 120$

Combine like terms. $3n + 6 = 120$

Subtract 6 from each side. $3n = 114$

Divide each side by 3. $n = 38$ 1^{st} integer

$n+2$ 2^{nd} integer

$38 + 2$

40

$n+4$ 3^{rd} integer

$38 + 4$

42

Step 6. Check.

$38 + 40 + 42 \stackrel{?}{=} 120$

$120 = 120$ ✓

Step 7. Answer the question. The three consecutive integers are 38, 40, and 42.

TRY IT : : 2.35 Find three consecutive even integers whose sum is 102.

TRY IT : : 2.36 Find three consecutive even integers whose sum is -24.

When a number problem is in a real life context, we still use the same strategies that we used for the previous examples.

EXAMPLE 2.19

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution

Step 1. Read the problem.

Step 2. Identify what you are looking for. How much does the husband earn?

Step 3. Name.

Choose a variable to represent the amount the husband earns. Let h = the amount the husband earns.

The wife earns \$16,000 less than twice that.

Step 4. Translate.

Restate the problem in one sentence with all the important information.

$2h - 16{,}000$ = the amount the wife earns

Together the husband and wife earn \$110,000.

Translate into an equation.

$$h + 2h - 16{,}000 = 110{,}000$$

Step 5. Solve the equation.

Combine like terms.

$$h + 2h - 16{,}000 = 110{,}000$$

Add 16,000 to both sides and simplify.

$$3h - 16{,}000 = 110{,}000$$

$$3h = 126{,}000$$

Divide each side by three.

$$h = 42{,}000$$

\$42,000 amount husband earns

$2h - 16{,}000$ amount wife earns

$2(42{,}000) - 16{,}000$

$84{,}000 - 16{,}000$

$68{,}000$

Step 6. Check:

If the wife earns \$68,000 and the husband earns \$42,000, is that \$110,000? Yes!

Step 7. Answer the question. The husband earns \$42,000 a year.

TRY IT : : 2.37

According to the National Automobile Dealers Association, the average cost of a car in 2014 was \$28,400. This was \$1,600 less than six times the cost in 1975. What was the average cost of a car in 1975?

TRY IT : : 2.38

US Census data shows that the median price of new home in the U.S. in November 2014 was \$280,900. This was \$10,700 more than 14 times the price in November 1964. What was the median price of a new home in November 1964?

Solve Percent Applications

There are several methods to solve percent equations. In algebra, it is easiest if we just translate English sentences into algebraic equations and then solve the equations. Be sure to change the given percent to a decimal before you use it in the equation.

EXAMPLE 2.20

Translate and solve:

ⓐ What number is 45% of 84?

ⓑ 8.5% of what amount is \$4.76?

ⓒ 168 is what percent of 112?

Solution

ⓐ

	What number	is	45%	of	84?
Translate into algebra. Let n = the number.	n	=	0.45	·	84
Multiply.	$n = 37.8$				
	37.8 is 45% of 84.				

ⓑ

	8.5%	of	what amount	is	$4.76?
Translate. Let n = the amount.	0.085	·	n	=	4.76
Multiply.			$0.085n$	=	4.76
Divide both sides by 0.085 and simplify.				$n = 56$	
	8.5% of $56 is $4.76				

ⓒ

	168	is	what percent	of	112?
We are asked to find percent, so we must have our result in percent form.					
Translate into algebra. Let p = the percent.	168	=	p	·	112
Multiply.	$168 = 112p$				
Divide both sides by 112 and simplify.	$1.5 = p$				
Convert to percent.	$150\% = p$				
	168 is 150% of 112.				

TRY IT : : 2.39

Translate and solve: ⓐ What number is 45% of 80? ⓑ 7.5% of what amount is $1.95? ⓒ 110 is what percent of 88?

TRY IT : : 2.40

Translate and solve: ⓐ What number is 55% of 60? ⓑ 8.5% of what amount is $3.06? ⓐ 126 is what percent of 72?

Now that we have a problem solving strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we will solve involve everyday situations, you can rely on your own experience.

EXAMPLE 2.21

The label on Audrey's yogurt said that one serving provided 12 grams of protein, which is 24% of the recommended daily amount. What is the total recommended daily amount of protein?

Solution

What are you asked to find?	What total amount of protein is recommended?
Choose a variable to represent it.	Let $a =$ total amount of protein.
Write a sentence that gives the information to find it.	12 g is 24% of the total amount
Translate into an equation.	$12 = 0.24 \cdot a$
Solve.	$50 = a$
Check: Does this make sense? Yes, 24% is about $\frac{1}{4}$ of the total and 12 is about $\frac{1}{4}$ of 50.	
Write a complete sentence to answer the question.	The amount of protein that is recommended is 50 g.

TRY IT : : 2.41

One serving of wheat square cereal has 7 grams of fiber, which is 28% of the recommended daily amount. What is the total recommended daily amount of fiber?

TRY IT : : 2.42

One serving of rice cereal has 190 mg of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Remember to put the answer in the form requested. In the next example we are looking for the percent.

EXAMPLE 2.22

Veronica is planning to make muffins from a mix. The package says each muffin will be 240 calories and 60 calories will be from fat. What percent of the total calories is from fat?

Solution

What are you asked to find?	What percent of the total calories is fat?
Choose a variable to represent it.	Let $p =$ percent of fat.
Write a sentence that gives the information to find it.	What percent of 240 is 60?
Translate the sentence into an equation.	$p \cdot 240 = 60$
Multiply.	$240p = 60$
Divide both sides by 240.	$p = 0.25$
Put in percent form.	$p = 25\%$
Check: does this make sense? Yes, 25% is one-fourth; 60 is one-fourth of 240. So, 25% makes sense.	
Write a complete sentence to answer the question.	Of the total calories in each muffin, 25% is fat.

TRY IT : : 2.43

Mitzi received some gourmet brownies as a gift. The wrapper said each 28% brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat? Round the answer to the nearest whole percent.

TRY IT : : 2.44

The mix Ricardo plans to use to make brownies says that each brownie will be 190 calories, and 76 calories are from fat. What percent of the total calories are from fat? Round the answer to the nearest whole percent.

It is often important in many fields—business, sciences, pop culture—to talk about how much an amount has increased or decreased over a certain period of time. This increase or decrease is generally expressed as a percent and called the **percent change**.

To find the percent change, first we find the amount of change, by finding the difference of the new amount and the original amount. Then we find what percent the amount of change is of the original amount.



HOW TO : : FIND PERCENT CHANGE.

Step 1. Find the amount of change.
$$\text{change} = \text{new amount} - \text{original amount}$$

Step 2. Find what percent the amount of change is of the original amount.
$$\text{change is what percent of the original amount?}$$

EXAMPLE 2.23

Recently, the California governor proposed raising community college fees from \$36 a unit to \$46 a unit. Find the percent change. (Round to the nearest tenth of a percent.)

Solution

Find the amount of change.	$46 - 36 = 10$
Find the percent.	Change is what percent of the original amount?
Let $p =$ the percent.	10 is what percent of 36?
Translate to an equation.	$10 = p \cdot 36$
Simplify.	$10 = 36p$
Divide both sides by 36.	$0.278 \approx p$
Change to percent form; round to the nearest tenth	$27.8\% \approx p$
Write a complete sentence to answer the question.	The new fees are approximately a 27.8% increase over the old fees.

Remember to round the division to the nearest thousandth in order to round the percent to the nearest tenth.

TRY IT : : 2.45

Find the percent change. (Round to the nearest tenth of a percent.) In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents.

TRY IT : : 2.46

Find the percent change. (Round to the nearest tenth of a percent.) In 1995, the standard bus fare in Chicago was $1.50. In 2008, the standard bus fare was 2.25.

Applications of discount and mark-up are very common in retail settings.

When you buy an item on sale, the original price has been discounted by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the amount of the discount. To determine the **amount of discount**, we multiply the discount rate by the original price.

The price a retailer pays for an item is called the **original cost**. The retailer then adds a **mark-up** to the original cost to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the original cost. To determine the amount of mark-up, multiply the mark-up rate by the original cost.

Discount

$$\text{amount of discount} = \text{discount rate} \cdot \text{original price}$$
$$\text{sale price} = \text{original amount} - \text{discount price}$$

The sale price should always be less than the original price.

Mark-up

$$\text{amount of mark-up} = \text{mark-up rate} \cdot \text{original price}$$
$$\text{list price} = \text{original cost} - \text{mark-up}$$

The list price should always be more than the original cost.

EXAMPLE 2.24

Liam's art gallery bought a painting at an original cost of $750. Liam marked the price up 40%. Find ⓐ the amount of mark-up and ⓑ the list price of the painting.

Solution

ⓐ

Identify what you are asked to find, and choose a variable to represent it.	What is the amount of mark-up? Let $m =$ the amount of mark-up.
Write a sentence that gives the information to find it.	The mark-up is 40% of the $750 original cost
Translate into an equation.	$m = 0.40 \times 750$
Solve the equation.	$m = 300$
Write a complete sentence.	The mark-up on the painting was $300.

ⓑ

Identify what you are asked to find, and choose a variable to represent it.	What is the list price? Let $p =$ the list price.
Write a sentence that gives the information to find it.	The list price is original cost plus the mark-up
Translate into an equation.	$p = 750 + 300$

Solve the equation.	$p = 1{,}050$
Check.	Is the list price more than the original cost? Is \$1,050 more than \$750? Yes.
Write a complete sentence.	The list price of the painting was \$1,050.

TRY IT : : 2.47

Find ⓐ the amount of mark-up and ⓑ the list price: Jim's music store bought a guitar at original cost \$1,200. Jim marked the price up 50%.

TRY IT : : 2.48

Find ⓐ the amount of mark-up and ⓑ the list price: The Auto Resale Store bought Pablo's Toyota for \$8,500. They marked the price up 35%.

Solve Simple Interest Applications

Interest is a part of our daily lives. From the interest earned on our savings to the interest we pay on a car loan or credit card debt, we all have some experience with interest in our lives.

The amount of money you initially deposit into a bank is called the **principal**, *P*, and the bank pays you **interest**, *I*. When you take out a loan, you pay interest on the amount you borrow, also called the principal.

In either case, the interest is computed as a certain percent of the principal, called the **rate of interest**, *r*. The rate of interest is usually expressed as a percent per year, and is calculated by using the decimal equivalent of the percent. The variable *t*, (for time) represents the number of years the money is saved or borrowed.

Interest is calculated as simple interest or compound interest. Here we will use simple interest.

Simple Interest

If an amount of money, *P*, called the principal, is invested or borrowed for a period of *t* years at an annual interest rate *r*, the amount of interest, *I*, earned or paid is

$$I = Prt \quad \text{where} \quad \begin{aligned} I &= \text{interest} \\ P &= \text{principal} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

Interest earned or paid according to this formula is called **simple interest**.

The formula we use to calculate interest is $I = Prt$. To use the formula we substitute in the values for variables that are given, and then solve for the unknown variable. It may be helpful to organize the information in a chart.

EXAMPLE 2.25

Areli invested a principal of \$950 in her bank account that earned simple interest at an interest rate of 3%. How much interest did she earn in five years?

Solution

$$\begin{aligned} I &= ? \\ P &= \$950 \\ r &= 3\% \\ t &= 5 \text{ years} \end{aligned}$$

Identify what you are asked to find, and choose a variable to represent it.	What is the simple interest? Let I = interest.
Write the formula.	$I = Prt$
Substitute in the given information.	$I = (950)(0.03)(5)$
Simplify.	$I = 142.5$
Check. Is \$142.50 a reasonable amount of interest on \$950? Yes.	
Write a complete sentence.	The interest is \$142.50.

TRY IT : : 2.49

Nathaly deposited \$12,500 in her bank account where it will earn 4% simple interest. How much interest will Nathaly earn in five years?

TRY IT : : 2.50

Susana invested a principal of \$36,000 in her bank account that earned simple interest at an interest rate of 6.5%. How much interest did she earn in three years?

There may be times when we know the amount of interest earned on a given principal over a certain length of time, but we do not know the rate.

EXAMPLE 2.26

Hang borrowed \$7,500 from her parents to pay her tuition. In five years, she paid them \$1,500 interest in addition to the \$7,500 she borrowed. What was the rate of simple interest?

Solution

$I = \$1500$
$P = \$7500$
$r = ?$
$t = 5$ years

Identify what you are asked to find, and choose a variable to represent it.	What is the rate of simple interest? Let r = rate of interest.
Write the formula.	$I = Prt$
Substitute in the given information.	$1{,}500 = (7{,}500)r(5)$
Multiply.	$1{,}500 = 37{,}500r$
Divide.	$0.04 = r$
Change to percent form.	$4\% = r$
Check. $I = Prt$ $1{,}500 \stackrel{?}{=} (7{,}500)(0.04)(5)$ $1{,}500 = 1{,}500$ ✓	
Write a complete sentence.	The rate of interest was 4%.

TRY IT : : 2.51

Jim lent his sister \$5,000 to help her buy a house. In three years, she paid him the \$5,000, plus \$900 interest. What was the rate of simple interest?

TRY IT : : 2.52

Loren lent his brother $3,000 to help him buy a car. In four years, his brother paid him back the $3,000 plus $660 in interest. What was the rate of simple interest?

In the next example, we are asked to find the principal—the amount borrowed.

EXAMPLE 2.27

Sean's new car loan statement said he would pay $4,866,25 in interest from a simple interest rate of 8.5% over five years. How much did he borrow to buy his new car?

Solution

$I = 4{,}866.25$
$P = ?$
$r = 8.5\%$
$t = 5$ years

Identify what you are asked to find and choose a variable to represent it.	What is the amount borrowed (the principal)? Let P = principal borrowed.
Write the formula.	$I = Prt$
Substitute in the given information.	$4{,}866.25 = P(0.085)(5)$
Multiply.	$4{,}866.25 = 0.425P$
Divide.	$11{,}450 = P$
Check.	$I = Prt$ $4{,}866.25 \stackrel{?}{=} (11{,}450)(0.085)(5)$ $4{,}866.25 = 4{,}866.25$ ✓
Write a complete sentence.	The principal was $11,450.

TRY IT : : 2.53

Eduardo noticed that his new car loan papers stated that with a 7.5% simple interest rate, he would pay $6,596.25 in interest over five years. How much did he borrow to pay for his car?

TRY IT : : 2.54

In five years, Gloria's bank account earned $2,400 interest at 5% simple interest. How much had she deposited in the account?

MEDIA : :

Access this online resource for additional instruction and practice with using a problem solving strategy.

- **Begining Arithmetic Problems (https://openstax.org/l/37begalgwordpro)**

2.2 EXERCISES

Practice Makes Perfect

Use a Problem Solving Strategy for Word Problems

81. List five positive thoughts you can say to yourself that will help you approach word problems with a positive attitude. You may want to copy them on a sheet of paper and put it in the front of your notebook, where you can read them often.

82. List five negative thoughts that you have said to yourself in the past that will hinder your progress on word problems. You may want to write each one on a small piece of paper and rip it up to symbolically destroy the negative thoughts.

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

83. There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

84. There are 18 Cub Scouts in Troop 645. The number of scouts is three more than five times the number of adult leaders. Find the number of adult leaders.

85. Huong is organizing paperback and hardback books for her club's used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong had 162 paperbacks. How many hardback books were there?

86. Jeff is lining up children's and adult bicycles at the bike shop where he works. The number of children's bicycles is nine less than three times the number of adult bicycles. There are 42 adult bicycles. How many children's bicycles are there?

Solve Number Word Problems

In the following exercises, solve each number word problem.

87. The difference of a number and 12 is three. Find the number.

88. The difference of a number and eight is four. Find the number.

89. The sum of three times a number and eight is 23. Find the number.

90. The sum of twice a number and six is 14. Find the number.

91. The difference of twice a number and seven is 17. Find the number.

92. The difference of four times a number and seven is 21. Find the number.

93. Three times the sum of a number and nine is 12. Find the number.

94. Six times the sum of a number and eight is 30. Find the number.

95. One number is six more than the other. Their sum is 42. Find the numbers.

96. One number is five more than the other. Their sum is 33. Find the numbers.

97. The sum of two numbers is 20. One number is four less than the other. Find the numbers.

98. The sum of two numbers is 27. One number is seven less than the other. Find the numbers.

99. One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.

100. One number is 11 less than another. If their sum is increased by eight, the result is 71. Find the numbers.

101. The sum of two numbers is 14. One number is two less than three times the other. Find the numbers.

102. The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.

103. The sum of two consecutive integers is 77. Find the integers.

104. The sum of two consecutive integers is 89. Find the integers.

105. The sum of three consecutive integers is 78. Find the integers.

106. The sum of three consecutive integers is 60. Find the integers.

107. Find three consecutive integers whose sum is -36.

108. Find three consecutive integers whose sum is -3.

109. Find three consecutive even integers whose sum is 258.

110. Find three consecutive even integers whose sum is 222.

111. Find three consecutive odd integers whose sum is -213.

112. Find three consecutive odd integers whose sum is -267.

113. Philip pays $1,620 in rent every month. This amount is $120 more than twice what his brother Paul pays for rent. How much does Paul pay for rent?

114. Marc just bought an SUV for $54,000. This is $7,400 less than twice what his wife paid for her car last year. How much did his wife pay for her car?

115. Laurie has $46,000 invested in stocks and bonds. The amount invested in stocks is $8,000 less than three times the amount invested in bonds. How much does Laurie have invested in bonds?

116. Erica earned a total of $50,450 last year from her two jobs. The amount she earned from her job at the store was $1,250 more than three times the amount she earned from her job at the college. How much did she earn from her job at the college?

Solve Percent Applications

In the following exercises, translate and solve.

117. ⓐ What number is 45% of 120? ⓑ 81 is 75% of what number? ⓐ What percent of 260 is 78?

118. ⓐ What number is 65% of 100? ⓑ 93 is 75% of what number? ⓐ What percent of 215 is 86?

119. ⓐ 250% of 65 is what number? ⓑ 8.2% of what amount is $2.87? ⓐ 30 is what percent of 20?

120. ⓐ 150% of 90 is what number? ⓑ 6.4% of what amount is $2.88? ⓐ 50 is what percent of 40?

In the following exercises, solve.

121. Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. Geneva wants to leave 16% of the total bill as a tip. How much should the tip be?

122. When Hiro and his co-workers had lunch at a restaurant near their work, the bill was $90.50. They want to leave 18% of the total bill as a tip. How much should the tip be?

123. One serving of oatmeal has 8 grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

124. One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?

125. A bacon cheeseburger at a popular fast food restaurant contains 2070 milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?

126. A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?

127. The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?

128. The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?

129. Emma gets paid $3,000 per month. She pays $750 a month for rent. What percent of her monthly pay goes to rent?

130. Dimple gets paid $3,200 per month. She pays $960 a month for rent. What percent of her monthly pay goes to rent?

In the following exercises, solve.

131. Tamanika received a raise in her hourly pay, from $15.50 to $17.36. Find the percent change.

132. Ayodele received a raise in her hourly pay, from $24.50 to $25.48. Find the percent change.

133. Annual student fees at the University of California rose from about $4,000 in 2000 to about $12,000 in 2010. Find the percent change.

134. The price of a share of one stock rose from $12.50 to $50. Find the percent change.

135. A grocery store reduced the price of a loaf of bread from $2.80 to $2.73. Find the percent change.

136. The price of a share of one stock fell from $8.75 to $8.54. Find the percent change.

137. Hernando's salary was $49,500 last year. This year his salary was cut to $44,055. Find the percent change.

138. In ten years, the population of Detroit fell from 950,000 to about 712,500. Find the percent change.

In the following exercises, find ⓐ the amount of discount and ⓑ the sale price.

139. Janelle bought a beach chair on sale at 60% off. The original price was $44.95.

140. Errol bought a skateboard helmet on sale at 40% off. The original price was $49.95.

In the following exercises, find ⓐ the amount of discount and ⓑ the discount rate (Round to the nearest tenth of a percent if needed.)

141. Larry and Donna bought a sofa at the sale price of $1,344. The original price of the sofa was $1,920.

142. Hiroshi bought a lawnmower at the sale price of $240. The original price of the lawnmower is $300.

In the following exercises, find ⓐ the amount of the mark-up and ⓑ the list price.

143. Daria bought a bracelet at original cost $16 to sell in her handicraft store. She marked the price up 45%. What was the list price of the bracelet?

144. Regina bought a handmade quilt at original cost $120 to sell in her quilt store. She marked the price up 55%. What was the list price of the quilt?

145. Tom paid $0.60 a pound for tomatoes to sell at his produce store. He added a 33% mark-up. What price did he charge his customers for the tomatoes?

146. Flora paid her supplier $0.74 a stem for roses to sell at her flower shop. She added an 85% mark-up. What price did she charge her customers for the roses?

Solve Simple Interest Applications

In the following exercises, solve.

147. Casey deposited $1,450 in a bank account that earned simple interest at an interest rate of 4%. How much interest was earned in two years?

148. Terrence deposited $5,720 in a bank account that earned simple interest at an interest rate of 6%. How much interest was earned in four years?

149. Robin deposited $31,000 in a bank account that earned simple interest at an interest rate of 5.2%. How much interest was earned in three years?

150. Carleen deposited $16,400 in a bank account that earned simple interest at an interest rate of 3.9% How much interest was earned in eight years?

151. Hilaria borrowed $8,000 from her grandfather to pay for college. Five years later, she paid him back the $8,000, plus $1,200 interest. What was the rate of simple interest?

152. Kenneth lent his niece $1,200 to buy a computer. Two years later, she paid him back the $1,200, plus $96 interest. What was the rate of simple interest?

153. Lebron lent his daughter $20,000 to help her buy a condominium. When she sold the condominium four years later, she paid him the $20,000, plus $3,000 interest. What was the rate of simple interest?

154. Pablo borrowed $50,000 to start a business. Three years later, he repaid the $50,000, plus $9,375 interest. What was the rate of simple interest?

155. In 10 years, a bank account that paid 5.25% simple interest earned $18,375 interest. What was the principal of the account?

156. In 25 years, a bond that paid 4.75% simple interest earned $2,375 interest. What was the principal of the bond?

157. Joshua's computer loan statement said he would pay $1,244.34 in simple interest for a three-year loan at 12.4%. How much did Joshua borrow to buy the computer?

158. Margaret's car loan statement said she would pay $7,683.20 in simple interest for a five-year loan at 9.8%. How much did Margaret borrow to buy the car?

Everyday Math

159. Tipping At the campus coffee cart, a medium coffee costs $1.65. MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

160. Tipping Four friends went out to lunch and the bill came to $53.75 They decided to add enough tip to make a total of $64, so that they could easily split the bill evenly among themselves. What percent tip did they leave?

Writing Exercises

161. What has been your past experience solving word problems? Where do you see yourself moving forward?

162. Without solving the problem "44 is 80% of what number" think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

163. After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.

164. Because of road construction in one city, commuters were advised to plan that their Monday morning commute would take 150% of their usual commuting time. Explain what this means.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.*

I can...	Confidently	With some help	No-I don't get it!
use a problem-solving strategy for word problems.			
solve number problems.			
solve percent applications.			
solve simple interest applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

2.3 Solve a Formula for a Specific Variable

Learning Objectives

By the end of this section, you will be able to:

- Solve a formula for a specific variable
- Use formulas to solve geometry applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $2(x+3)$ when $x=5$.
 If you missed this problem, review **Example 1.6**.
2. The length of a rectangle is three less than the width. Let *w* represent the width. Write an expression for the length of the rectangle.
 If you missed this problem, review **Example 1.10**.
3. Evaluate $\frac{1}{2}bh$ when $b=14$ and $h=9$.
 If you missed this problem, review **Example 1.33**.

Solve a Formula for a Specific Variable

We have all probably worked with some geometric formulas in our study of mathematics. Formulas are used in so many fields, it is important to recognize formulas and be able to manipulate them easily.

It is often helpful to solve a formula for a specific variable. If you need to put a formula in a spreadsheet, it is not unusual to have to solve it for a specific variable first. We isolate that variable on one side of the equals sign with a coefficient of one and all other variables and constants are on the other side of the equal sign.

Geometric formulas often need to be solved for another variable, too. The formula $V=\frac{1}{3}\pi r^2 h$ is used to find the volume of a right circular cone when given the radius of the base and height. In the next example, we will solve this formula for the height.

EXAMPLE 2.28

Solve the formula $V=\frac{1}{3}\pi r^2 h$ for *h*.

Solution

Write the formula.	$V=\frac{1}{3}\pi r^2 h$
Remove the fraction on the right.	$3\cdot V=3\cdot\frac{1}{3}\pi r^2 h$
Simplify.	$3V=\pi r^2 h$
Divide both sides by πr^2.	$\frac{3V}{\pi r^2}=h$

We could now use this formula to find the height of a right circular cone when we know the volume and the radius of the base, by using the formula $h=\frac{3V}{\pi r^2}$.

TRY IT : : 2.55 Use the formula $A=\frac{1}{2}bh$ to solve for *b*.

TRY IT : : 2.56 Use the formula $A=\frac{1}{2}bh$ to solve for *h*.

In the sciences, we often need to change temperature from Fahrenheit to Celsius or vice versa. If you travel in a foreign country, you may want to change the Celsius temperature to the more familiar Fahrenheit temperature.

EXAMPLE 2.29

Solve the formula $C = \frac{5}{9}(F - 32)$ for *F*.

Solution

Write the formula.	$C = \frac{5}{9}(F - 32)$
Remove the fraction on the right.	$\frac{9}{5}C = \frac{9}{5} \cdot \frac{5}{9}(F - 32)$
Simplify.	$\frac{9}{5}C = (F - 32)$
Add 32 to both sides.	$\frac{9}{5}C + 32 = F$

We can now use the formula $F = \frac{9}{5}C + 32$ to find the Fahrenheit temperature when we know the Celsius temperature.

TRY IT : : 2.57 Solve the formula $F = \frac{9}{5}C + 32$ for *C*.

TRY IT : : 2.58 Solve the formula $A = \frac{1}{2}h(b + B)$ for *b*.

The next example uses the formula for the surface area of a right cylinder.

EXAMPLE 2.30

Solve the formula $S = 2\pi r^2 + 2\pi rh$ for *h*.

Solution

Write the formula.	$S = 2\pi r^2 + 2\pi rh$
Isolate the *h* term by subtracting $2\pi r^2$ from each side.	$S - 2\pi r^2 = 2\pi r^2 - 2\pi r^2 + 2\pi rh$
Simplify.	$S - 2\pi r^2 = 2\pi rh$
Solve for *h* by dividing both sides by $2\pi r$.	$\frac{S - 2\pi r^2}{2\pi r} = \frac{2\pi rh}{2\pi r}$
Simplify.	$\frac{S - 2\pi r^2}{2\pi r} = h$

TRY IT : : 2.59 Solve the formula $A = P + Prt$ for *t*.

TRY IT : : 2.60 Solve the formula $A = P + Prt$ for *r*.

Sometimes we might be given an equation that is solved for *y* and need to solve it for *x*, or vice versa. In the following example, we're given an equation with both *x* and *y* on the same side and we'll solve it for *y*.

EXAMPLE 2.31

Solve the formula $8x + 7y = 15$ for y.

Solution

We will isolate y on one side of the equation.	$8x + 7y = 15$
Subtract $6x$ from both sides to isolate the term with y.	$8x - 8x + 7y = 15 - 8x$
Simplify.	$7y = 15 - 8x$
Divide both sides by 7 to make the coefficient of y one.	$\frac{7y}{7} = \frac{15 - 8x}{7}$
Simplify.	$y = \frac{15 - 8x}{7}$

TRY IT : : 2.61 Solve the formula $4x + 7y = 9$ for y.

TRY IT : : 2.62 Solve the formula $5x + 8y = 1$ for y.

Use Formulas to Solve Geometry Applications

In this objective we will use some common geometry formulas. We will adapt our problem solving strategy so that we can solve geometry applications. The geometry formula will name the variables and give us the equation to solve.

In addition, since these applications will all involve shapes of some sort, most people find it helpful to draw a figure and label it with the given information. We will include this in the first step of the problem solving strategy for geometry applications.

HOW TO : : SOLVE GEOMETRY APPLICATIONS.

Step 1. **Read** the problem and make sure all the words and ideas are understood.

Step 2. **Identify** what you are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it. Draw the figure and label it with the given information.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

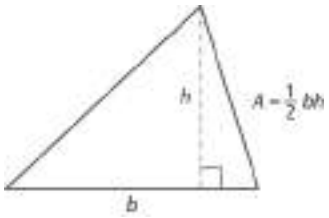
Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

When we solve geometry applications, we often have to use some of the properties of the figures. We will review those properties as needed.

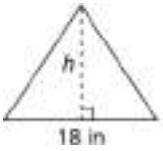
The next example involves the area of a triangle. The area of a triangle is one-half the base times the height. We can write this as $A = \frac{1}{2}bh$, where b = length of the base and h = height.



EXAMPLE 2.32

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	height of a triangle
Step 3. Name.	
Choose a variable to represent it.	Let $h =$ the height.
Draw the figure and label it with the given information.	Area = 126 sq. in.
	
Step 4. Translate.	
Write the appropriate formula.	$A = \frac{1}{2}bh$
Substitute in the given information.	$126 = \frac{1}{2} \cdot 18 \cdot h$
Step 5. Solve the equation.	$126 = 9h$
Divide both sides by 9.	$14 = h$
Step 6. Check.	
$A = \frac{1}{2}bh$ $126 \stackrel{?}{=} \frac{1}{2} \cdot 18 \cdot 14$ $126 = 126$ ✓	
Step 7. Answer the question.	The height of the triangle is 14 inches.

TRY IT : : 2.63

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

TRY IT : : 2.64

A triangular tent door has area 15 square feet. The height is five feet. What is the length of the base?

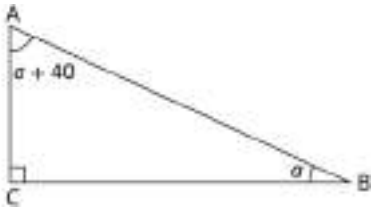
In the next example, we will work with a right triangle. To solve for the measure of each angle, we need to use two triangle properties. In any triangle, the sum of the measures of the angles is $180°$. We can write this as a formula: $m\angle A + m\angle B + m\angle C = 180$. Also, since the triangle is a right triangle, we remember that a right triangle has one $90°$ angle.

Here, we will have to define one angle in terms of another. We will wait to draw the figure until we write expressions for all the angles we are looking for.

EXAMPLE 2.33

The measure of one angle of a right triangle is 40 degrees more than the measure of the smallest angle. Find the measures of all three angles.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the measures of all three angles
Step 3. Name. Choose a variable to represent it.	Let a = 1^{st} angle. $a+40$ = 2^{nd} angle 90 = 3^{rd} angle (the right angle)
Draw the figure and label it with the given information.	
Step 4. Translate.	
Write the appropriate formula.	$m\angle A + m\angle B + m\angle C = 180$
Substitute into the formula.	$a + (a+40) + 90 = 180$
Step 5. Solve the equation.	$2a + 130 = 180$ $2a = 50$ $a = 25$ first angle $a + 40$ second angle $25 + 40$ 65 90 third angle
Step 6. Check. $25 + 65 + 90 \stackrel{?}{=} 180$ $180 = 180$ ✓	
Step 7. Answer the question.	The three angles measure $25°$, $65°$, and $90°$.

TRY IT : : 2.65

The measure of one angle of a right triangle is 50 more than the measure of the smallest angle. Find the measures of all three angles.

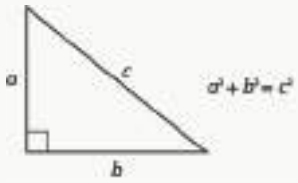
TRY IT : : 2.66

The measure of one angle of a right triangle is 30 more than the measure of the smallest angle. Find the measures of all three angles.

The next example uses another important geometry formula. The **Pythagorean Theorem** tells how the lengths of the three sides of a right triangle relate to each other. Writing the formula in every exercise and saying it aloud as you write it may help you memorize the Pythagorean Theorem.

The Pythagorean Theorem

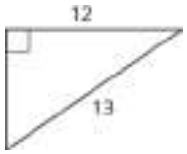
In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.



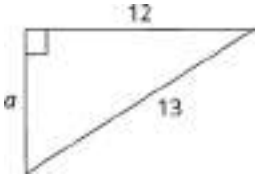
We will use the Pythagorean Theorem in the next example.

EXAMPLE 2.34

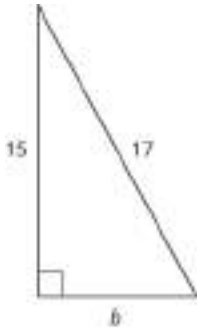
Use the Pythagorean Theorem to find the length of the other leg in



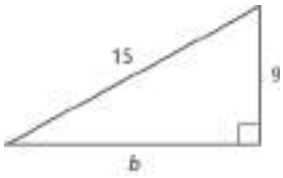
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the leg of the triangle
Step 3. Name.	
Choose a variable to represent it.	Let a = the leg of the triangle.
Label side a.	
Step 4. Translate.	
Write the appropriate formula. Substitute.	$a^2+b^2 = c^2$ $a^2+12^2 = 13^2$
Step 5. Solve the equation. Isolate the variable term. Use the definition of square root. Simplify.	$a^2+144 = 169$ $a^2 = 25$ $a = \sqrt{25}$ $a = 5$
Step 6. Check. $5^2+12^2 \stackrel{?}{=} 13^2$ $25+144 \stackrel{?}{=} 169$ $169 = 169$ ✓	
Step 7. Answer the question.	The length of the leg is 5.

TRY IT : : 2.67 Use the Pythagorean Theorem to find the length of the leg in the figure.



TRY IT : : 2.68 Use the Pythagorean Theorem to find the length of the leg in the figure.

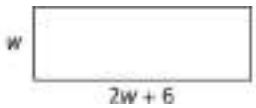


The next example is about the perimeter of a rectangle. Since the perimeter is just the distance around the rectangle, we find the sum of the lengths of its four sides—the sum of two lengths and two widths. We can write is as $P = 2L + 2W$ where L is the length and W is the width. To solve the example, we will need to define the length in terms of the width.

EXAMPLE 2.35

The length of a rectangle is six centimeters more than twice the width. The perimeter is 96 centimeters. Find the length and width.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and the width
Step 3. Name. Choose a variable to represent the width. The length is six more than twice the width.	Let $w =$ width. $2w + 6 =$ length  $P = 96$ cm
Step 4. Translate.	
Write the appropriate formula.	$P = 2L + 2W$
Substitute in the given information.	$96 = 2(2w + 6) + 2w$

Step 5. Solve the equation.

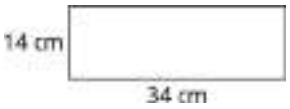
$$96 = 4w + 12 + 2w$$
$$96 = 6w + 12$$
$$84 = 6w$$
$$14 = w \text{ (width)}$$

$2w + 6$ (length)

$2(14) + 6$

34 The length is 34 cm.

Step 6. Check.



$$P = 2L + 2W$$
$$96 \stackrel{?}{=} 2 \cdot 34 + 2 \cdot 14$$
$$96 = 96 \checkmark$$

Step 7. Answer the question. The length is 34 cm and the width is 14 cm.

TRY IT : : 2.69

The length of a rectangle is seven more than twice the width. The perimeter is 110 inches. Find the length and width.

TRY IT : : 2.70

The width of a rectangle is eight yards less than twice the length. The perimeter is 86 yards. Find the length and width.

The next example is about the perimeter of a triangle. Since the perimeter is just the distance around the triangle, we find the sum of the lengths of its three sides. We can write this as $P = a + b + c$, where *a*, *b*, and *c* are the lengths of the sides.

EXAMPLE 2.36

One side of a triangle is three inches more than the first side. The third side is two inches more than twice the first. The perimeter is 29 inches. Find the length of the three sides of the triangle.

Solution

Step 1. Read the problem.

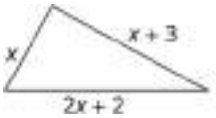
Step 2. Identify what we are looking for. the lengths of the three sides of a triangle

Step 3. Name. Choose a variable to represent the length of the first side.

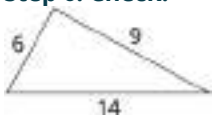
Let x = length of 1st side.

$x + 3$ = length of 2nd side

$2x + 2$ = length of 3rd side



Perimeter is 29 in.

Step 4. Translate. Write the appropriate formula. Substitute in the given information.	$P = a + b + c$ $29 = x + (x + 3) + (2x + 2)$
Step 5. Solve the equation.	$29 = 4x + 5$ $24 = 4x$ $6 = x$ length of first side $x + 3$ length of second side $6 + 3$ 9 $2x + 2$ length of second side $2 \cdot 6 + 2$ 14
Step 6. Check.  $29 \stackrel{?}{=} 6 + 9 + 14$ $29 = 29$ ✓	
Step 7. Answer the question.	The lengths of the sides of the triangle are 6, 9, and 14 inches.

TRY IT : : 2.71

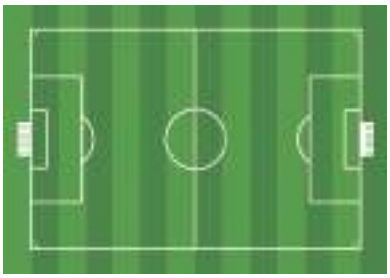
One side of a triangle is seven inches more than the first side. The third side is four inches less than three times the first. The perimeter is 28 inches. Find the length of the three sides of the triangle.

TRY IT : : 2.72

One side of a triangle is three feet less than the first side. The third side is five feet less than twice the first. The perimeter is 20 feet. Find the length of the three sides of the triangle.

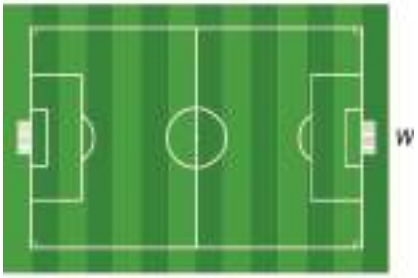
EXAMPLE 2.37

The perimeter of a rectangular soccer field is 360 feet. The length is 40 feet more than the width. Find the length and width.



Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and width of the soccer field

Step 3. Name. Choose a variable to represent it. The length is 40 feet more than the width. Draw the figure and label it with the given information.	Let w = width. $w + 40 =$ length Perimeter = 360 feet 
Step 4. Translate. Write the appropriate formula and substitute.	$P = 2L + 2W$ $360 = 2(w + 40) + 2w$
Step 5. Solve the equation.	$360 = 2w + 80 + 2w$ $360 = 4w + 80$ $280 = 4w$ $70 = w$ the width of the field $w + 40$ the length of the field $70 + 40$ 110
Step 6. Check.	$P = 2L + 2W$ $360 \stackrel{?}{=} 2(110) + 2(70)$ $360 = 360$ ✓
Step 7. Answer the question.	The length of the soccer field is 110 feet and the width is 70 feet.

TRY IT : : 2.73

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

TRY IT : : 2.74

The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.

Applications of these geometric properties can be found in many everyday situations as shown in the next example.

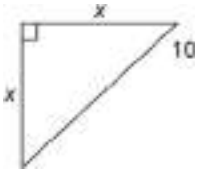
EXAMPLE 2.38

Kelvin is building a gazebo and wants to brace each corner by placing a 10” piece of wood diagonally as shown.



How far from the corner should he fasten the wood if wants the distances from the corner to be equal? Approximate to the nearest tenth of an inch.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the distance from the corner that the bracket should be attached
Step 3. Name. Choose a variable to represent it. Draw the figure and label it with the given information.	Let $x =$ the distance from the corner. 
Step 4. Translate. Write the appropriate formula and substitute.	$a^2+b^2=c^2$ $x^2+x^2=10^2$
Step 5. Solve the equation. Isolate the variable. Use the definition of square root. Simplify. Approximate to the nearest tenth.	$2x^2 = 100$ $x^2 = 50$ $x = \sqrt{50}$ $x \approx 7.1$
Step 6. Check. $a^2+b^2 = c^2$ $(7.1)^2+(7.1)^2 \approx 10^2$ Yes.	
Step 7. Answer the question.	Kelvin should fasten each piece of wood approximately 7.1" from the corner.

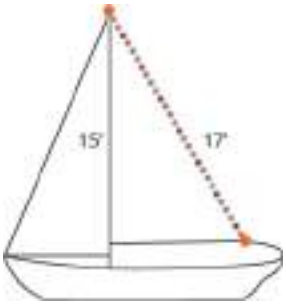
TRY IT : : 2.75

John puts the base of a 13-foot ladder five feet from the wall of his house as shown in the figure. How far up the wall does the ladder reach?



TRY IT : : 2.76

Randy wants to attach a 17-foot string of lights to the top of the 15 foot mast of his sailboat, as shown in the figure. How far from the base of the mast should he attach the end of the light string?



MEDIA : :

Access this online resource for additional instruction and practice with solving for a variable in literal equations.

- **Solving Literal Equations (https://openstax.org/l/37literalequat)**

2.3 EXERCISES

Practice Makes Perfect

Solve a Formula for a Specific Variable

In the following exercises, solve the given formula for the specified variable.

165. Solve the formula $C = \pi d$ for *d*.

166. Solve the formula $C = \pi d$ for π.

167. Solve the formula $V = LWH$ for *L*.

168. Solve the formula $V = LWH$ for *H*.

169. Solve the formula $A = \frac{1}{2}bh$ for *b*.

170. Solve the formula $A = \frac{1}{2}bh$ for *h*.

171. Solve the formula $A = \frac{1}{2}d_1 d_2$ for d_1.

172. Solve the formula $A = \frac{1}{2}d_1 d_2$ for d_2.

173. Solve the formula $A = \frac{1}{2}h(b_1 + b_2)$ for b_1.

174. Solve the formula $A = \frac{1}{2}h(b_1 + b_2)$ for b_2.

175. Solve the formula $h = 54t + \frac{1}{2}at^2$ for *a*.

176. Solve the formula $h = 48t + \frac{1}{2}at^2$ for *a*.

177. Solve $180 = a + b + c$ for *a*.

178. Solve $180 = a + b + c$ for *c*.

179. Solve the formula $A = \frac{1}{2}pl + B$ for *p*.

180. Solve the formula $A = \frac{1}{2}pl + B$ for *l*.

181. Solve the formula $P = 2L + 2W$ for *L*.

182. Solve the formula $P = 2L + 2W$ for *W*.

In the following exercises, solve for the formula for y.

183. Solve the formula $8x + y = 15$ for *y*.

184. Solve the formula $9x + y = 13$ for *y*.

185. Solve the formula $-4x + y = -6$ for *y*.

186. Solve the formula $-5x + y = -1$ for *y*.

187. Solve the formula $x - y = -4$ for *y*.

188. Solve the formula $x - y = -3$ for *y*.

189. Solve the formula $4x + 3y = 7$ for *y*.

190. Solve the formula $3x + 2y = 11$ for *y*.

191. Solve the formula $2x + 3y = 12$ for *y*.

192. Solve the formula $5x + 2y = 10$ for *y*.

193. Solve the formula $3x - 2y = 18$ for *y*.

194. Solve the formula $4x - 3y = 12$ for *y*.

Use Formulas to Solve Geometry Applications

In the following exercises, solve using a geometry formula.

195. A triangular flag has area 0.75 square feet and height 1.5 foot. What is its base?

196. A triangular window has area 24 square feet and height six feet. What is its base?

197. What is the base of a triangle with area 207 square inches and height 18 inches?

198. What is the height of a triangle with area 893 square inches and base 38 inches?

199. The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

200. The measure of the smallest angle of a right triangle is $20°$ less than the measure of the next larger angle. Find the measures of all three angles.

201. The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

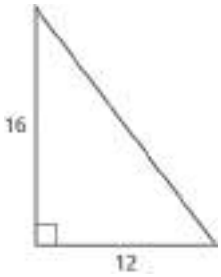
202. The angles in a triangle are such that one angle is 20 more than the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.

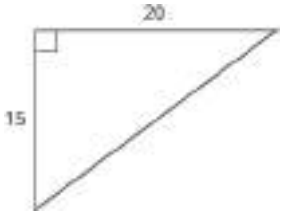
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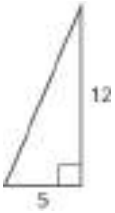
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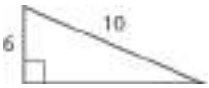


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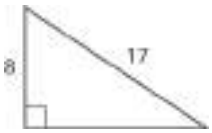


In the following exercises, use the Pythagorean Theorem to find the length of the leg. Round to the nearest tenth if necessary.

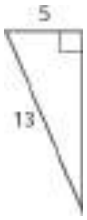
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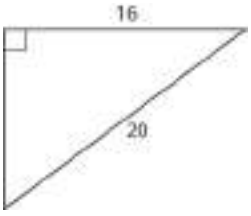
208.



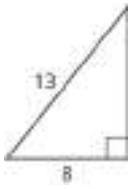
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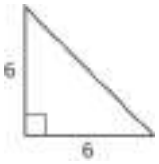
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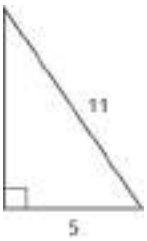
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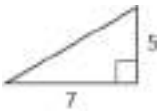
212.



213.



214.



In the following exercises, solve using a geometry formula.

215. The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

216. The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

217. The width of the rectangle is 0.7 meters less than the length. The perimeter of a rectangle is 52.6 meters. Find the dimensions of the rectangle.

218. The length of the rectangle is 1.1 meters less than the width. The perimeter of a rectangle is 49.4 meters. Find the dimensions of the rectangle.

219. The perimeter of a rectangle of 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.

220. The length of the rectangle is three times the width. The perimeter of a rectangle is 72 feet. Find the length and width of the rectangle.

221. The length of the rectangle is three meters less than twice the width. The perimeter of a rectangle is 36 meters. Find the dimensions of the rectangle.

222. The length of a rectangle is five inches more than twice the width. The perimeter is 34 inches. Find the length and width.

223. The perimeter of a triangle is 39 feet. One side of the triangle is one foot longer than the second side. The third side is two feet longer than the second side. Find the length of each side.

224. The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

225. One side of a triangle is twice the smallest side. The third side is five feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.

226. One side of a triangle is three times the smallest side. The third side is three feet more than the shortest side. The perimeter is 13 feet. Find the lengths of all three sides.

227. The perimeter of a rectangular field is 560 yards. The length is 40 yards more than the width. Find the length and width of the field.

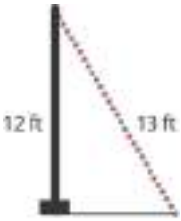
228. The perimeter of a rectangular atrium is 160 feet. The length is 16 feet more than the width. Find the length and width of the atrium.

229. A rectangular parking lot has perimeter 250 feet. The length is five feet more than twice the width. Find the length and width of the parking lot.

230. A rectangular rug has perimeter 240 inches. The length is 12 inches more than twice the width. Find the length and width of the rug.

In the following exercises, solve. Approximate answers to the nearest tenth, if necessary.

231. A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display as shown. How far from the base of the pole should the end of the string of lights be anchored?



232. Pam wants to put a banner across her garage door diagonally, as shown, to congratulate her son for his college graduation. The garage door is 12 feet high and 16 feet wide. How long should the banner be to fit the garage door?



233. Chi is planning to put a diagonal path of paving stones through her flower garden as shown. The flower garden is a square with side 10 feet. What will the length of the path be?



234. Brian borrowed a 20-foot extension ladder to use when he paints his house. If he sets the base of the ladder six feet from the house as shown, how far up will the top of the ladder reach?



Everyday Math

235. Converting temperature While on a tour in Greece, Tatyana saw that the temperature was 40° Celsius. Solve for *F* in the formula $C = \frac{5}{9}(F - 32)$ to find the Fahrenheit temperature.

236. Converting temperature Yon was visiting the United States and he saw that the temperature in Seattle one day was 50° Fahrenheit. Solve for *C* in the formula $F = \frac{9}{5}C + 32$ to find the Celsius temperature.

237. Christa wants to put a fence around her triangular flowerbed. The sides of the flowerbed are six feet, eight feet and 10 feet. How many feet of fencing will she need to enclose her flowerbed?

238. Jose just removed the children's play set from his back yard to make room for a rectangular garden. He wants to put a fence around the garden to keep the dog out. He has a 50-foot roll of fence in his garage that he plans to use. To fit in the backyard, the width of the garden must be 10 feet. How long can he make the other side?

Writing Exercises

239. If you need to put tile on your kitchen floor, do you need to know the perimeter or the area of the kitchen? Explain your reasoning.

240. If you need to put a fence around your backyard, do you need to know the perimeter or the area of the backyard? Explain your reasoning.

241. Look at the two figures below.



ⓐ Which figure looks like it has the larger area? Which looks like it has the larger perimeter?

ⓑ Now calculate the area and perimeter of each figure. Which has the larger area? Which has the larger perimeter?

ⓒ Were the results of part (b) the same as your answers in part (a)? Is that surprising to you?

242. Write a geometry word problem that relates to your life experience, then solve it and explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve a formula for a specific variable.			
use formulas to solve geometry applications.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

2.4 Solve Mixture and Uniform Motion Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve coin word problems
- Solve ticket and stamp word problems
- Solve mixture word problems
- Solve uniform motion applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $0.25x + 0.10(x + 4)$.
 If you missed this problem, review Example 1.52.
2. The number of adult tickets is three more than twice the number of children tickets. Let c represent the number of children tickets. Write an expression for the number of adult tickets.
 If you missed this problem, review Example 1.11.
3. Convert 4.2% to a decimal.
 If you missed this problem, review Example 1.40.

Solve Coin Word Problems

Using algebra to find the number of nickels and pennies in a piggy bank may seem silly. You may wonder why we just don't open the bank and count them. But this type of problem introduces us to some techniques that will be useful as we move forward in our study of mathematics.



If we have a pile of dimes, how would we determine its value? If we count the number of dimes, we'll know how many we have—the *number* of dimes. But this does not tell us the *value* of all the dimes. Say we counted 23 dimes, how much are they worth? Each dime is worth \$0.10—that is the *value* of one dime. To find the total value of the pile of 23 dimes, multiply 23 by \$0.10 to get \$2.30.

The number of dimes times the value of each dime equals the total value of the dimes.

$$\begin{aligned} number \cdot value &= total\ value \\ 23 \cdot \$0.10 &= \$2.30 \end{aligned}$$

This method leads to the following model.

Total Value of Coins

For the same type of coin, the total value of a number of coins is found by using the model

$$number \cdot value = total\ value$$

- *number* is the number of coins
- *value* is the value of each coin
- *total value* is the total value of all the coins

If we had several types of coins, we could continue this process for each type of coin, and then we would know the total value of each type of coin. To get the total value of *all* the coins, add the total value of each type of coin.

EXAMPLE 2.39

Jesse has $3.02 worth of pennies and nickels in his piggy bank. The number of nickels is three more than eight times the number of pennies. How many nickels and how many pennies does Jesse have?

Solution

Step 1. Read the problem. Determine the types of coins involved. Create a table. Write in the value of each type of coin.	pennies and nickels Pennies are worth $0.10. Nickels are worth $0.05.
Step 2. Identify what we are looking for.	the number of pennies and nickels
Step 3. Name. Represent the number of each type of coin using variables. The number of nickels is defined in terms of the number of pennies, so start with pennies. The number of nickels is three more than eight times the number of pennies.	Let $p =$ number of pennies. $8p + 3 =$ number of nickels
In the chart, multiply the number and the value to get the total value of each type of coin.	

Type	Number	Value ($)	Total Value ($)
pennies	p	0.01	$0.01p$
nickels	$8p + 3$	0.05	$0.05(8p + 3)$
			$3.02

Step 4. Translate. Write the equation by adding the total value of all the types of coins.	$0.01p + 0.05(8p + 3) = 3.02$
Step 5. Solve the equation.	$0.01p + 0.40p + 0.15 = 3.02$ $0.41p + 0.15 = 3.02$ $0.41p = 2.87$ $p = 7$ pennies
How many nickels?	$8p + 3$ $8(7) + 3$ 59 nickels
Step 6. Check the answer in the problem and make sure it makes sense. Jesse has 7 pennies and 59 nickels. Is the total value $\$3.02$? $7(0.01) + 59(0.05) \stackrel{?}{=} 3.02$ $3.02 = 3.02$ ✓	

TRY IT : : 2.77

Jesse has $6.55 worth of quarters and nickels in his pocket. The number of nickels is five more than two times the number of quarters. How many nickels and how many quarters does Jesse have?

TRY IT : : 2.78

Elane has $7.00 total in dimes and nickels in her coin jar. The number of dimes that Elane has is seven less than three times the number of nickels. How many of each coin does Elane have?

The steps for solving a coin word problem are summarized below.

HOW TO : : SOLVE COIN WORD PROBLEMS.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
- Determine the types of coins involved.
- Create a table to organize the information.
 - Label the columns "type," "number," "value," and "total value."
 - List the types of coins.
 - Write in the value of each type of coin.
 - Write in the total value of all the coins.

Type	Number •	Value ($) =	Total Value ($)

Step 2. **Identify** what you are looking for.

Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.
- Use variable expressions to represent the number of each type of coin and write them in the table.
- Multiply the number times the value to get the total value of each type of coin.

Step 4. **Translate** into an equation.
- It may be helpful to restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.
- Write the equation by adding the total values of all the types of coins.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

Solve Ticket and Stamp Word Problems

Problems involving tickets or stamps are very much like coin problems. Each type of ticket and stamp has a value, just like each type of coin does. So to solve these problems, we will follow the same steps we used to solve coin problems.

EXAMPLE 2.40

Danny paid $15.75 for stamps. The number of 49-cent stamps was five less than three times the number of 35-cent stamps. How many 49-cent stamps and how many 35-cent stamps did Danny buy?

Solution

Step 1. Determine the types of stamps involved.	49-cent stamps and 35-cent stamps
Step 2. Identify we are looking for.	the number of 49-cent stamps and the number of 35-cent stamps
Step 3. Write variable expressions to represent the number of each type of stamp.	Let x = number of 35-cent stamps.
"The number of 49-cent stamps was five less than three times the number of 35-cent stamps."	$3x - 5 =$ number of 49-cent stamps

Type	Number	· Value($)	= Total Value ($)
49 cent stamps	$3x-5$	0.49	$0.49(3x-5)$
35 cent stamps	x	0.35	$0.35x$
			15.75

Step 4. Write the equation from the total values.

$$0.49(3x-5)+0.35x=15.75$$

Step 5. Solve the equation.

$$\begin{aligned}1.47x-2.45+0.35x&=15.75\\1.82x-2.45&=15.75\\1.82x&=18.2\end{aligned}$$

How many 49-cent stamps?

$x=10$ 35-cent stamps

$3x-5$

$3(10)-5$

25 49-cent stamps

Step 6. Check.

$$\begin{aligned}10(0.35)+25(0.49)&\stackrel{?}{=}15.75\\3.50+12.25&\stackrel{?}{=}15.75\\15.75&=15.75\checkmark\end{aligned}$$

Step 7. Answer the question with a complete sentence.

Danny bought ten 35-cent stamps and twenty-five 49-cent stamps.

TRY IT : : 2.79

Eric paid $19.88 for stamps. The number of 49-cent stamps was eight more than twice the number of 35-cent stamps. How many 49-cent stamps and how many 35-cent stamps did Eric buy?

TRY IT : : 2.80

Kailee paid $14.74 for stamps. The number of 49-cent stamps was four less than three times the number of 20-cent stamps. How many 49-cent stamps and how many 20-cent stamps did Kailee buy?

In most of our examples so far, we have been told that one quantity is four more than twice the other, or something similar. In our next example, we have to relate the quantities in a different way.

Suppose Aniket sold a total of 100 tickets. Each ticket was either an adult ticket or a child ticket. If he sold 20 child tickets, how many adult tickets did he sell?

Did you say "80"? How did you figure that out? Did you subtract 20 from 100?

If he sold 45 child tickets, how many adult tickets did he sell?

Did you say "55"? How did you find it? By subtracting 45 from 100?

Now, suppose Aniket sold x child tickets. Then how many adult tickets did he sell? To find out, we would follow the same logic we used above. In each case, we subtracted the number of child tickets from 100 to get the number of adult tickets. We now do the same with x.

We have summarized this in the table.

Child tickets	Adult tickets
20	80
45	55
75	25
x	$100-x$

We will apply this technique in the next example.

EXAMPLE 2.41

A whale-watching ship had 40 paying passengers on board. The total revenue collected from tickets was $1,196. Full-fare passengers paid $32 each and reduced-fare passengers paid $26 each. How many full-fare passengers and how many reduced-fare passengers were on the ship?

Solution

Step 1. Determine the types of tickets involved.	full-fare tickets and reduced-fare tickets
Step 2. Identify what we are looking for.	the number of full-fare tickets and reduced-fare tickets
Step 3. Name. Represent the number of each type of ticket using variables.	Let f = the number of full-fare tickets. $40 - f =$ the number of reduced-fare tickets
We know the total number of tickets sold was 40. This means the number of reduced-fare tickets is 40 less the number of full-fare tickets. Multiply the number times the value to get the total value of each type of ticket.	

Type	Number	· Value ($)	= Total Value ($)
Full-fare	f	32	$32f$
Reduced-fare	$40 - f$	26	$26(40 - f)$
			1,196

Step 4. Translate. Write the equation by adding the total values of each type of ticket.	$32f + 26(40 - f) = 1{,}196$
Step 5. Solve the equation.	$32f + 1{,}040 - 26f = 1{,}196$ $6f = 156$ $f = 26$ full-fare tickets
How many reduced-fare?	$40 - f$ $40 - 26$ 14 reduced-fare tickets
Step 6. Check the answer. There were 26 full-fare tickets at $32 each and 14 reduced-fare tickets at $26 each. Is the total value $116? $26 \cdot 32 = 832$ $14 \cdot 26 = 364$ $1{,}196$ ✓	
Step 7. Answer the question.	They sold 26 full-fare and 14 reduced-fare tickets.

TRY IT : : 2.81

During her shift at the museum ticket booth, Leah sold 115 tickets for a total of $1,163. Adult tickets cost $12 and student tickets cost $5. How many adult tickets and how many student tickets did Leah sell?

TRY IT : : 2.82

Galen sold 810 tickets for his church's carnival for a total revenue of $2,820. Children's tickets cost $3 each and adult tickets cost $5 each. How many children's tickets and how many adult tickets did he sell?

Solve Mixture Word Problems

Now we'll solve some more general applications of the mixture model. In mixture problems, we are often mixing two

quantities, such as raisins and nuts, to create a mixture, such as trail mix. In our tables we will have a row for each item to be mixed as well as one for the final mixture.

EXAMPLE 2.42

Henning is mixing raisins and nuts to make 25 pounds of trail mix. Raisins cost $4.50 a pound and nuts cost $8 a pound. If Henning wants his cost for the trail mix to be $6.60 a pound, how many pounds of raisins and how many pounds of nuts should he use?

Solution

Step 1. Determine what is being mixed.

The 25 pounds of trail mix will come from mixing raisins and nuts.

Step 2. Identify what we are looking for.

the number of pounds of raisins and nuts

Step 3. Represent the number of each type of ticket using variables.

As before, we fill in a chart to organize our information.
We enter the price per pound for each item.
We multiply the number times the value to get the total value.

Let $x =$ number of pounds of raisins.
$25 - x =$ number of pounds of nuts

Type	Number of pounds	· Price per Pound ($)	= Total Value ($)
Raisins	x	4.50	$4.5x$
Nuts	$25 - x$	8	$8(25 - x)$
Trail mix	25	6.60	25(6.60)

Notice that the last column in the table gives the information for the total amount of the mixture.

Step 4. Translate into an equation.

The value of the raisins plus the value of the nuts will be the value of the trail mix.

Step 5. Solve the equation.

$$4.5x + 8(25 - x) = 25(6.60)$$

$$4.5x + 200 - 8x = 165$$
$$-3.5x = -35$$
$$x = 10 \text{ pounds of raisins}$$

Find the number of pounds of nuts.

$$25 - x$$
$$25 - 10$$
$$15 \text{ pounds of nuts}$$

Step 6. Check.

$$4.5(10) + 8(15) \stackrel{?}{=} 25(6.60)$$
$$45 + 120 \stackrel{?}{=} 165$$
$$165 = 165 \checkmark$$

Step 7. Answer the question.

Henning mixed ten pounds of raisins with 15 pounds of nuts.

TRY IT : : 2.83

Orlando is mixing nuts and cereal squares to make a party mix. Nuts sell for $7 a pound and cereal squares sell for $4 a pound. Orlando wants to make 30 pounds of party mix at a cost of $6.50 a pound, how many pounds of nuts and how many pounds of cereal squares should he use?

TRY IT : : 2.84

Becca wants to mix fruit juice and soda to make a punch. She can buy fruit juice for $3 a gallon and soda for $4 a gallon. If she wants to make 28 gallons of punch at a cost of $3.25 a gallon, how many gallons of fruit juice and how many gallons of soda should she buy?

Solve Uniform Motion Applications

When you are driving down the interstate using your cruise control, the speed of your car stays the same—it is uniform. We call a problem in which the speed of an object is constant a uniform motion application. We will use the distance, rate, and time formula, $D = rt$, to compare two scenarios, such as two vehicles travelling at different rates or in opposite directions.

Our problem solving strategies will still apply here, but we will add to the first step. The first step will include drawing a diagram that shows what is happening in the example. Drawing the diagram helps us understand what is happening so that we will write an appropriate equation. Then we will make a table to organize the information, like we did for the coin, ticket, and stamp applications.

The steps are listed here for easy reference:



HOW TO : : SOLVE A UNIFORM MOTION APPLICATION.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
- Draw a diagram to illustrate what is happening.
- Create a table to organize the information.
 - Label the columns rate, time, distance.
 - List the two scenarios.
 - Write in the information you know.

	Rate	• Time	= Distance

Step 2. **Identify** what you are looking for.

Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.
- Complete the chart.
- Use variable expressions to represent that quantity in each row.
- Multiply the rate times the time to get the distance.

Step 4. **Translate** into an equation.
- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

EXAMPLE 2.43

Wayne and Dennis like to ride the bike path from Riverside Park to the beach. Dennis's speed is seven miles per hour faster than Wayne's speed, so it takes Wayne two hours to ride to the beach while it takes Dennis 1.5 hours for the ride. Find the speed of both bikers.

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.
- Draw a diagram to illustrate what it happening. Shown below is a sketch of what is happening in the example.



- Create a table to organize the information.
 - Label the columns “Rate,” “Time,” and “Distance.”
 - List the two scenarios.
 - Write in the information you know.

	Rate (mph)	·	Time (hrs)	=	Distance (miles)
Dennis			1.5		
Wayne			2		

Step 2. Identify what you are looking for.

You are asked to find the speed of both bikers.

Notice that the distance formula uses the word “rate,” but it is more common to use “speed” when we talk about vehicles in everyday English.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

- Complete the chart
- Use variable expressions to represent that quantity in each row.
 We are looking for the speed of the bikers. Let’s let r represent Wayne’s speed. Since Dennis’ speed is 7 mph faster, we represent that as $r+7$

 $r+7 = \text{Dennis' speed}$

 $r = \text{Wayne's speed}$

 Fill in the speeds into the chart.

	Rate (mph)	·	Time (hrs)	=	Distance (miles)
Dennis	$r+7$		1.5		
Wayne	r		2		

- Multiply the rate times the time to get the distance.

	Rate (mph)	·	Time (hrs)	=	Distance (miles)
Dennis	$r+7$		1.5		$1.5(r+7)$
Wayne	r		2		$2r$

Step 4. Translate into an equation.

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.
 The equation to model this situation will come from the relation between the distances. Look at the diagram we drew above. How is the distance travelled by Dennis related to the distance travelled by Wayne?
 Since both bikers leave from Riverside and travel to the beach, they travel the same distance. So we write:

	distance traveled by Dennis	=	distance traveled by Wayne
Translate to an equation.	$1.5(r+7)$	=	$2r$

Step 5. Solve the equation using algebra techniques.

Now solve this equation.	$1.5(r+7) = 2r$ $1.5r + 10.5 = 2r$ $10.5 = 0.5r$ $21 = r$
	So Wayne's speed is 21 mph.
Find Dennis' speed.	$r + 7$ $21 + 7$ 28
	Dennis' speed 28 mph.

Step 6. Check the answer in the problem and make sure it makes sense.

Dennis $28 \text{ mph}(1.5 \text{ hours}) = 42 \text{ miles}$

Wayne $21 \text{ mph}(2 \text{ hours}) = 42 \text{ miles}$ ✓

Step 7. Answer the question with a complete sentence.

Wayne rode at 21 mph and Dennis rode at 28 mph.

TRY IT : : 2.85

An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in four hours and the local train takes five hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.

TRY IT : : 2.86

Jeromy can drive from his house in Cleveland to his college in Chicago in 4.5 hours. It takes his mother six hours to make the same drive. Jeromy drives 20 miles per hour faster than his mother. Find Jeromy's speed and his mother's speed.

In Example 2.43, we had two bikers traveling the same distance. In the next example, two people drive toward each other until they meet.

EXAMPLE 2.44

Carina is driving from her home in Anaheim to Berkeley on the same day her brother is driving from Berkeley to Anaheim, so they decide to meet for lunch along the way in Buttonwillow. The distance from Anaheim to Berkeley is 410 miles. It takes Carina three hours to get to Buttonwillow, while her brother drives four hours to get there. Carina's average speed is 15 miles per hour faster than her brother's average speed. Find Carina's and her brother's average speeds.

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.

- Draw a diagram to illustrate what it happening. Below shows a sketch of what is happening in the example.



- Create a table to organize the information.
 - Label the columns rate, time, distance.
 - List the two scenarios.
 - Write in the information you know.

	Rate (mph)	· Time (hrs)	= Distance (miles)
Carina		3	
Brother		4	
			410

Step 2. Identify what we are looking for.

We are asked to find the average speeds of Carina and her brother.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

- Complete the chart.
- Use variable expressions to represent that quantity in each row.
 We are looking for their average speeds. Let's let r represent the average speed of Carina. Since the brother's speed is 15 mph faster, we represent that as $r + 15$.
 Fill in the speeds into the chart.
- Multiply the rate times the time to get the distance.

	Rate (mph)	· Time (hrs)	= Distance (miles)
Carina	r	3	$3r$
Brother	$r+15$	4	$4(r+15)$
			410

Step 4. Translate into an equation.

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.
 Again, we need to identify a relationship between the distances in order to write an equation. Look at the diagram we created above and notice the relationship between the distance Carina traveled and the distance her brother traveled.
 The distance Carina traveled plus the distance her brother travel must add up to 410 miles. So we write:

distance traveled by Carina + distance traveled by her brother = 410

Translate to an equation. $3r + 4(r + 15) = 410$

Step 5. Solve the equation using algebra techniques.

Now solve this equation.

$$\begin{aligned} 3r + 4(r + 15) &= 410 \\ 3r + 4r + 60 &= 410 \\ 7r + 60 &= 410 \\ 7r &= 350 \\ r &= 50 \end{aligned}$$

So Carina's speed was 50 mph.

Her brother's speed is $r + 10$.

$$\begin{aligned} &r + 15 \\ &50 + 15 \\ &65 \end{aligned}$$

Her brother's speed was 65 mph.

Step 6. Check the answer in the problem and make sure it makes sense.

$$\begin{aligned} \text{Carina drove } 50 \text{ mph}(3 \text{ hours}) &= 150 \text{ miles} \\ \text{Her brother drove } 65 \text{ mph}(4 \text{ hours}) &= \underline{260 \text{ miles}} \\ &410 \text{ miles } ✓ \end{aligned}$$

Step 7. Answer the question with a complete sentence.

Carina drove 50 mph and her brother 65 mph.

TRY IT : : 2.87

Christopher and his parents live 115 miles apart. They met at a restaurant between their homes to celebrate his mother's birthday. Christopher drove one and a half hours while his parents drove one hour to get to the restaurant. Christopher's average speed was ten miles per hour faster than his parents' average speed. What were the average speeds of Christopher and of his parents as they drove to the restaurant?

TRY IT : : 2.88

Ashley goes to college in Minneapolis, 234 miles from her home in Sioux Falls. She wants her parents to bring her more winter clothes, so they decide to meet at a restaurant on the road between Minneapolis and Sioux Falls. Ashley and her parents both drove two hours to the restaurant. Ashley's average speed was seven miles per hour faster than her parents' average speed. Find Ashley's and her parents' average speed.

As you read the next example, think about the relationship of the distances traveled. Which of the previous two examples is more similar to this situation?

EXAMPLE 2.45

Two truck drivers leave a rest area on the interstate at the same time. One truck travels east and the other one travels west. The truck traveling west travels at 70 mph and the truck traveling east has an average speed of 60 mph. How long will they travel before they are 325 miles apart?

Solution

Step 1. Read the problem. Make all the words and ideas are understood.

- Draw a diagram to illustrate what it happening.



- Create a table to organize the information.
 - Label the columns rate, time, distance.
 - List the two scenarios.
 - Write in the information you know.

	Rate (mph)	· Time (hrs)	= Distance (miles)
West	70		
East	60		
			325

Step 2. Identify what we are looking for.

We are asked to find the amount of time the trucks will travel until they are 325 miles apart.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

- Complete the chart.
- Use variable expressions to represent that quantity in each row.
 We are looking for the time travelled. Both trucks will travel the same amount of time.
 Let's call the time t. Since their speeds are different, they will travel different distances.

- Multiply the rate times the time to get the distance.

	Rate (mph)	· Time (hrs)	= Distance (miles)
West	70	t	$70t$
East	60	t	$60t$
			325

Step 4. Translate into an equation.

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.
 We need to find a relation between the distances in order to write an equation. Looking at the diagram, what is the relationship between the distances each of the trucks will travel?
 The distance travelled by the truck going west plus the distance travelled by the truck going east must add up to 325 miles. So we write:

	distance traveled by westbound truck	+	distance traveled by eastbound truck	= 325
Translate to an equation.	$70t$	+	$60t$	= 325

Step 5. Solve the equation using algebra techniques.

$$\begin{aligned} \text{Now solve this equation} \quad 70t + 60t &= 325 \\ 130t &= 325 \\ t &= 2.5 \end{aligned}$$

So it will take the trucks 2.5 hours to be 325 miles apart.

Step 6. Check the answer in the problem and make sure it makes sense.

$$\begin{aligned} \text{Truck going West} \quad 70\text{ mph}(2.5\text{ hours}) &= 175\text{ miles} \\ \text{Truck going East} \quad 60\text{ mph}(2.5\text{hours}) &= \underline{150\text{ miles}} \\ & 325\text{ miles} \checkmark \end{aligned}$$

Step 7. Answer the question with a complete sentence.
It will take the trucks 2.5 hours to be 325 miles apart.

TRY IT : : 2.89

Pierre and Monique leave their home in Portland at the same time. Pierre drives north on the turnpike at a speed of 75 miles per hour while Monique drives south at a speed of 68 miles per hour. How long will it take them to be 429 miles apart?

TRY IT : : 2.90

Thanh and Nhat leave their office in Sacramento at the same time. Thanh drives north on I-5 at a speed of 72 miles per hour. Nhat drives south on I-5 at a speed of 76 miles per hour. How long will it take them to be 330 miles apart?

It is important to make sure that the units match when we use the distance rate and time formula. For instance, if the rate is in miles per hour, then the time must be in hours.

EXAMPLE 2.46

When Naoko walks to school, it takes her 30 minutes. If she rides her bike, it takes her 15 minutes. Her speed is three miles per hour faster when she rides her bike than when she walks. What is her speed walking and her speed riding her bike?

Solution

First, we draw a diagram that represents the situation to help us see what is happening.



We are asked to find her speed walking and riding her bike. Let's call her walking speed r. Since her biking speed is three miles per hour faster, we will call that speed $r+3$. We write the speeds in the chart.

The speed is in miles per hour, so we need to express the times in hours, too, in order for the units to be the same. Remember, 1 hour is 60 minutes. So:

$$30 \text{ minutes is } \frac{30}{60} \text{ or } \frac{1}{2} \text{ hour}$$
$$15 \text{ minutes is } \frac{15}{60} \text{ or } \frac{1}{4} \text{ hour}$$

We write the times in the chart.

Next, we multiply rate times time to fill in the distance column.

	Rate (mph)	· Time (hrs)	= Distance (miles)
Walk	r	$\frac{1}{2}$	$\frac{1}{2}r$
Bike	$r+3$	$\frac{1}{4}$	$\frac{1}{4}(r+3)$

The equation will come from the fact that the distance from Naoko's home to her school is the same whether she is walking or riding her bike.
So we say:

	distance walked = distance covered by bike
Translate to an equation.	$\frac{1}{2}r = \frac{1}{4}(r+3)$
Solve this equation.	$\frac{1}{2}r = \frac{1}{4}(r+3)$
Clear the fractions by multiplying by the LCD of all the fractions in the equation.	$8 \cdot \frac{1}{2}r = 8 \cdot \frac{1}{4}(r+3)$
Simplify.	$4r = 2(r+3)$ $4r = 2r+6$ $2r = 6$ $r = 3$ mph walking speed $r+3$ biking speed $3+3$
	6

Let's check if this works.

Walk 3 mph (0.5 hour) = 1.5 miles

Bike 6 mph (0.25 hour) = 1.5 miles

Yes, either way Naoko travels 1.5 miles to school.

Naoko's walking speed is 3 mph and her speed riding her bike is 6 mph.

TRY IT : : 2.91

Suzy takes 50 minutes to hike uphill from the parking lot to the lookout tower. It takes her 30 minutes to hike back down to the parking lot. Her speed going downhill is 1.2 miles per hour faster than her speed going uphill. Find Suzy's uphill and downhill speeds.

TRY IT : : 2.92

Llewyn takes 45 minutes to drive his boat upstream from the dock to his favorite fishing spot. It takes him 30 minutes to drive the boat back downstream to the dock. The boat's speed going downstream is four miles per hour faster than its speed going upstream. Find the boat's upstream and downstream speeds.

In the distance, rate and time formula, time represents the actual amount of elapsed time (in hours, minutes, etc.). If a problem gives us starting and ending times as clock times, we must find the elapsed time in order to use the formula.

EXAMPLE 2.47

Cruz is training to compete in a triathlon. He left his house at 6:00 and ran until 7:30. Then he rode his bike until 9:45. He covered a total distance of 51 miles. His speed when biking was 1.6 times his speed when running. Find Cruz's biking and running speeds.

Solution

A diagram will help us model this trip.



Next, we create a table to organize the information. We know the total distance is 51 miles. We are looking for the rate of speed for each part of the trip. The rate while biking is 1.6 times the rate of running. If we let r = the rate running, then the rate biking is $1.6r$.

The times here are given as clock times. Cruz started from home at 6:00 a.m. and started biking at 7:30 a.m. So he spent 1.5 hours running. Then he biked from 7:30 a.m until 9:45 a.m. So he spent 2.25 hours biking.

Now, we multiply the rates by the times.

	Rate (mph)	· Time (hrs)	= Distance (miles)
run	r	1.5	$1.5r$
bike	$1.6r$	2.25	$2.25(1.6r)$
			51

By looking at the diagram, we can see that the sum of the distance running and the distance biking is 255 miles.

	distance running + distance biking = 51
Translate to an equation.	$1.5r + 2.25(1.6r) = 51$
Solve this equation.	$1.5r + 2.25(1.6r) = 51$ $1.5r + 3.6r = 51$ $5.1r = 51$ $r = 10$ mph running $1.6r$ biking speed $1.6(10)$ 16 mph
Check. Run 10 mph (1.5 hours) = 15 mi Bike 16 mph (2.25 hours) = 36 mi 51 mi	

TRY IT : : 2.93

Hamilton loves to travel to Las Vegas, 255 miles from his home in Orange County. On his last trip, he left his house at 2:00 p.m. The first part of his trip was on congested city freeways. At 4:00 pm, the traffic cleared and he was able to drive through the desert at a speed 1.75 times faster than when he drove in the congested area. He arrived in Las Vegas at 6:30 p.m. How fast was he driving during each part of his trip?

TRY IT : : 2.94

Phuong left home on his bicycle at 10:00. He rode on the flat street until 11:15, then rode uphill until 11:45. He rode a total of 31 miles. His speed riding uphill was 0.6 times his speed on the flat street. Find his speed biking uphill and on the flat street.

2.4 EXERCISES

Practice Makes Perfect

Solve Coin Word Problems

In the following exercises, solve each coin word problem.

243. Michaela has $2.05 in dimes and nickels in her change purse. She has seven more dimes than nickels. How many coins of each type does she have?

244. Liliana has $2.10 in nickels and quarters in her backpack. She has 12 more nickels than quarters. How many coins of each type does she have?

245. In a cash drawer there is $125 in $5 and $10 bills. The number of $10 bills is twice the number of $5 bills. How many of each type of bill is in the drawer?

246. Sumanta has $175 in $5 and $10 bills in his drawer. The number of $5 bills is three times the number of $10 bills. How many of each are in the drawer?

247. Chi has $11.30 in dimes and quarters. The number of dimes is three more than three times the number of quarters. How many of each are there?

248. Alison has $9.70 in dimes and quarters. The number of quarters is eight more than four times the number of dimes. How many of each coin does she have?

249. Mukul has $3.75 in quarters, dimes and nickels in his pocket. He has five more dimes than quarters and nine more nickels than quarters. How many of each coin are in his pocket?

250. Vina has $4.70 in quarters, dimes and nickels in her purse. She has eight more dimes than quarters and six more nickels than quarters. How many of each coin are in her purse?

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

251. The first day of a water polo tournament the total value of tickets sold was $17,610. One-day passes sold for $20 and tournament passes sold for $30. The number of tournament passes sold was 37 more than the number of day passes sold. How many day passes and how many tournament passes were sold?

252. At the movie theater, the total value of tickets sold was $2,612.50. Adult tickets sold for $10 each and senior/child tickets sold for $7.50 each. The number of senior/child tickets sold was 25 less than twice the number of adult tickets sold. How many senior/child tickets and how many adult tickets were sold?

253. Julie went to the post office and bought both $0.41 stamps and $0.26 postcards. She spent $51.40. The number of stamps was 20 more than twice the number of postcards. How many of each did she buy?

254. Jason went to the post office and bought both $0.41 stamps and $0.26 postcards and spent $10.28 The number of stamps was four more than twice the number of postcards. How many of each did he buy?

255. Hilda has $210 worth of $10 and $12 stock shares. The number of $10 shares is five more than twice the number of $12 shares. How many of each type of share does she have?

256. Mario invested $475 in $45 and $25 stock shares. The number of $25 shares was five less than three times the number of $45 shares. How many of each type of share did he buy?

257. The ice rink sold 95 tickets for the afternoon skating session, for a total of $828. General admission tickets cost $10 each and youth tickets cost $8 each. How many general admission tickets and how many youth tickets were sold?

258. For the 7:30 show time, 140 movie tickets were sold. Receipts from the $13 adult tickets and the $10 senior tickets totaled $1,664. How many adult tickets and how many senior tickets were sold?

259. The box office sold 360 tickets to a concert at the college. The total receipts were $4,170. General admission tickets cost $15 and student tickets cost $10. How many of each kind of ticket was sold?

260. Last Saturday, the museum box office sold 281 tickets for a total of $3,954. Adult tickets cost $15 and student tickets cost $12. How many of each kind of ticket was sold?

Solve Mixture Word Problems

In the following exercises, solve each mixture word problem.

261. Macario is making 12 pounds of nut mixture with macadamia nuts and almonds. Macadamia nuts cost $9 per pound and almonds cost $5.25 per pound. How many pounds of macadamia nuts and how many pounds of almonds should Macario use for the mixture to cost $6.50 per pound to make?

262. Carmen wants to tile the floor of his house. He will need 1,000 square feet of tile. He will do most of the floor with a tile that costs $1.50 per square foot, but also wants to use an accent tile that costs $9.00 per square foot. How many square feet of each tile should he plan to use if he wants the overall cost to be $3 per square foot?

263. Riley is planning to plant a lawn in his yard. He will need nine pounds of grass seed. He wants to mix Bermuda seed that costs $4.80 per pound with Fescue seed that costs $3.50 per pound. How much of each seed should he buy so that the overall cost will be $4.02 per pound?

264. Vartan was paid $25,000 for a cell phone app that he wrote and wants to invest it to save for his son's education. He wants to put some of the money into a bond that pays 4% annual interest and the rest into stocks that pay 9% annual interest. If he wants to earn 7.4% annual interest on the total amount, how much money should he invest in each account?

265. Vern sold his 1964 Ford Mustang for $55,000 and wants to invest the money to earn him 5.8% interest per year. He will put some of the money into Fund A that earns 3% per year and the rest in Fund B that earns 10% per year. How much should he invest into each fund if he wants to earn 5.8% interest per year on the total amount?

266. Dominic pays 7% interest on his $15,000 college loan and 12% interest on his $11,000 car loan. What average interest rate does he pay on the total $26,000 he owes? (Round your answer to the nearest tenth of a percent.)

267. Liam borrowed a total of $35,000 to pay for college. He pays his parents 3% interest on the $8,000 he borrowed from them and pays the bank 6.8% on the rest. What average interest rate does he pay on the total $35,000? (Round your answer to the nearest tenth of a percent.)

Solve Uniform Motion Applications

In the following exercises, solve.

268. Lilah is moving from Portland to Seattle. It takes her three hours to go by train. Mason leaves the train station in Portland and drives to the train station in Seattle with all Lilah's boxes in his car. It takes him 2.4 hours to get to Seattle, driving at 15 miles per hour faster than the speed of the train. Find Mason's speed and the speed of the train.

269. Kathy and Cheryl are walking in a fundraiser. Kathy completes the course in 4.8 hours and Cheryl completes the course in eight hours. Kathy walks two miles per hour faster than Cheryl. Find Kathy's speed and Cheryl's speed.

270. Two busses go from Sacramento to San Diego. The express bus makes the trip in 6.8 hours and the local bus takes 10.2 hours for the trip. The speed of the express bus is 25 mph faster than the speed of the local bus. Find the speed of both busses.

271. A commercial jet and a private airplane fly from Denver to Phoenix. It takes the commercial jet 5.8% hours for the flight, and it takes the private airplane 1.8 hours. The speed of the commercial jet is 210 miles per hour faster than the speed of the private airplane. Find the speed of both airplanes.

272. Saul drove his truck three hours from Dallas towards Kansas City and stopped at a truck stop to get dinner. At the truck stop he met Erwin, who had driven four hours from Kansas City towards Dallas. The distance between Dallas and Kansas City is 542 miles, and Erwin's speed was eight miles per hour slower than Saul's speed. Find the speed of the two truckers.

273. Charlie and Violet met for lunch at a restaurant between Memphis and New Orleans. Charlie had left Memphis and drove 4.8 hours towards New Orleans. Violet had left New Orleans and drove two hours towards Memphis, at a speed 10 miles per hour faster than Charlie's speed. The distance between Memphis and New Orleans is 394 miles. Find the speed of the two drivers.

274. Sisters Helen and Anne live 332 miles apart. For Thanksgiving, they met at their other sister's house partway between their homes. Helen drove 3.2 hours and Anne drove 2.8 hours. Helen's average speed was four miles per hour faster than Anne's. Find Helen's average speed and Anne's average speed.

275. Ethan and Leo start riding their bikes at the opposite ends of a 65-mile bike path. After Ethan has ridden 1.5 hours and Leo has ridden two hours, they meet on the path. Ethan's speed is six miles per hour faster than Leo's speed. Find the speed of the two bikers.

276. Elvira and Aletheia live 3.1 miles apart on the same street. They are in a study group that meets at a coffee shop between their houses. It took Elvira half an hour and Aletheia two-thirds of an hour to walk to the coffee shop. Aletheia's speed is 0.6 miles per hour slower than Elvira's speed. Find both women's walking speeds.

277. DaMarcus and Fabian live 23 miles apart and play soccer at a park between their homes. DaMarcus rode his bike for three-quarters of an hour and Fabian rode his bike for half an hour to get to the park. Fabian's speed was six miles per hour faster than DaMarcus' speed. Find the speed of both soccer players.

278. Cindy and Richard leave their dorm in Charleston at the same time. Cindy rides her bicycle north at a speed of 18 miles per hour. Richard rides his bicycle south at a speed of 14 miles per hour. How long will it take them to be 96 miles apart?

279. Matt and Chris leave their uncle's house in Phoenix at the same time. Matt drives west on I-60 at a speed of 76 miles per hour. Chris drives east on I-60 at a speed of 82 miles per hour. How many hours will it take them to be 632 miles apart?

280. Two busses leave Billings at the same time. The Seattle bus heads west on I-90 at a speed of 73 miles per hour while the Chicago bus heads east at a speed of 79 miles an hour. How many hours will it take them to be 532 miles apart?

281. Two boats leave the same dock in Cairo at the same time. One heads north on the Mississippi River while the other heads south. The northbound boat travels four miles per hour. The southbound boat goes eight miles per hour. How long will it take them to be 54 miles apart?

282. Lorena walks the path around the park in 30 minutes. If she jogs, it takes her 20 minutes. Her jogging speed is 1.5 miles per hour faster than her walking speed. Find Lorena's walking speed and jogging speed.

283. Julian rides his bike uphill for 45 minutes, then turns around and rides back downhill. It takes him 15 minutes to get back to where he started. His uphill speed is 3.2 miles per hour slower than his downhill speed. Find Julian's uphill and downhill speed.

284. Cassius drives his boat upstream for 45 minutes. It takes him 30 minutes to return downstream. His speed going upstream is three miles per hour slower than his speed going downstream. Find his upstream and downstream speeds.

285. It takes Darline 20 minutes to drive to work in light traffic. To come home, when there is heavy traffic, it takes her 36 minutes. Her speed in light traffic is 24 miles per hour faster than her speed in heavy traffic. Find her speed in light traffic and in heavy traffic.

286. At 1:30, Marlon left his house to go to the beach, a distance of 7.6 miles. He rode his skateboard until 2:15, and then walked the rest of the way. He arrived at the beach at 3:00. Marlon's speed on his skateboard is 2.5 times his walking speed. Find his speed when skateboarding and when walking.

287. Aaron left at 9:15 to drive to his mountain cabin 108 miles away. He drove on the freeway until 10:45 and then drove on a mountain road. He arrived at 11:05. His speed on the freeway was three times his speed on the mountain road. Find Aaron's speed on the freeway and on the mountain road.

288. Marisol left Los Angeles at 2:30 to drive to Santa Barbara, a distance of 95 miles. The traffic was heavy until 3:20. She drove the rest of the way in very light traffic and arrived at 4:20. Her speed in heavy traffic was 40 miles per hour slower than her speed in light traffic. Find her speed in heavy traffic and in light traffic.

289. Lizette is training for a marathon. At 7:00 she left her house and ran until 8:15 then she walked until 11:15. She covered a total distance of 19 miles. Her running speed was five miles per hour faster than her walking speed. Find her running and walking speeds.

Everyday Math

290. John left his house in Irvine at 8:35 a.m. to drive to a meeting in Los Angeles, 45 miles away. He arrived at the meeting at 9:50 a.m.. At 6:30 p.m. he left the meeting and drove home. He arrived home at 7:18 p.m.

ⓐ What was his average speed on the drive from Irvine to Los Angeles?

ⓑ What was his average speed on the drive from Los Angeles to Irvine?

ⓒ What was the total time he spent driving to and from this meeting?

291. Sarah wants to arrive at her friend's wedding at 3:00. The distance from Sarah's house to the wedding is 95 miles. Based on usual traffic patterns, Sarah predicts she can drive the first 15 miles at 60 miles per hour, the next 10 miles at 30 miles per hour, and the remainder of the drive at 70 miles per hour.

ⓐ How long will it take Sarah to drive the first 15 miles?

ⓑ How long will it take Sarah to drive the next 10 miles?

ⓒ How long will it take Sarah to drive the rest of the trip?

ⓓ What time should Sarah leave her house?

Writing Exercises

292. Suppose you have six quarters, nine dimes, and four pennies. Explain how you find the total value of all the coins.

293. Do you find it helpful to use a table when solving coin problems? Why or why not?

294. In the table used to solve coin problems, one column is labeled "number" and another column is labeled "value." What is the difference between the "number" and the "value"?

295. When solving a uniform motion problem, how does drawing a diagram of the situation help you?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve coin word problems.			
solve ticket and stamp word problems.			
solve mixture word problems.			
solve uniform motion applications.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

2.5 Solve Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve linear inequalities
- Translate words to an inequality and solve
- Solve applications with linear inequalities

Be Prepared!

Before you get started, take this readiness quiz.

1. Translate from algebra to English: $15 > x$.
 If you missed this problem, review **Example 1.3**.
2. Translate to an algebraic expression: 15 is less than *x*.
 If you missed this problem, review **Example 1.8**.

Graph Inequalities on the Number Line

What number would make the inequality $x > 3$ true? Are you thinking, "*x* could be four"? That's correct, but *x* could be 6, too, or 37, or even 3.001. Any number greater than three is a solution to the inequality $x > 3$.

We show all the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of three, to show that all numbers greater than three are solutions. Because the number three itself is not a solution, we put an open parenthesis at three.

We can also represent inequalities using **interval notation**. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "**infinity**." It is not an actual number.

Figure 2.2 shows both the number line and the interval notation.

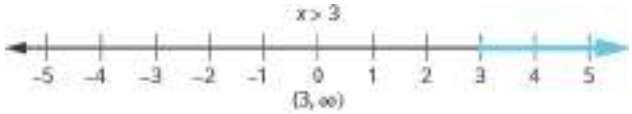


Figure 2.2 The inequality $x > 3$ is graphed on this number line and written in interval notation.

We use the left parenthesis symbol, (, to show that the endpoint of the inequality is not included. The left bracket symbol, [, shows that the endpoint is included.

The inequality $x \leq 1$ means all numbers less than or equal to one. Here we need to show that one is a solution, too. We do that by putting a bracket at $x = 1$. We then shade in all the numbers to the left of one, to show that all numbers less than one are solutions. See **Figure 2.3**.

There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity." **Figure 2.3** shows both the number line and interval notation.



Figure 2.3 The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Inequalities, Number Lines, and Interval Notation



The notation for inequalities on a number line and in interval notation use the same symbols to express the endpoints of intervals.

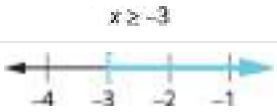
EXAMPLE 2.48

Graph each inequality on the number line and write in interval notation.

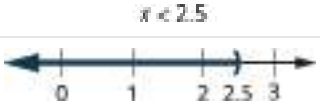
ⓐ $x \geq -3$ ⓑ $x < 2.5$ ⓒ $x \leq -\frac{3}{5}$

Solution

ⓐ

Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

ⓑ

Shade to the left of 2.5 and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

ⓒ

Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, -\frac{3}{5}\right]$

TRY IT : : 2.95

Graph each inequality on the number line and write in interval notation: ⓐ $x > 2$ ⓑ $x \leq -1.5$ ⓒ $x \geq \frac{3}{4}$.

TRY IT : : 2.96

Graph each inequality on the number line and write in interval notation: ⓐ $x \leq -4$ ⓑ $x \geq 0.5$ ⓒ $x < -\frac{2}{3}$.

What numbers are greater than two but less than five? Are you thinking say, $2.5,\ 3,\ 3\frac{2}{3},\ 4,\ 4,\ 99$? We can represent all the numbers between two and five with the inequality $2 < x < 5$. We can show $2 < x < 5$ on the number line by shading all the numbers between two and five. Again, we use the parentheses to show the numbers two and five are not included. See Figure 2.4.

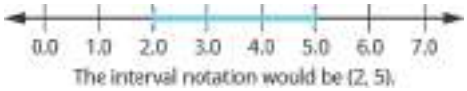


Figure 2.4

EXAMPLE 2.49

Graph each inequality on the number line and write in interval notation.

ⓐ $-3 < x < 4$ ⓑ $-6 \leq x < -1$ ⓒ $0 \leq x \leq 2.5$

Solution

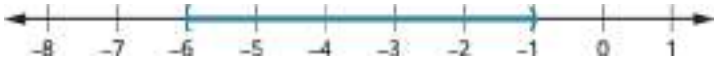
ⓐ

$-3 < x < 4$

Shade between -3 and 4. Put a parentheses at -3 and 4.	
Write in interval notation.	$(-3, 4)$

ⓑ

$-6 \leq x < -1$

Shade between -6 and -1. Put a bracket at -6, and a parenthesis at -1.	
Write in interval notation.	$[-6, 1)$

ⓒ

$0 \leq x \leq 2.5$

Shade between 0 and 2.5. Put a bracket at 0 and at 2.5.	
Write in interval notation.	$[0, 2.5]$

TRY IT : : 2.97 Graph each inequality on the number line and write in interval notation:

ⓐ $-2 < x < 1$ ⓑ $-5 \leq x < -4$ ⓒ $1 \leq x \leq 4.25$

TRY IT : : 2.98 Graph each inequality on the number line and write in interval notation:

ⓐ $-6 < x < 2$ ⓑ $-3 \leq x < -1$ ⓒ $2.5 \leq x \leq 6$

Solve Linear Inequalities

A linear inequality is much like a linear equation—but the equal sign is replaced with an inequality sign. A **linear**

inequality is an inequality in one variable that can be written in one of the forms, $ax+b<c$, $ax+b\le c$, $ax+b>c$, or $ax+b\ge c$.

Linear Inequality

A linear inequality is an inequality in one variable that can be written in one of the following forms where *a*, *b*, and *c* are real numbers and $a\neq 0$:

$$ax+b<c,\qquad ax+b\le c,\qquad ax+b>c,\qquad ax+b\ge c.$$

When we solved linear equations, we were able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example:

$$\begin{array}{ll} -4<2 & -4<2 \\ -4-5<2-5 & -4+7<2+7 \\ -9<-3 \text{ True} & 3<9 \text{ True} \end{array}$$

Notice that the inequality sign stayed the same.

This leads us to the Addition and Subtraction Properties of Inequality.

Addition and Subtraction Property of Inequality

For any numbers *a*, *b*, and *c*, if $a<b$, then

$$\begin{array}{ll} a+c<b+c & a-c<b-c \\ a+c>b+c & a-c>b-c \end{array}$$

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.

What happens to an inequality when we divide or multiply both sides by a constant?

Let's first multiply and divide both sides by a positive number.

$$\begin{array}{ll} 10<15 & 10<15 \\ 10(5)<15(5) & \frac{10}{5}<\frac{15}{5} \\ 50<75 \text{ True} & 2<3 \text{ True} \end{array}$$

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

$$\begin{array}{ll} 10<15 & 10<15 \\ 10(-5)\,?\,15(-5) & \frac{10}{-5}\,?\,\frac{15}{-5} \\ -50\,?\,-75 & -2\,?\,-3 \\ -50>-75 & -2>-3 \end{array}$$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

This gives us the Multiplication and Division Property of Inequality.

Multiplication and Division Property of Inequality

For any numbers *a*, *b*, and *c*,

multiply or divide by a positive

if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

multiply or divide by a negative

if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

When we divide or multiply an inequality by a:

- positive number, the inequality stays the same.
- negative number, the inequality reverses.

Sometimes when solving an inequality, as in the next example, the variable ends upon the right. We can rewrite the inequality in reverse to get the variable to the left.

$x > a$ has the same meaning as $a < x$

Think about it as "If Xander is taller than Andy, then Andy is shorter than Xander."

EXAMPLE 2.50

Solve each inequality. Graph the solution on the number line, and write the solution in interval notation.

ⓐ $x - \frac{3}{8} \leq \frac{3}{4}$ ⓑ $9y < 54$ ⓒ $-15 < \frac{3}{5}z$

Solution

ⓐ

	$x - \frac{3}{8} \leq \frac{3}{4}$
Add $\frac{3}{8}$ to both sides of the inequality.	$x - \frac{3}{8} + \frac{3}{8} \leq \frac{3}{4} + \frac{3}{8}$
Simplify.	$x \leq \frac{9}{8}$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, \frac{9}{8}]$

ⓑ

	$9y < 54$
Divide both sides of the inequality by 9; since 9 is positive, the inequality stays the same.	$\frac{9y}{9} < \frac{54}{9}$
Simplify.	$y < 6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

ⓒ

	$-15 < \frac{3}{5}x$
Multiply both sides of the inequality by $\frac{5}{3}$. Since $\frac{5}{3}$ is positive, the inequality stays the same.	$\frac{5}{3}(-15) < \frac{5}{3}\left(\frac{3}{5}x\right)$
Simplify.	$-25 < x$
Rewrite with the variable on the left.	$x > -25$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-25, \infty)$

TRY IT : : 2.99

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

ⓐ $p - \frac{3}{4} \geq \frac{1}{6}$ ⓑ $9c > 72$ ⓒ $24 \leq \frac{3}{8}m$

TRY IT : : 2.100

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

ⓐ $r - \frac{1}{3} \leq \frac{7}{12}$ ⓑ $12d \leq 60$ ⓒ $-24 < \frac{4}{3}n$

Be careful when you multiply or divide by a negative number—remember to reverse the inequality sign.

EXAMPLE 2.51

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

ⓐ $-13m \geq 65$ ⓑ $\frac{n}{-2} \geq 8$

Solution

ⓐ

	$-13m \geq 65$
Divide both sides of the inequality by -13. Since -13 is a negative, the inequality reverses.	$\frac{-13m}{-13} \leq \frac{65}{-13}$
Simplify.	$m \leq -5$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -5]$

ⓑ

	$\frac{n}{-2} \geq 8$
Multiply both sides of the inequality by -2. Since -2 is a negative, the inequality reverses.	$-2\left(\frac{n}{-2}\right) \leq -2(8)$
Simplify.	$n \leq -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

TRY IT : : 2.101

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

ⓐ $-8q < 32$ ⓑ $\frac{k}{-12} \leq 15.$

TRY IT : : 2.102

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

ⓐ $-7r \leq -70$ ⓑ $\frac{u}{-4} \geq -16.$

Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but make sure to pay close attention when we multiply or divide to isolate the variable.

EXAMPLE 2.52

Solve the inequality $6y \leq 11y + 17,$ graph the solution on the number line, and write the solution in interval notation.

Solution

	$6y \leq 11y + 17$
Subtract $11y$ from both sides to collect the variables on the left.	$6y - 11y \leq 11y - 11y + 17$
Simplify.	$-5y \leq 17$
Divide both sides of the inequality by -5, and reverse the inequality.	$\frac{-5y}{-5} \geq \frac{17}{-5}$
Simplify.	$y \geq \frac{17}{-5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

TRY IT : : 2.103

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $3q \geq 7q - 23.$

TRY IT : : 2.104

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $6x < 10x + 19$.

When solving inequalities, it is usually easiest to collect the variables on the side where the coefficient of the variable is largest. This eliminates negative coefficients and so we don't have to multiply or divide by a negative—which means we don't have to remember to reverse the inequality sign.

EXAMPLE 2.53

Solve the inequality $8p + 3(p - 12) > 7p - 28$, graph the solution on the number line, and write the solution in interval notation.

Solution

	$8p + 3(p - 12) > 7p - 28$
Simplify each side as much as possible.	
Distribute.	$8p + 3p - 36 > 7p - 28$
Combine like terms.	$11p - 36 > 7p - 28$
Subtract $7p$ from both sides to collect the variables on the left, since $11 > 7$.	$11p - 36 - 7p > 7p - 28 - 7p$
Simplify.	$4p - 36 > -28$
Add 36 to both sides to collect the constants on the right.	$4p - 36 + 36 > -28 + 36$
Simplify.	$4p > 8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4} > \frac{8}{4}$
Simplify.	$p > 2$
Graph the solution on the number line.	
Write the solution in interval notation.	$(2, \infty)$

TRY IT : : 2.105

Solve the inequality $9y + 2(y + 6) > 5y - 24$, graph the solution on the number line, and write the solution in interval notation.

TRY IT : : 2.106

Solve the inequality $6u + 8(u - 1) > 10u + 32$, graph the solution on the number line, and write the solution in interval notation.

Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

EXAMPLE 2.54

Solve the inequality $8x - 2(5 - x) < 4(x + 9) + 6x$, graph the solution on the number line, and write the solution in interval notation.

Solution

Simplify each side as much as possible.	$8x - 2(5 - x) < 4(x + 9) + 6x$
Distribute.	$8x - 10 + 2x < 4x + 36 + 6x$
Combine like terms.	$10x - 10 < 10x + 36$
Subtract $10x$ from both sides to collect the variables on the left.	$10x - 10 - 10x < 10x + 36 - 10x$
Simplify.	$-10 < 36$
The x's are gone, and we have a true statement.	The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, \infty)$

TRY IT : : 2.107

Solve the inequality $4b - 3(3 - b) > 5(b - 6) + 2b$, graph the solution on the number line, and write the solution in interval notation.

TRY IT : : 2.108

Solve the inequality $9h - 7(2 - h) < 8(h + 11) + 8h$, graph the solution on the number line, and write the solution in interval notation.

We can clear fractions in inequalities much as we did in equations. Again, be careful with the signs when multiplying or dividing by a negative.

EXAMPLE 2.55

Solve the inequality $\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

Solution

	$\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a - \frac{1}{8}a\right) > 24\left(\frac{5}{24}a + \frac{3}{4}\right)$
Simplify.	$8a - 3a > 5a + 18$
Combine like terms.	$5a > 5a + 18$
Subtract $5a$ from both sides to collect the variables on the left.	$5a - 5a > 5a - 5a + 18$

Simplify.	$0 > 18$
The statement is false.	The inequality is a contradiction. There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

TRY IT : : 2.109

Solve the inequality $\frac{1}{4}x - \frac{1}{12}x > \frac{1}{6}x + \frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

TRY IT : : 2.110

Solve the inequality $\frac{2}{5}z - \frac{1}{3}z < \frac{1}{15}z - \frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like “more than” and “less than.” But others are not as obvious. Table 2.60 shows some common phrases that indicate inequalities.

$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Table 2.60

EXAMPLE 2.56

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Twenty-seven less than x is at least 48.

Solution

	Twenty-seven less than x is at least 48.
Translate.	$x - 27 \geq 48$
Solve—add 27 to both sides.	$x - 27 + 27 \geq 48 + 27$
Simplify.	$x \geq 75$
Graph on the number line.	
Write in interval notation.	$[75, \infty)$

TRY IT : : 2.111

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Nineteen less than p is no less than 47.

TRY IT : : 2.112

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Four more than a is at most 15.

Solve Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications with linear inequalities is very much like the one we used when we solved applications with equations.

We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality. Then, we will solve the inequality.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

EXAMPLE 2.57

Dawn won a mini-grant of $4,000 to buy tablet computers for her classroom. The tablets she would like to buy cost $254.12 each, including tax and delivery. What is the maximum number of tablets Dawn can buy?

Solution

Step 1. Read the problem.

Step 2. Identify what you are looking for. the maximum number of tablets Dawn can buy

Step 3. Name what you are looking for.

Choose a variable to represent that quantity. Let n = the number of tablets.

Step 4. Translate. Write a sentence that gives the information to find it

Translate into an inequality. \$254.12 times the number of tablets is no more than \$4,000.

$$254.12n \leq 4000$$

Step 5. Solve the inequality.

$$n \leq 15.74$$

But n must be a whole number of tablets, so round to 15.

$$n \leq 15$$

Step 6. Check the answer in the problem and make sure it makes sense.

Rounding down the price to \$250, 15 tablets would cost \$3,750, while 16 tablets would be \$4,000. So a maximum of 15 tablets at \$254.12 seems reasonable.

Step 7. Answer the question with a complete sentence. Dawn can buy a maximum of 15 tablets.

TRY IT : : 2.113

Angie has \$20 to spend on juice boxes for her son's preschool picnic. Each pack of juice boxes costs \$2.63. What is the maximum number of packs she can buy?

TRY IT : : 2.114

Daniel wants to surprise his girlfriend with a birthday party at her favorite restaurant. It will cost \$42.75 per person for dinner, including tip and tax. His budget for the party is \$500. What is the maximum number of people Daniel can have at the party?

EXAMPLE 2.58

Taleisha's phone plan costs her \$28.80 a month plus \$0.20 per text message. How many text messages can she send/receive and keep her monthly phone bill no more than \$50?

Solution

Step 1. Read the problem.

Step 2. Identify what you are looking for.

the number of text messages Taleisha can make

Step 3. Name what you are looking for.

Choose a variable to represent that quantity.

Let t = the number of text messages.

Step 4. Translate Write a sentence that gives the information to find it

Translate into an inequality.

$28.80 plus $0.20 times the number of text messages is less than or equal to $50.

$28.80 + 0.20t \leq 50$

Step 5. Solve the inequality.

$$0.2t \leq 21.2$$
$$t \leq 106 \text{ text messages}$$

Step 6. Check the answer in the problem and make sure it makes sense.

Yes, $28.80 + 0.20(106) = 50$.

Step 7. Write a sentence that answers the question.

Taleisha can send/receive no more than 106 text messages to keep her bill no more than $50.

TRY IT : : 2.115

Sergio and Lizeth have a very tight vacation budget. They plan to rent a car from a company that charges $75 a week plus $0.25 a mile. How many miles can they travel during the week and still keep within their $200 budget?

TRY IT : : 2.116

Rameen's heating bill is $5.42 per month plus $1.08 per therm. How many therms can Rameen use if he wants his heating bill to be a maximum of $87.50.

Profit is the money that remains when the costs have been subtracted from the revenue. In the next example, we will find the number of jobs a small businesswoman needs to do every month in order to make a certain amount of profit.

EXAMPLE 2.59

Felicity has a calligraphy business. She charges $2.50 per wedding invitation. Her monthly expenses are $650. How many invitations must she write to earn a profit of at least $2,800 per month?

Solution

Step 1. Read the problem.

Step 2. Identify what you are looking for. the number of invitations Felicity needs to write

Step 3. Name what you are looking for. Choose a variable to represent it. Let j = the number of invitations.

Step 4. Translate. Write a sentence that gives the information to find it

$2.50 times the number of invitations minus $650 is at least $2,800.

Translate into an inequality.

$$2.50j - 650 \geq 2{,}800$$

Step 5. Solve the inequality.

$$2.5j \geq 3{,}450$$
$$j \geq 1{,}380 \text{ invitations}$$

Step 6. Check the answer in the problem and make sure it makes sense.

If Felicity wrote 1400 invitations, her profit ould be $2.50(1400) - 650$, or $2,850. This is more than $2800.

Step 7. Write a sentence that answers the question. Felicity must write at least 1,380 invitations.

TRY IT : : 2.117

Caleb has a pet sitting business. He charges $32 per hour. His monthly expenses are $2,272. How many hours must he work in order to earn a profit of at least $800 per month?

TRY IT : : 2.118

Elliot has a landscape maintenance business. His monthly expenses are $1,100. If he charges $60 per job, how many jobs must he do to earn a profit of at least $4,000 a month?

There are many situations in which several quantities contribute to the total expense. We must make sure to account for all the individual expenses when we solve problems like this.

EXAMPLE 2.60

Malik is planning a six-day summer vacation trip. He has $840 in savings, and he earns $45 per hour for tutoring. The trip will cost him $525 for airfare, $780 for food and sightseeing, and $95 per night for the hotel. How many hours must he tutor to have enough money to pay for the trip?

Solution

Step 1. Read the problem.

Step 2. Identify what you are looking for. — the number of hours Malik must tutor

Step 3. Name what you are looking for.

Choose a variable to represent that quantity. — Let h = the number of hours.

Step 4. Translate. Write a sentence that gives the information to find it — The expenses must be less than or equal to the income. The cost of airfare plus the cost of food and sightseeing and the hotel bill must be less than the savings plus the amount earned tutoring.

Translate into an inequality. — $525 + 780 + 95(6) \leq 840 + 45h$

Step 5. Solve the inequality.

$$\begin{aligned} 1{,}875 &\leq 840 + 45h \\ 1{,}035 &\leq 45h \\ 23 &\leq h \\ h &\geq 23 \end{aligned}$$

Step 6. Check the answer in the problem and make sure it makes sense.
We substitute 23 into the inequality.

$$\begin{aligned} 1{,}875 &\leq 840 + 45h \\ 1{,}875 &\leq 840 + 45(23) \\ 1{,}875 &\leq 1875 \end{aligned}$$

Step 7. Write a sentence that answers the question. — Malik must tutor at least 23 hours.

TRY IT : : 2.119

Brenda's best friend is having a destination wedding and the event will last three days. Brenda has $500 in savings and can earn $15 an hour babysitting. She expects to pay $350 airfare, $375 for food and entertainment and $60 a night for her share of a hotel room. How many hours must she babysit to have enough money to pay for the trip?

TRY IT : : 2.120

Josue wants to go on a 10-night road trip with friends next spring. It will cost him $180 for gas, $450 for food, and $49 per night to share a motel room. He has $520 in savings and can earn $30 per driveway shoveling snow. How many driveways must he shovel to have enough money to pay for the trip?

2.5 EXERCISES

Practice Makes Perfect

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line and write in interval notation.

296.
ⓐ $x < -2$
ⓑ $x \geq -3.5$
ⓒ $x \leq \frac{2}{3}$

297.
ⓐ $x > 3$
ⓑ $x \leq -0.5$
ⓒ $x \geq \frac{1}{3}$

298.
ⓐ $x \geq -4$
ⓑ $x < 2.5$
ⓒ $x > -\frac{3}{2}$

299.
ⓐ $x \leq 5$
ⓑ $x \geq -1.5$
ⓒ $x < -\frac{7}{3}$

300.
ⓐ $-5 < x < 2$
ⓑ $-3 \leq x < -1$
ⓒ $0 \leq x \leq 1.5$

301.
ⓐ $-2 < x < 0$
ⓑ $-5 \leq x < -3$
ⓒ $0 \leq x \leq 3.5$

302.
ⓐ $-1 < x < 3$
ⓑ $-3 < x \leq -2$
ⓒ $-1.25 \leq x \leq 0$

303.
ⓐ $-4 < x < 2$
ⓑ $-5 < x \leq -2$
ⓒ $-3.75 \leq x \leq 0$

Solve Linear Inequalities

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

304.
ⓐ $a + \frac{3}{4} \geq \frac{7}{10}$
ⓑ $8x > 72$
ⓒ $20 > \frac{2}{5}h$

305.
ⓐ $b + \frac{7}{8} \geq \frac{1}{6}$
ⓑ $6y < 48$
ⓒ $40 < \frac{5}{8}k$

306.
ⓐ $f - \frac{13}{20} < -\frac{5}{12}$
ⓑ $9t \geq -27$
ⓒ $\frac{7}{6}j \geq 42$

307.
ⓐ $g - \frac{11}{12} < -\frac{5}{18}$
ⓑ $7s < -28$
ⓒ $\frac{9}{4}g \leq 36$

308.
ⓐ $-5u \geq 65$
ⓑ $\frac{a}{-3} \leq 9$

309.
ⓐ $-8v \leq 96$
ⓑ $\frac{b}{-10} \geq 30$

310.
ⓐ $-9c < 126$
ⓑ $-25 < \frac{p}{-5}$

311.
ⓐ $-7d > 105$
ⓑ $-18 > \frac{q}{-6}$

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

312. $4v \geq 9v - 40$

313. $5u \leq 8u - 21$

314. $13q < 7q - 29$

315. $9p > 14p - 18$

316. $12x + 3(x + 7) > 10x - 24$

317. $9y + 5(y + 3) < 4y - 35$

318. $6h - 4(h - 1) \leq 7h - 11$

319. $4k - (k - 2) \geq 7k - 26$

320. $8m - 2(14 - m) \geq 7(m - 4) + 3m$

321. $6n - 12(3 - n) \leq 9(n - 4) + 9n$

322. $\frac{3}{4}b - \frac{1}{3}b < \frac{5}{12}b - \frac{1}{2}$

323. $9u + 5(2u - 5) \geq 12(u - 1) + 7u$

324. $\frac{2}{3}g - \frac{1}{2}(g - 14) \leq \frac{1}{6}(g + 42)$

325. $\frac{4}{5}h - \frac{2}{3}(h - 9) \geq \frac{1}{15}(2h + 90)$

326. $\frac{5}{6}a - \frac{1}{4}a > \frac{7}{12}a + \frac{2}{3}$

327. $12v + 3(4v - 1) \leq 19(v - 2) + 5v$

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

328. $15k \leq -40$

329. $35k \geq -77$

330. $23p - 2(6 - 5p) > 3(11p - 4)$

331. $18q - 4(10 - 3q) < 5(6q - 8)$

332. $-\frac{9}{4}x \geq -\frac{5}{12}$

333. $-\frac{21}{8}y \leq -\frac{15}{28}$

334. $c + 34 < -99$

335. $d + 29 > -61$

336. $\frac{m}{18} \geq -4$

337. $\frac{n}{13} \leq -6$

Translate to an Inequality and Solve

In the following exercises, translate and solve. Then graph the solution on the number line and write the solution in interval notation.

338. Three more than h is no less than 25.

339. Six more than k exceeds 25.

340. Ten less than w is at least 39.

341. Twelve less than x is no less than 21.

342. Negative five times r is no more than 95.

343. Negative two times s is lower than 56.

344. Nineteen less than b is at most -22.

345. Fifteen less than a is at least -7.

Solve Applications with Linear Inequalities

In the following exercises, solve.

346. Alan is loading a pallet with boxes that each weighs 45 pounds. The pallet can safely support no more than 900 pounds. How many boxes can he safely load onto the pallet?

347. The elevator in Yehire's apartment building has a sign that says the maximum weight is 2100 pounds. If the average weight of one person is 150 pounds, how many people can safely ride the elevator?

348. Andre is looking at apartments with three of his friends. They want the monthly rent to be no more than $2,360. If the roommates split the rent evenly among the four of them, what is the maximum rent each will pay?

349. Arleen got a $20 gift card for the coffee shop. Her favorite iced drink costs $3.79. What is the maximum number of drinks she can buy with the gift card?

350. Teegan likes to play golf. He has budgeted $60 next month for the driving range. It costs him $10.55 for a bucket of balls each time he goes. What is the maximum number of times he can go to the driving range next month?

351. Ryan charges his neighbors $17.50 to wash their car. How many cars must he wash next summer if his goal is to earn at least $1,500?

352. Keshad gets paid $2,400 per month plus 6% of his sales. His brother earns $3,300 per month. For what amount of total sales will Keshad's monthly pay be higher than his brother's monthly pay?

353. Kimuyen needs to earn $4,150 per month in order to pay all her expenses. Her job pays her $3,475 per month plus 4% of her total sales. What is the minimum Kimuyen's total sales must be in order for her to pay all her expenses?

354. Andre has been offered an entry-level job. The company offered him $48,000 per year plus 3.5% of his total sales. Andre knows that the average pay for this job is $62,000. What would Andre's total sales need to be for his pay to be at least as high as the average pay for this job?

355. Nataly is considering two job offers. The first job would pay her $83,000 per year. The second would pay her $66,500 plus 15% of her total sales. What would her total sales need to be for her salary on the second offer be higher than the first?

356. Jake's water bill is $24.80 per month plus $2.20 per ccf (hundred cubic feet) of water. What is the maximum number of ccf Jake can use if he wants his bill to be no more than $60?

357. Kiyoshi's phone plan costs $17.50 per month plus $0.15 per text message. What is the maximum number of text messages Kiyoshi can use so the phone bill is no more than $56.60?

358. Marlon's TV plan costs $49.99 per month plus $5.49 per first-run movie. How many first-run movies can he watch if he wants to keep his monthly bill to be a maximum of $100?

359. Kellen wants to rent a banquet room in a restaurant for her cousin's baby shower. The restaurant charges $350 for the banquet room plus $32.50 per person for lunch. How many people can Kellen have at the shower if she wants the maximum cost to be $1,500?

360. Moshde runs a hairstyling business from her house. She charges $45 for a haircut and style. Her monthly expenses are $960. She wants to be able to put at least $1,200 per month into her savings account order to open her own salon. How many "cut & styles" must she do to save at least $1,200 per month?

361. Noe installs and configures software on home computers. He charges $125 per job. His monthly expenses are $1,600. How many jobs must he work in order to make a profit of at least $2,400?

362. Katherine is a personal chef. She charges $115 per four-person meal. Her monthly expenses are $3,150. How many four-person meals must she sell in order to make a profit of at least $1,900?

363. Melissa makes necklaces and sells them online. She charges $88 per necklace. Her monthly expenses are $3,745. How many necklaces must she sell if she wants to make a profit of at least $1,650?

364. Five student government officers want to go to the state convention. It will cost them $110 for registration, $375 for transportation and food, and $42 per person for the hotel. There is $450 budgeted for the convention in the student government savings account. They can earn the rest of the money they need by having a car wash. If they charge $5 per car, how many cars must they wash in order to have enough money to pay for the trip?

365. Cesar is planning a four-day trip to visit his friend at a college in another state. It will cost him $198 for airfare, $56 for local transportation, and $45 per day for food. He has $189 in savings and can earn $35 for each lawn he mows. How many lawns must he mow to have enough money to pay for the trip?

366. Alonzo works as a car detailer. He charges $175 per car. He is planning to move out of his parents' house and rent his first apartment. He will need to pay $120 for application fees, $950 for security deposit, and first and last months' rent at $1,140 per month. He has $1,810 in savings. How many cars must he detail to have enough money to rent the apartment?

367. Eun-Kyung works as a tutor and earns $60 per hour. She has $792 in savings. She is planning an anniversary party for her parents. She would like to invite 40 guests. The party will cost her $1,520 for food and drinks and $150 for the photographer. She will also have a favor for each of the guests, and each favor will cost $7.50. How many hours must she tutor to have enough money for the party?

Everyday Math

368. Maximum load on a stage In 2014, a high school stage collapsed in Fullerton, California, when 250 students got on stage for the finale of a musical production. Two dozen students were injured. The stage could support a maximum of 12,750 pounds. If the average weight of a student is assumed to be 140 pounds, what is the maximum number of students who could safely be on the stage?

369. Maximum weight on a boat In 2004, a water taxi sank in Baltimore harbor and five people drowned. The water taxi had a maximum capacity of 3,500 pounds (25 people with average weight 140 pounds). The average weight of the 25 people on the water taxi when it sank was 168 pounds per person. What should the maximum number of people of this weight have been?

370. Wedding budget Adele and Walter found the perfect venue for their wedding reception. The cost is $9850 for up to 100 guests, plus $38 for each additional guest. How many guests can attend if Adele and Walter want the total cost to be no more than $12,500?

371. Shower budget Penny is planning a baby shower for her daughter-in-law. The restaurant charges $950 for up to 25 guests, plus $31.95 for each additional guest. How many guests can attend if Penny wants the total cost to be no more than $1,500?

Writing Exercises

372. Explain why it is necessary to reverse the inequality when solving $-5x > 10$.

373. Explain why it is necessary to reverse the inequality when solving $\frac{n}{-3} < 12$.

374. Find your last month's phone bill and the hourly salary you are paid at your job. Calculate the number of hours of work it would take you to earn at least enough money to pay your phone bill by writing an appropriate inequality and then solving it. Do you feel this is an appropriate number of hours? Is this the appropriate phone plan for you?

375. Find out how many units you have left, after this term, to achieve your college goal and estimate the number of units you can take each term in college. Calculate the number of terms it will take you to achieve your college goal by writing an appropriate inequality and then solving it. Is this an acceptable number of terms until you meet your goal? What are some ways you could accelerate this process?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
graph inequalities on the number line.			
solve linear inequalities.			
translate words to an inequality and solve.			
solve applications with linear inequalities.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

2.6 Solve Compound Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve compound inequalities with "and"
- Solve compound inequalities with "or"
- Solve applications with compound inequalities

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{2}{5}(x+10)$.
 If you missed this problem, review Example 1.51.
2. Simplify: $-(x-4)$.
 If you missed this problem, review Example 1.54.

Solve Compound Inequalities with "and"

Now that we know how to solve linear inequalities, the next step is to look at compound inequalities. A **compound inequality** is made up of two inequalities connected by the word "and" or the word "or." For example, the following are compound inequalities.

$$x+3>-4 \quad \text{and} \quad 4x-5\leq 3$$
$$2(y+1)<0 \quad \text{or} \quad y-5\geq -2$$

Compound Inequality

A **compound inequality** is made up of two inequalities connected by the word "and" or the word "or."

To solve a compound inequality means to find all values of the variable that make the compound inequality a true statement. We solve compound inequalities using the same techniques we used to solve linear inequalities. We solve each inequality separately and then consider the two solutions.

To solve a compound inequality with the word "and," we look for all numbers that make *both* inequalities true. To solve a compound inequality with the word "or," we look for all numbers that make *either* inequality true.

Let's start with the compound inequalities with "and." Our solution will be the numbers that are solutions to *both* inequalities known as the intersection of the two inequalities. Consider the intersection of two streets—the part where the streets overlap—belongs to both streets.



To find the solution of the compound inequality, we look at the graphs of each inequality and then find the numbers that belong to both graphs—where the graphs overlap.

For the compound inequality $x>-3$ and $x\leq 2$, we graph each inequality. We then look for where the graphs "overlap". The numbers that are shaded on both graphs, will be shaded on the graph of the solution of the compound inequality. See Figure 2.5.

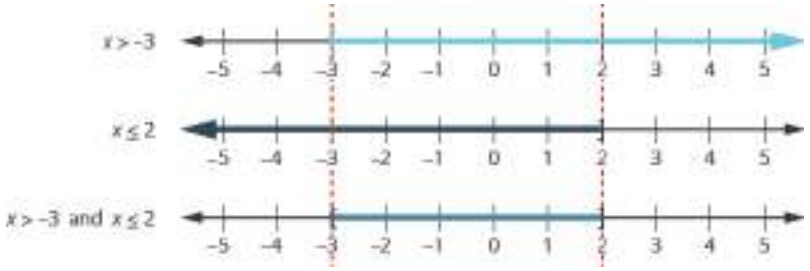


Figure 2.5

We can see that the numbers between -3 and 2 are shaded on both of the first two graphs. They will then be shaded on the solution graph.

The number -3 is not shaded on the first graph and so since it is not shaded on both graphs, it is not included on the solution graph.

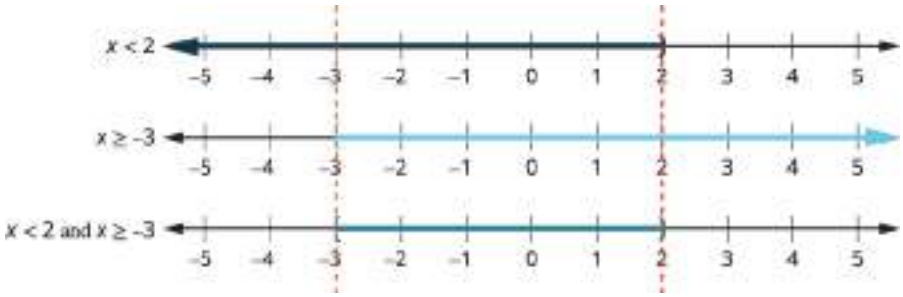
The number two is shaded on both the first and second graphs. Therefore, it is be shaded on the solution graph.

This is how we will show our solution in the next examples.

EXAMPLE 2.61

Solve $6x - 3 < 9$ and $2x + 7 \geq 3$. Graph the solution and write the solution in interval notation.

Solution

	$6x - 3 < 9$	and	$2x + 9 \geq 3$
Step 1. Solve each inequality.	$6x - 3 < 9$		$2x + 9 \geq 3$
	$6x < 12$		$2x \geq -6$
	$x < 2$	and	$x \geq -3$
Step 2. Graph each solution. Then graph the numbers that make both inequalities true. The final graph will show all the numbers that make both inequalities true—the numbers shaded on *both* of the first two graphs.			
Step 3. Write the solution in interval notation.	$[-3, 2)$		

All the numbers that make both inequalities true are the solution to the compound inequality.

TRY IT : : 2.121

Solve the compound inequality. Graph the solution and write the solution in interval notation: $4x - 7 < 9$ and $5x + 8 \geq 3$.

TRY IT : : 2.122

Solve the compound inequality. Graph the solution and write the solution in interval notation: $3x - 4 < 5$ and $4x + 9 \geq 1$.

HOW TO : : SOLVE A COMPOUND INEQUALITY WITH "AND."

Step 1. Solve each inequality.

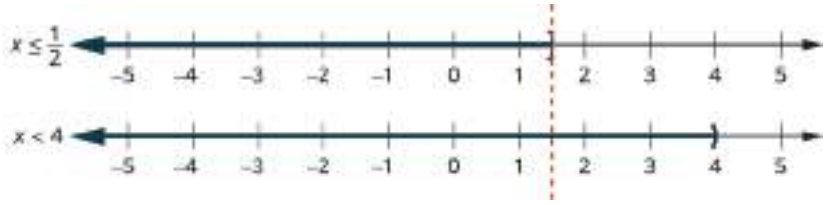
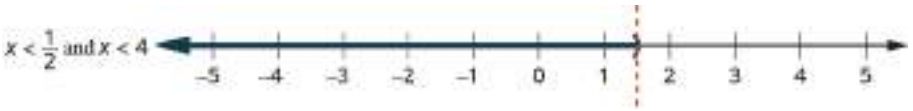
Step 2. Graph each solution. Then graph the numbers that make *both* inequalities true. This graph shows the solution to the compound inequality.

Step 3. Write the solution in interval notation.

EXAMPLE 2.62

Solve $3(2x + 5) \leq 18$ and $2(x - 7) < -6$. Graph the solution and write the solution in interval notation.

Solution

	$3(2x + 5) \leq 18$	and	$2(x - 7) < -6$
Solve each inequality.	$6x + 15 \leq 18$		$2x - 14 < -6$
	$6x \leq 3$		$2x < 8$
	$x \leq \frac{1}{2}$	and	$x < 4$
Graph each solution.			
Graph the numbers that make both inequalities true.			
Write the solution in interval notation.	$(-\infty, \frac{1}{2}]$		

TRY IT : : 2.123

Solve the compound inequality. Graph the solution and write the solution in interval notation: $2(3x + 1) \leq 20$ and $4(x - 1) < 2$.

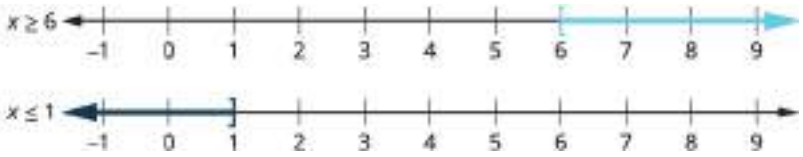
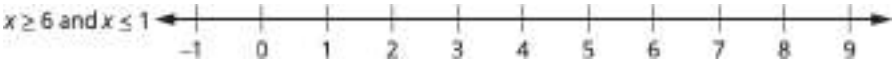
TRY IT : : 2.124

Solve the compound inequality. Graph the solution and write the solution in interval notation: $5(3x - 1) \leq 10$ and $4(x + 3) < 8$.

EXAMPLE 2.63

Solve $\frac{1}{3}x - 4 \geq -2$ and $-2(x-3) \geq 4$. Graph the solution and write the solution in interval notation.

Solution

	$\frac{1}{3}x - 4 \geq -2$	and	$-2(x-3) \geq 4$
Solve each inequality.	$\frac{1}{3}x - 4 \geq -2$		$-2x + 6 \geq 4$
	$\frac{1}{3}x \geq 2$		$-2x \geq -2$
	$x \geq 6$	and	$x \leq 1$
Graph each solution.			
Graph the numbers that make both inequalities true.			
	There are no numbers that make both inequalities true. This is a contradiction so there is no solution.		

TRY IT : : 2.125

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{1}{4}x - 3 \geq -1$ and $-3(x-2) \geq 2$.

TRY IT : : 2.126

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{1}{5}x - 5 \geq -3$ and $-4(x-1) \geq -2$.

Sometimes we have a compound inequality that can be written more concisely. For example, $a < x$ and $x < b$ can be written simply as $a < x < b$ and then we call it a double inequality. The two forms are equivalent.

Double Inequality

A double inequality is a compound inequality such as $a < x < b$. It is equivalent to $a < x$ and $x < b$.

Other forms:	$a < x < b$	is equivalent to	$a < x$	and	$x < b$
	$a \leq x \leq b$	is equivalent to	$a \leq x$	and	$x \leq b$
	$a > x > b$	is equivalent to	$a > x$	and	$x > b$
	$a \geq x \geq b$	is equivalent to	$a \geq x$	and	$x \geq b$

To solve a double inequality we perform the same operation on all three "parts" of the double inequality with the goal of isolating the variable in the center.

EXAMPLE 2.64

Solve $-4 \leq 3x - 7 < 8$. Graph the solution and write the solution in interval notation.

Solution

	$-4 \leq 3x - 7 < 8$
Add 7 to all three parts.	$-4+7 \leq 3x-7+7 < 8+7$
Simplify.	$3 \leq 3x < 15$
Divide each part by three.	$\frac{3}{3} \leq \frac{3x}{3} < \frac{15}{3}$
Simplify.	$1 \leq x < 5$
Graph the solution.	
Write the solution in interval notation.	$[1, 5)$

When written as a double inequality, $1 \leq x < 5$, it is easy to see that the solutions are the numbers caught between one and five, including one, but not five. We can then graph the solution immediately as we did above.

Another way to graph the solution of $1 \leq x < 5$ is to graph both the solution of $x \geq 1$ and the solution of $x < 5$. We would then find the numbers that make both inequalities true as we did in previous examples.

TRY IT : : 2.127

Solve the compound inequality. Graph the solution and write the solution in interval notation: $-5 \leq 4x - 1 < 7$.

TRY IT : : 2.128

Solve the compound inequality. Graph the solution and write the solution in interval notation: $-3 < 2x - 5 \leq 1$.

Solve Compound Inequalities with "or"

To solve a compound inequality with "or", we start out just as we did with the compound inequalities with "and"—we solve the two inequalities. Then we find all the numbers that make *either* inequality true.

Just as the United States is the union of all of the 50 states, the solution will be the union of all the numbers that make either inequality true. To find the solution of the compound inequality, we look at the graphs of each inequality, find the numbers that belong to either graph and put all those numbers together.

To write the solution in interval notation, we will often use the union symbol, $\cup$ to show the union of the solutions shown in the graphs.

HOW TO : : SOLVE A COMPOUND INEQUALITY WITH "OR."

Step 1. Solve each inequality.

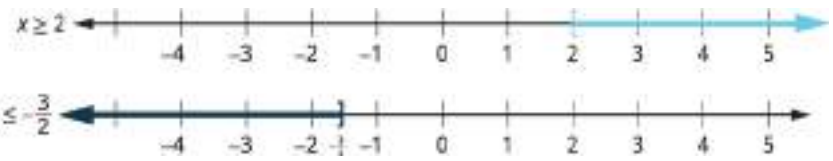
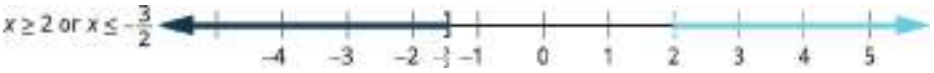
Step 2. Graph each solution. Then graph the numbers that make either inequality true.

Step 3. Write the solution in interval notation.

EXAMPLE 2.65

Solve $5 - 3x \leq -1$ or $8 + 2x \leq 5$. Graph the solution and write the solution in interval notation.

Solution

	$5-3x\le -1$	or	$8+2x\le 5$
Solve each inequality.	$5-3x\le -1$		$8+2x\le 5$
	$-3x\le -6$		$2x\le -3$
	$x\ge 2$	or	$x\le -\frac{3}{2}$
Graph each solution.			
Graph numbers that make either inequality true.			
	$\left(-\infty, -\frac{3}{2}\right] \cup [2, \infty)$		

TRY IT : : 2.129

Solve the compound inequality. Graph the solution and write the solution in interval notation: $1-2x\le -3$ or $7+3x\le 4$.

TRY IT : : 2.130

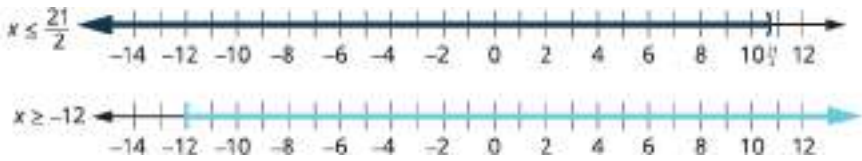
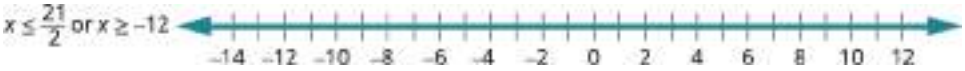
Solve the compound inequality. Graph the solution and write the solution in interval notation: $2-5x\le -3$ or $5+2x\le 3$.

EXAMPLE 2.66

Solve $\frac{2}{3}x-4\le 3$ or $\frac{1}{4}(x+8)\ge -1$. Graph the solution and write the solution in interval notation.

Solution

	$\frac{2}{3}x-4\le 3$	or	$\frac{1}{4}(x+8)\ge -1$
Solve each inequality.	$3\left(\frac{2}{3}x-4\right)\le 3(3)$		$4\cdot\frac{1}{4}(x+8)\ge 4\cdot(-1)$
	$2x-12\le 9$		$x+8\ge -4$
	$2x\le 21$		$x\ge -12$
	$x\le \frac{21}{2}$		

	$x \le \frac{21}{2}$ or	$x \ge -12$
Graph each solution.		
Graph numbers that make either inequality true.		
	The solution covers all real numbers.	
	$(-\infty, \infty)$	

TRY IT : : 2.131

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{3}{5}x - 7 \le -1$ or $\frac{1}{3}(x + 6) \ge -2$.

TRY IT : : 2.132

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{3}{4}x - 3 \le 3$ or $\frac{2}{5}(x + 10) \ge 0$.

Solve Applications with Compound Inequalities

Situations in the real world also involve compound inequalities. We will use the same problem solving strategy that we used to solve linear equation and inequality applications.

Recall the problem solving strategies are to first read the problem and make sure all the words are understood. Then, identify what we are looking for and assign a variable to represent it. Next, restate the problem in one sentence to make it easy to translate into a compound inequality. Last, we will solve the compound inequality.

EXAMPLE 2.67

Due to the drought in California, many communities have tiered water rates. There are different rates for Conservation Usage, Normal Usage and Excessive Usage. The usage is measured in the number of hundred cubic feet (hcf) the property owner uses.

During the summer, a property owner will pay $24.72 plus $1.54 per hcf for Normal Usage. The bill for Normal Usage would be between or equal to $57.06 and $171.02. How many hcf can the owner use if he wants his usage to stay in the normal range?

Solution

Identify what we are looking for.	The number of hcf he can use and stay in the “normal usage” billing range.
Name what we are looking for.	Let $x =$ the number of hcf he can use.
Translate to an inequality.	Bill is \$24.72 plus \$1.54 times the number of hcf he uses or $24.72 + 1.54x$.
	His bill will be between or equal to \$57.06 and \$171.02 $57.06 \le 24.74 + 1.54x \le 171.02$
Solve the inequality.	$57.06 \le 24.74 + 1.54x \le 171.02$ $57.06 - 24.72 \le 24.74 - 24.72 + 1.54x \le 171.02 - 24.72$ $32.34 \le 1.54x \le 146.3$ $\frac{32.34}{1.54} \le \frac{1.54x}{1.54} \le \frac{146.3}{1.54}$ $21 \le x \le 95$
Answer the question.	The property owner can use 21–95 hcf and still fall within the “normal usage” billing range.

TRY IT : : 2.133

Due to the drought in California, many communities now have tiered water rates. There are different rates for Conservation Usage, Normal Usage and Excessive Usage. The usage is measured in the number of hundred cubic feet (hcf) the property owner uses.

During the summer, a property owner will pay \$24.72 plus \$1.32 per hcf for Conservation Usage. The bill for Conservation Usage would be between or equal to \$31.32 and \$52.12. How many hcf can the owner use if she wants her usage to stay in the conservation range?

TRY IT : : 2.134

Due to the drought in California, many communities have tiered water rates. There are different rates for Conservation Usage, Normal Usage and Excessive Usage. The usage is measured in the number of hundred cubic feet (hcf) the property owner uses.

During the winter, a property owner will pay \$24.72 plus \$1.54 per hcf for Normal Usage. The bill for Normal Usage would be between or equal to \$49.36 and \$86.32. How many hcf will he be allowed to use if he wants his usage to stay in the normal range?

MEDIA : :

Access this online resource for additional instruction and practice with solving compound inequalities.

- **Compound inequalities (https://openstax.org/l/37compinequalit)**

2.6 EXERCISES

Practice Makes Perfect

Solve Compound Inequalities with "and"

In the following exercises, solve each inequality, graph the solution, and write the solution in interval notation.

376. $x < 3$ and $x \geq 1$

377. $x \leq 4$ and $x > -2$

378. $x \geq -4$ and $x \leq -1$

379. $x > -6$ and $x < -3$

380. $5x - 2 < 8$ and $6x + 9 \geq 3$

381. $4x - 1 < 7$ and $2x + 8 \geq 4$

382. $4x + 6 \leq 2$ and $2x + 1 \geq -5$

383. $4x - 2 \leq 4$ and $7x - 1 > -8$

384. $2x - 11 < 5$ and $3x - 8 > -5$

385. $7x - 8 < 6$ and $5x + 7 > -3$

386. $4(2x - 1) \leq 12$ and $2(x + 1) < 4$

387. $5(3x - 2) \leq 5$ and $3(x + 3) < 3$

388. $3(2x - 3) > 3$ and $4(x + 5) \geq 4$

389. $-3(x + 4) < 0$ and $-1(3x - 1) \leq 7$

390. $\frac{1}{2}(3x - 4) \leq 1$ and $\frac{1}{3}(x + 6) \leq 4$

391. $\frac{3}{4}(x - 8) \leq 3$ and $\frac{1}{5}(x - 5) \leq 3$

392. $5x - 2 \leq 3x + 4$ and $3x - 4 \geq 2x + 1$

393. $\frac{3}{4}x - 5 \geq -2$ and $-3(x + 1) \geq 6$

394. $\frac{2}{3}x - 6 \geq -4$ and $-4(x + 2) \geq 0$

395. $\frac{1}{2}(x - 6) + 2 < -5$ and $4 - \frac{2}{3}x < 6$

396. $-5 \leq 4x - 1 < 7$

397. $-3 < 2x - 5 \leq 1$

398. $5 < 4x + 1 < 9$

399. $-1 < 3x + 2 < 8$

400. $-8 < 5x + 2 \leq -3$

401. $-6 \leq 4x - 2 < -2$

Solve Compound Inequalities with "or"

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

402. $x \leq -2$ or $x > 3$

403. $x \leq -4$ or $x > -3$

404. $x < 2$ or $x \geq 5$

405. $x < 0$ or $x \geq 4$

406. $2 + 3x \leq 4$ or $5 - 2x \leq -1$

407. $4 - 3x \leq -2$ or $2x - 1 \leq -5$

408. $2(3x - 1) < 4$ or $3x - 5 > 1$

409. $3(2x - 3) < -5$ or $4x - 1 > 3$

410. $\frac{3}{4}x - 2 > 4$ or $4(2 - x) > 0$

411. $\frac{2}{3}x - 3 > 5$ or $3(5 - x) > 6$

412. $3x - 2 > 4$ or $5x - 3 \leq 7$

413. $2(x + 3) \geq 0$ or $3(x + 4) \leq 6$

414. $\frac{1}{2}x - 3 \leq 4$ or $\frac{1}{3}(x - 6) \geq -2$

415. $\frac{3}{4}x + 2 \leq -1$ or $\frac{1}{2}(x + 8) \geq -3$

Mixed practice

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

416. $3x + 7 \leq 1$ and $2x + 3 \geq -5$

417. $6(2x - 1) > 6$ and $5(x + 2) \geq 0$

418. $4 - 7x \geq -3$ or $5(x - 3) + 8 > 3$

419. $\frac{1}{2}x - 5 \leq 3$ or $\frac{1}{4}(x - 8) \geq -3$

420. $-5 \leq 2x - 1 < 7$

421. $\frac{1}{5}(x - 5) + 6 < 4$ and $3 - \frac{2}{3}x < 5$

422. $4x - 2 > 6$ or $3x - 1 \leq -2$

423. $6x - 3 \leq 1$ and $5x - 1 > -6$

424. $-2(3x - 4) \leq 2$ and $-4(x - 1) < 2$

425. $-5 \leq 3x - 2 \leq 4$

Solve Applications with Compound Inequalities

In the following exercises, solve.

426. Penelope is playing a number game with her sister June. Penelope is thinking of a number and wants June to guess it. Five more than three times her number is between 2 and 32. Write a compound inequality that shows the range of numbers that Penelope might be thinking of.

427. Gregory is thinking of a number and he wants his sister Lauren to guess the number. His first clue is that six less than twice his number is between four and forty-two. Write a compound inequality that shows the range of numbers that Gregory might be thinking of.

428. Andrew is creating a rectangular dog run in his back yard. The length of the dog run is 18 feet. The perimeter of the dog run must be at least 42 feet and no more than 72 feet. Use a compound inequality to find the range of values for the width of the dog run.

429. Elouise is creating a rectangular garden in her back yard. The length of the garden is 12 feet. The perimeter of the garden must be at least 36 feet and no more than 48 feet. Use a compound inequality to find the range of values for the width of the garden.

Everyday Math

430. Blood Pressure A person's blood pressure is measured with two numbers. The systolic blood pressure measures the pressure of the blood on the arteries as the heart beats. The diastolic blood pressure measures the pressure while the heart is resting.

ⓐ Let x be your systolic blood pressure. Research and then write the compound inequality that shows you what a normal systolic blood pressure should be for someone your age.

ⓑ Let y be your diastolic blood pressure. Research and then write the compound inequality that shows you what a normal diastolic blood pressure should be for someone your age.

431. Body Mass Index (BMI) is a measure of body fat is determined using your height and weight.

ⓐ Let x be your BMI. Research and then write the compound inequality to show the BMI range for you to be considered normal weight.

ⓑ Research a BMI calculator and determine your BMI. Is it a solution to the inequality in part (a)?

Writing Exercises

432. In your own words, explain the difference between the properties of equality and the properties of inequality.

433. Explain the steps for solving the compound inequality $2 - 7x \geq -5$ or $4(x - 3) + 7 > 3$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve compound inequalities with "and".			
solve compound inequalities with "or".			
solve applications with compound inequalities.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

2.7 Solve Absolute Value Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve absolute value equations
- Solve absolute value inequalities with "less than"
- Solve absolute value inequalities with "greater than"
- Solve applications with absolute value

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate: $-|7|$.
 If you missed this problem, review **Example 1.12**.
2. Fill in $<$, $>$, or $=$ for each of the following pairs of numbers.
 ⓐ $|-8|___-|-8|$ ⓑ $12___-|-12|$ ⓒ $|-6|___-6$ ⓓ $-(-15)___-|-15|$
 If you missed this problem, review **Example 1.12**.
3. Simplify: $14-2|8-3(4-1)|$.
 If you missed this problem, review **Example 1.13**.

Solve Absolute Value Equations

As we prepare to solve absolute value equations, we review our definition of **absolute value**.

Absolute Value

The absolute value of a number is its distance from zero on the number line.

The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

We learned that both a number and its opposite are the same distance from zero on the number line. Since they have the same distance from zero, they have the same absolute value. For example:

-5 is 5 units away from 0, so $|-5| = 5$.

5 is 5 units away from 0, so $|5| = 5$.

Figure 2.6 illustrates this idea.

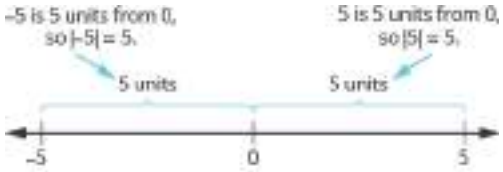


Figure 2.6 The numbers 5 and -5 are both five units away from zero.

For the equation $|x| = 5$, we are looking for all numbers that make this a true statement. We are looking for the numbers whose distance from zero is 5. We just saw that both 5 and -5 are five units from zero on the number line. They are the solutions to the equation.

$$\begin{array}{ll} \text{If} & |x| = 5 \\ \text{then} & x = -5 \text{ or } x = 5 \end{array}$$

The solution can be simplified to a single statement by writing $x = \pm 5$. This is read, "x is equal to positive or negative 5".

We can generalize this to the following property for absolute value equations.

Absolute Value Equations

For any algebraic expression, u, and any positive real number, a,

$$\text{if} \quad |u| = a$$
$$\text{then} \quad u = -a \text{ or } u = a$$

Remember that an absolute value cannot be a negative number.

EXAMPLE 2.68

Solve: ⓐ $|x| = 8$ ⓑ $|y| = -6$ ⓒ $|z| = 0$

Solution

ⓐ

Write the equivalent equations.

$$|x| = 8$$
$$x = -8 \text{ or } x = 8$$
$$x = \pm 8$$

ⓑ

$$|y| = -6$$

No solution

Since an absolute value is always positive, there are no solutions to this equation.

ⓒ

$$|z| = 0$$

Write the equivalent equations. $z = -0$ or $z = 0$

Since $-0 = 0$, $z = 0$

Both equations tell us that $z = 0$ and so there is only one solution.

TRY IT : : 2.135 Solve: ⓐ $|x| = 2$ ⓑ $|y| = -4$ ⓒ $|z| = 0$

TRY IT : : 2.136 Solve: ⓐ $|x| = 11$ ⓑ $|y| = -5$ ⓒ $|z| = 0$

To solve an absolute value equation, we first isolate the absolute value expression using the same procedures we used to solve linear equations. Once we isolate the absolute value expression we rewrite it as the two equivalent equations.

EXAMPLE 2.69 HOW TO SOLVE ABSOLUTE VALUE EQUATIONS

Solve $|5x - 4| - 3 = 8$.

Solution

Step 1. Isolate the absolute value expression.	Add 3 to both sides.	$\|5x-4\|-3=8$ $\|5x-4\|=11$
Step 2. Write the equivalent equations.		$5x-4=-11$ or $5x-4=11$
Step 3. Solve each equation.	Add 4 to each side. Divide each side by 5.	$5x=-7$ $\quad 5x=15$ $x=-\frac{7}{5}$ or $x=3$

Step 4. Check each solution.	Substitute 3 and $-\frac{7}{5}$ into the original equation.	$\lvert 5x-4\rvert -3=8$ $x=3$ $\lvert 5\cdot 3-4\rvert -3\stackrel{?}{=}8$ $\lvert 15-4\rvert -3\stackrel{?}{=}8$ $\lvert 11\rvert -3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8$ ✓ $\lvert 5x-4\rvert -3=8$ $x=-\frac{7}{5}$ $\left\lvert 5\left(-\frac{7}{5}\right)-4\right\rvert -3\stackrel{?}{=}8$ $\lvert -7-4\rvert -3\stackrel{?}{=}8$ $\lvert -11\rvert -3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8$ ✓

TRY IT : : 2.137 Solve: $|3x-5|-1=6$.

TRY IT : : 2.138 Solve: $|4x-3|-5=2$.

The steps for solving an absolute value equation are summarized here.

HOW TO : : SOLVE ABSOLUTE VALUE EQUATIONS.

Step 1. Isolate the absolute value expression.
Step 2. Write the equivalent equations.
Step 3. Solve each equation.
Step 4. Check each solution.

EXAMPLE 2.70

Solve $2|x-7|+5=9$.

Solution

	$2\lvert x-7\rvert +5=9$
Isolate the absolute value expression.	$2\lvert x-7\rvert =4$
	$\lvert x-7\rvert =2$
Write the equivalent equations.	$x-7=-2$ or $x-7=2$
Solve each equation.	$x=5$ or $x=9$

Check:

$2|x-7|+5=9$ | $2|x-7|+5=9$

$2|5-7|+5\stackrel{?}{=}9$ | $2|9-7|+5\stackrel{?}{=}9$

$2|-2|+5\stackrel{?}{=}9$ | $2|2|+5\stackrel{?}{=}9$

$2\cdot 2+5\stackrel{?}{=}9$ | $2\cdot 2+5\stackrel{?}{=}9$

$4+5\stackrel{?}{=}9$ | $4+5\stackrel{?}{=}9$

$9=9$ ✓ | $9=9$ ✓

TRY IT : : 2.139 Solve: $3|x-4|-4=8$.

TRY IT : : 2.140 Solve: $2|x-5|+3=9$.

Remember, an absolute value is always positive!

EXAMPLE 2.71

Solve: $\left|\frac{2}{3}x-4\right|+11=3$.

Solution

$$\left|\frac{2}{3}x-4\right|+11=3$$

Isolate the absolute value term. $\left|\frac{2}{3}x-4\right|=-8$

An absolute value cannot be negative. No solution

TRY IT : : 2.141 Solve: $\left|\frac{3}{4}x-5\right|+9=4$.

TRY IT : : 2.142 Solve: $\left|\frac{5}{6}x+3\right|+8=6$.

Some of our absolute value equations could be of the form $|u|=|v|$ where u and v are algebraic expressions. For example, $|x-3|=|2x+1|$.

How would we solve them? If two algebraic expressions are equal in absolute value, then they are either equal to each other or negatives of each other. The property for absolute value equations says that for any algebraic expression, u, and a positive real number, a, if $|u|=a$, then $u=-a$ or $u=a$.

This tell us that

if $|u|=|v|$

then $u=-v$ or $u=v$

This leads us to the following property for equations with two absolute values.

Equations with Two Absolute Values

For any algebraic expressions, u and v,

if $|u|=|v|$

then $u=-v$ or $u=v$

When we take the opposite of a quantity, we must be careful with the signs and to add parentheses where needed.

EXAMPLE 2.72

Solve: $|5x-1|=|2x+3|$.

Solution

$$|5x-1|=|2x+3|$$

Write the equivalent equations.

$$5x-1=-(2x+3) \quad \text{or} \quad 5x-1=2x+3$$

Solve each equation.

$$5x-1=-2x-3 \quad \text{or} \quad 3x-1=3$$

$$7x-1=-3 \qquad\qquad 3x=4$$

$$7x=-2 \qquad\qquad x=\frac{4}{3}$$

$$x=-\frac{2}{7} \quad \text{or} \quad x=\frac{4}{3}$$

Check.
We leave the check to you.

TRY IT : : 2.143 Solve: $|7x-3|=|3x+7|$.

TRY IT : : 2.144 Solve: $|6x-5|=|3x+4|$.

Solve Absolute Value Inequalities with "Less Than"

Let's look now at what happens when we have an absolute value inequality. Everything we've learned about solving inequalities still holds, but we must consider how the absolute value impacts our work.

Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line. For the equation $|x|=5$, we saw that both 5 and -5 are five units from zero on the number line. They are the solutions to the equation.

$$|x|=5$$
$$x=-5 \quad \text{or} \quad x=5$$

What about the inequality $|x|\leq 5$? Where are the numbers whose distance is less than or equal to 5? We know -5 and 5 are both five units from zero. All the numbers between -5 and 5 are less than five units from zero. See Figure 2.7.

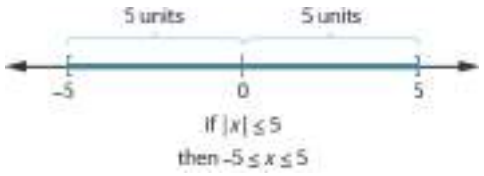


Figure 2.7

In a more general way, we can see that if $|u|\leq a$, then $-a\leq u\leq a$. See Figure 2.8.

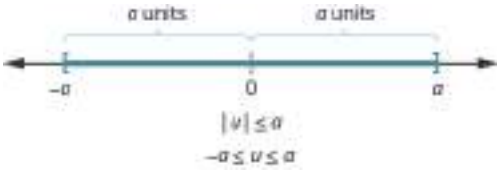


Figure 2.8

This result is summarized here.

Absolute Value Inequalities with $<$ or $\leq$

For any algebraic expression, *u*, and any positive real number, *a*,

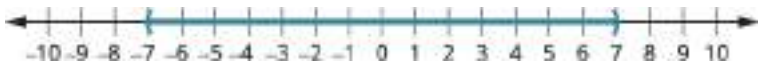
if	$\lvert u\rvert < a$,	then $-a < u < a$
if	$\lvert u\rvert \leq a$,	then $-a \leq u \leq a$

After solving an inequality, it is often helpful to check some points to see if the solution makes sense. The graph of the solution divides the number line into three sections. Choose a value in each section and substitute it in the original inequality to see if it makes the inequality true or not. While this is not a complete check, it often helps verify the solution.

EXAMPLE 2.73

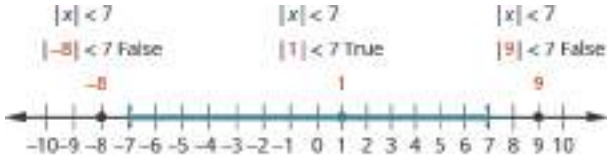
Solve $\lvert x\rvert < 7$. Graph the solution and write the solution in interval notation.

Solution

	$\lvert x\rvert < 7$
Write the equivalent inequality.	$-7 < x < 7$
Graph the solution.	
Write the solution using interval notation.	$(-7, 7)$

Check:

To verify, check a value in each section of the number line showing the solution. Choose numbers such as -8, 1, and 9.



TRY IT : : 2.145 Graph the solution and write the solution in interval notation: $\lvert x\rvert < 9$.

TRY IT : : 2.146 Graph the solution and write the solution in interval notation: $\lvert x\rvert < 1$.

EXAMPLE 2.74

Solve $\lvert 5x - 6\rvert \leq 4$. Graph the solution and write the solution in interval notation.

Solution

Step 1. Isolate the absolute value expression. It is isolated.	$\lvert 5x - 6\rvert \leq 4$
Step 2. Write the equivalent compound inequality.	$-4 \leq 5x - 6 \leq 4$
Step 3. Solve the compound inequality.	$2 \leq 5x \leq 10$ $\frac{2}{5} \leq x \leq 2$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$\left[\frac{2}{5}, 2\right]$

Check:
The check is left to you.

TRY IT : : 2.147 Solve $|2x-1| \leq 5$. Graph the solution and write the solution in interval notation:

TRY IT : : 2.148 Solve $|4x-5| \leq 3$. Graph the solution and write the solution in interval notation:

HOW TO : : SOLVE ABSOLUTE VALUE INEQUALITIES WITH < OR ≤.

Step 1. Isolate the absolute value expression.
Step 2. Write the equivalent compound inequality.

$|u| < a$ is equivalent to $-a < u < a$
$|u| \leq a$ is equivalent to $-a \leq u \leq a$

Step 3. Solve the compound inequality.
Step 4. Graph the solution
Step 5. Write the solution using interval notation.

Solve Absolute Value Inequalities with "Greater Than"

What happens for absolute value inequalities that have "greater than"? Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line.

We started with the inequality $|x| \leq 5$. We saw that the numbers whose distance is less than or equal to five from zero on the number line were -5 and 5 and all the numbers between -5 and 5. See Figure 2.9.

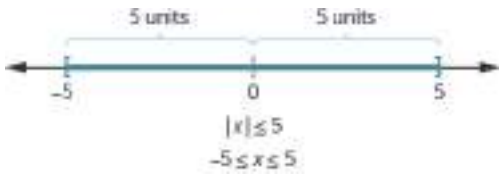


Figure 2.9

Now we want to look at the inequality $|x| \geq 5$. Where are the numbers whose distance from zero is greater than or equal to five?

Again both -5 and 5 are five units from zero and so are included in the solution. Numbers whose distance from zero is greater than five units would be less than -5 and greater than 5 on the number line. See Figure 2.10.

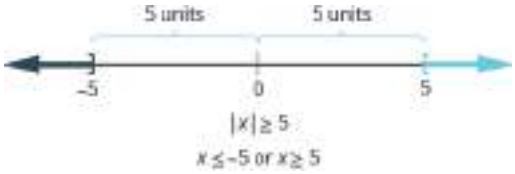


Figure 2.10

In a more general way, we can see that if $|u| \geq a$, then $u \leq -a$ or $u \leq a$. See Figure 2.11.

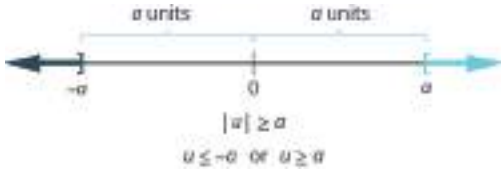


Figure 2.11

This result is summarized here.

Absolute Value Inequalities with > or ≥

For any algebraic expression, u, and any positive real number, a,

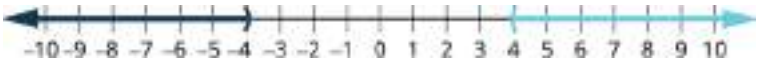
if $|u| > a$, then $u < -a$ or $u > a$

if $|u| \geq a$, then $u \leq -a$ or $u \geq a$

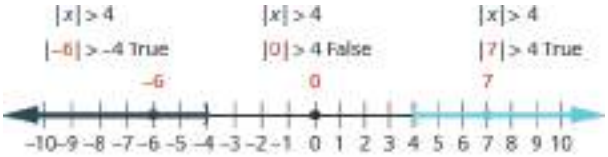
EXAMPLE 2.75

Solve $|x| > 4$. Graph the solution and write the solution in interval notation.

Solution

	$\|x\| > 4$
Write the equivalent inequality.	$x < -4$ or $x > 4$
Graph the solution.	
Write the solution using interval notation.	$(-\infty, -4) \cup (4, \infty)$
Check:	

To verify, check a value in each section of the number line showing the solution. Choose numbers such as -6, 0, and 7.



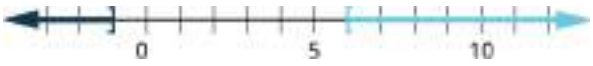
TRY IT : : 2.149 Solve $|x| > 2$. Graph the solution and write the solution in interval notation.

TRY IT : : 2.150 Solve $|x| > 1$. Graph the solution and write the solution in interval notation.

EXAMPLE 2.76

Solve $|2x - 3| \geq 5$. Graph the solution and write the solution in interval notation.

Solution

	$\lvert 2x-3\rvert \geq 5$
Step 1. Isolate the absolute value expression. It is isolated.	
Step 2. Write the equivalent compound inequality.	$2x-3 \leq -5 \text{ or } 2x-3 \geq 5$
Step 3. Solve the compound inequality.	$2x \leq -2 \text{ or } 2x \geq 8$ $x \leq -1 \text{ or } x \geq 4$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$(-\infty, -1] \cup [4, \infty)$
Check: The check is left to you.	

TRY IT : : 2.151 Solve $\lvert 4x-3\rvert \geq 5$. Graph the solution and write the solution in interval notation.

TRY IT : : 2.152 Solve $\lvert 3x-4\rvert \geq 2$. Graph the solution and write the solution in interval notation.

HOW TO : : SOLVE ABSOLUTE VALUE INEQUALITIES WITH > OR ≥.

Step 1. Isolate the absolute value expression.

Step 2. Write the equivalent compound inequality.

$\lvert u\rvert > a$ is equivalent to $u < -a \text{ or } u > a$

$\lvert u\rvert \geq a$ is equivalent to $u \leq -a \text{ or } u \geq a$

Step 3. Solve the compound inequality.

Step 4. Graph the solution

Step 5. Write the solution using interval notation.

Solve Applications with Absolute Value

Absolute value inequalities are often used in the manufacturing process. An item must be made with near perfect specifications. Usually there is a certain *tolerance* of the difference from the specifications that is allowed. If the difference from the specifications exceeds the tolerance, the item is rejected.

$$\lvert \text{actual-ideal}\rvert \leq \text{tolerance}$$

EXAMPLE 2.77

The ideal diameter of a rod needed for a machine is 60 mm. The actual diameter can vary from the ideal diameter by 0.075 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Solution

Use an absolute value inequality to express this situation.	Let $x =$ the actual measurement. $\lvert\text{actual-ideal}\rvert \leq \text{tolerance}$
	$\lvert x - 60\rvert \leq 0.075$
Rewrite as a compound inequality.	$-0.075 \leq x - 60 \leq 0.075$
Solve the inequality.	$59.925 \leq x \leq 60.075$
Answer the question.	The diameter of the rod can be between 59.925 mm and 60.075 mm.

TRY IT : : 2.153

The ideal diameter of a rod needed for a machine is 80 mm. The actual diameter can vary from the ideal diameter by 0.009 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

TRY IT : : 2.154

The ideal diameter of a rod needed for a machine is 75 mm. The actual diameter can vary from the ideal diameter by 0.05 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

MEDIA : :

Access this online resource for additional instruction and practice with solving linear absolute value equations and inequalities.

- **Solving Linear Absolute Value Equations and Inequalities (https://openstax.org/l/37solvlinabsol)**

2.7 EXERCISES

Practice Makes Perfect

Solve Absolute Value Equations

In the following exercises, solve.

434. ⓐ $|x| = 6$ ⓑ $|y| = -3$ ⓒ $|z| = 0$

435. ⓐ $|x| = 4$ ⓑ $|y| = -5$ ⓒ $|z| = 0$

436. ⓐ $|x| = 7$ ⓑ $|y| = -11$ ⓒ $|z| = 0$

437. ⓐ $|x| = 3$ ⓑ $|y| = -1$ ⓒ $|z| = 0$

438. $|2x - 3| - 4 = 1$

439. $|4x - 1| - 3 = 0$

440. $|3x - 4| + 5 = 7$

441. $|4x + 7| + 2 = 5$

442. $4|x - 1| + 2 = 10$

443. $3|x - 4| + 2 = 11$

444. $3|4x - 5| - 4 = 11$

445. $3|x + 2| - 5 = 4$

446. $-2|x - 3| + 8 = -4$

447. $-3|x - 4| + 4 = -5$

448. $\left|\frac{3}{4}x - 3\right| + 7 = 2$

449. $\left|\frac{3}{5}x - 2\right| + 5 = 2$

450. $\left|\frac{1}{2}x + 5\right| + 4 = 1$

451. $\left|\frac{1}{4}x + 3\right| + 3 = 1$

452. $|3x - 2| = |2x - 3|$

453. $|4x + 3| = |2x + 1|$

454. $|6x - 5| = |2x + 3|$

455. $|6 - x| = |3 - 2x|$

Solve Absolute Value Inequalities with "less than"

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

456. $|x| < 5$

457. $|x| < 1$

458. $|x| \leq 8$

459. $|x| \leq 3$

460. $|3x - 3| \leq 6$

461. $|2x - 5| \leq 3$

462. $|2x + 3| + 5 < 4$

463. $|3x - 7| + 3 < 1$

464. $|4x - 3| < 1$

465. $|6x - 5| < 7$

466. $|x - 4| \leq -1$

467. $|5x + 1| \leq -2$

Solve Absolute Value Inequalities with "greater than"

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

468. $|x| > 3$

469. $|x| > 6$

470. $|x| \geq 2$

471. $|x| \geq 5$

472. $|3x - 8| > -1$

473. $|x - 5| > -2$

474. $|3x - 2| > 4$

475. $|2x - 1| > 5$

476. $|x + 3| \geq 5$

477. $|x - 7| \geq 1$

478. $3|x| + 4 \geq 1$

479. $5|x| + 6 \geq 1$

In the following exercises, solve. For each inequality, also graph the solution and write the solution in interval notation.

480. $2|x + 6| + 4 = 8$

481. $|6x - 5| = |2x + 3|$

482. $|3x - 4| \geq 2$

483. $|2x - 5| + 2 = 3$

484. $|4x - 3| < 5$

485. $|3x + 1| - 3 = 7$

486. $|7x + 2| + 8 < 4$

487. $5|2x - 1| - 3 = 7$

488. $|8 - x| = |4 - 3x|$

489. $|x - 7| > -3$

Solve Applications with Absolute Value

In the following exercises, solve.

490. A chicken farm ideally produces 200,000 eggs per day. But this total can vary by as much as 25,000 eggs. What is the maximum and minimum expected production at the farm?

491. An organic juice bottler ideally produces 215,000 bottle per day. But this total can vary by as much as 7,500 bottles. What is the maximum and minimum expected production at the bottling company?

492. In order to insure compliance with the law, Miguel routinely overshoots the weight of his tortillas by 0.5 gram. He just received a report that told him that he could be losing as much as $100,000 per year using this practice. He now plans to buy new equipment that guarantees the thickness of the tortilla within 0.005 inches. If the ideal thickness of the tortilla is 0.04 inches, what thickness of tortillas will be guaranteed?

493. At Lilly's Bakery, the ideal weight of a loaf of bread is 24 ounces. By law, the actual weight can vary from the ideal by 1.5 ounces. What range of weight will be acceptable to the inspector without causing the bakery being fined?

Writing Exercises

494. Write a graphical description of the absolute value of a number.

495. In your own words, explain how to solve the absolute value inequality, $|3x - 2| \geq 4$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve absolute value equations.			
solve absolute value inequalities with "less than".			
solve absolute value inequalities with "greater than".			
solve applications with absolute value.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

CHAPTER 2 REVIEW

KEY TERMS

compound inequality A compound inequality is made up of two inequalities connected by the word "and" or the word "or."

conditional equation An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity An equation that is true for any value of the variable is called an Identity. The solution of an identity is all real numbers.

linear equation A linear equation is an equation in one variable that can be written, where a and b are real numbers and $a \neq 0$, as $ax + b = 0$.

solution of an equation A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

KEY CONCEPTS

2.1 Use a General Strategy to Solve Linear Equations

- **How to determine whether a number is a solution to an equation**

 Step 1. Substitute the number in for the variable in the equation.

 Step 2. Simplify the expressions on both sides of the equation.

 Step 3. Determine whether the resulting equation is true.
 If it is true, the number is a solution.
 If it is not true, the number is not a solution.

- **How to Solve Linear Equations Using a General Strategy**

 Step 1. Simplify each side of the equation as much as possible.
 Use the Distributive Property to remove any parentheses.
 Combine like terms.

 Step 2. Collect all the variable terms on one side of the equation.
 Use the Addition or Subtraction Property of Equality.

 Step 3. Collect all the constant terms on the other side of the equation.
 Use the Addition or Subtraction Property of Equality.

 Step 4. Make the coefficient of the variable term equal to 1.
 Use the Multiplication or Division Property of Equality.
 State the solution to the equation.

 Step 5. Check the solution.
 Substitute the solution into the original equation to make sure the result is a true statement.

- **How to Solve Equations with Fraction or Decimal Coefficients**

 Step 1. Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.

 Step 2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.

 Step 3. Solve using the General Strategy for Solving Linear Equations.

2.2 Use a Problem Solving Strategy

- **How To Use a Problem Solving Strategy for Word Problems**

 Step 1. **Read** the problem. Make sure all the words and ideas are understood.

 Step 2. **Identify** what you are looking for.

 Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.

 Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5. **Solve** the equation using proper algebra techniques.

Step 6. **Check** the answer in the problem to make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

- **How To Find Percent Change**

 Step 1. Find the amount of change
 $$\text{change} = \text{new amount} - \text{original amount}$$

 Step 2. Find what percent the amount of change is of the original amount.
 $$\text{change is what percent of the original amount?}$$

- **Discount**
 $$\text{amount of discount} = \text{discount rate} \cdot \text{original price}$$
 $$\text{sale price} = \text{original amount} - \text{discount}$$

- **Mark-up**
 $$\text{amount of mark-up} = \text{mark-up rate} \cdot \text{original cost}$$
 $$\text{list price} = \text{original cost} + \text{mark up}$$

- **Simple Interest**
 If an amount of money, P, called the principal, is invested or borrowed for a period of t years at an annual interest rate r, the amount of interest, I, earned or paid is:
 $$I = Prt \quad \text{where} \quad \begin{aligned} I &= \text{interest} \\ P &= \text{principal} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

2.3 Solve a Formula for a Specific Variable

- **How To Solve Geometry Applications**

 Step 1. **Read** the problem and make sure all the words and ideas are understood.

 Step 2. **Identify** what you are looking for.

 Step 3. **Name** what you are looking for by choosing a variable to represent it. Draw the figure and label it with the given information.

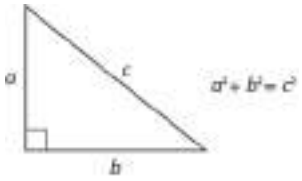
 Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

 Step 5. **Solve** the equation using good algebra techniques.

 Step 6. **Check** the answer in the problem and make sure it makes sense.

 Step 7. **Answer** the question with a complete sentence.

- **The Pythagorean Theorem**
 - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.



2.4 Solve Mixture and Uniform Motion Applications

- **Total Value of Coins**
 For the same type of coin, the total value of a number of coins is found by using the model
 $$number \cdot value = total\ value$$
 - *number* is the number of coins

- *value* is the value of each coin
- *total value* is the total value of all the coins

- **How to solve coin word problems.**

 Step 1. **Read** the problem. Make sure all the words and ideas are understood.
 Determine the types of coins involved.
 Create a table to organize the information.
 Label the columns "type," "number," "value," "total value."
 List the types of coins.
 Write in the value of each type of coin.
 Write in the total value of all the coins.

Type	Number	• Value ($)	= Total Value ($)

 Step 2. **Identify** what you are looking for.

 Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.
 Use variable expressions to represent the number of each type of coin and write them in the table.
 Multiply the number times the value to get the total value of each type of coin.

 Step 4. **Translate** into an equation.
 It may be helpful to restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.
 Write the equation by adding the total values of all the types of coins.

 Step 5. **Solve** the equation using good algebra techniques.

 Step 6. **Check** the answer in the problem and make sure it makes sense.

 Step 7. **Answer** the question with a complete sentence.

- **How To Solve a Uniform Motion Application**

 Step 1. **Read** the problem. Make sure all the words and ideas are understood.
 Draw a diagram to illustrate what it happening.
 Create a table to organize the information.
 Label the columns rate, time, distance.
 List the two scenarios.
 Write in the information you know.

	Rate	• Time	= Distance

 Step 2. **Identify** what you are looking for.

 Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.
 Complete the chart.
 Use variable expressions to represent that quantity in each row.
 Multiply the rate times the time to get the distance.

 Step 4. **Translate** into an equation.
 Restate the problem in one sentence with all the important information.
 Then, translate the sentence into an equation.

 Step 5. **Solve** the equation using good algebra techniques.

 Step 6. **Check** the answer in the problem and make sure it makes sense.

 Step 7. **Answer** the question with a complete sentence.

2.5 Solve Linear Inequalities

- **Inequalities, Number Lines, and Interval Notation**

 $x > a$ $x \geq a$ $x < a$ $x \leq a$



- **Linear Inequality**
 - A **linear inequality** is an inequality in one variable that can be written in one of the following forms where *a*, *b*, and *c* are real numbers and $a \neq 0$:

$$ax + b < c, \qquad ax + b \leq c, \qquad ax + b > c, \qquad ax + b \geq c.$$

- **Addition and Subtraction Property of Inequality**
 - For any numbers *a*, *b*, and *c*, if $a < b$, then

$$a + c < b + c \qquad a - c < b - c$$
$$a + c > b + c \qquad a - c > b - c$$

 - We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.
- **Multiplication and Division Property of Inequality**
 - For any numbers *a*, *b*, and *c*,

multiply or divide by a **positive**

if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

multiply or divide by a **negative**

if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

- **Phrases that indicate inequalities**

>	≥	<	≤
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

2.6 Solve Compound Inequalities

- **How to solve a compound inequality with "and"**

 Step 1. Solve each inequality.

 Step 2. Graph each solution. Then graph the numbers that make *both* inequalities true. This graph shows the solution to the compound inequality.

 Step 3. Write the solution in interval notation.

- **Double Inequality**

- A **double inequality** is a compound inequality such as $a < x < b$. It is equivalent to $a < x$ and $x < b$.

 Other forms:

 $a < x < b$ is equivalent to $a < x$ and $x < b$

 $a \leq x \leq b$ is equivalent to $a \leq x$ and $x \leq b$

 $a > x > b$ is equivalent to $a > x$ and $x > b$

 $a \geq x \geq b$ is equivalent to $a \geq x$ and $x \geq b$

- **How to solve a compound inequality with "or"**

 Step 1. Solve each inequality.

 Step 2. Graph each solution. Then graph the numbers that make either inequality true.

 Step 3. Write the solution in interval notation.

2.7 Solve Absolute Value Inequalities

- **Absolute Value**
 The absolute value of a number is its distance from 0 on the number line.
 The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.
 Absolute values are always greater than or equal to zero.
- **Absolute Value Equations**
 For any algebraic expression, *u*, and any positive real number, *a*,
 if $|u| = a$
 then $u = -a$ or $u = a$
 Remember that an absolute value cannot be a negative number.
- **How to Solve Absolute Value Equations**

 Step 1. Isolate the absolute value expression.

 Step 2. Write the equivalent equations.

 Step 3. Solve each equation.

 Step 4. Check each solution.
- **Equations with Two Absolute Values**
 For any algebraic expressions, *u* and *v*,
 if $|u| = |v|$
 then $u = -v$ or $u = v$
- **Absolute Value Inequalities with** $<$ or $\leq$

 For any algebraic expression, *u*, and any positive real number, *a*,
 if $|u| < a$, then $-a < u < a$
 if $|u| \leq a$, then $-a \leq u \leq a$
- **How To Solve Absolute Value Inequalities with** $<$ or $\leq$

 Step 1. Isolate the absolute value expression.

 Step 2. Write the equivalent compound inequality.

 $|u| < a$ is equivalent to $-a < u < a$

 $|u| \leq a$ is equivalent to $-a \leq u \leq a$

 Step 3. Solve the compound inequality.

 Step 4. Graph the solution

 Step 5. Write the solution using interval notation
- **Absolute Value Inequalities with** $>$ or $\geq$

 For any algebraic expression, *u*, and any positive real number, *a*,
 if $|u| > a$, then $u < -a$ or $u > a$
 if $|u| \geq a$, then $u \leq -a$ or $u \geq a$
- **How To Solve Absolute Value Inequalities with** $>$ or $\geq$

 Step 1. Isolate the absolute value expression.

Step 2. Write the equivalent compound inequality.

$|u| > a$ is equivalent to $u < -a$ or $u > a$

$|u| \geq a$ is equivalent to $u \leq -a$ or $u \geq a$

Step 3. Solve the compound inequality.

Step 4. Graph the solution

Step 5. Write the solution using interval notation

REVIEW EXERCISES

2.1 Use a General Strategy to Solve Linear Equations

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, determine whether each number is a solution to the equation.

496. $10x - 1 = 5x,\ x = \frac{1}{5}$

497. $-12n + 5 = 8n,\ n = -\frac{5}{4}$

In the following exercises, solve each linear equation.

498. $6(x + 6) = 24$

499. $-(s + 4) = 18$

500. $23 - 3(y - 7) = 8$

501. $\frac{1}{3}(6m + 21) = m - 7$

502. $4(3.5y + 0.25) = 365$

503. $0.25(q - 8) = 0.1(q + 7)$

504. $8(r - 2) = 6(r + 10)$

505. $5 + 7(2 - 5x) = 2(9x + 1) - (13x - 57)$

506. $(9n + 5) - (3n - 7) = 20 - (4n - 2)$

507. $2[-16 + 5(8k - 6)] = 8(3 - 4k) - 32$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

508. $17y - 3(4 - 2y) = 11(y - 1) + 12y - 1$

509. $9u + 32 = 15(u - 4) - 3(2u + 21)$

510. $-8(7m + 4) = -6(8m + 9)$

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation.

511. $\frac{2}{5}n - \frac{1}{10} = \frac{7}{10}$

512. $\frac{3}{4}a - \frac{1}{3} = \frac{1}{2}a + \frac{5}{6}$

513. $\frac{1}{2}(k + 3) = \frac{1}{3}(k + 16)$

514. $\frac{5y - 1}{3} + 4 = \frac{-8y + 4}{6}$

515. $0.8x - 0.3 = 0.7x + 0.2$

516. $0.10d + 0.05(d - 4) = 2.05$

2.2 Use a Problem-Solving Strategy

Use a Problem Solving Strategy for Word Problems

In the following exercises, solve using the problem solving strategy for word problems.

517. Three-fourths of the people at a concert are children. If there are 87 children, what is the total number of people at the concert?

518. There are nine saxophone players in the band. The number of saxophone players is one less than twice the number of tuba players. Find the number of tuba players.

Solve Number Word Problems

In the following exercises, solve each number word problem.

519. The sum of a number and three is forty-one. Find the number.

520. One number is nine less than another. Their sum is negative twenty-seven. Find the numbers.

521. One number is two more than four times another. Their sum is negative thirteen. Find the numbers.

522. The sum of two consecutive integers is -135. Find the numbers.

523. Find three consecutive even integers whose sum is 234.

524. Find three consecutive odd integers whose sum is 51.

525. Koji has $5,502 in his savings account. This is $30 less than six times the amount in his checking account. How much money does Koji have in his checking account?

Solve Percent Applications

In the following exercises, translate and solve.

526. What number is 67% of 250?

527. 12.5% of what number is 20?

528. What percent of 125 is 150?

In the following exercises, solve.

529. The bill for Dino's lunch was $19.45. He wanted to leave 20% of the total bill as a tip. How much should the tip be?

530. Dolores bought a crib on sale for $350. The sale price was 40% of the original price. What was the original price of the crib?

531. Jaden earns $2,680 per month. He pays $938 a month for rent. What percent of his monthly pay goes to rent?

532. Angel received a raise in his annual salary from $55,400 to $56,785. Find the percent change.

533. Rowena's monthly gasoline bill dropped from $83.75 last month to $56.95 this month. Find the percent change.

534. Emmett bought a pair of shoes on sale at 40% off from an original price of $138. Find ⓐ the amount of discount and ⓑ the sale price.

535. Lacey bought a pair of boots on sale for $95. The original price of the boots was $200. Find ⓐ the amount of discount and ⓑ the discount rate. (Round to the nearest tenth of a percent, if needed.)

536. Nga and Lauren bought a chest at a flea market for $50. They re-finished it and then added a 350% mark-up. Find ⓐ the amount of the mark-up and ⓑ the list price.

Solve Simple Interest Applications

In the following exercises, solve.

537. Winston deposited $3,294 in a bank account with interest rate 2.6% How much interest was earned in five years?

538. Moira borrowed $4,500 from her grandfather to pay for her first year of college. Three years later, she repaid the $4,500 plus $243 interest. What was the rate of interest?

539. Jaime's refrigerator loan statement said he would pay $1,026 in interest for a four-year loan at 13.5%. How much did Jaime borrow to buy the refrigerator?

2.3 Solve a formula for a Specific Variable

Solve a Formula for a Specific Variable

In the following exercises, solve the formula for the specified variable.

540. Solve the formula $V = LWH$ for L.

541. Solve the formula $A = \frac{1}{2}d_1 d_2$ for d_2.

542. Solve the formula $h = 48t + \frac{1}{2}at^2$ for t.

543. Solve the formula $4x - 3y = 12$ for y.

Use Formulas to Solve Geometry Applications

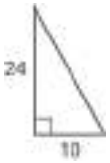
In the following exercises, solve using a geometry formula.

544. What is the height of a triangle with area 67.5 square meters and base 9 meters?

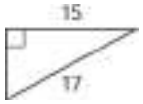
545. The measure of the smallest angle in a right triangle is $45°$ less than the measure of the next larger angle. Find the measures of all three angles.

546. The perimeter of a triangle is 97 feet. One side of the triangle is eleven feet more than the smallest side. The third side is six feet more than twice the smallest side. Find the lengths of all sides.

547. Find the length of the hypotenuse.



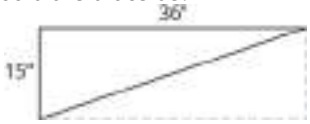
548. Find the length of the missing side. Round to the nearest tenth, if necessary.



549. Sergio needs to attach a wire to hold the antenna to the roof of his house, as shown in the figure. The antenna is eight feet tall and Sergio has 10 feet of wire. How far from the base of the antenna can he attach the wire? Approximate to the nearest tenth, if necessary.



550. Seong is building shelving in his garage. The shelves are 36 inches wide and 15 inches tall. He wants to put a diagonal brace across the back to stabilize the shelves, as shown. How long should the brace be?



551. The length of a rectangle is 12 cm more than the width. The perimeter is 74 cm. Find the length and the width.

552. The width of a rectangle is three more than twice the length. The perimeter is 96 inches. Find the length and the width.

553. The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

2.4 Solve Mixture and Uniform Motion Applications

Solve Coin Word Problems

In the following exercises, solve.

554. Paulette has $140 in $5 and $10 bills. The number of $10 bills is one less than twice the number of $5 bills. How many of each does she have?

555. Lenny has $3.69 in pennies, dimes, and quarters. The number of pennies is three more than the number of dimes. The number of quarters is twice the number of dimes. How many of each coin does he have?

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

556. Tickets for a basketball game cost $2 for students and $5 for adults. The number of students was three less than 10 times the number of adults. The total amount of money from ticket sales was $619. How many of each ticket were sold?

557. 125 tickets were sold for the jazz band concert for a total of $1,022. Student tickets cost $6 each and general admission tickets cost $10 each. How many of each kind of ticket were sold?

558. Yumi spent $34.15 buying stamps. The number of $0.56 stamps she bought was 10 less than four times the number of $0.41 stamps. How many of each did she buy?

Solve Mixture Word Problems

In the following exercises, solve.

559. Marquese is making 10 pounds of trail mix from raisins and nuts. Raisins cost $3.45 per pound and nuts cost $7.95 per pound. How many pounds of raisins and how many pounds of nuts should Marquese use for the trail mix to cost him $6.96 per pound?

560. Amber wants to put tiles on the backsplash of her kitchen counters. She will need 36 square feet of tile. She will use basic tiles that cost $8 per square foot and decorator tiles that cost $20 per square foot. How many square feet of each tile should she use so that the overall cost of the backsplash will be $10 per square foot?

561. Enrique borrowed $23,500 to buy a car. He pays his uncle 2% interest on the $4,500 he borrowed from him, and he pays the bank 11.5% interest on the rest. What average interest rate does he pay on the total $23,500? (Round your answer to the nearest tenth of a percent.)

Solve Uniform Motion Applications

In the following exercises, solve.

562. When Gabe drives from Sacramento to Redding it takes him 2.2 hours. It takes Elsa two hours to drive the same distance. Elsa's speed is seven miles per hour faster than Gabe's speed. Find Gabe's speed and Elsa's speed.

563. Louellen and Tracy met at a restaurant on the road between Chicago and Nashville. Louellen had left Chicago and drove 3.2 hours towards Nashville. Tracy had left Nashville and drove 4 hours towards Chicago, at a speed one mile per hour faster than Louellen's speed. The distance between Chicago and Nashville is 472 miles. Find Louellen's speed and Tracy's speed.

564. Two busses leave Amarillo at the same time. The Albuquerque bus heads west on the I-40 at a speed of 72 miles per hour, and the Oklahoma City bus heads east on the I-40 at a speed of 78 miles per hour. How many hours will it take them to be 375 miles apart?

565. Kyle rowed his boat upstream for 50 minutes. It took him 30 minutes to row back downstream. His speed going upstream is two miles per hour slower than his speed going downstream. Find Kyle's upstream and downstream speeds.

566. At 6:30, Devon left her house and rode her bike on the flat road until 7:30. Then she started riding uphill and rode until 8:00. She rode a total of 15 miles. Her speed on the flat road was three miles per hour faster than her speed going uphill. Find Devon's speed on the flat road and riding uphill.

567. Anthony drove from New York City to Baltimore, which is a distance of 192 miles. He left at 3:45 and had heavy traffic until 5:30. Traffic was light for the rest of the drive, and he arrived at 7:30. His speed in light traffic was four miles per hour more than twice his speed in heavy traffic. Find Anthony's driving speed in heavy traffic and light traffic.

2.5 Solve Linear Inequalities

Graph Inequalities on the Number Line

In the following exercises, graph the inequality on the number line and write in interval notation.

568. $x < -1$

569. $x \geq -2.5$

570. $x \leq \frac{5}{4}$

571. $x > 2$

572. $-2 < x < 0$

573. $-5 \leq x < -3$

574. $0 \leq x \leq 3.5$

Solve Linear Inequalities

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

575. $n - 12 \leq 23$

576. $a + \frac{2}{3} \geq \frac{7}{12}$

577. $9x > 54$

578. $\frac{q}{-2} \geq -24$

579. $6p > 15p - 30$

580. $9h - 7(h - 1) \leq 4h - 23$

581. $5n - 15(4 - n) < 10(n - 6) + 10n$

582. $\frac{3}{8}a - \frac{1}{12}a > \frac{5}{12}a + \frac{3}{4}$

Translate Words to an Inequality and Solve

In the following exercises, translate and solve. Then write the solution in interval notation and graph on the number line.

583. Five more than z is at most 19.

584. Three less than c is at least 360.

585. Nine times n exceeds 42.

586. Negative two times a is no more than eight.

Solve Applications with Linear Inequalities

In the following exercises, solve.

587. Julianne has a weekly food budget of $231 for her family. If she plans to budget the same amount for each of the seven days of the week, what is the maximum amount she can spend on food each day?

588. Rogelio paints watercolors. He got a $100 gift card to the art supply store and wants to use it to buy 12" × 16" canvases. Each canvas costs $10.99. What is the maximum number of canvases he can buy with his gift card?

589. Briana has been offered a sales job in another city. The offer was for $42,500 plus 8% of her total sales. In order to make it worth the move, Briana needs to have an annual salary of at least $66,500. What would her total sales need to be for her to move?

590. Renee's car costs her $195 per month plus $0.09 per mile. How many miles can Renee drive so that her monthly car expenses are no more than $250?

591. Costa is an accountant. During tax season, he charges $125 to do a simple tax return. His expenses for buying software, renting an office, and advertising are $6,000. How many tax returns must he do if he wants to make a profit of at least $8,000?

592. Jenna is planning a five-day resort vacation with three of her friends. It will cost her $279 for airfare, $300 for food and entertainment, and $65 per day for her share of the hotel. She has $550 saved towards her vacation and can earn $25 per hour as an assistant in her uncle's photography studio. How many hours must she work in order to have enough money for her vacation?

2.6 Solve Compound Inequalities

Solve Compound Inequalities with "and"

In each of the following exercises, solve each inequality, graph the solution, and write the solution in interval notation.

593. $x \leq 5$ and $x > -3$

594. $4x - 2 \leq 4$ and $7x - 1 > -8$

595. $5(3x - 2) \leq 5$ and $4(x + 2) < 3$

596. $\frac{3}{4}(x - 8) \leq 3$ and $\frac{1}{5}(x - 5) \leq 3$

597. $\frac{3}{4}x - 5 \geq -2$ and $-3(x + 1) \geq 6$

598. $-5 \leq 4x - 1 < 7$

Solve Compound Inequalities with "or"

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

599. $5 - 2x \leq -1$ or $6 + 3x \leq 4$

600. $3(2x - 3) < -5$ or $4x - 1 > 3$

601. $\frac{3}{4}x - 2 > 4$ or $4(2 - x) > 0$

602. $2(x + 3) \geq 0$ or $3(x + 4) \leq 6$

603. $\frac{1}{2}x - 3 \leq 4$ or $\frac{1}{3}(x - 6) \geq -2$

Solve Applications with Compound Inequalities

In the following exercises, solve.

604. Liam is playing a number game with his sister Audry. Liam is thinking of a number and wants Audry to guess it. Five more than three times her number is between 2 and 32. Write a compound inequality that shows the range of numbers that Liam might be thinking of.

605. Elouise is creating a rectangular garden in her back yard. The length of the garden is 12 feet. The perimeter of the garden must be at least 36 feet and no more than 48 feet. Use a compound inequality to find the range of values for the width of the garden.

2.7 Solve Absolute Value Inequalities

Solve Absolute Value Equations

In the following exercises, solve.

606. $|x| = 8$

607. $|y| = -14$

608. $|z| = 0$

609. $|3x - 4| + 5 = 7$

610. $4|x - 1| + 2 = 10$

611. $-2|x - 3| + 8 = -4$

612. $\left|\frac{1}{2}x+5\right|+4=1$

613. $|6x-5|=|2x+3|$

Solve Absolute Value Inequalities with "less than"

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

614. $|x|\leq 8$

615. $|2x-5|\leq 3$

616. $|6x-5|<7$

617. $|5x+1|\leq -2$

Solve Absolute Value Inequalities with "greater than"

In the following exercises, solve. Graph the solution and write the solution in interval notation.

618. $|x|>6$

619. $|x|\geq 2$

620. $|x-5|>-2$

621. $|x-7|\geq 1$

622. $3|x|+4\geq 1$

Solve Applications with Absolute Value

In the following exercises, solve.

623. A craft beer brewer needs 215,000 bottle per day. But this total can vary by as much as 5,000 bottles. What is the maximum and minimum expected usage at the bottling company?

624. At Fancy Grocery, the ideal weight of a loaf of bread is 16 ounces. By law, the actual weight can vary from the ideal by 1.5 ounces. What range of weight will be acceptable to the inspector without causing the bakery being fined?

PRACTICE TEST

In the following exercises, solve each equation.

625. $-5(2x+1)=45$

626. $\frac{1}{4}(12m+28)=6+2(3m+1)$

627. $8(3a+5)-7(4a-3)=20-3a$

628. $0.1d+0.25(d+8)=4.1$

629. $14n-3(4n+5)=-9+2(n-8)$

630. $3(3u+2)+4[6-8(u-1)]=3(u-2)$

631. $\frac{3}{4}x-\frac{2}{3}=\frac{1}{2}x+\frac{5}{6}$

632. $|3x-4|=8$

633. $|2x-1|=|4x+3|$

634. Solve the formula $x+2y=5$ for y.

In the following exercises, graph the inequality on the number line and write in interval notation.

635. $x\geq -3.5$

636. $x<\frac{11}{4}$

637. $-2\leq x<5$

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

638. $8k\geq 5k-120$

639. $3c-10(c-2)<5c+16$

640. $\frac{3}{4}x-5\geq -2$ and $-3(x+1)\geq 6$

641. $3(2x-3)<-5$ or $4x-1>3$

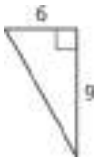
642. $\frac{1}{2}x-3\leq 4$ or $\frac{1}{3}(x-6)\geq -2$

643. $|4x-3|\geq 5$

In the following exercises, translate to an equation or inequality and solve.

644. Four less than twice x is 16.

645. Find the length of the missing side.



646. One number is four more than twice another. Their sum is -47. Find the numbers.

647. The sum of two consecutive odd integers is -112. Find the numbers.

648. Marcus bought a television on sale for \$626.50 The original price of the television was \$895. Find ⓐ the amount of discount and ⓑ the discount rate.

649. Bonita has \$2.95 in dimes and quarters in her pocket. If she has five more dimes than quarters, how many of each coin does she have?

650. Kim is making eight gallons of punch from fruit juice and soda. The fruit juice costs \$6.04 per gallon and the soda costs \$4.28 per gallon. How much fruit juice and how much soda should she use so that the punch costs \$5.71 per gallon?

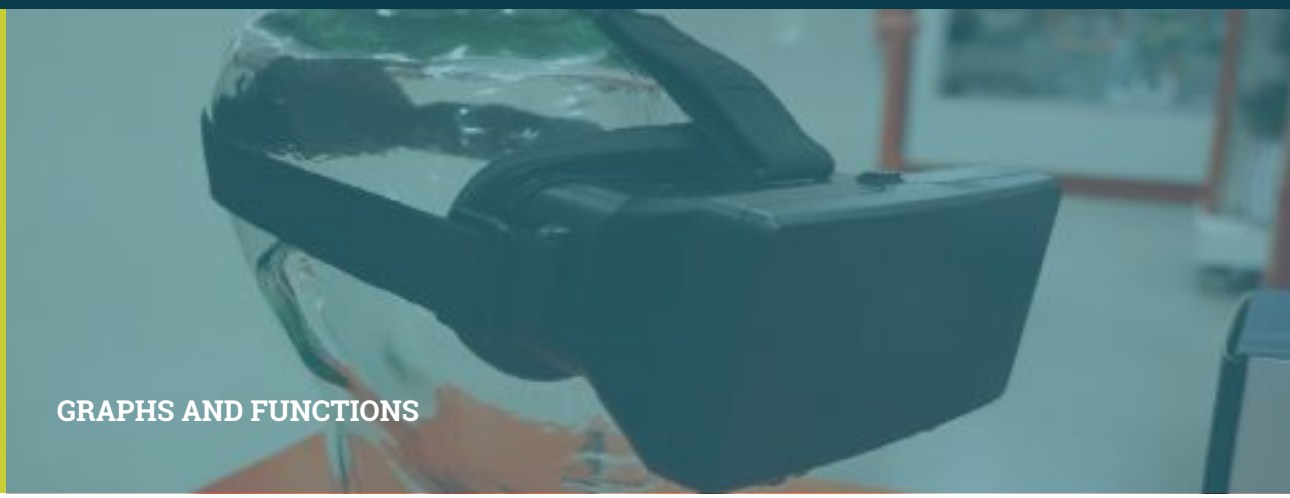
651. The measure of one angle of a triangle is twice the measure of the smallest angle. The measure of the third angle is three times the measure of the smallest angle. Find the measures of all three angles.

652. The length of a rectangle is five feet more than four times the width. The perimeter is 60 feet. Find the dimensions of the rectangle.

653. Two planes leave Dallas at the same time. One heads east at a speed of 428 miles per hour. The other plane heads west at a speed of 382 miles per hour. How many hours will it take them to be 2,025 miles apart?

654. Leon drove from his house in Cincinnati to his sister's house in Cleveland, a distance of 252 miles. It took him $4\frac{1}{2}$ hours. For the first half hour, he had heavy traffic, and the rest of the time his speed was five miles per hour less than twice his speed in heavy traffic. What was his speed in heavy traffic?

655. Sara has a budget of $1,000 for costumes for the 18 members of her musical theater group. What is the maximum she can spend for each costume?



3 GRAPHS AND FUNCTIONS

Figure 3.1 This odd-looking headgear provides the user with a virtual world. (credit: fill/Pixabay)

Chapter Outline

3.1 Graph Linear Equations in Two Variables
3.2 Slope of a Line
3.3 Find the Equation of a Line
3.4 Graph Linear Inequalities in Two Variables
3.5 Relations and Functions
3.6 Graphs of Functions

Introduction

Imagine visiting a faraway city or even outer space from the comfort of your living room. It could be possible using virtual reality. This technology creates realistic images that make you feel as if you are truly immersed in the scene and even enable you to interact with them. It is being developed for fun applications, such as video games, but also for architects to plan buildings, car companies to design prototypes, the military to train, and medical students to learn.

Developing virtual reality devices requires modeling the environment using graphs and mathematical relationships. In this chapter, you will graph different relationships and learn ways to describe and analyze graphs.

3.1 Graph Linear Equations in Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Graph a linear equation by plotting points
- Graph vertical and horizontal lines
- Find the x- and y-intercepts
- Graph a line using the intercepts

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $5x - 4$ when $x = -1$.
 If you missed this problem, review Example 1.6.
2. Evaluate $3x - 2y$ when $x = 4,\ y = -3$.
 If you missed this problem, review Example 1.21.

3. Solve for y: $8 - 3y = 20$.

 If you missed this problem, review **Example 2.2**.

Plot Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular coordinate system. The rectangular coordinate system is also called the *xy*-plane or the "coordinate plane."

The rectangular coordinate system is formed by two intersecting number lines, one horizontal and one vertical. The horizontal number line is called the *x*-axis. The vertical number line is called the *y*-axis. These axes divide a plane into four regions, called quadrants. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See **Figure 3.2**.

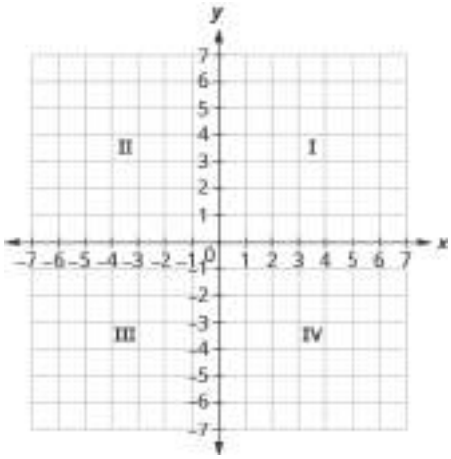
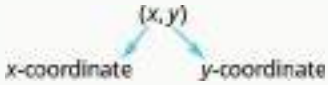


Figure 3.2

In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the *x*-coordinate of the point, and the second number is the *y*-coordinate of the point. The phrase "ordered pair" means that the order is important.

Ordered Pair

An **ordered pair**, (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the *x*-coordinate. The second number is the *y*-coordinate.



What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the **origin**.

The Origin

The point $(0, 0)$ is called the **origin**. It is the point where the *x*-axis and *y*-axis intersect.

We use the coordinates to locate a point on the *xy*-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the *x*-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the *y*-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$. See **Figure 3.3**.

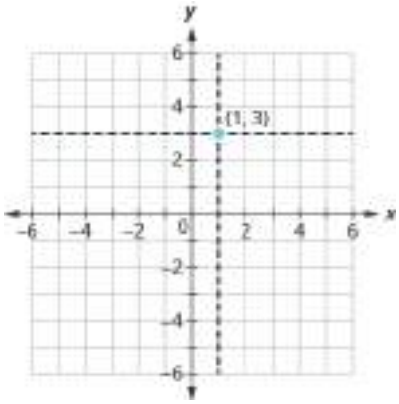


Figure 3.3

Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

When one of the coordinate is zero, the point lies on one of the axes. In Figure 3.4 the point $(0, 4)$ is on the *y*-axis and the point $(-2, 0)$ is on the *x*-axis.

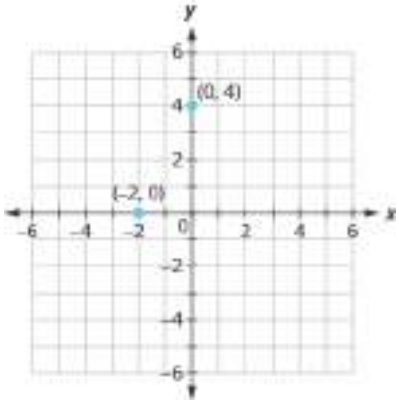


Figure 3.4

Points on the Axes

Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates $(a, 0)$.

Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates $(0, b)$.

EXAMPLE 3.1

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-5, 4)$ ⓑ $(-3, -4)$ ⓒ $(2, -3)$ ⓓ $(0, -1)$ ⓔ $\left(3, \frac{5}{2}\right)$.

Solution

The first number of the coordinate pair is the *x*-coordinate, and the second number is the *y*-coordinate. To plot each point, sketch a vertical line through the *x*-coordinate and a horizontal line through the *y*-coordinate. Their intersection is the point.

ⓐ Since $x = -5$, the point is to the left of the *y*-axis. Also, since $y = 4$, the point is above the *x*-axis. The point $(-5, 4)$ is in Quadrant II.

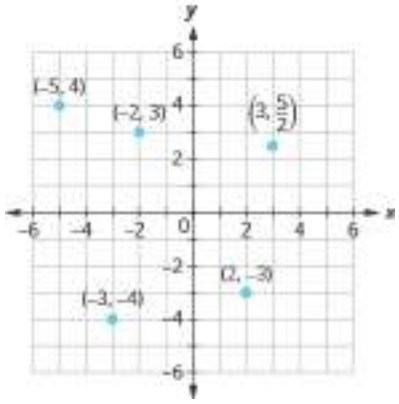
ⓑ Since $x = -3$, the point is to the left of the *y*-axis. Also, since $y = -4$, the point is below the *x*-axis. The point

$(-3, -4)$ is in Quadrant III.

ⓒ Since $x = 2,$ the point is to the right of the *y*-axis. Since $y = -3,$ the point is below the *x*-axis. The point $(2, -3)$ is in Quadrant IV.

ⓓ Since $x = 0,$ the point whose coordinates are $(0, -1)$ is on the *y*-axis.

ⓔ Since $x = 3,$ the point is to the right of the *y*-axis. Since $y = \frac{5}{2},$ the point is above the *x*-axis. (It may be helpful to write $\frac{5}{2}$ as a mixed number or decimal.) The point $\left(3, \frac{5}{2}\right)$ is in Quadrant I.



TRY IT : : 3.1

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-2, 1)$ ⓑ $(-3, -1)$ ⓒ $(4, -4)$ ⓓ $(-4, 4)$ ⓔ $\left(-4, \frac{3}{2}\right)$

TRY IT : : 3.2

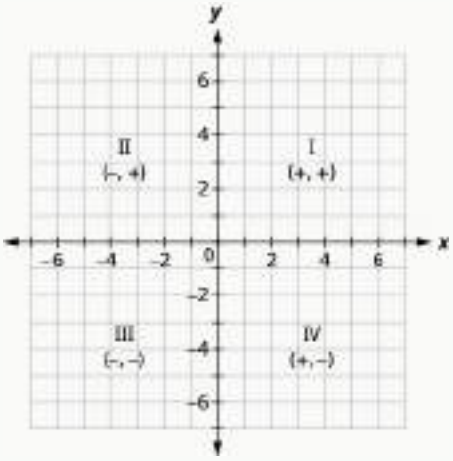
Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-4, 1)$ ⓑ $(-2, 3)$ ⓒ $(2, -5)$ ⓓ $(-2, 5)$ ⓔ $\left(-3, \frac{5}{2}\right)$

The signs of the *x*-coordinate and *y*-coordinate affect the location of the points. You may have noticed some patterns as you graphed the points in the previous example. We can summarize sign patterns of the quadrants in this way:

Quadrants

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$



Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax + By = C$. An equation of this form is called a **linear equation** in two variables.

Linear Equation

An equation of the form $Ax + By = C$, where *A* and *B* are not both zero, is called a **linear equation** in two variables.

Here is an example of a linear equation in two variables, *x* and *y*.

$$Ax + By = C$$
$$x + 4y = 8$$
$$A = 1,\ B = 4,\ C = 8$$

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the Addition Property of Equality and rewrite it in $Ax + By = C$ form.

	$y = -3x + 5$
Add to both sides.	$y + 3x = 3x + 5 + 3x$
Simplify.	$y + 3x = 5$
Use the Commutative Property to put it in $Ax + By = C$ form.	$3x + y = 5$

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax + By = C$. When an equation is in the form $Ax + By = C$, we say it is in **standard form of a linear equation**.

Standard Form of Linear Equation

A linear equation is in **standard form** when it is written $Ax + By = C$.

Most people prefer to have *A*, *B*, and *C* be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for *x* there is a corresponding *y* value. This pair of values is a **solution** to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of *x* and *y* into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Solution of a Linear Equation in Two Variables

An ordered pair (x, y) is a **solution** of the linear equation $Ax + By = C$, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

Linear equations have infinitely many solutions. We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of the equation. It is an example of the saying, "A picture is worth a thousand words." The line shows you *all* the solutions to that equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the graph of the equation. Points *not* on the line are not solutions!

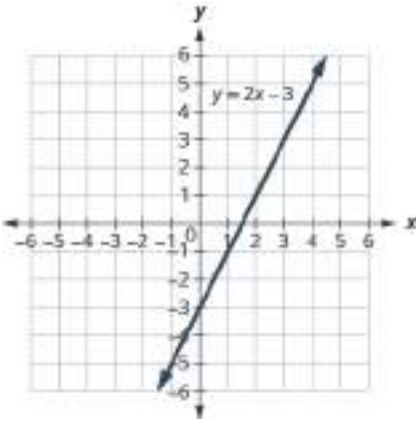
Graph of a Linear Equation

The graph of a linear equation $Ax + By = C$ is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

EXAMPLE 3.2

The graph of $y = 2x - 3$ is shown.



For each ordered pair, decide:

ⓐ Is the ordered pair a solution to the equation?

ⓑ Is the point on the line?

A: $(0, -3)$ B: $(3, 3)$ C: $(2, -3)$ D: $(-1, -5)$

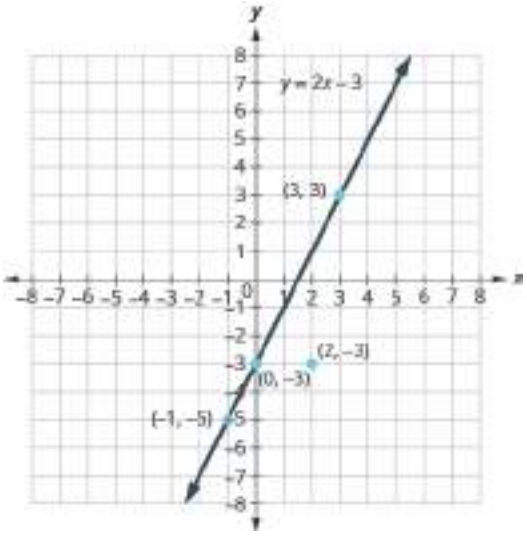
Solution

Substitute the *x*- and *y*-values into the equation to check if the ordered pair is a solution to the equation.

ⓐ

A: (0, –3)	B: (3, 3)	C: (2, –3)	D: (–1, –5)
$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
$-3 \stackrel{?}{=} 2(0) - 3$	$3 \stackrel{?}{=} 2(3) - 3$	$-3 \stackrel{?}{=} 2(2) - 3$	$-5 \stackrel{?}{=} 2(-1) - 3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
(0, –3) is a solution.	(3, 3) is a solution.	(2, –3) is not a solution.	(–1, –5) is a solution.

ⓑ Plot the points $(0, -3)$, $(3, 3)$, $(2, -3)$, and $(-1, -5)$.



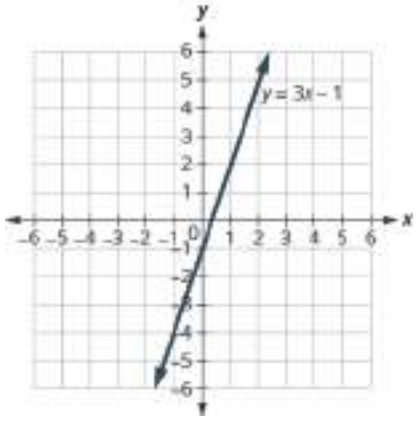
The points $(0, 3)$, $(3, -3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line. The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

TRY IT : : 3.3

Use graph of $y = 3x - 1$. For each ordered pair, decide:

ⓐ Is the ordered pair a solution to the equation?
ⓑ Is the point on the line?

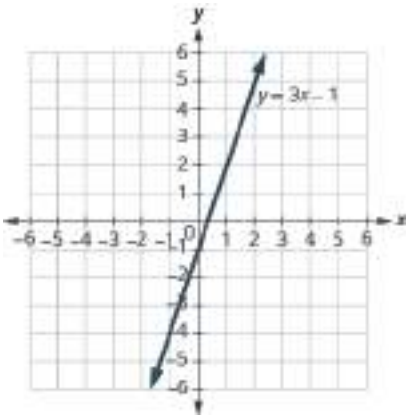
A $(0, -1)$ B $(2, 5)$



TRY IT : : 3.4 Use graph of $y = 3x - 1$. For each ordered pair, decide:

ⓐ Is the ordered pair a solution to the equation?
ⓑ Is the point on the line?

A $(3, -1)$ B $(-1, -4)$



Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The first method we will use is called plotting points, or the Point-Plotting Method. We find three points whose coordinates are solutions to the equation and then plot them in a rectangular coordinate system. By connecting these points in a line, we have the graph of the linear equation.

EXAMPLE 3.3 HOW TO GRAPH A LINEAR EQUATION BY PLOTTING POINTS

Graph the equation $y = 2x + 1$ by plotting points.

Solution

Step 1. Find three points whose coordinates are solutions to the equation.

You can choose any values for x or y.

In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.

$y = 2x + 1$

$x = 0$
$y = 2x + 1$
$y = 2 \cdot 0 + 1$
$y = 0 + 1$
$y = 1$

$x = 1$
$y = 2x + 1$
$y = 2 \cdot 1 + 1$
$y = 2 + 1$
$y = 3$

$x = -2$
$y = 2x + 1$
$y = 2(-2) + 1$
$y = -4 + 1$
$y = -3$

Organize the solutions in a table.

Put the three solutions in a table.

$y = 2x + 1$

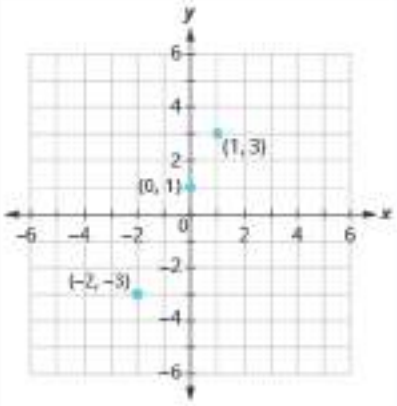
x	y	(x, y)
0	1	(0, 1)
1	3	(1, 3)
−2	−3	(−2, −3)

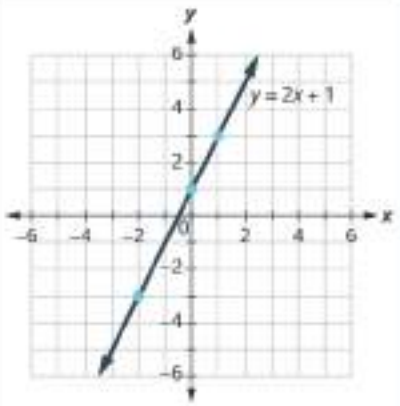
Step 2. Plot the points in a rectangular coordinate system.

Check that the points line up. If they do not, carefully check your work!

Plot:
(0, 1), (1, 3), (−2, −3).

Do the points line up?
Yes, the points line up.



Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.	This line is the graph of $y = 2x + 1$.	

TRY IT : : 3.5 Graph the equation by plotting points: $y = 2x - 3$.

TRY IT : : 3.6 Graph the equation by plotting points: $y = -2x + 4$.

The steps to take when graphing a linear equation by plotting points are summarized here.

HOW TO : : GRAPH A LINEAR EQUATION BY PLOTTING POINTS.

Step 1. Find three points whose coordinates are solutions to the equation. Organize them in a table.

Step 2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.

Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between these illustrations.



(a) (b)

When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the arithmetic is easier if we make "good" choices for the values of x. This way we will avoid fractional answers, which are hard to graph precisely.

EXAMPLE 3.4

Graph the equation: $y = \frac{1}{2}x + 3$.

Solution

Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will

choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of x? By choosing multiples of 2 the multiplication by $\frac{1}{2}$ simplifies to a whole number

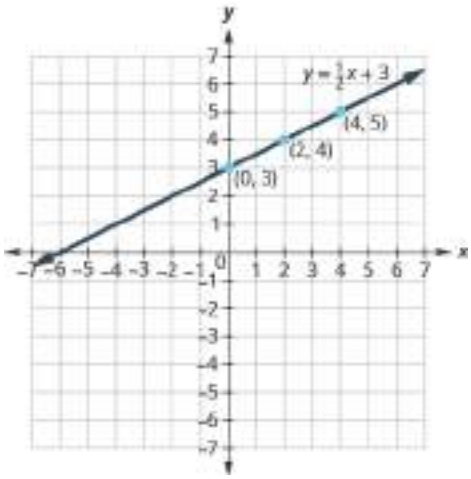
$x=0$	$x=2$	$x=4$
$y=\frac{1}{2}x+3$	$y=\frac{1}{2}x+3$	$y=\frac{1}{2}x+3$
$y=\frac{1}{2}(0)+3$	$y=\frac{1}{2}(2)+3$	$y=\frac{1}{2}(4)+3$
$y=0+3$	$y=1+3$	$y=2+3$
$y=3$	$y=4$	$y=5$

The points are shown in Table 3.1.

$y=\frac{1}{2}x+3$		
x	y	(x, y)
0	3	(0, 3)
2	4	(2, 4)
4	5	(4, 5)

Table 3.1

Plot the points, check that they line up, and draw the line.



TRY IT : : 3.7 Graph the equation: $y=\frac{1}{3}x-1$.

TRY IT : : 3.8 Graph the equation: $y=\frac{1}{4}x+2$.

Graph Vertical and Horizontal Lines

Some linear equations have only one variable. They may have just x and no y, or just y without an x. This changes how we make a table of values to get the points to plot.

Let's consider the equation $x=-3$. This equation has only one variable, x. The equation says that x is *always* equal to

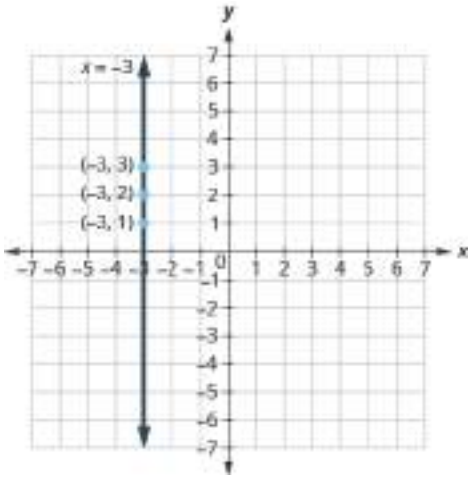
$-3,$ so its value does not depend on *y*. No matter what is the value of *y*, the value of *x* is always -3.

So to make a table of values, write -3 in for all the *x*-values. Then choose any values for *y*. Since *x* does not depend on *y*, you can choose any numbers you like. But to fit the points on our coordinate graph, we'll use 1, 2, and 3 for the *y*-coordinates. See Table 3.2.

$x = -3$		
x	**y**	**(x, y)**
-3	1	$(-3, 1)$
-3	2	$(-3, 2)$
-3	3	$(-3, 3)$

Table 3.2

Plot the points from the table and connect them with a straight line. Notice that we have graphed a **vertical line**.

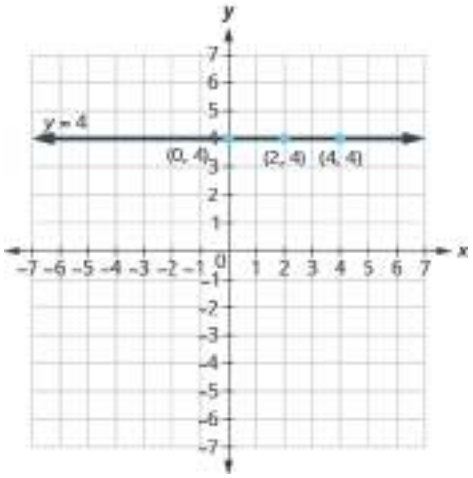


What if the equation has *y* but no *x*? Let's graph the equation $y = 4.$ This time the *y*-value is a constant, so in this equation, *y* does not depend on *x*. Fill in 4 for all the *y*'s in Table 3.3 and then choose any values for *x*. We'll use 0, 2, and 4 for the *x*-coordinates.

$y = 4$		
x	**y**	**(x, y)**
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

Table 3.3

In this figure, we have graphed a **horizontal line** passing through the *y*-axis at 4.



Vertical and Horizontal Lines

A **vertical line** is the graph of an equation of the form $x = a$.

The line passes through the *x*-axis at $(a, 0)$.

A **horizontal line** is the graph of an equation of the form $y = b$.

The line passes through the *y*-axis at $(0, b)$.

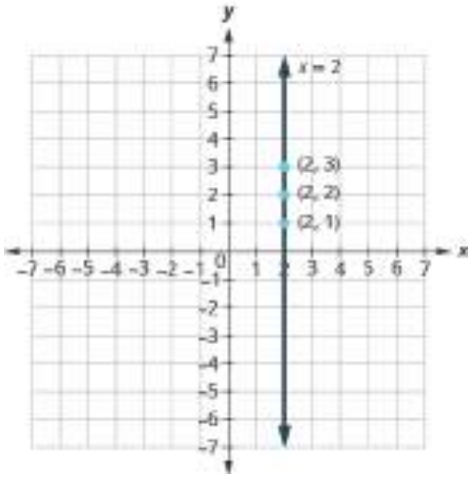
EXAMPLE 3.5

Graph: ⓐ $x = 2$ ⓑ $y = -1$.

Solution

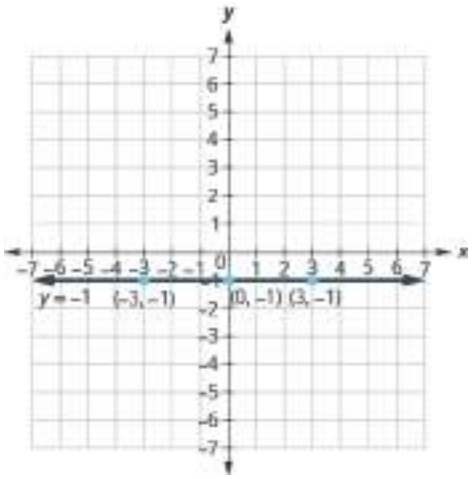
ⓐ The equation has only one variable, *x*, and *x* is always equal to 2. We create a table where *x* is always 2 and then put in any values for *y*. The graph is a vertical line passing through the *x*-axis at 2.

$x = 2$		
x	**y**	**(x, y)**
2	1	(2, 1)
2	2	(2, 2)
2	3	(2, 3)



ⓑ Similarly, the equation $y = -1$ has only one variable, *y*. The value of *y* is constant. All the ordered pairs in the next table have the same *y*-coordinate. The graph is a horizontal line passing through the *y*-axis at -1.

$y = -1$		
x	***y***	**(*x*, *y*)**
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$



TRY IT : : 3.9 Graph the equations: ⓐ $x = 5$ ⓑ $y = -4$.

TRY IT : : 3.10 Graph the equations: ⓐ $x = -2$ ⓑ $y = 3$.

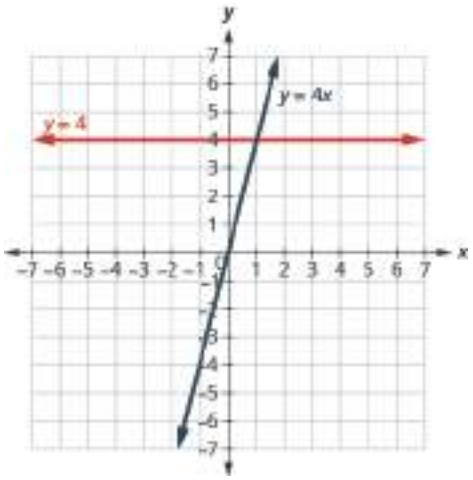
What is the difference between the equations $y = 4x$ and $y = 4$?

The equation $y = 4x$ has both *x* and *y*. The value of *y* depends on the value of *x*, so the *y* -coordinate changes according

to the value of *x*. The equation $y = 4$ has only one variable. The value of *y* is constant, it does not depend on the value of *x*, so the *y*-coordinate is always 4.

$y = 4x$		
x	y	(x, y)
0	0	(0, 0)
1	4	(1, 4)
2	8	(2, 8)

$y = 4$		
x	y	(x, y)
0	4	(0, 4)
1	4	(1, 4)
2	4	(2, 4)



Notice, in the graph, the equation $y = 4x$ gives a slanted line, while $y = 4$ gives a horizontal line.

EXAMPLE 3.6

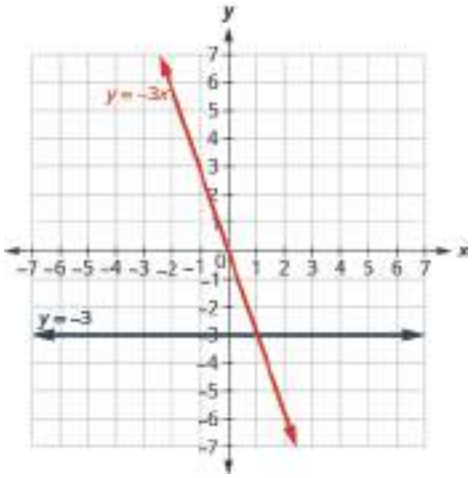
Graph $y = -3x$ and $y = -3$ in the same rectangular coordinate system.

Solution

We notice that the first equation has the variable *x*, while the second does not. We make a table of points for each equation and then graph the lines. The two graphs are shown.

$y = -3x$		
x	y	(x, y)
0	0	(0, 0)
1	−3	(1, −3)
2	−6	(2, −6)

$y = -3$		
x	y	(x, y)
0	−3	(0, −3)
1	−3	(1, −3)
2	−3	(2, −3)



TRY IT : : 3.11 Graph the equations in the same rectangular coordinate system: $y = -4x$ and $y = -4$.

TRY IT : : 3.12 Graph the equations in the same rectangular coordinate system: $y = 3$ and $y = 3x$.

Find *x*- and *y*-intercepts

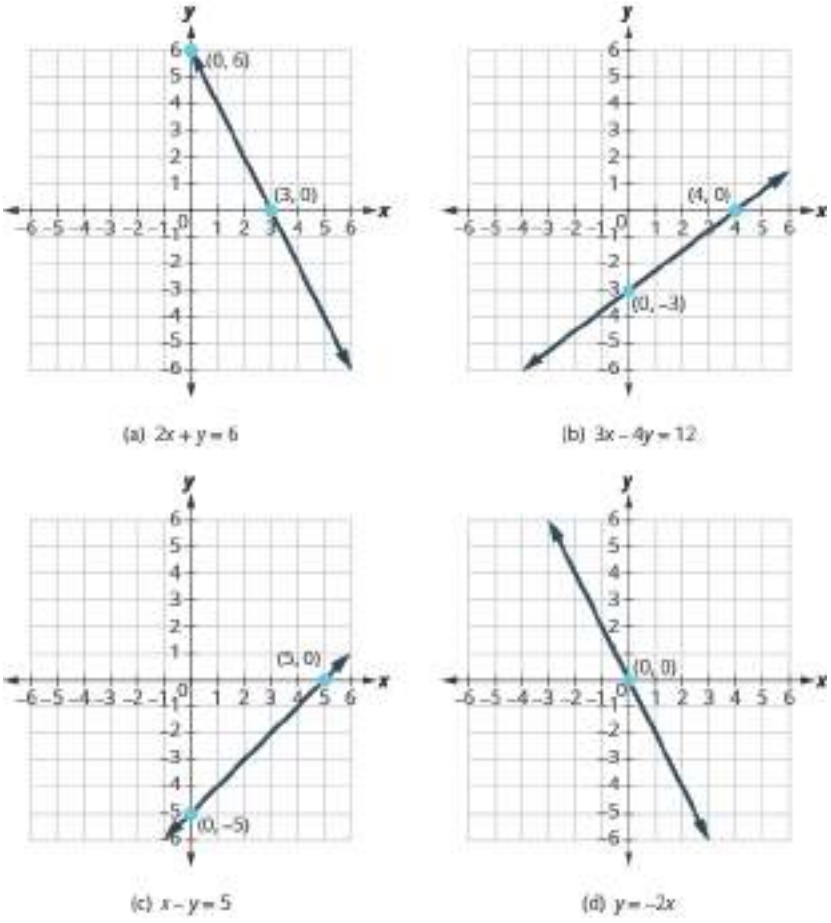
Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the *x*-axis and the *y*-axis. These points are called the **intercepts of a line**.

Intercepts of a Line

The points where a line crosses the *x*-axis and the *y*-axis are called the **intercepts of the line**.

Let's look at the graphs of the lines.



(a) $2x + y = 6$ (b) $3x - 4y = 12$

(c) $x - y = 5$ (d) $y = -2x$

First, notice where each of these lines crosses the *x*-axis. See Table 3.6.

Now, let's look at the points where these lines cross the *y*-axis.

Figure	The line crosses the *x*-axis at:	Ordered pair for this point	The line crosses the *y*-axis at:	Ordered pair for this point
Figure (a)	3	$(3, 0)$	6	$(0, 6)$
Figure (b)	4	$(4, 0)$	-3	$(0, -3)$
Figure (c)	5	$(5, 0)$	-5	$(0, 5)$
Figure (d)	0	$(0, 0)$	0	$(0, 0)$
General Figure	a	$(a, 0)$	b	$(0, b)$

Table 3.6

Do you see a pattern?

For each line, the *y*-coordinate of the point where the line crosses the *x*-axis is zero. The point where the line crosses the *x*-axis has the form $(a, 0)$ and is called the *x-intercept* of the line. The *x*-intercept occurs when *y* is zero.

In each line, the *x*-coordinate of the point where the line crosses the *y*-axis is zero. The point where the line crosses the *y*-axis has the form $(0, b)$ and is called the *y-intercept* of the line. The *y*-intercept occurs when *x* is zero.

x-intercept and *y*-intercept of a Line

The *x*-intercept is the point $(a, 0)$ where the line crosses the *x*-axis.

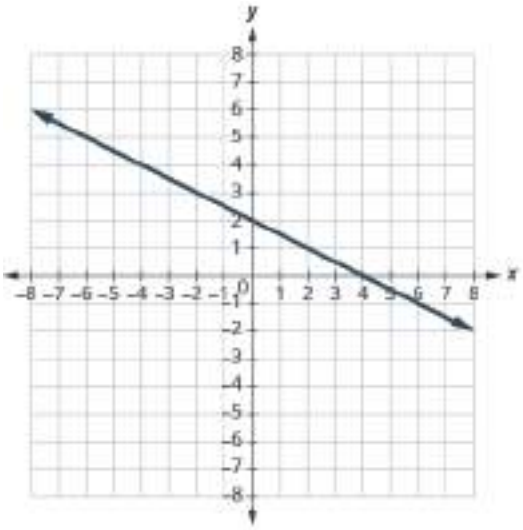
The *y*-intercept is the point $(0, b)$ where the line crosses the *y*-axis.

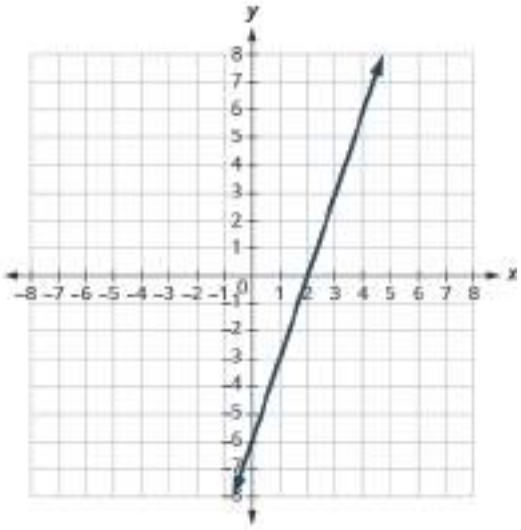
- The *x*-intercept occurs when *y* is zero.
- The *y*-intercept occurs when *x* is zero.

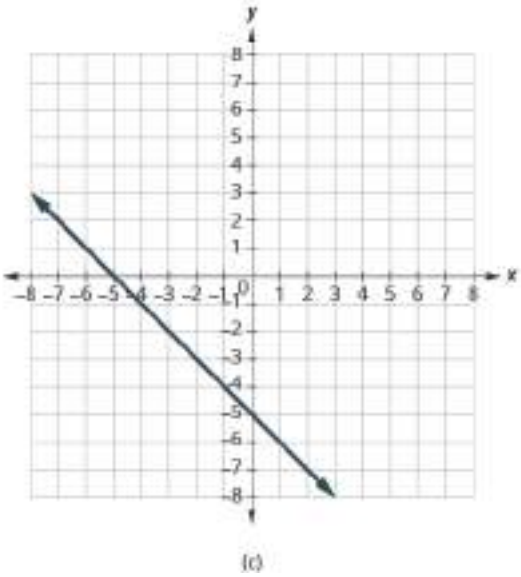
x	y
a	0
0	b

EXAMPLE 3.7

Find the *x*- and *y*-intercepts on each graph shown.







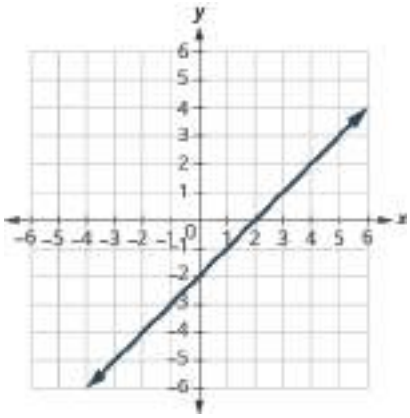
Solution

ⓐ The graph crosses the x-axis at the point $(4, 0)$. The x-intercept is $(4, 0)$.
The graph crosses the y-axis at the point $(0, 2)$. The y-intercept is $(0, 2)$.

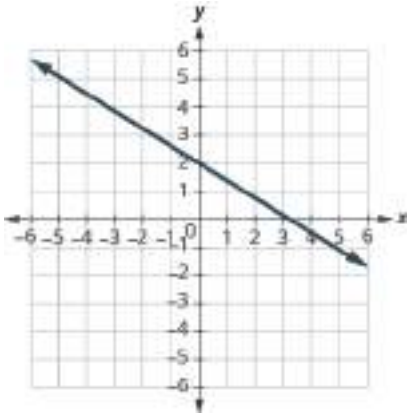
ⓑ The graph crosses the x-axis at the point $(2, 0)$. The x-intercept is $(2, 0)$.
The graph crosses the y-axis at the point $(0, -6)$. The y-intercept is $(0, -6)$.

ⓒ The graph crosses the x-axis at the point $(-5, 0)$. The x-intercept is $(-5, 0)$.
The graph crosses the y-axis at the point $(0, -5)$. The y-intercept is $(0, -5)$.

TRY IT : : 3.13 Find the *x*- and *y*-intercepts on the graph.



TRY IT : : 3.14 Find the *x*- and *y*-intercepts on the graph.



Recognizing that the *x*-intercept occurs when *y* is zero and that the *y*-intercept occurs when *x* is zero, gives us a method to find the intercepts of a line from its equation. To find the *x*-intercept, let $y = 0$ and solve for *x*. To find the *y*-intercept, let $x = 0$ and solve for *y*.

Find the *x*- and *y*-intercepts from the Equation of a Line

Use the equation of the line. To find:

- the *x*-intercept of the line, let $y = 0$ and solve for *x*.
- the *y*-intercept of the line, let $x = 0$ and solve for *y*.

EXAMPLE 3.8

Find the intercepts of $2x + y = 8$.

Solution

We will let $y = 0$ to find the *x*-intercept, and let $x = 0$ to find the *y*-intercept. We will fill in a table, which reminds us of what we need to find.

$2x+y=8$	
x	y
	0
0	

x-intercept

y-intercept

To find the *x*-intercept, let $y = 0$.	
	$2x + y = 8$
Let $y = 0$.	$2x + 0 = 8$
Simplify.	$2x = 8$
	$x = 4$
The *x*-intercept is:	$(4, 0)$
To find the *y*-intercept, let $x = 0$.	
	$2x + y = 8$
Let $x = 0$.	$2 \cdot 0 + y = 8$
Simplify.	$0 + y = 8$
	$y = 8$
The *y*-intercept is:	$(0, 8)$

The intercepts are the points $(4, 0)$ and $(0, 8)$ as shown in the table.

$2x + y = 8$	
x	***y***
4	0
0	8

TRY IT : : 3.15 Find the intercepts: $3x + y = 12$.

TRY IT : : 3.16 Find the intercepts: $x + 4y = 8$.

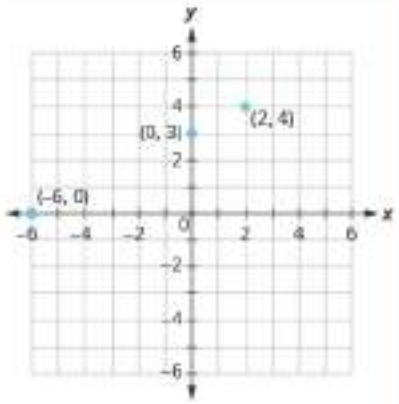
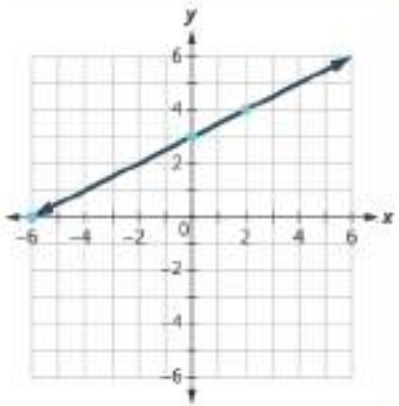
Graph a Line Using the Intercepts

To graph a linear equation by plotting points, you need to find three points whose coordinates are solutions to the equation. You can use the *x*- and *y*- intercepts as two of your three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

EXAMPLE 3.9 HOW TO GRAPH A LINE USING THE INTERCEPTS

Graph $-x + 2y = 6$ using the intercepts.

Solution

Step 1. Find the x- and y-intercepts of the line. Let $y = 0$ and solve for x. Let $x = 0$ and solve for y.	Find the x-intercept. Find the y-intercept.	Let $y = 0$ $-x + 2y = 6$ $-x + 2(0) = 6$ $-x = 6$ $x = -6$ The x-intercept is $(-6, 0)$. Let $x = 0$. $-x + 2y = 6$ $-0 + 2y = 6$ $2y = 6$ $y = 3$ The y-intercept is $(0, 3)$.
Step 2. Find another solution to the equation.	We'll use $x = 2$.	Let $x = 2$. $-x + 2y = 6$ $-2 + 2y = 6$ $2y = 8$ $y = 4$ A third point is $(2, 4)$.
Step 3. Plot the three points. Check that the points line up.	x \| y \| (x, y) -6 \| 0 \| $(-6, 0)$ 0 \| 3 \| $(0, 3)$ 2 \| 4 \| $(2, 4)$	
Step 4. Draw the line.	See the graph.	

TRY IT : : 3.17 Graph using the intercepts: $x - 2y = 4$.

TRY IT : : 3.18 Graph using the intercepts: $-x+3y=6$.

The steps to graph a linear equation using the intercepts are summarized here.



HOW TO : : GRAPH A LINEAR EQUATION USING THE INTERCEPTS.

Step 1. Find the *x*- and *y*-intercepts of the line.
- Let $y=0$ and solve for *x*.
- Let $x=0$ and solve for *y*.

Step 2. Find a third solution to the equation.

Step 3. Plot the three points and check that they line up.

Step 4. Draw the line.

EXAMPLE 3.10

Graph $4x-3y=12$ using the intercepts.

Solution

Find the intercepts and a third point.

x-intercept, let $y=0$

$$4x-3y=12$$
$$4x-3(0)=12$$
$$4x=12$$
$$x=3$$

y-intercept, let $x=0$

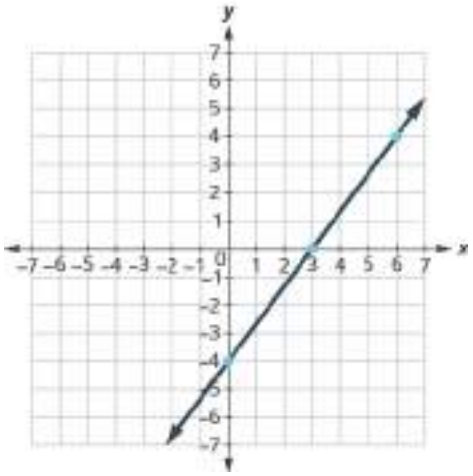
$$4x-3y=12$$
$$4(0)-3y=12$$
$$-3y=12$$
$$y=-4$$

third point, let $y=4$

$$4x-3y=12$$
$$4x-3(4)=12$$
$$4x-12=12$$
$$4x=24$$
$$x=6$$

We list the points in the table and show the graph.

$4x-3y=12$		
x	y	(x, y)
3	0	$(3, 0)$
0	-4	$(0, -4)$
6	4	$(6, 4)$



TRY IT : : 3.19 Graph using the intercepts: $5x - 2y = 10$.

TRY IT : : 3.20 Graph using the intercepts: $3x - 4y = 12$.

When the line passes through the origin, the *x*-intercept and the *y*-intercept are the same point.

EXAMPLE 3.11

Graph $y = 5x$ using the intercepts.

Solution

x-intercept	y-intercept
Let $y = 0$.	Let $x = 0$.
$y = 5x$	$y = 5x$
$0 = 5x$	$y = 5 \cdot 0$
$0 = x$	$y = 0$
$(0, 0)$	$(0, 0)$

This line has only one intercept. It is the point $(0, 0)$.

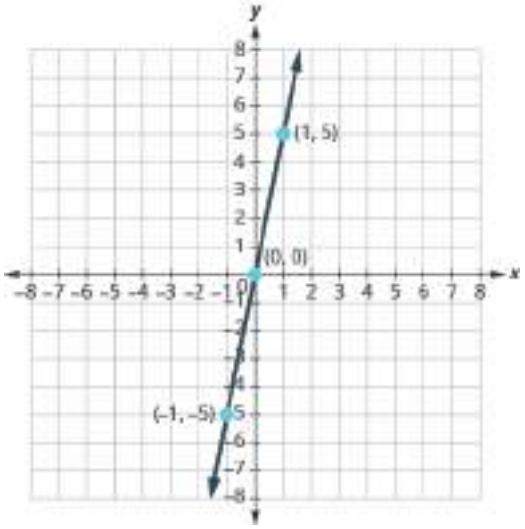
To ensure accuracy, we need to plot three points. Since the *x*- and *y*-intercepts are the same point, we need *two* more points to graph the line.

Let $x = 1$.	Let $x = -1$.
$y = 5x$	$y = 5x$
$y = 5 \cdot 1$	$y = 5(-1)$
$y = 5$	$y = -5$

The resulting three points are summarized in the table.

$y = 5x$		
x	y	(x, y)
0	0	$(0, 0)$
1	5	$(1, 5)$
-1	-5	$(-1, -5)$

Plot the three points, check that they line up, and draw the line.



TRY IT : : 3.21 Graph using the intercepts: $y = 4x$.

TRY IT : : 3.22 Graph the intercepts: $y = -x$.

3.1 EXERCISES

Practice Makes Perfect

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

1. ⓐ $(-4, 2)$ ⓑ $(-1, -2)$ ⓒ $(3, -5)$ ⓓ $(-3, 0)$ ⓔ $\left(\frac{5}{3}, 2\right)$

2. ⓐ $(-2, -3)$ ⓑ $(3, -3)$ ⓒ $(-4, 1)$ ⓓ $(4, -1)$ ⓔ $\left(\frac{3}{2}, 1\right)$

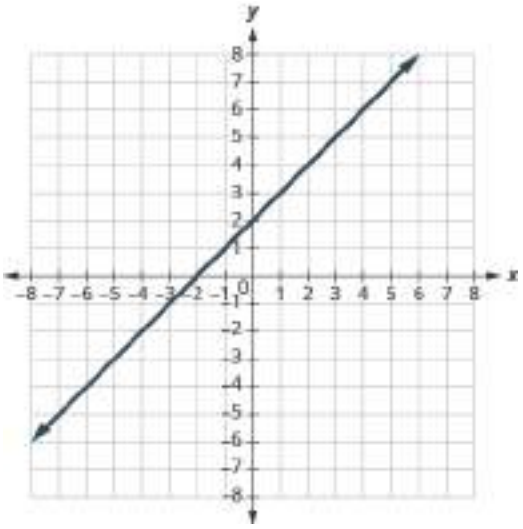
3. ⓐ $(3, -1)$ ⓑ $(-3, 1)$ ⓒ $(-2, 0)$ ⓓ $(-4, -3)$ ⓔ $\left(1, \frac{14}{5}\right)$

4. ⓐ $(-1, 1)$ ⓑ $(-2, -1)$ ⓒ $(2, 0)$ ⓓ $(1, -4)$ ⓔ $\left(3, \frac{7}{2}\right)$

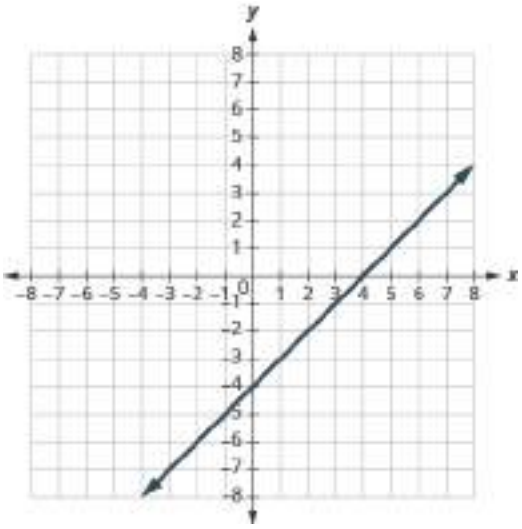
In the following exercises, for each ordered pair, decide

ⓐ is the ordered pair a solution to the equation? ⓑ is the point on the line?

5. $y = x + 2$;
A: $(0, 2)$; B: $(1, 2)$; C: $(-1, 1)$; D: $(-3, -1)$.

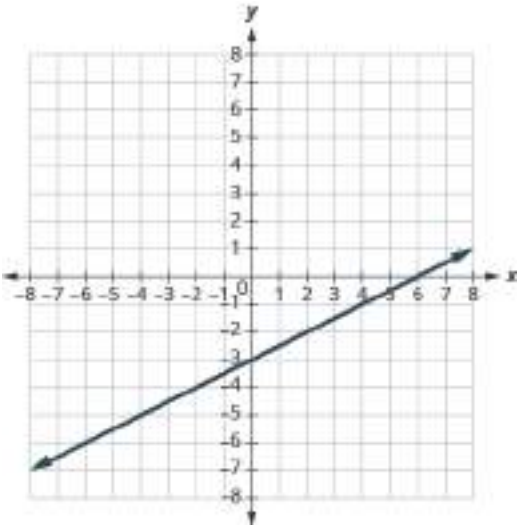


6. $y = x - 4$;
A: $(0, -4)$; B: $(3, -1)$; C: $(2, 2)$; D: $(1, -5)$.



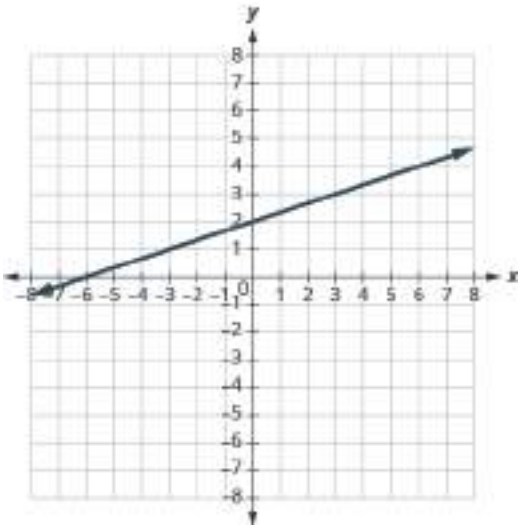
7. $y = \frac{1}{2}x - 3$;

A: $(0, -3)$; B: $(2, -2)$; C: $(-2, -4)$; D: $(4, 1)$



8. $y = \frac{1}{3}x + 2$;

A: $(0, 2)$; B: $(3, 3)$; C: $(-3, 2)$; D: $(-6, 0)$.



Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

9. $y = x + 2$

10. $y = x - 3$

11. $y = 3x - 1$

12. $y = -2x + 2$

13. $y = -x - 3$

14. $y = -x - 2$

15. $y = 2x$

16. $y = -2x$

17. $y = \frac{1}{2}x + 2$

18. $y = \frac{1}{3}x - 1$

19. $y = \frac{4}{3}x - 5$

20. $y = \frac{3}{2}x - 3$

21. $y = -\frac{2}{5}x + 1$

22. $y = -\frac{4}{5}x - 1$

23. $y = -\frac{3}{2}x + 2$

24. $y = -\frac{5}{3}x + 4$

Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

25. ⓐ $x = 4$ ⓑ $y = 3$

26. ⓐ $x = 3$ ⓑ $y = 1$

27. ⓐ $x = -2$ ⓑ $y = -5$

28. ⓐ $x = -5$ ⓑ $y = -2$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

29. $y = 2x$ and $y = 2$

30. $y = 5x$ and $y = 5$

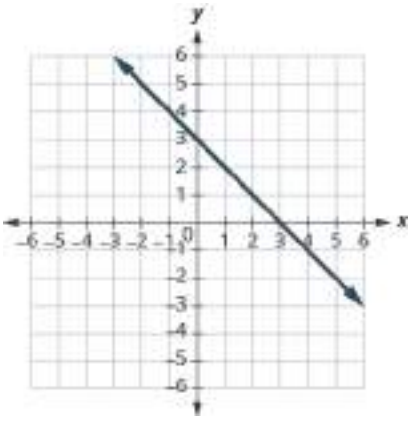
31. $y = -\frac{1}{2}x$ and $y = -\frac{1}{2}$

32. $y = -\frac{1}{3}x$ and $y = -\frac{1}{3}$

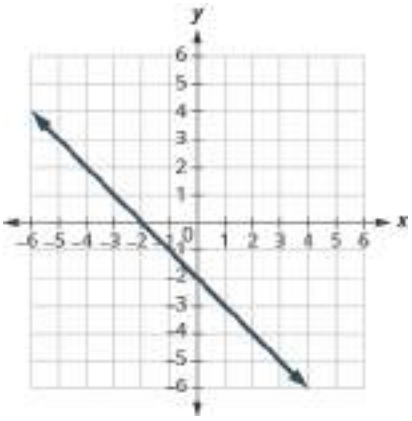
Find *x*- and *y*-Intercepts

In the following exercises, find the x- and y-intercepts on each graph.

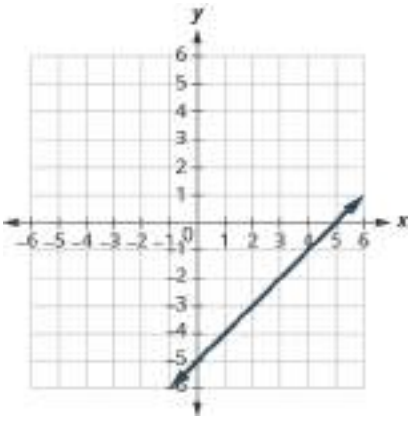
33.



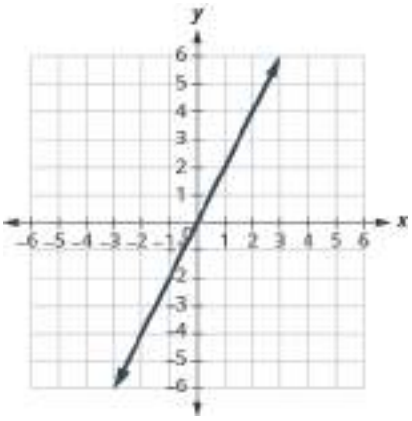
34.



35.



36.



In the following exercises, find the intercepts for each equation.

37. $x - y = 5$

38. $x - y = -4$

39. $3x + y = 6$

40. $x - 2y = 8$

41. $4x - y = 8$

42. $5x - y = 5$

43. $2x + 5y = 10$

44. $3x - 2y = 12$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

45. $-x + 4y = 8$

46. $x + 2y = 4$

47. $x + y = -3$

48. $x - y = -4$

49. $4x + y = 4$

50. $3x + y = 3$

51. $3x - y = -6$

52. $2x - y = -8$

53. $2x + 4y = 12$

54. $3x - 2y = 6$

55. $2x - 5y = -20$

56. $3x - 4y = -12$

57. $y = -2x$

58. $y = 5x$

59. $y = x$

60. $y = -x$

Mixed Practice

In the following exercises, graph each equation.

61. $y = \frac{3}{2}x$

62. $y = -\frac{2}{3}x$

63. $y = -\frac{1}{2}x + 3$

64. $y = \frac{1}{4}x - 2$

65. $4x + y = 2$

66. $5x + 2y = 10$

67. $y = -1$

68. $x = 3$

Writing Exercises

69. Explain how you would choose three x-values to make a table to graph the line $y = \frac{1}{5}x - 2$.

70. What is the difference between the equations of a vertical and a horizontal line?

71. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $4x + y = -4$? Why?

72. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y = \frac{2}{3}x - 2$? Why?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
plot points on a rectangular coordinate system.			
graph a linear equation by plotting points.			
graph vertical and horizontal lines.			
find x- and y-intercepts.			
graph a line using the intercepts.			

ⓑ *If most of your checks were:*

Confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

With some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

No, I don't get it. This is a warning sign and you must address it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

3.2 Slope of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find the slope of a line
- Graph a line given a point and the slope
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept
- Use slopes to identify parallel and perpendicular lines

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{(1-4)}{(8-2)}$.
 If you missed this problem, review Example 1.30.
2. Divide: $\frac{0}{4}, \frac{4}{0}$.
 If you missed this problem, review Example 1.49.
3. Simplify: $\frac{15}{-3}, \frac{-15}{3}, \frac{-15}{-3}$.
 If you missed this problem, review Example 1.30.

Find the Slope of a Line

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter.

In mathematics, the measure of the steepness of a line is called the *slope* of the line.

The concept of slope has many applications in the real world. In construction the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope. and as you ski or jog down a hill, you definitely experience slope.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The *rise* is the amount the vertical distance changes while the *run* measures the horizontal change, as shown in this illustration. Slope is a rate of change. See Figure 3.5.

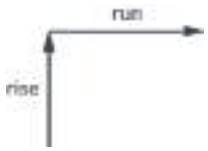


Figure 3.5

Slope of a Line

The slope of a line is $m = \frac{\text{rise}}{\text{run}}$.

The rise measures the vertical change and the run measures the horizontal change.

To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices and one side is horizontal and one side is vertical.

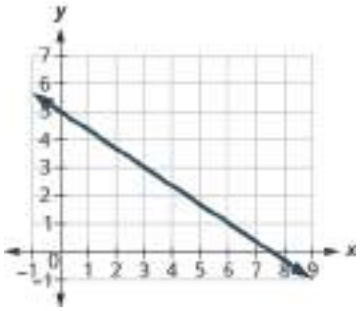
To find the slope of the line, we measure the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the *rise* and the horizontal distance is called the *run*,

HOW TO : : FIND THE SLOPE OF A LINE FROM ITS GRAPH USING $m = \frac{\text{rise}}{\text{run}}$.

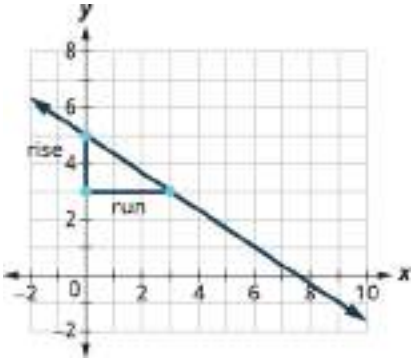
Step 1. Locate two points on the line whose coordinates are integers.
Step 2. Starting with one point, sketch a right triangle, going from the first point to the second point.
Step 3. Count the rise and the run on the legs of the triangle.
Step 4. Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

EXAMPLE 3.12

Find the slope of the line shown.

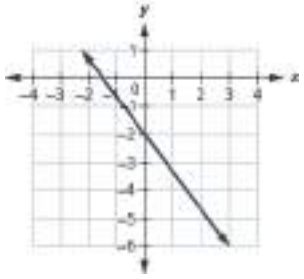


Solution

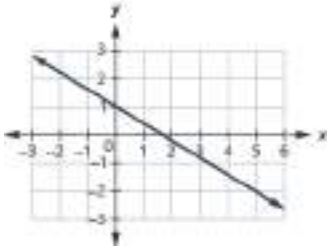
Locate two points on the graph whose coordinates are integers.	$(0, 5)$ and $(3, 3)$
Starting at $(0, 5)$, sketch a right triangle to $(3, 3)$ as shown in this graph.	
Count the rise— since it goes down, it is negative.	The rise is -2.
Count the run.	The run is 3.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-2}{3}$
Simplify.	$m = -\frac{2}{3}$
	The slope of the line is $-\frac{2}{3}$.

So y decreases by 2 units as x increases by 3 units.

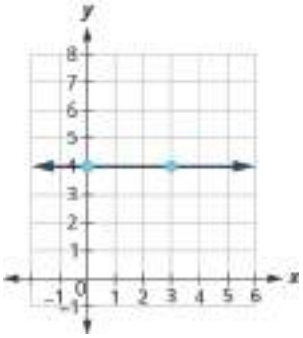
TRY IT : : 3.23 Find the slope of the line shown.



TRY IT : : 3.24 Find the slope of the line shown.



How do we find the slope of horizontal and vertical lines? To find the slope of the horizontal line, $y = 4,$ we could graph the line, find two points on it, and count the rise and the run. Let's see what happens when we do this, as shown in the graph below.



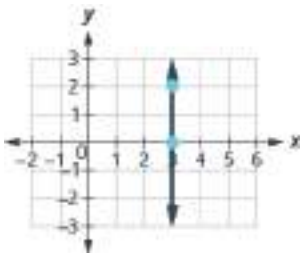
What is the rise? The rise is 0.
What is the run? The run is 3.
What is the slope? $m = \frac{\text{rise}}{\text{run}}$

$$m = \frac{0}{3}$$

$$m = 0$$

The slope of the horizontal line $y = 4$ is 0.

Let's also consider a vertical line, the line $x = 3,$ as shown in the graph.



What is the rise? The rise is 2.

What is the run? The run is 0.

What is the slope? $m = \frac{\text{rise}}{\text{run}}$

$m = \frac{2}{0}$

The slope is undefined since division by zero is undefined. So we say that the slope of the vertical line $x = 3$ is undefined.

All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

Slope of a Horizontal and Vertical Line

The slope of a horizontal line, $y = b$, is 0.

The slope of a vertical line, $x = a$, is undefined.

EXAMPLE 3.13

Find the slope of each line: ⓐ $x = 8$ ⓑ $y = -5$.

Solution

ⓐ $x = 8$

This is a vertical line. Its slope is undefined.

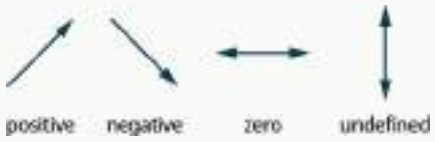
ⓑ $y = -5$

This is a horizontal line. It has slope 0.

TRY IT : : 3.25 Find the slope of the line: $x = -4$.

TRY IT : : 3.26 Find the slope of the line: $y = 7$.

Quick Guide to the Slopes of Lines



Sometimes we'll need to find the slope of a line between two points when we don't have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

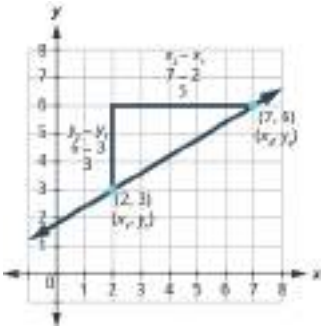
We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

(x_1, y_1) read "x sub 1, y sub 1"
(x_2, y_2) read "x sub 2, y sub 2"

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$, as shown in this graph.



Since we have two points, we will use subscript notation. $\begin{matrix}(x_1, y_1)\\(2, 3)\end{matrix}\begin{matrix}(x_2, y_2)\\(7, 6)\end{matrix}$

$$m = \frac{\text{rise}}{\text{run}}$$

On the graph, we counted the rise of 3 and the run of 5. $$m = \frac{3}{5}$$

Notice that the rise of 3 can be found by subtracting the y-coordinates, 6 and 3, and the run of 5 can be found by subtracting the x-coordinates 7 and 2.

We rewrite the rise and run by putting in the coordinates. $$m = \frac{6-3}{7-2}$$

But 6 is y_2, the y-coordinate of the second point and 3 is y_1, the y-coordinate of the fir t point. So we can rewrite the slope using subscript notation. $$m = \frac{y_2 - y_1}{7-2}$$

Also 7 is the x-coordinate of the second point and 2 is the x-coordinate of the fir t point. So again we rewrite the slope using subscript notation. $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

We've shown that $m = \frac{y_2 - y_1}{x_2 - x_1}$ is really another version of $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

Slope of a line between two points

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope is:

y of the second point minus y of the fir t point
over
x of the second point minus x of the fir t point.

EXAMPLE 3.14

Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution

We'll call $(-2, -3)$ point #1 and $(-7, 4)$ point #2.

$$\begin{pmatrix} x_1, & y_1 \\ -2, & -3 \end{pmatrix}\begin{pmatrix} x_2, & y_2 \\ -7, & 4 \end{pmatrix}$$

Use the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute the values.
y of the second point minus y of the first point
x of the second point minus x of the first point

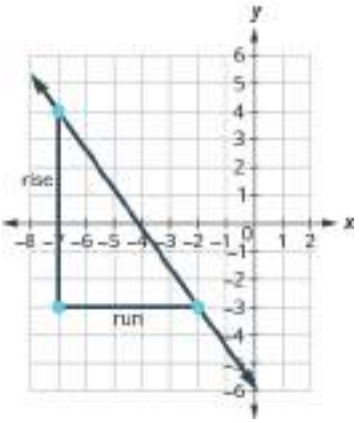
$$m = \frac{4 - (-3)}{-7 - (-2)}$$

Simplify.

$$m = \frac{7}{-5}$$

$$m = -\frac{7}{5}$$

Let's verify this slope on the graph shown.



$$m = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{7}{-5}$$

$$m = -\frac{7}{5}$$

TRY IT : : 3.27

Use the slope formula to find the slope of the line through the pair of points: $(-3, 4)$ and $(2, -1)$.

TRY IT : : 3.28

Use the slope formula to find the slope of the line through the pair of points: $(-2, 6)$ and $(-3, -4)$.

Graph a Line Given a Point and the Slope

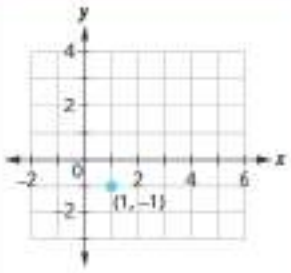
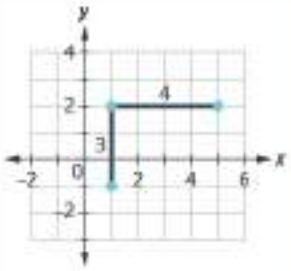
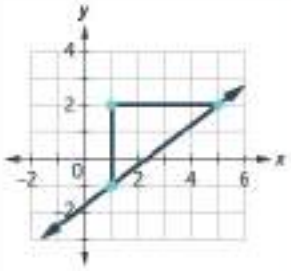
Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

We can also graph a line when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

EXAMPLE 3.15 HOW TO GRAPH A LINE GIVEN A POINT AND THE SLOPE

Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Solution

Step 1. Plot the given point.	Plot (1, −1).	
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ rise = 3 run = 4
Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, −1) and count the rise and the run. Up 3 units, right 4 units.	
Step 4. Connect the points with a line.	Connect the two points with a line.	

You can check your work by finding a third point. Since the slope is $m = \frac{3}{4}$, it can also be written as $m = \frac{-3}{-4}$ (negative divided by negative is positive!). Go back to $(1, -1)$ and count out the rise, -3, and the run, -4.

TRY IT : : 3.29 Graph the line passing through the point $(2, -2)$ with the slope $m = \frac{4}{3}$.

TRY IT : : 3.30 Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

HOW TO : : GRAPH A LINE GIVEN A POINT AND THE SLOPE.

Step 1. Plot the given point.

Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.

Step 3. Starting at the given point, count out the rise and run to mark the second point.

Step 4. Connect the points with a line.

Graph a Line Using its Slope and Intercept

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using one point and the slope of the line. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

See **Figure 3.6**. Let's look at the graph of the equation $y = \frac{1}{2}x + 3$ and find its slope and *y*-intercept.

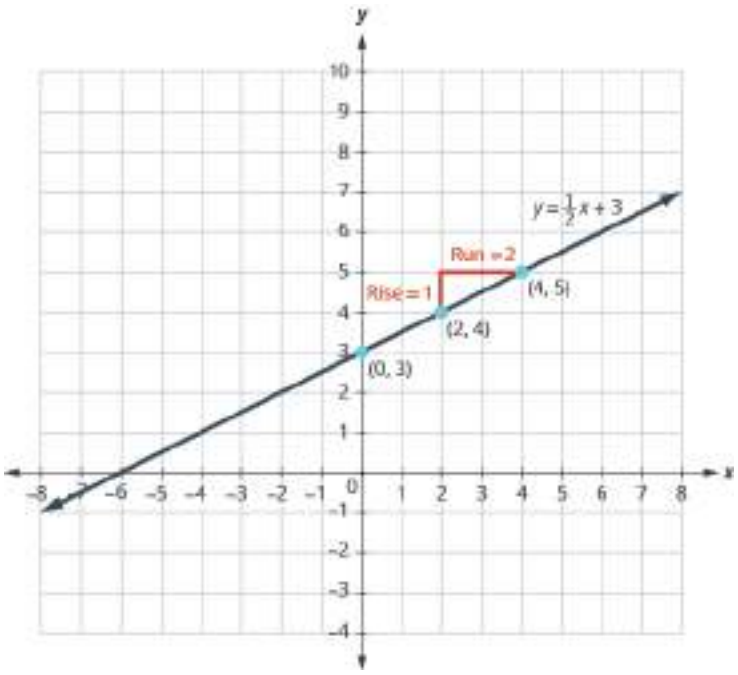


Figure 3.6

The red lines in the graph show us the rise is 1 and the run is 2. Substituting into the slope formula:

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{1}{2}$$

The *y*-intercept is $(0, 3)$.

Look at the equation of this line.

$$y = \frac{1}{2}x + 3$$

Look at the slope and *y*-intercept.

$$\text{slope } m = \frac{1}{2} \text{ and } y\text{-intercept } (0, 3).$$

When a linear equation is solved for *y*, the coefficient of the *x* term is the slope and the constant term is the *y*-coordinate

of the *y*-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in slope–intercept form. Sometimes the slope–intercept form is called the "*y*-form."

$m = \frac{1}{2}$; *y*-intercept is $(0, 3)$

$y = \frac{1}{2}x + 3$

$y = mx + b$

Slope Intercept Form of an Equation of a Line

The slope–intercept form of an equation of a line with slope *m* and *y*-intercept, $(0, b)$ is $y = mx + b$.

Let's practice finding the values of the slope and *y*-intercept from the equation of a line.

EXAMPLE 3.16

Identify the slope and *y*-intercept of the line from the equation:

ⓐ $y = -\frac{4}{7}x - 2$ ⓑ $x + 3y = 9$

Solution

ⓐ We compare our equation to the slope−intercept form of the equation.

Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{4}{7}x - 2$
Identify the slope.	$m = -\frac{4}{7}$
Identify the *y*-intercept.	*y*-intercept is $(0, -2)$

ⓑ When an equation of a line is not given in slope−intercept form, our first step will be to solve the equation for *y*.

Solve for *y*.	$x + 3y = 9$
Subtract *x* from each side.	$3y = -x + 9$
Divide both sides by 3.	$\frac{3y}{3} = \frac{-x + 9}{3}$
Simplify.	$y = -\frac{1}{3}x + 3$
Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{1}{3}x + 3$
Identify the slope.	$m = -\frac{1}{3}$
Identify the *y*-intercept.	*y*-intercept is $(0, 3)$

TRY IT : : 3.31 Identify the slope and *y*-intercept from the equation of the line.

ⓐ $y = \frac{2}{5}x - 1$ ⓑ $x + 4y = 8$

TRY IT : : 3.32 Identify the slope and *y*-intercept from the equation of the line.

ⓐ $y = -\frac{4}{3}x + 1$ ⓑ $3x + 2y = 12$

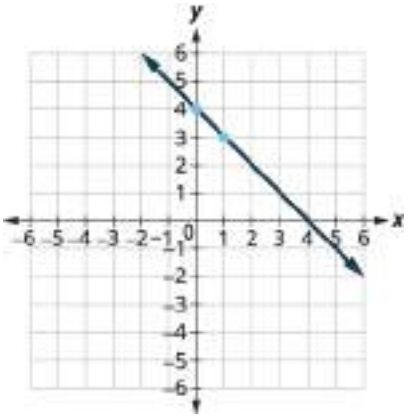
We have graphed a line using the slope and a point. Now that we know how to find the slope and *y*-intercept of a line from its equation, we can use the *y*-intercept as the point, and then count out the slope from there.

EXAMPLE 3.17

Graph the line of the equation $y = -x + 4$ using its slope and *y*-intercept.

Solution

	$y = mx + b$
The equation is in slope–intercept form.	$y = -x + 4$
Identify the slope and *y*-intercept.	$m = -1$ *y*-intercept is $(0, 4)$
Plot the *y*-intercept.	See the graph.
Identify the rise over the run.	$m = \frac{-1}{1}$
Count out the rise and run to mark the second point.	rise -1, run 1

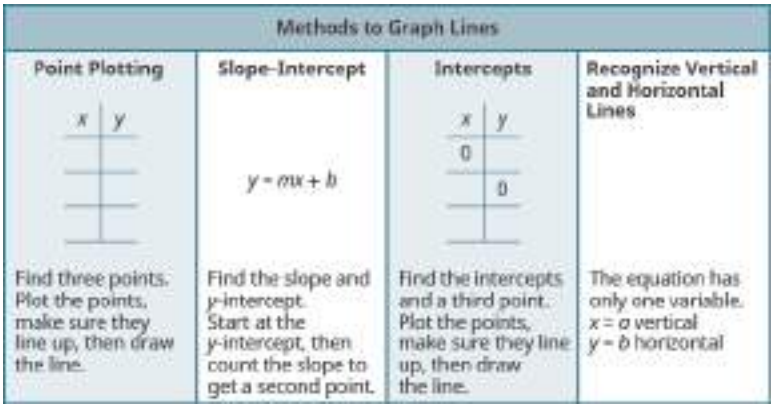


Draw the line as shown in the graph.

TRY IT : : 3.33 Graph the line of the equation $y = -x - 3$ using its slope and *y*-intercept.

TRY IT : : 3.34 Graph the line of the equation $y = -x - 1$ using its slope and *y*-intercept.

Now that we have graphed lines by using the slope and *y*-intercept, let's summarize all the methods we have used to graph lines.



Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier.

Generally, plotting points is not the most efficient way to graph a line. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are five equations we graphed in this chapter, and the method we used to graph each of them.

	Equation	Method
#1	$x = 2$	Vertical line
#2	$y = -1$	Horizontal line
#3	$-x + 2y = 6$	Intercepts
#4	$4x - 3y = 12$	Intercepts
#5	$y = -x + 4$	Slope–intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both *x* and *y* are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the *x*- intercept and $x = 0$ to find the *y*-intercept, and then found a third point by choosing another value for *x* or *y*.

Equation #5 is written in slope–intercept form. After identifying the slope and *y*-intercept from the equation we used them to graph the line.

This leads to the following strategy.

Strategy for Choosing the Most Convenient Method to Graph a Line

Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
 - $x = a$ is a vertical line passing through the *x*-axis at *a*.
 - $y = b$ is a horizontal line passing through the *y*-axis at *b*.
- If *y* is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and *y*-intercept.
 - Identify the slope and *y*-intercept and then graph.
- If the equation is of the form $Ax + By = C$, find the intercepts.
 - Find the *x*- and *y*-intercepts, a third point, and then graph.

EXAMPLE 3.18

Determine the most convenient method to graph each line:

ⓐ $y = 5$ ⓑ $4x - 5y = 20$ ⓒ $x = -3$ ⓓ $y = -\frac{5}{9}x + 8$

Solution

ⓐ $y = 5$

This equation has only one variable, *y*. Its graph is a horizontal line crossing the *y*-axis at 5.

ⓑ $4x - 5y = 20$

This equation is of the form $Ax + By = C$. The easiest way to graph it will be to find the intercepts and one more point.

ⓒ $x = -3$

There is only one variable, *x*. The graph is a vertical line crossing the *x*-axis at -3.

ⓓ $y = -\frac{5}{9}x + 8$

Since this equation is in $y = mx + b$ form, it will be easiest to graph this line by using the slope and *y*-intercepts.

TRY IT : : 3.35 Determine the most convenient method to graph each line:

ⓐ $3x + 2y = 12$ ⓑ $y = 4$ ⓒ $y = \frac{1}{5}x - 4$ ⓓ $x = -7$.

TRY IT : : 3.36 Determine the most convenient method to graph each line:

ⓐ $x = 6$ ⓑ $y = -\frac{3}{4}x + 1$ ⓒ $y = -8$ ⓓ $4x - 3y = -1$.

Graph and Interpret Applications of Slope–Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so you can see how equations written in slope–intercept form relate to real world situations.

Usually, when a linear equation models uses real-world data, different letters are used for the variables, instead of using only *x* and *y*. The variable names remind us of what quantities are being measured.

Also, we often will need to extend the axes in our rectangular coordinate system to bigger positive and negative numbers to accommodate the data in the application.

EXAMPLE 3.19

The equation $F = \frac{9}{5}C + 32$ is used to convert temperatures, *C*, on the Celsius scale to temperatures, *F*, on the Fahrenheit scale.

ⓐ Find the Fahrenheit temperature for a Celsius temperature of 0.

ⓑ Find the Fahrenheit temperature for a Celsius temperature of 20.

ⓒ Interpret the slope and *F*-intercept of the equation.

ⓓ Graph the equation.

Solution

ⓐ

Find the Fahrenheit temperature for a Celsius temperature of 0.	$F = \frac{9}{5}C + 32$
Find F when $C = 0$.	$F = \frac{9}{5}(0) + 32$
Simplify.	$F = 32$

ⓑ

Find the Fahrenheit temperature for a Celsius temperature of 20.	$F = \frac{9}{5}C + 32$
Find F when $C = 20$.	$F = \frac{9}{5}(20) + 32$
Simplify.	$F = 36 + 32$
Simplify.	$F = 68$

ⓒ

Interpret the slope and *F*-intercept of the equation.
Even though this equation uses *F* and *C*, it is still in slope–intercept form.

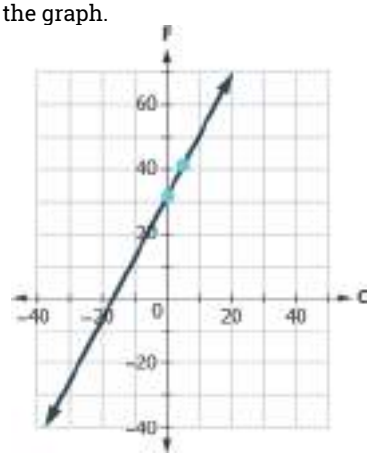
$$y = mx + b$$

$$F = mC + b$$

$$F = \frac{9}{5}C + 32$$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (*F*) increases 9 degrees when the temperature Celsius (*C*) increases 5 degrees.
The *F*-intercept means that when the temperature is $0°$ on the Celsius scale, it is $32°$ on the Fahrenheit scale.

ⓓ Graph the equation.
We'll need to use a larger scale than our usual. Start at the *F*-intercept $(0, 32)$, and then count out the rise of 9 and the run of 5 to get a second point as shown in the graph.



TRY IT : : 3.37

The equation $h = 2s + 50$ is used to estimate a woman's height in inches, *h*, based on her shoe size, *s*.

ⓐ Estimate the height of a child who wears women's shoe size 0.

ⓑ Estimate the height of a woman with shoe size 8.

ⓒ Interpret the slope and *h*-intercept of the equation.

ⓓ Graph the equation.

TRY IT : : 3.38

The equation $T = \frac{1}{4}n + 40$ is used to estimate the temperature in degrees Fahrenheit, *T*, based on the number of cricket chirps, *n*, in one minute.

ⓐ Estimate the temperature when there are no chirps.

ⓑ Estimate the temperature when the number of chirps in one minute is 100.

ⓒ Interpret the slope and *T*-intercept of the equation.

ⓓ Graph the equation.

The cost of running some types business have two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labor needed to produce each item.

EXAMPLE 3.20

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, *C*, in dollars and the number of miles, *m*, that he drives.

ⓐ Find Sam's cost for a week when he drives 0 miles.

ⓑ Find the cost for a week when he drives 250 miles.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Solution

ⓐ

Find Sam's cost for a week when he drives 0 miles.	$C = 0.5m + 60$
Find C when $m = 0$.	$C = 0.5(0) + 60$
Simplify.	$C = 60$
	Sam's costs are \$60 when he drives 0 miles.

ⓑ

Find the cost for a week when he drives 250 miles.	$C = 0.5m + 60$
Find C when $m = 250$.	$C = 0.5(250) + 60$
Simplify.	$C = 185$
	Sam's costs are \$185 when he drives 250 miles.

ⓒ Interpret the slope and *C*-intercept of the equation.

$y = mx + b$

$C = 0.5n + 60$

The slope, 0.5, means that the weekly cost, *C*, increases by \$0.50 when the number of miles driven, *n,* increases by 1. The *C*-intercept means that when the number of miles driven is 0, the weekly cost is \$60.

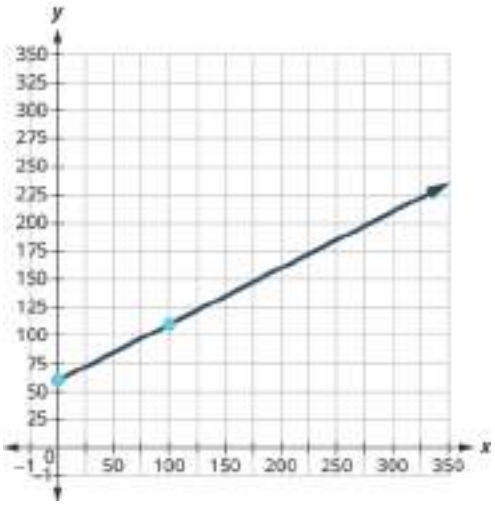
ⓓ Graph the equation.

We'll need to use a larger scale than our usual. Start at the *C*-intercept $(0, 60)$.

To count out the slope *m* = 0.5, we rewrite it as an equivalent fraction that will make our graphing easier.

	$m = 0.5$
Rewrite as a fraction.	$m = \frac{0.5}{1}$
Multiply numerator and denominator by 100.	$m = \frac{0.5(100)}{1(100)}$
Simplify.	$m = \frac{50}{100}$

So to graph the next point go up 50 from the intercept of 60 and then to the right 100. The second point will be (100, 110).



TRY IT : : 3.39

Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, *C*, in dollars and the number of pizzas, *p*, that she sells.

ⓐ Find Stella's cost for a week when she sells no pizzas.

ⓑ Find the cost for a week when she sells 15 pizzas.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

TRY IT : : 3.40

Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, *C*, in dollars and the number of wedding invitations, *n*, that she writes.

ⓐ Find Loreen's cost for a week when she writes no invitations.

ⓑ Find the cost for a week when she writes 75 invitations.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Use Slopes to Identify Parallel and Perpendicular Lines

Two lines that have the same slope are called **parallel lines**. Parallel lines have the same steepness and never intersect.

We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different *y*-intercepts are called parallel lines. See Figure 3.7.

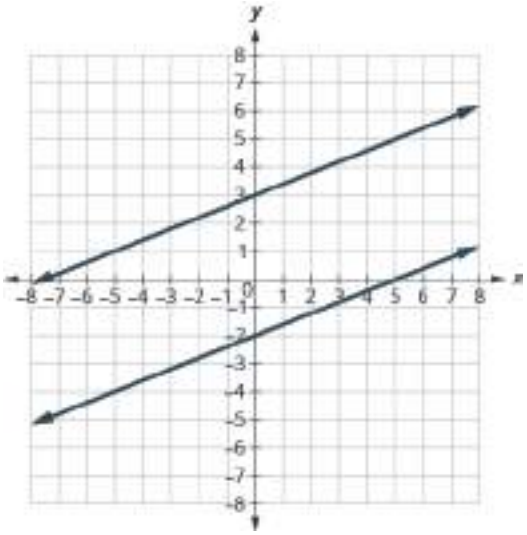


Figure 3.7

Verify that both lines have the same slope, $m = \frac{2}{5}$, and different *y*-intercepts.

What about vertical lines? The slope of a vertical line is undefined, so vertical lines don't fit in the definition above. We say that vertical lines that have different *x*-intercepts are parallel, like the lines shown in this graph.

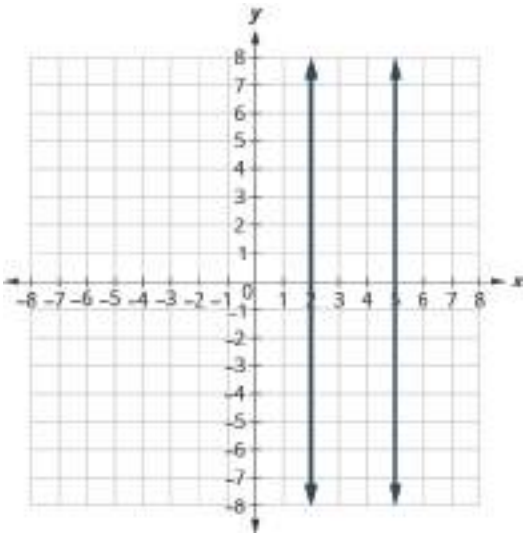


Figure 3.8

Parallel Lines

Parallel lines are lines in the same plane that do not intersect.

- Parallel lines have the same slope and different *y*-intercepts.
- If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
- Parallel vertical lines have different *x*-intercepts.

Since parallel lines have the same slope and different *y*-intercepts, we can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

EXAMPLE 3.21

Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $3x-2y=6$ and $y=\frac{3}{2}x+1$ ⓑ $y=2x-3$ and $-6x+3y=-9$.

Solution

ⓐ

	$3x-2y$	$=6$	and	$y=\frac{3}{2}x+1$
	$-2y$	$=-3x+6$		
Solve the first equation for y.	$\frac{-2y}{-2}$	$=\frac{-3x+6}{-2}$		
The equation is now in slope–intercept form.	y	$=\frac{3}{2}x-3$		
The equation of the second line is already in slope–intercept form.				$y=\frac{3}{2}x+1$
	y	$=\frac{3}{2}x-3$		$y=\frac{3}{2}x+1$
Identify the slope and y-intercept of both lines.	y	$=mx+b$		$y=mx+b$
	m	$=\frac{3}{2}$		$y=\frac{3}{2}$
	y-intercept is $(0,-3)$			y-intercept is $(0,1)$

The lines have the same slope and different y-intercepts and so they are parallel.
You may want to graph the lines to confirm whether they are parallel.

ⓑ

	y	$=2x-3$	and	$-6x+3y=-9$
The first equation is already in slope–intercept form.	y	$=2x-3$		
				$-6x+3y=-9$
				$3y=6x-9$
Solve the second equation for y.				$\frac{3y}{3}=\frac{6x-9}{3}$
				$y=2x-3$
The second equation is now in slope–intercept form.				$y=2x-3$
	y	$=2x-3$		$y=2x-3$
Identify the slope and y-intercept of both lines.	y	$=mx+b$		$y=mx+b$
	m	$=2$		$m=2$
	y-intercept is $(0,-3)$			y-intercept is $(0,-3)$

The lines have the same slope, but they also have the same y-intercepts. Their equations represent the same line and we say the lines are coincident. They are not parallel; they are the same line.

TRY IT : : 3.41 Use slopes and y-intercepts to determine if the lines are parallel:

ⓐ $2x+5y=5$ and $y=-\frac{2}{5}x-4$ ⓑ $y=-\frac{1}{2}x-1$ and $x+2y=-2$.

TRY IT : : 3.42 Use slopes and y-intercepts to determine if the lines are parallel:

ⓐ $4x-3y=6$ and $y=\frac{4}{3}x-1$ ⓑ $y=\frac{3}{4}x-3$ and $3x-4y=12$.

EXAMPLE 3.22

Use slopes and y-intercepts to determine if the lines are parallel:

ⓐ $y = -4$ and $y = 3$ ⓑ $x = -2$ and $x = -5$.

Solution

ⓐ $y = -4$ and $y = 3$

We recognize right away from the equations that these are horizontal lines, and so we know their slopes are both 0. Since the horizontal lines cross the *y*-axis at $y = -4$ and at $y = 3$, we know the *y*-intercepts are $(0, -4)$ and $(0, 3)$. The lines have the same slope and different *y*-intercepts and so they are parallel.

ⓑ $x = -2$ and $x = -5$

We recognize right away from the equations that these are vertical lines, and so we know their slopes are undefined. Since the vertical lines cross the *x*-axis at $x = -2$ and $x = -5$, we know the *y*-intercepts are $(-2, 0)$ and $(-5, 0)$. The lines are vertical and have different *x*-intercepts and so they are parallel.

TRY IT : : 3.43 Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y = 8$ and $y = -6$ ⓑ $x = 1$ and $x = -5$.

TRY IT : : 3.44 Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y = 1$ and $y = -5$ ⓑ $x = 8$ and $x = -6$.

Let's look at the lines whose equations are $y = \frac{1}{4}x - 1$ and $y = -4x + 2$, shown in Figure 3.9.

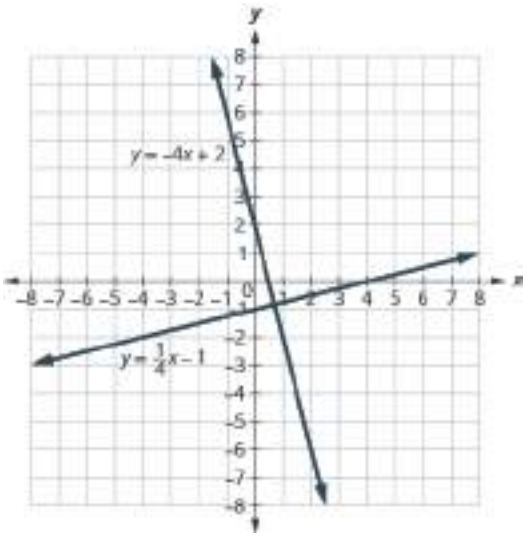


Figure 3.9

These lines lie in the same plane and intersect in right angles. We call these lines perpendicular.

If we look at the slope of the first line, $m_1 = \frac{1}{4}$, and the slope of the second line, $m_2 = -4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.

$$m_1 \cdot m_2$$
$$\frac{1}{4}(-4)$$
$$-1$$

This is always true for **perpendicular lines** and leads us to this definition.

Perpendicular Lines

Perpendicular lines are lines in the same plane that form a right angle.

- If m_1 and m_2 are the slopes of two perpendicular lines, then:
 - their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$.
 - the product of their slopes is -1, $m_1 \cdot m_2 = -1$.
- A vertical line and a horizontal line are always perpendicular to each other.

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation, and then see if the slopes are opposite reciprocals. If the product of the slopes is -1, the lines are perpendicular.

EXAMPLE 3.23

Use slopes to determine if the lines are perpendicular:

ⓐ $y = -5x - 4$ and $x - 5y = 5$ ⓑ $7x + 2y = 3$ and $2x + 7y = 5$

Solution

ⓐ

The first equation is in slope–intercept form. $y = -5x - 4$

Solve the second equation for y.

$$\begin{aligned} x - 5y &= 5 \\ -5y &= -x + 5 \\ \frac{-5y}{-5} &= \frac{-x + 5}{-5} \\ y &= \frac{1}{5}x - 1 \end{aligned}$$

Identify the slope of each line.

$$\begin{aligned} y &= -5x - 4 \\ y &= mx + b \\ m_1 &= -5 \end{aligned} \qquad \begin{aligned} y &= \frac{1}{5}x - 1 \\ y &= mx + b \\ m_2 &= \frac{1}{5} \end{aligned}$$

The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes, Since $-5\left(\frac{1}{5}\right) = -1$, it checks.

ⓑ

Solve the equations for y.

$$\begin{aligned} 7x + 2y &= 3 \\ 2y &= -7x + 3 \\ \frac{2y}{2} &= \frac{-7x + 3}{2} \\ y &= -\frac{7}{2}x + \frac{3}{2} \end{aligned} \qquad \begin{aligned} 2x + 7y &= 5 \\ 7y &= -2x + 5 \\ \frac{7y}{7} &= \frac{-2x + 5}{7} \\ y &= -\frac{2}{7}x + \frac{5}{7} \end{aligned}$$

Identify the slope of each line.

$$\begin{aligned} y &= mx + b \\ m_1 &= -\frac{7}{2} \end{aligned} \qquad \begin{aligned} y &= mx + b \\ m_1 &= -\frac{2}{7} \end{aligned}$$

The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

TRY IT : : 3.45 Use slopes to determine if the lines are perpendicular:

ⓐ $y = -3x + 2$ and $x - 3y = 4$ ⓑ $5x + 4y = 1$ and $4x + 5y = 3$.

TRY IT : : 3.46 Use slopes to determine if the lines are perpendicular:

ⓐ $y = 2x - 5$ and $x + 2y = -6$ ⓑ $2x - 9y = 3$ and $9x - 2y = 1$.

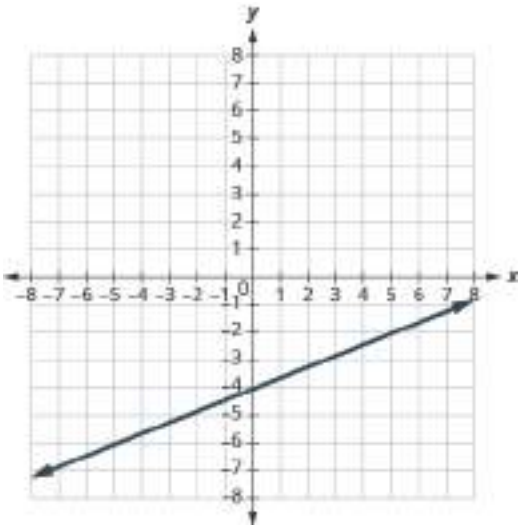
3.2 EXERCISES

Practice Makes Perfect

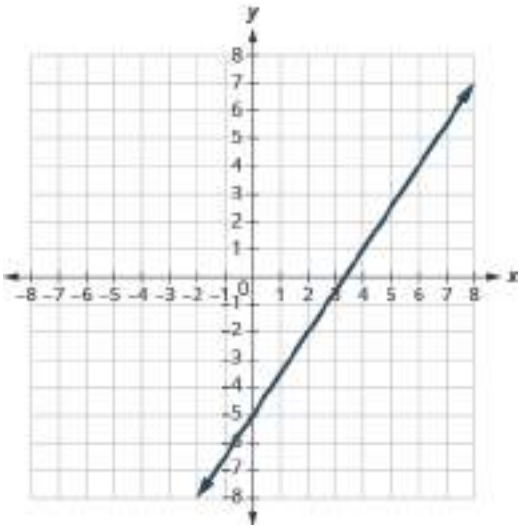
Find the Slope of a Line

In the following exercises, find the slope of each line shown.

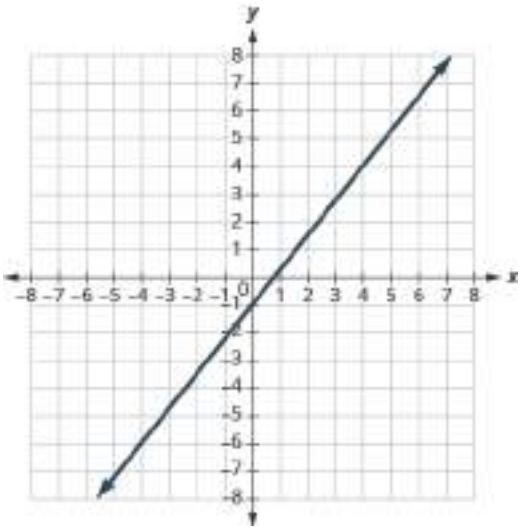
73.



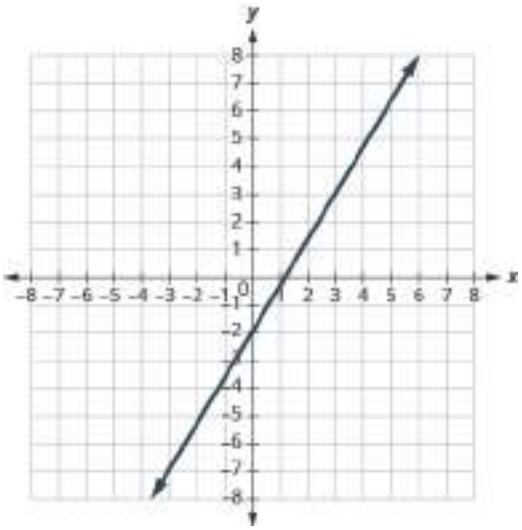
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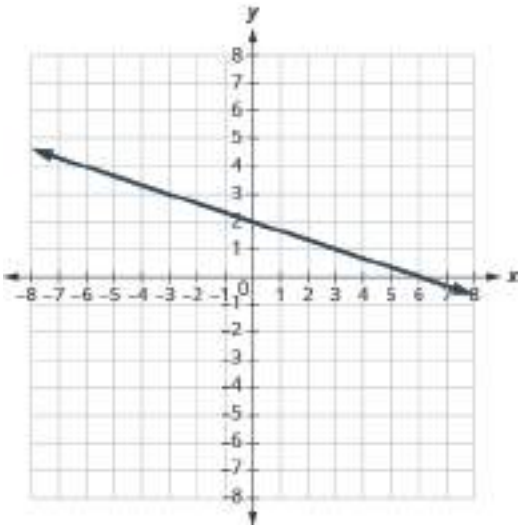
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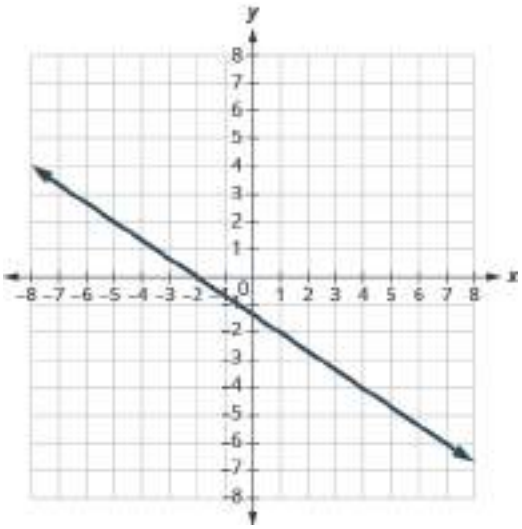
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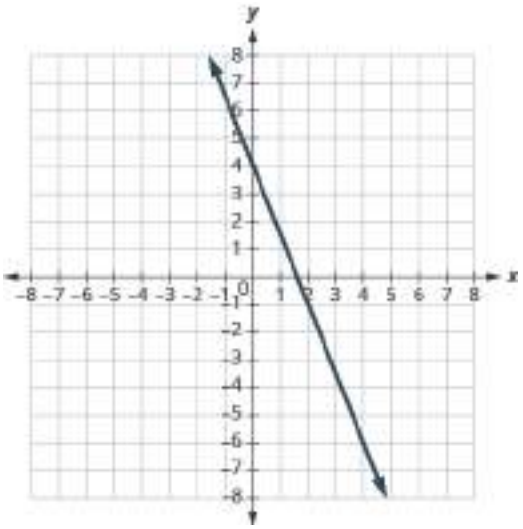
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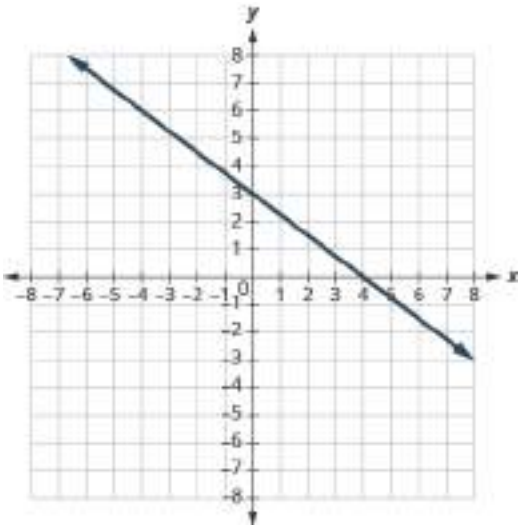
78.



79.



80.



In the following exercises, find the slope of each line.

81. $y = 3$

82. $y = -2$

83. $x = -5$

84. $x = 4$

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

85. (2, 5), (4, 0)

86. (3, 6), (8, 0)

87. (−3, 3), (4, −5)

88. (−2, 4), (3, −1)

89. (−1, −2), (2, 5)

90. (−2, −1), (6, 5)

91. (4, −5), (1, −2)

92. (3, −6), (2, −2)

Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

93. $(2, 5)$; $m = -\frac{1}{3}$

94. $(1, 4)$; $m = -\frac{1}{2}$

95. $(-1, -4)$; $m = \frac{4}{3}$

96. $(-3, -5)$; $m = \frac{3}{2}$

97. y-intercept 3; $m = -\frac{2}{5}$

98. x-intercept -2; $m = \frac{3}{4}$

99. $(-4, 2)$; $m = 4$

100. $(1, 5)$; $m = -3$

Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and y-intercept of each line.

101. $y = -7x + 3$

102. $y = 4x - 10$

103. $3x + y = 5$

104. $4x + y = 8$

105. $6x + 4y = 12$

106. $8x + 3y = 12$

107. $5x - 2y = 6$

108. $7x - 3y = 9$

In the following exercises, graph the line of each equation using its slope and y-intercept.

109. $y = 3x - 1$

110. $y = 2x - 3$

111. $y = -x + 3$

112. $y = -x - 4$

113. $y = -\frac{2}{5}x - 3$

114. $y = -\frac{3}{5}x + 2$

115. $3x - 2y = 4$

116. $3x - 4y = 8$

Choose the Most Convenient Method to Graph a Line

In the following exercises, determine the most convenient method to graph each line.

117. $x = 2$

118. $y = 5$

119. $y = -3x + 4$

120. $x - y = 5$

121. $x - y = 1$

122. $y = \frac{2}{3}x - 1$

123. $3x - 2y = -12$

124. $2x - 5y = -10$

Graph and Interpret Applications of Slope–Intercept

125. The equation $P = 31 + 1.75w$ models the relation between the amount of Tuyet's monthly water bill payment, *P*, in dollars, and the number of units of water, *w*, used.

ⓐ Find Tuyet's payment for a month when 0 units of water are used.

ⓑ Find Tuyet's payment for a month when 12 units of water are used.

ⓒ Interpret the slope and *P*-intercept of the equation.

ⓓ Graph the equation.

126. The equation $P = 28 + 2.54w$ models the relation between the amount of R and y's monthly water bill payment, *P*, in dollars, and the number of units of water, *w*, used.

ⓐ Find the payment for a month when R and y used 0 units of water.

ⓑ Find the payment for a month when R and y used 15 units of water.

ⓒ Interpret the slope and *P*-intercept of the equation.

ⓓ Graph the equation.

127. Bruce drives his car for his job. The equation $R = 0.575m + 42$ models the relation between the amount in dollars, *R*, that he is reimbursed and the number of miles, *m*, he drives in one day.

ⓐ Find the amount Bruce is reimbursed on a day when he drives 0 miles.

ⓑ Find the amount Bruce is reimbursed on a day when he drives 220 miles.

ⓒ Interpret the slope and *R*-intercept of the equation.

ⓓ Graph the equation.

128. Janelle is planning to rent a car while on vacation. The equation $C = 0.32m + 15$ models the relation between the cost in dollars, *C*, per day and the number of miles, *m*, she drives in one day.

ⓐ Find the cost if Janelle drives the car 0 miles one day.

ⓑ Find the cost on a day when Janelle drives the car 400 miles.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

129. Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation $S = 400 + 0.15c$ models the relation between her weekly salary, *S*, in dollars and the amount of her sales, *c*, in dollars.

ⓐ Find Cherie's salary for a week when her sales were $0.

ⓑ Find Cherie's salary for a week when her sales were $3,600.

ⓒ Interpret the slope and *S*-intercept of the equation.

ⓓ Graph the equation.

130. Patel's weekly salary includes a base pay plus commission on his sales. The equation $S = 750 + 0.09c$ models the relation between his weekly salary, *S*, in dollars and the amount of his sales, *c*, in dollars.

ⓐ Find Patel's salary for a week when his sales were 0.

ⓑ Find Patel's salary for a week when his sales were 18,540.

ⓒ Interpret the slope and *S*-intercept of the equation.

ⓓ Graph the equation.

131. Costa is planning a lunch banquet. The equation $C = 450 + 28g$ models the relation between the cost in dollars, *C*, of the banquet and the number of guests, *g*.

ⓐ Find the cost if the number of guests is 40.

ⓑ Find the cost if the number of guests is 80.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

132. Margie is planning a dinner banquet. The equation $C = 750 + 42g$ models the relation between the cost in dollars, *C* of the banquet and the number of guests, *g*.

ⓐ Find the cost if the number of guests is 50.

ⓑ Find the cost if the number of guests is 100.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

133. $y = \frac{3}{4}x - 3; \quad 3x - 4y = -2$

134. $3x - 4y = -2; \quad y = \frac{3}{4}x - 3$

135. $2x - 4y = 6; \quad x - 2y = 3$

136. $8x + 6y = 6; \quad 12x + 9y = 12$

137. $x = 5; \quad x = -6$

138. $x = -3; \quad x = -2$

139. $4x - 2y = 5; \quad 3x + 6y = 8$

140. $8x - 2y = 7; \quad 3x + 12y = 9$

141. $3x - 6y = 12; \quad 6x - 3y = 3$

142. $9x - 5y = 4; \quad 5x + 9y = -1$

143. $7x - 4y = 8; \quad 4x + 7y = 14$

144. $5x - 2y = 11; \quad 5x - y = 7$

145. $3x - 2y = 8; \quad 2x + 3y = 6$

146. $2x + 3y = 5; \quad 3x - 2y = 7$

147. $3x - 2y = 1; \quad 2x - 3y = 2$

148. $2x + 4y = 3; \quad 6x + 3y = 2$

149. $y = 2; \quad y = 6$

150. $y = -1; \quad y = 2$

Writing Exercises

151. How does the graph of a line with slope $m = \frac{1}{2}$ differ from the graph of a line with slope $m = 2$?

152. Why is the slope of a vertical line "undefined"?

153. Explain how you can graph a line given a point and its slope.

154. Explain in your own words how to decide which method to use to graph a line.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find the slope of a line.			
graph a line given a point and the slope.			
graph a line using its slope and intercept.			
choose the most convenient method to graph a line.			
graph and interpret applications of slope-intercept.			
use slopes to identify parallel and perpendicular lines.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

3.3 Find the Equation of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $\frac{2}{5}(x+15)$.
 If you missed this problem, review **Example 1.50**.
2. Simplify: $-3(x-(-2))$.
 If you missed this problem, review **Example 1.53**.
3. Solve for y: $y-3=-2(x+1)$.
 If you missed this problem, review **Example 2.31**.

How do online companies know that "you may also like" a particular item based on something you just ordered? How can economists know how a rise in the minimum wage will affect the unemployment rate? How do medical researchers create drugs to target cancer cells? How can traffic engineers predict the effect on your commuting time of an increase or decrease in gas prices? It's all mathematics.

The physical sciences, social sciences, and the business world are full of situations that can be modeled with linear equations relating two variables. To create a mathematical model of a linear relation between two variables, we must be able to find the equation of the line. In this section, we will look at several ways to write the equation of a line. The specific method we use will be determined by what information we are given.

Find an Equation of the Line Given the Slope and y-Intercept

We can easily determine the slope and intercept of a line if the equation is written in slope-intercept form, $y=mx+b$.

Now we will do the reverse—we will start with the slope and y-intercept and use them to find the equation of the line.

EXAMPLE 3.24

Find the equation of a line with slope -9 and y-intercept $(0,-4)$.

Solution

Since we are given the slope and y-intercept of the line, we can substitute the needed values into the slope-intercept form, $y=mx+b$.

Name the slope.	$m=-9$
Name the y-intercept.	y-intercept $(0,-4)$
Substitute the values into $y=mx+b$.	$y=mx+b$
	$y=-9x+(-4)$
	$y=-9x-4$

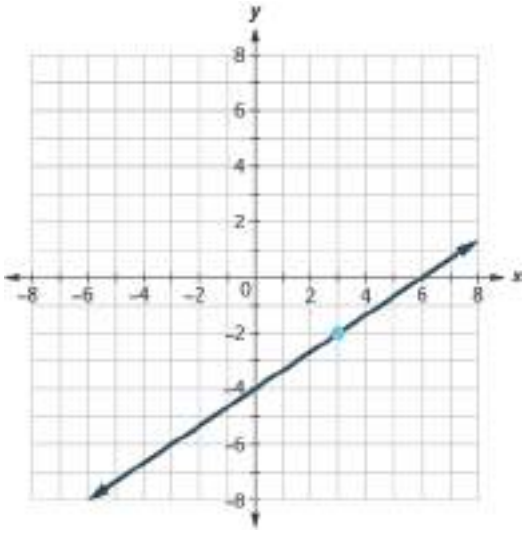
TRY IT : : 3.47 Find the equation of a line with slope $\frac{2}{5}$ and y-intercept $(0,4)$.

TRY IT : : 3.48 Find the equation of a line with slope -1 and y-intercept $(0, -3)$.

Sometimes, the slope and intercept need to be determined from the graph.

EXAMPLE 3.25

Find the equation of the line shown in the graph.



Solution

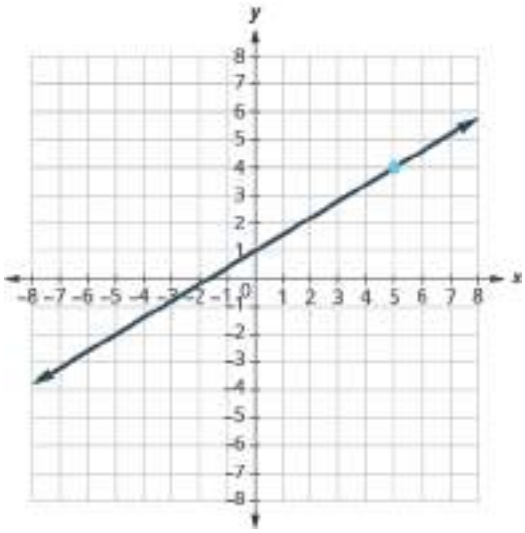
We need to find the slope and y-intercept of the line from the graph so we can substitute the needed values into the slope-intercept form, $y = mx + b$.

To find the slope, we choose two points on the graph.

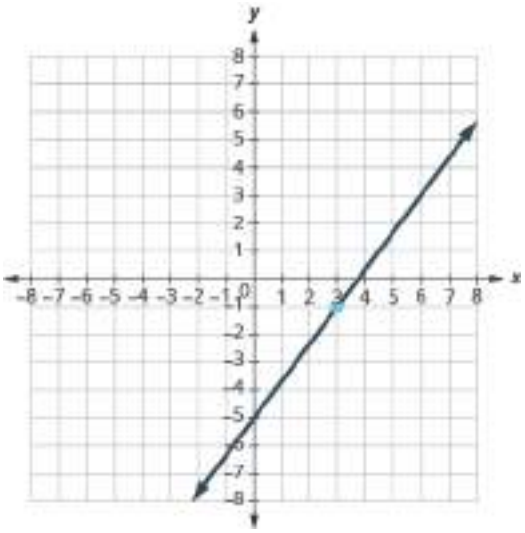
The y-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Find the slope, by counting the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{3}$
Find the y-intercept.	y-intercept $(0, -4)$
Substitute the values into $y = mx + b$.	$y = mx + b$
	$y = \frac{2}{3}x - 4$

TRY IT : : 3.49 Find the equation of the line shown in the graph.



TRY IT : : 3.50 Find the equation of the line shown in the graph.



Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope-intercept form of the equation works well when you are given the slope and *y*-intercept or when you read them off a graph. But what happens when you have another point instead of the *y*-intercept?

We are going to use the slope formula to derive another form of an equation of the line.

Suppose we have a line that has slope *m* and that contains some specific point (x_1, y_1) and some other point, which we will just call (x, y). We can write the slope of this line and then change it to a different form.

$$m = \frac{y - y_1}{x - x_1}$$

Multiply both sides of the equation by $x - x_1$. $$m(x - x_1) = \left(\frac{y - y_1}{x - x_1}\right)(x - x_1)$$

Simplify. $$m(x - x_1) = y - y_1$$

Rewrite the equation with the y terms on the left. $$y - y_1 = m(x - x_1)$$

This format is called the **point-slope form** of an equation of a line.

Point-slope Form of an Equation of a Line

The **point-slope form** of an equation of a line with slope m and containing the point (x_1, y_1) is:

$$y - y_1 = m(x - x_1)$$

We can use the point-slope form of an equation to find an equation of a line when we know the slope and at least one point. Then, we will rewrite the equation in slope-intercept form. Most applications of linear equations use the the slope-intercept form.

EXAMPLE 3.26 HOW TO FIND AN EQUATION OF A LINE GIVEN A POINT AND THE SLOPE

Find an equation of a line with slope $m = -\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope-intercept form.

Solution

Step 1. Identify the slope.	The slope is given.	$m = -\frac{1}{3}$
Step 2. Identify the point.	The point is given.	(x_1, y_1) $(6, -4)$
Step 3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - (-4) = -\frac{1}{3}(x - 6)$ $y + 4 = -\frac{1}{3}x + 2$
Step 4. Write the equation in slope-intercept form.		$y = -\frac{1}{3}x - 2$

TRY IT : : 3.51 Find the equation of a line with slope $m = -\frac{2}{5}$ and containing the point $(10, -5)$.

TRY IT : : 3.52 Find the equation of a line with slope $m = -\frac{3}{4}$, and containing the point $(4, -7)$.

We list the steps for easy reference.



HOW TO : : TO FIND AN EQUATION OF A LINE GIVEN THE SLOPE AND A POINT.

Step 1. Identify the slope.
Step 2. Identify the point.
Step 3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
Step 4. Write the equation in slope-intercept form.

EXAMPLE 3.27

Find an equation of a horizontal line that contains the point $(-2, -6)$. Write the equation in slope-intercept form.

Solution

Every horizontal line has slope 0. We can substitute the slope and points into the point-slope form, $y - y_1 = m(x - x_1)$.

Identify the slope.	$m = 0$
Identify the point.	$\begin{pmatrix} x_1 & y_1 \\ -2, & -6 \end{pmatrix}$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$
	$y - (-6) = 0(x - (-2))$
Simplify.	$y + 6 = 0$
	$y = -6$
Write in slope-intercept form.	It is in *y*-form, but could be written $y = 0x - 6$.

Did we end up with the form of a horizontal line, $y = a$?

TRY IT : : 3.53 Find the equation of a horizontal line containing the point $(-3, 8)$.

TRY IT : : 3.54 Find the equation of a horizontal line containing the point $(-1, 4)$.

Find an Equation of the Line Given Two Points

When real-world data is collected, a linear model can be created from two data points. In the next example we'll see how to find an equation of a line when just two points are given.

So far, we have two options for finding an equation of a line: slope-intercept or point-slope. When we start with two points, it makes more sense to use the point-slope form.

But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find the equation.

EXAMPLE 3.28 HOW TO FIND THE EQUATION OF A LINE GIVEN TWO POINTS

Find an equation of a line that contains the points $(-3, -1)$ and $(2, -2)$ Write the equation in slope-intercept form.

Solution

Step 1. Find the slope using the given points.	Find the slope of the line through (–3, –1) and (2, –2).	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{-2 - (-1)}{2 - (-3)}$ $m = \frac{-1}{5}$ $m = -\frac{1}{5}$
Step 2. Choose one point.	Choose either point.	$\begin{pmatrix} x_1 & y_1 \\ 2, & -2 \end{pmatrix}$
Step 3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - (-2) = -\frac{1}{5}(x - 2)$ $y + 2 = -\frac{1}{5}x + \frac{2}{5}$

Step 4. Write the equation in slope-intercept form.		$y=-\frac{1}{5}x-\frac{8}{5}$

TRY IT : : 3.55 Find the equation of a line containing the points $(-2, -4)$ and $(1, -3)$.

TRY IT : : 3.56 Find the equation of a line containing the points $(-4, -3)$ and $(1, -5)$.

The steps are summarized here.

HOW TO : : TO FIND AN EQUATION OF A LINE GIVEN TWO POINTS.

Step 1. Find the slope using the given points. $m=\frac{y_2-y_1}{x_2-x_1}$

Step 2. Choose one point.

Step 3. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.

Step 4. Write the equation in slope-intercept form.

EXAMPLE 3.29

Find an equation of a line that contains the points $(-3, 5)$ and $(-3, 4)$. Write the equation in slope-intercept form.

Solution

Again, the first step will be to find the slope.

Find the slope of the line through $(-3, 5)$ and $(-3, 4)$.

$$m = \frac{y_2-y_1}{x_2-x_1}$$
$$m = \frac{4-5}{-3-(-3)}$$
$$m = \frac{-1}{0}$$

The slope is undefined

This tells us it is a vertical line. Both of our points have an *x*-coordinate of -2. So our equation of the line is $x=-2$. Since there is no *y*, we cannot write it in slope-intercept form.

You may want to sketch a graph using the two given points. Does your graph agree with our conclusion that this is a vertical line?

TRY IT : : 3.57 Find the equation of a line containing the points $(5, 1)$ and $(5, -4)$.

TRY IT : : 3.58 Find the equaion of a line containing the points $(-4, 4)$ and $(-4, 3)$.

We have seen that we can use either the slope-intercept form or the point-slope form to find an equation of a line. Which form we use will depend on the information we are given.

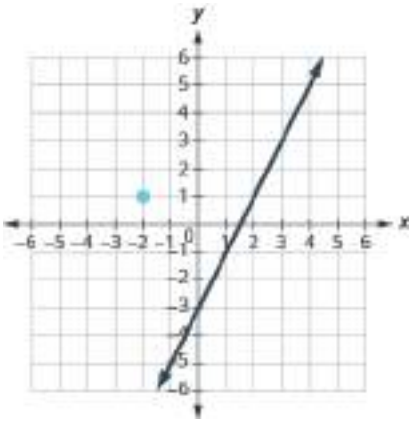
To Write an Equation of a Line		
If given:	**Use:**	**Form:**
Slope and *y*-intercept	slope-intercept	$y = mx + b$
Slope and a point	point-slope	$y - y_1 = m(x - x_1)$
Two points	point-slope	$y - y_1 = m(x - x_1)$

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point-slope equation.

First, let's look at this graphically.

This graph shows $y = 2x - 3$. We want to graph a line parallel to this line and passing through the point $(-2, 1)$.

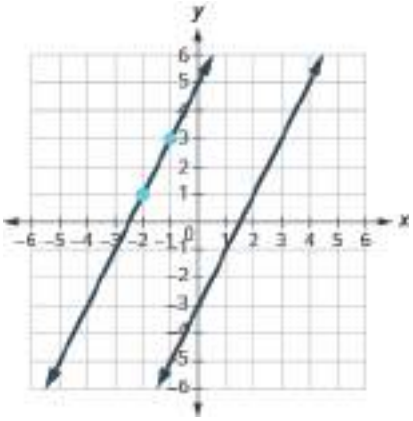


We know that parallel lines have the same slope. So the second line will have the same slope as $y = 2x - 3$. That slope is $m_{||} = 2$. We'll use the notation $m_{||}$ to represent the slope of a line parallel to a line with slope *m*. (Notice that the subscript || looks like two parallel lines.)

The second line will pass through $(-2, 1)$ and have $m = 2$.

To graph the line, we start at $(-2, 1)$ and count out the rise and run.

With $m = 2$ (or $m = \frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line, as shown in the graph.



Do the lines appear parallel? Does the second line pass through $(-2, 1)$?

We were asked to graph the line, now let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point-slope form.

EXAMPLE 3.30 HOW TO FIND THE EQUATION OF A LINE PARALLEL TO A GIVEN LINE AND A POINT

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Solution

Step 1. Find the slope of the given line.	The line is in slope-intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the parallel line.	Parallel lines have the same slope.	$m_{\parallel} = 2$
Step 3. Identify the point.	The given point is $(-2, 1)$.	$\begin{pmatrix} x_1 & y_1 \\ -2 & 1 \end{pmatrix}$
Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = 2(x - (-2))$ $y - 1 = 2(x + 2)$ $y - 1 = 2x + 4$
Step 5. Write the equation in slope-intercept form.		$y = 2x + 5$

Look at graph with the parallel lines shown previously. Does this equation make sense? What is the *y*-intercept of the line? What is the slope?

TRY IT : : 3.59

Find an equation of a line parallel to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

TRY IT : : 3.60

Find an equation of a line parallel to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$.

Write the equation in slope-intercept form.

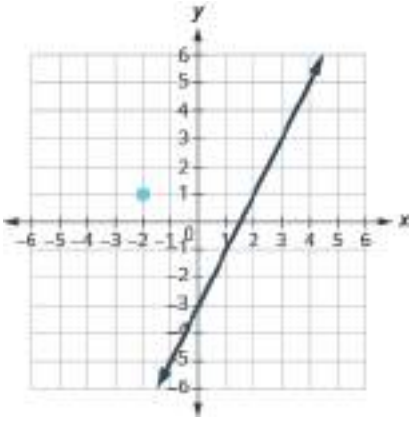
HOW TO : : FIND AN EQUATION OF A LINE PARALLEL TO A GIVEN LINE.

Step 1. Find the slope of the given line.
Step 2. Find the slope of the parallel line.
Step 3. Identify the point.
Step 4. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.
Step 5. Write the equation in slope-intercept form.

Find an Equation of a Line Perpendicular to a Given Line

Now, let's consider perpendicular lines. Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point-slope equation, like we did with parallel lines.

This graph shows $y = 2x - 3$. Now, we want to graph a line perpendicular to this line and passing through $(-2, 1)$.



We know that perpendicular lines have slopes that are negative reciprocals.

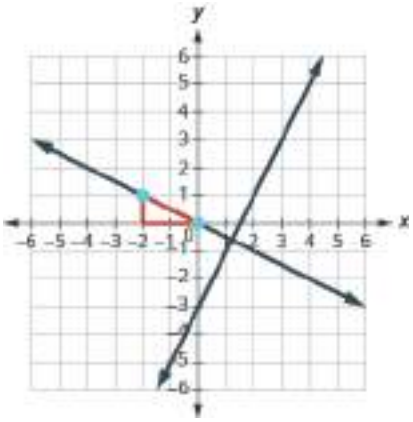
We'll use the notation $m_{\perp}$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

$$y = 2x - 3 \quad \text{perpendicular line}$$

$$m = 2 \qquad m_{\perp} = -\frac{1}{2}$$

We now know the perpendicular line will pass through $(-2,\ 1)$ with $m_{\perp} = -\frac{1}{2}$.

To graph the line, we will start at $(-2,\ 1)$ and count out the rise -1 and the run 2. Then we draw the line.



Do the lines appear perpendicular? Does the second line pass through $(-2,\ 1)$?

We were asked to graph the line, now, let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. In this example we know one point, and can find the slope, so we will use the point-slope form.

EXAMPLE 3.31 HOW TO FIND THE EQUATION OF A LINE PERPENDICULAR TO A GIVEN LINE AND A POINT

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2,\ 1)$. Write the equation in slope-intercept form.

Solution

Step 1. Find the slope of the given line.	The line is in slope-intercept form, $y = 2x - 3$.	$m = 2$

Step 2. Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative reciprocals.	$m_{\perp}=-\frac{1}{2}$
Step 3. Identify the point.	The given point is $(-2, 1)$.	(x_1, y_1) $(-2, 1)$
Step 4. Substitute the values into the point–slope form, $y-y_1=m(x-x_1)$.	Simplify.	$y-y_1=m(x-x_1)$ $y-1=-\frac{1}{2}(x-(-2))$ $y-1=-\frac{1}{2}(x+2)$ $y-1=-\frac{1}{2}x-1$
Step 5. Write the equation in slope–intercept form.		$y=-\frac{1}{2}x$

TRY IT : : 3.61

Find an equation of a line perpendicular to the line $y=3x+1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

TRY IT : : 3.62

Find an equation of a line perpendicular to the line $y=\frac{1}{2}x-3$ that contains the point $(6, 4)$. Write the equation in slope-intercept form.

HOW TO : : FIND AN EQUATION OF A LINE PERPENDICULAR TO A GIVEN LINE.

Step 1. Find the slope of the given line.
Step 2. Find the slope of the perpendicular line.
Step 3. Identify the point.
Step 4. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
Step 5. Write the equation in slope-intercept form.

EXAMPLE 3.32

Find an equation of a line perpendicular to $x=5$ that contains the point $(3, -2)$. Write the equation in slope-intercept form.

Solution

Again, since we know one point, the point-slope option seems more promising than the slope-intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x=5$. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_{\perp}=0$.

Identify the point.	$(3, -2)$
Identify the slope of the perpendicular line.	$m_{\perp} = 0$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$
	$y - (-2) = 0(x - 3)$
Simplify.	$y + 2 = 0$
	$y = -2$

Sketch the graph of both lines. On your graph, do the lines appear to be perpendicular?

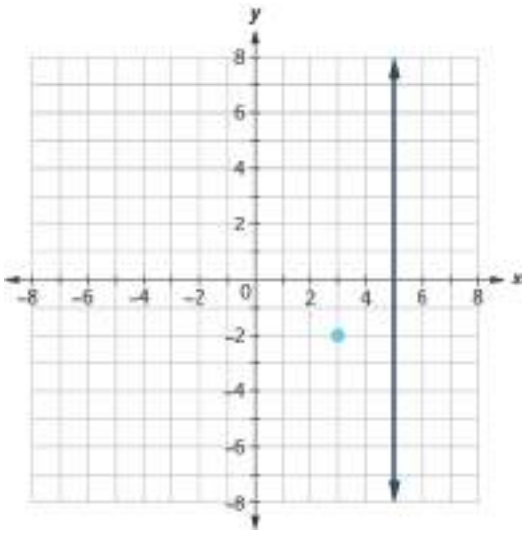
TRY IT : : 3.63

Find an equation of a line that is perpendicular to the line $x = 4$ that contains the point $(4, -5)$. . Write the equation in slope-intercept form.

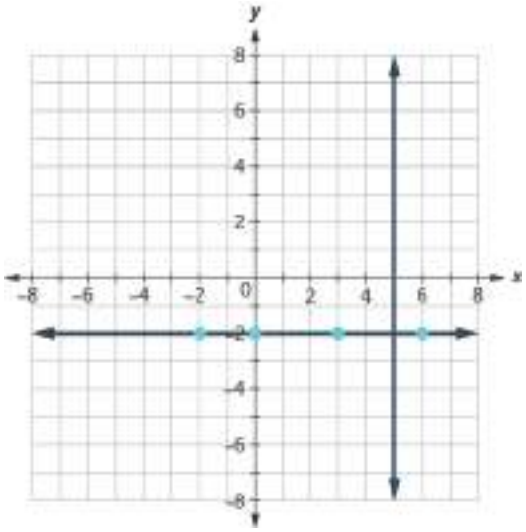
TRY IT : : 3.64

Find an equation of a line that is perpendicular to the line $x = 2$ that contains the point $(2, -1)$. Write the equation in slope-intercept form.

In **Example 3.32**, we used the point-slope form to find the equation. We could have looked at this in a different way. We want to find a line that is perpendicular to $x = 5$ that contains the point $(3, -2)$. This graph shows us the line $x = 5$ and the point $(3, -2)$.



We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3, -2)$.



Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have *y*-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x = 5$ is $y = -2$.

EXAMPLE 3.33

Find an equation of a line that is perpendicular to $y = -3$ that contains the point $(-3, 5)$. Write the equation in slope-intercept form.

Solution

The line $y = -3$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x = a$. Since the perpendicular line is vertical and passes through $(-3, 5)$, every point on it has an *x*-coordinate of -3. The equation of the perpendicular line is $x = -3$

You may want to sketch the lines. Do they appear perpendicular?

TRY IT : : 3.65

Find an equation of a line that is perpendicular to the line $y = 1$ that contains the point $(-5, 1)$. Write the equation in slope-intercept form.

TRY IT : : 3.66

Find an equation of a line that is perpendicular to the line $y = -5$ that contains the point $(-4, -5)$. Write the equation in slope-intercept form.

MEDIA : :

Access these online resources for additional instruction and practice with finding the equation of a line.

- **Write an Equation of Line Given its slope and Y-Intercept (https://openstax.org/l/37slopeycept)**
- **Using Point Slope Form to Write the Equation of a Line, Find the equation given slope and point (https://openstax.org/l/37slopepoint)**
- **Find the equation given two points (https://openstax.org/l/37twoptspline)**
- **Find the equation of perpendicular and parallel lines (https://openstax.org/l/37perpenpara)**

3.3 EXERCISES

Practice Makes Perfect

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope-intercept form.

155. slope 3 and y-intercept $(0, 5)$

156. slope 8 and y-intercept $(0, -6)$

157. slope -3 and y-intercept $(0, -1)$

158. slope -1 and y-intercept $(0, 3)$

159. slope $\frac{1}{5}$ and y-intercept $(0, -5)$

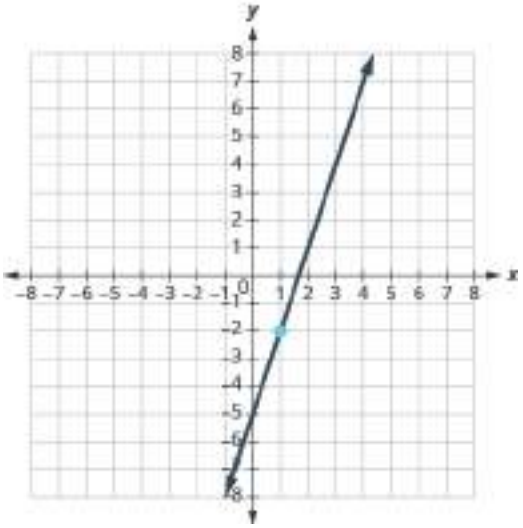
160. slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

161. slope 0 and y-intercept $(0, -1)$

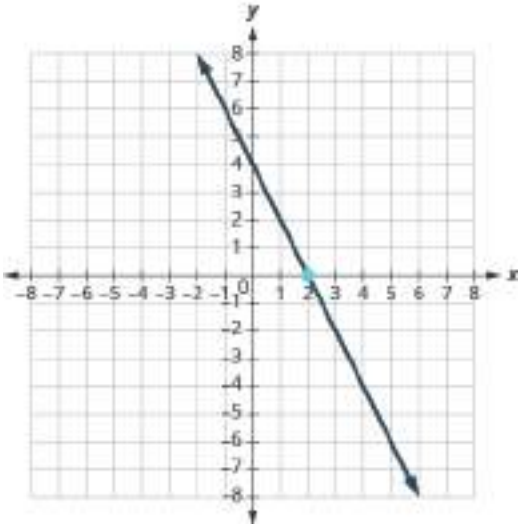
162. slope -4 and y-intercept $(0, 0)$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope-intercept form.

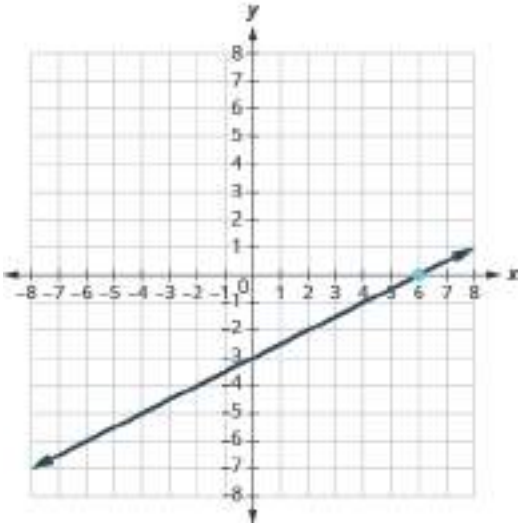
163.



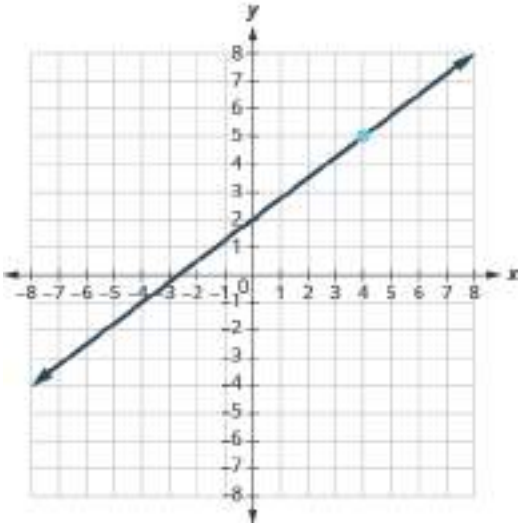
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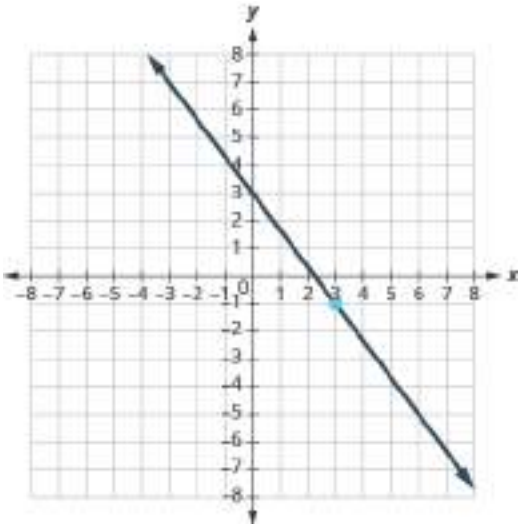
165.



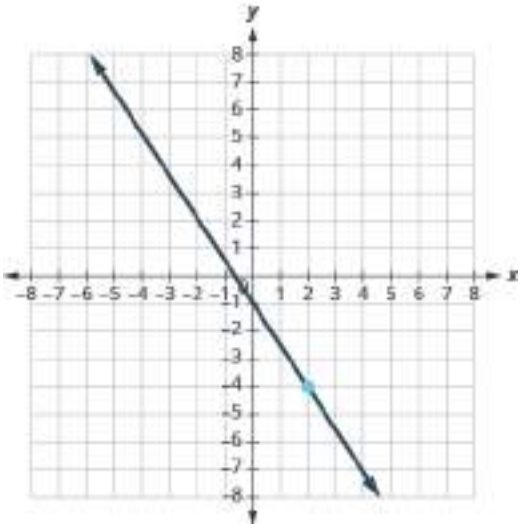
166.



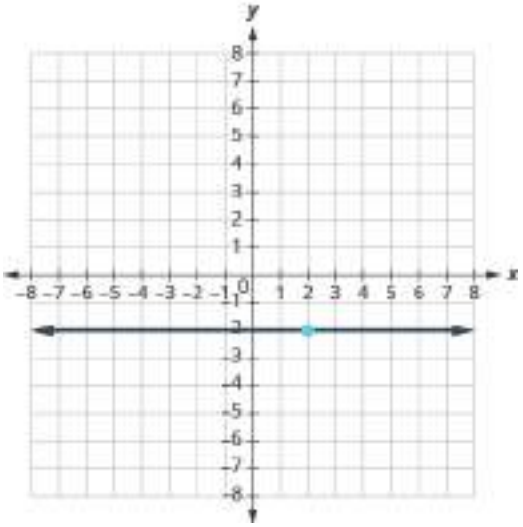
167.



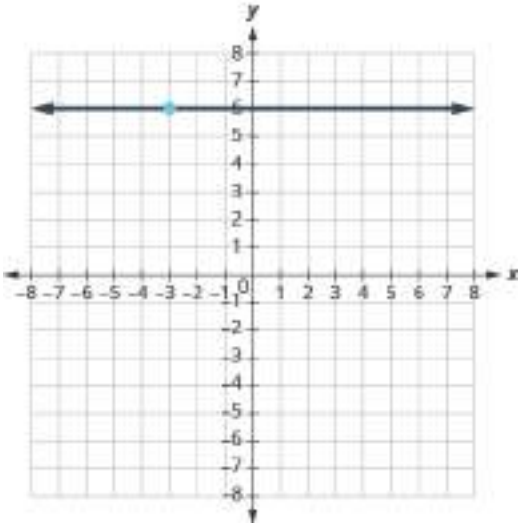
168.



169.



170.



Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope-intercept form.

171. $m = \frac{5}{8}$, point $(8, 3)$

172. $m = \frac{5}{6}$, point $(6, 7)$

173. $m = -\frac{3}{5}$, point $(10, -5)$

174. $m = -\frac{3}{4}$, point $(8, -5)$

175. $m = -\frac{3}{2}$, point $(-4, -3)$

176. $m = -\frac{5}{2}$, point $(-8, -2)$

177. $m = -7$, point $(-1, -3)$

178. $m = -4$, point $(-2, -3)$

179. Horizontal line containing $(-2, 5)$

180. Horizontal line containing $(-2, -3)$

181. Horizontal line containing $(-1, -7)$

182. Horizontal line containing $(4, -8)$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope-intercept form.

183. $(2, 6)$ and $(5, 3)$

184. $(4, 3)$ and $(8, 1)$

185. $(-3, -4)$ and $(5 - 2)$.

186. $(-5, -3)$ and $(4, -6)$.

187. $(-1, 3)$ and $(-6, -7)$.

188. $(-2, 8)$ and $(-4, -6)$.

189. $(0, 4)$ and $(2, -3)$.

190. $(0, -2)$ and $(-5, -3)$.

191. $(7, 2)$ and $(7, -2)$.

192. $(-2, 1)$ and $(-2, -4)$.

193. $(3, -4)$ and $(5, -4)$.

194. $(-6, -3)$ and $(-1, -3)$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope-intercept form.

195. line $y = 4x + 2$,
point $(1, 2)$

196. line $y = -3x - 1$,
point $(2, -3)$.

197. line $2x - y = 6$,
point $(3, 0)$.

198. line $2x + 3y = 6$,
point $(0, 5)$.

199. line $x = -4$,
point $(-3, -5)$.

200. line $x - 2 = 0$,
point $(1, -2)$

201. line $y = 5$,
point $(2, -2)$

202. line $y + 2 = 0$,
point $(3, -3)$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope-intercept form.

203. line $y = -2x + 3$,
point $(2, 2)$

204. line $y = -x + 5$,
point $(3, 3)$

205. line $y = \frac{3}{4}x - 2$,
point $(-3, 4)$

206. line $y = \frac{2}{3}x - 4$,
point $(2, -4)$

207. line $2x - 3y = 8$,
point $(4, -1)$

208. line $4x - 3y = 5$,
point $(-3, 2)$

209. line $2x + 5y = 6$,
point $(0, 0)$

210. line $4x + 5y = -3$,
point $(0, 0)$

211. line $x = 3$,
point $(3, 4)$

212. line $x = -5$,
point $(1, -2)$

213. line $x = 7$,
point $(-3, -4)$

214. line $x = -1$,
point $(-4, 0)$

215. line $y - 3 = 0$,
point $(-2, -4)$

216. line $y - 6 = 0$,
point $(-5, -3)$

217. line *y*-axis,
point $(3, 4)$

218. line *y*-axis,
point $(2, 1)$

Mixed Practice

In the following exercises, find the equation of each line. Write the equation in slope-intercept form.

219. Containing the points $(4, 3)$ and $(8, 1)$

220. Containing the points $(-2, 0)$ and $(-3, -2)$

221. $m = \frac{1}{6}$, containing point $(6, 1)$

222. $m = \frac{5}{6}$, containing point $(6, 7)$

223. Parallel to the line $4x + 3y = 6$, containing point $(0, -3)$

224. Parallel to the line $2x + 3y = 6$, containing point $(0, 5)$

225. $m = -\frac{3}{4}$, containing point $(8, -5)$

226. $m = -\frac{3}{5}$, containing point $(10, -5)$

227. Perpendicular to the line $y - 1 = 0$, point $(-2, 6)$

228. Perpendicular to the line *y*-axis, point $(-6, 2)$

229. Parallel to the line $x = -3$, containing point $(-2, -1)$

230. Parallel to the line $x = -4$, containing point $(-3, -5)$

231. Containing the points $(-3, -4)$ and $(2, -5)$

232. Containing the points $(-5, -3)$ and $(4, -6)$

233. Perpendicular to the line $x - 2y = 5$, point $(-2, 2)$

234. Perpendicular to the line $4x + 3y = 1$, point $(0, 0)$

Writing Exercises

235. Why are all horizontal lines parallel?

236. Explain in your own words why the slopes of two perpendicular lines must have opposite signs.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find the equation of the line given the slope and *y*-intercept.			
find an equation of the line given the slope and a point.			
find an equation of the line given two points.			
find an equation of a line parallel to a given line.			
find an equation of a line perpendicular to a given line.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

3.4 Graph Linear Inequalities in Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Verify solutions to an inequality in two variables.
- Recognize the relation between the solutions of an inequality and its graph.
- Graph linear inequalities in two variables
- Solve applications using linear inequalities in two variables

Be Prepared!

Before you get started, take this readiness quiz.

1. Graph $x > 2$ on a number line.
 If you missed this problem, review **Example 2.48**.
2. Solve: $4x + 3 > 23$.
 If you missed this problem, review **Example 2.52**.
3. Translate: $8 < x > 3$.
 If you missed this problem, review **Example 2.56**.

Verify Solutions to an Inequality in Two Variables

Previously we learned to solve inequalities with only one variable. We will now learn about inequalities containing two variables. In particular we will look at **linear inequalities** in two variables which are very similar to linear equations in two variables.

Linear inequalities in two variables have many applications. If you ran a business, for example, you would want your revenue to be greater than your costs—so that your business made a profit.

Linear Inequality

A **linear inequality** is an inequality that can be written in one of the following forms:

$$Ax + By > C \quad Ax + By \geq C \quad Ax + By < C \quad Ax + By \leq C$$

Where *A* and *B* are not both zero.

Recall that an inequality with one variable had many solutions. For example, the solution to the inequality $x > 3$ is any number greater than 3. We showed this on the number line by shading in the number line to the right of 3, and putting an open parenthesis at 3. See **Figure 3.10**.

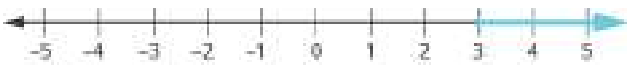


Figure 3.10

Similarly, linear inequalities in two variables have many solutions. Any ordered pair (x, y) that makes an inequality true when we substitute in the values is a **solution to a linear inequality**.

Solution to a Linear Inequality

An ordered pair (x, y) is a **solution to a linear inequality** if the inequality is true when we substitute the values of *x* and *y*.

EXAMPLE 3.34

Determine whether each ordered pair is a solution to the inequality $y > x + 4$:

ⓐ $(0, 0)$ ⓑ $(1, 6)$ ⓒ $(2, 6)$ ⓓ $(-5, -15)$ ⓔ $(-8, 12)$

Solution

ⓐ

$(0, 0)$	$y > x + 4$
Substitute 0 for x and 0 for y.	$0 \stackrel{?}{>} 0 + 4$
Simplify.	$0 \not> 4$
	So, $(0, 0)$ is not a solution to $y > x + 4$.

ⓑ

$(1, 6)$	$y > x + 4$
Substitute 1 for x and 6 for y.	$6 \stackrel{?}{>} 1 + 4$
Simplify.	$6 > 5$
	So, $(1, 6)$ is a solution to $y > x + 4$.

ⓒ

$(2, 6)$	$y > x + 4$
Substitute 2 for x and 6 for y.	$6 \stackrel{?}{>} 2 + 4$
Simplify.	$6 \not> 6$
	So, $(2, 6)$ is not a solution to $y > x + 4$.

ⓓ

$(-5, -15)$	$y > x + 4$
Substitute -5 for x and -15 for y.	$-15 \stackrel{?}{>} -5 + 4$
Simplify.	$-15 \not> -1$
	So, $(-5, -15)$ is not a solution to $y > x + 4$.

ⓔ

$(-8,\ 12)$	$y > x + 4$
Substitute -8 for x and 12 for y.	$12 \stackrel{?}{>} -8 + 4$
Simplify.	$12 > -4$
	So, $(-8,\ 12)$ is a solution to $y > x + 4$.

TRY IT : : 3.67 Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

ⓐ $(0,\ 0)$ ⓑ $(4,\ 9)$ ⓒ $(-2,\ 1)$ ⓓ $(-5,\ -3)$ ⓔ $(5,\ 1)$

TRY IT : : 3.68 Determine whether each ordered pair is a solution to the inequality $y < x + 1$:

ⓐ $(0,\ 0)$ ⓑ $(8,\ 6)$ ⓒ $(-2,\ -1)$ ⓓ $(3,\ 4)$ ⓔ $(-1,\ -4)$

Recognize the Relation Between the Solutions of an Inequality and its Graph

Now, we will look at how the solutions of an inequality relate to its graph.

Let's think about the number line in shown previously again. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3. On the other side of 3 all the numbers are greater than 3. See **Figure 3.11**.

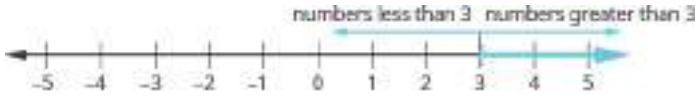


Figure 3.11 The solution to $x > 3$ is the shaded part of the number line to the right of $x = 3$.

Similarly, the line $y = x + 4$ separates the plane into two regions. On one side of the line are points with $y < x + 4$. On the other side of the line are the points with $y > x + 4$. We call the line $y = x + 4$ a **boundary line**.

Boundary Line

The line with equation $Ax + By = C$ is the **boundary line** that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

For an inequality in one variable, the endpoint is shown with a parenthesis or a bracket depending on whether or not a is included in the solution:



Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to show whether or not it the line is included in the solution.

$Ax + By < C$	$Ax + By \leq C$
$Ax + By > C$	$Ax + By \geq C$
Boundary line is $Ax + By = C$	Boundary line is $Ax + By = C$
Boundary line is not included in solution.	Boundary line is included in solution.
Boundary line is dashed.	**Boundary line is solid.**

Now, let's take a look at what we found in **Example 3.34**. We'll start by graphing the line $y = x + 4$, and then we'll plot the five points we tested, as shown in the graph. See **Figure 3.12**.

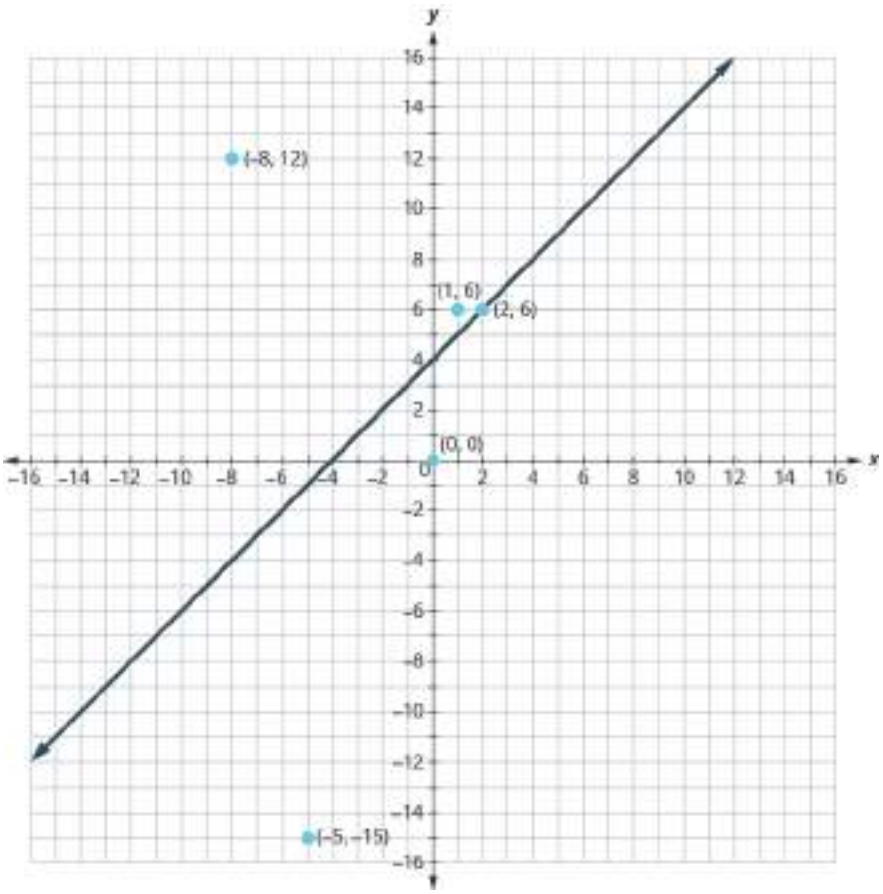


Figure 3.12

In **Example 3.34** we found that some of the points were solutions to the inequality $y > x + 4$ and some were not.

Which of the points we plotted are solutions to the inequality $y > x + 4$?

The points $(1, 6)$ and $(-8, 12)$ are solutions to the inequality $y > x + 4$. Notice that they are both on the same side of the boundary line $y = x + 4$.

The two points $(0, 0)$ and $(-5, -15)$ are on the other side of the boundary line $y = x + 4$, and they are not solutions to the inequality $y > x + 4$. For those two points, $y < x + 4$.

What about the point $(2, 6)$? Because $6 = 2 + 4$, the point is a solution to the equation $y = x + 4$, but not a solution to the inequality $y > x + 4$. So the point $(2, 6)$ is on the boundary line.

Let's take another point above the boundary line and test whether or not it is a solution to the inequality $y > x + 4$. The point $(0, 10)$ clearly looks to above the boundary line, doesn't it? Is it a solution to the inequality?

$$\begin{aligned} y &> x + 4 \\ 10 &\stackrel{?}{>} 0 + 4 \\ 10 &> 4 \end{aligned}$$

So, $(0, 10)$ is a solution to $y > x + 4$.

Any point you choose above the boundary line is a solution to the inequality $y > x + 4$. All points above the boundary line are solutions.

Similarly, all points below the boundary line, the side with $(0, 0)$ and $(-5, -15)$, are not solutions to $y > x + 4$, as shown in **Figure 3.13**.

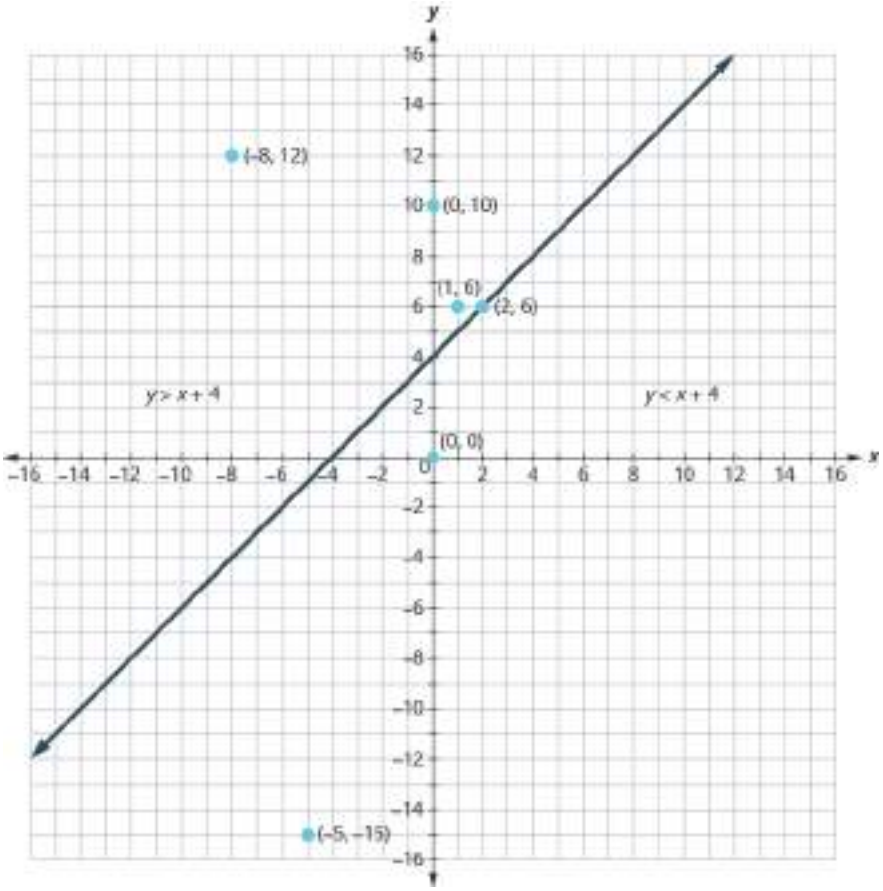
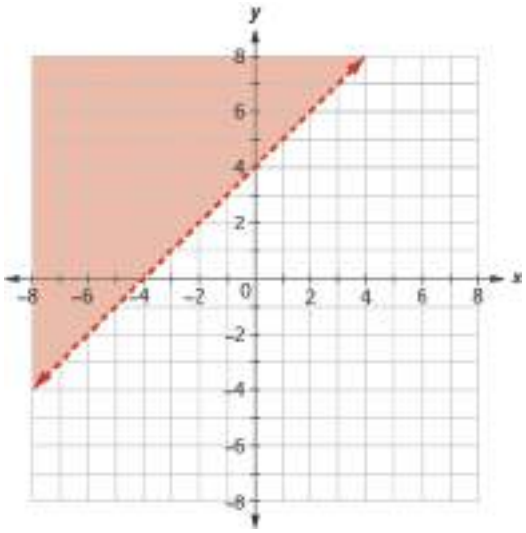


Figure 3.13

The graph of the inequality $y > x + 4$ is shown in below.

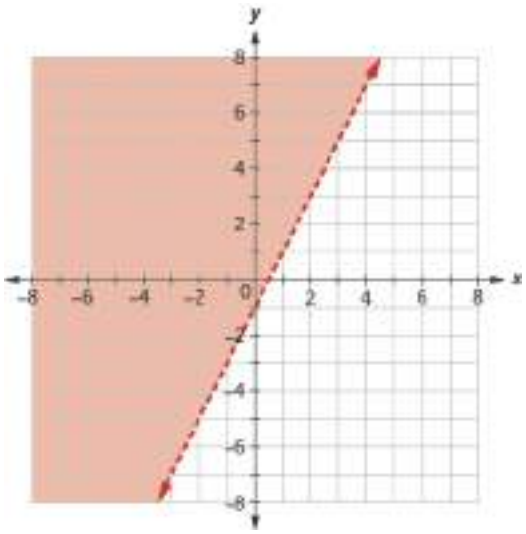
The line $y = x + 4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y > x + 4$.

The points on the boundary line, those where $y = x + 4,$ are not solutions to the inequality $y > x + 4,$ so the line itself is not part of the solution. We show that by making the line dashed, not solid.



EXAMPLE 3.35

The boundary line shown in this graph is $y = 2x - 1$. Write the inequality shown by the graph.



Solution

The line $y = 2x - 1$ is the boundary line. On one side of the line are the points with $y > 2x - 1$ and on the other side of the line are the points with $y < 2x - 1$.

Let's test the point $(0, 0)$ and see which inequality describes its position relative to the boundary line.

At $(0, 0)$, which inequality is true: $y > 2x - 1$ or $y < 2x - 1$?

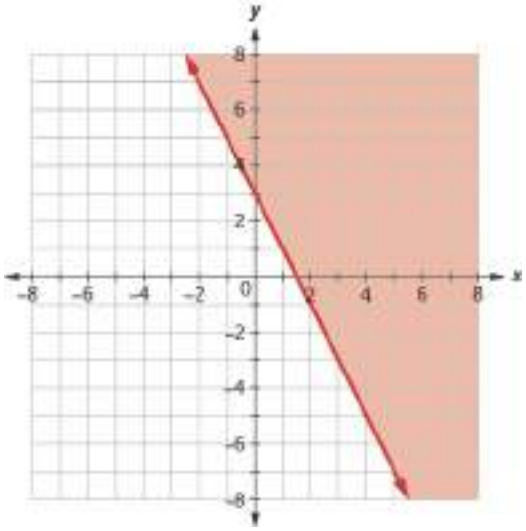
$y > 2x - 1$	$y < 2x - 1$
$0 \stackrel{?}{>} 2 \cdot 0 - 1$	$0 \stackrel{?}{<} 2 \cdot 0 - 1$
$0 > -1$ True	$0 < -1$ False

Since, $y > 2x - 1$ is true, the side of the line with $(0, 0)$, is the solution. The shaded region shows the solution of the inequality $y > 2x - 1$.

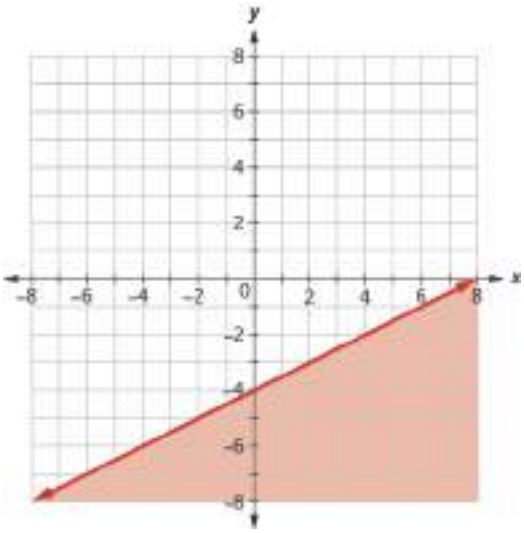
Since the boundary line is graphed with a solid line, the inequality includes the equal sign.
The graph shows the inequality $y \geq 2x - 1$.

We could use any point as a test point, provided it is not on the line. Why did we choose $(0, 0)$? Because it's the easiest to evaluate. You may want to pick a point on the other side of the boundary line and check that $y < 2x - 1$.

TRY IT : : 3.69 Write the inequality shown by the graph with the boundary line $y = -2x + 3$.

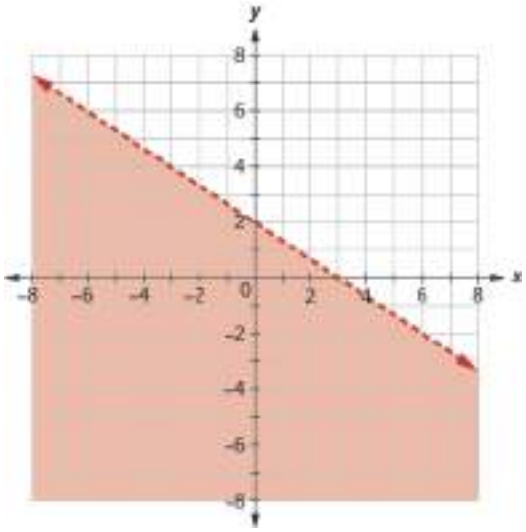


TRY IT : : 3.70 Write the inequality shown by the graph with the boundary line $y = \frac{1}{2}x - 4$.



EXAMPLE 3.36

The boundary line shown in this graph is $2x + 3y = 6$. Write the inequality shown by the graph.



Solution

The line $2x + 3y = 6$ is the boundary line. On one side of the line are the points with $2x + 3y > 6$ and on the other side of the line are the points with $2x + 3y < 6$.

Let's test the point $(0, 0)$ and see which inequality describes its side of the boundary line.

At $(0, 0)$, which inequality is true: $2x + 3y > 6$ or $2x + 3y < 6$?

$$\begin{array}{rcl} 2x+3y & > & 6 \\ 2(0)+3(0) & \overset{?}{>} & 6 \\ 0 & > & 6 \text{ False} \end{array} \qquad \begin{array}{rcl} 2x+3y & < & 6 \\ 2(0)+3(0) & \overset{?}{<} & 6 \\ 0 & < & 6 \text{ True} \end{array}$$

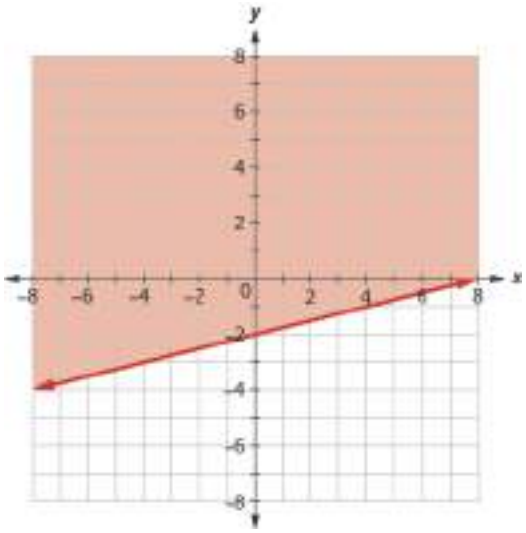
So the side with $(0, 0)$ is the side where $2x + 3y < 6$.

(You may want to pick a point on the other side of the boundary line and check that $2x + 3y > 6$.)

Since the boundary line is graphed as a dashed line, the inequality does not include an equal sign.
The shaded region shows the solution to the inequality $2x + 3y < 6$.

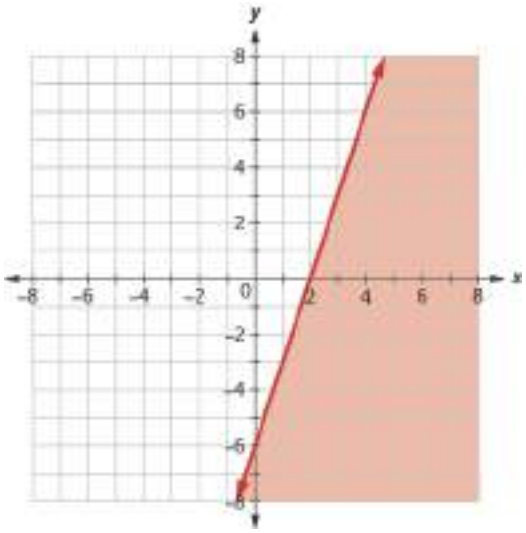
TRY IT : : 3.71

Write the inequality shown by the shaded region in the graph with the boundary line $x - 4y = 8$.



TRY IT : : 3.72

Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.



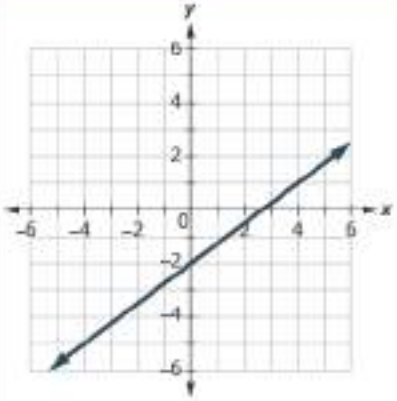
Graph Linear Inequalities in Two Variables

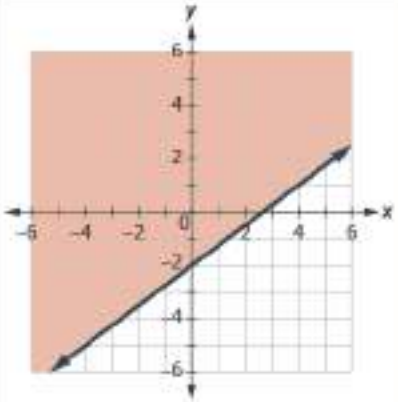
Now that we know what the graph of a linear inequality looks like and how it relates to a boundary equation we can use this knowledge to graph a given linear inequality.

EXAMPLE 3.37 HOW TO GRAPH A LINEAR EQUATION IN TWO VARIABLES

Graph the linear inequality $y \geq \frac{3}{4}x - 2$.

Solution

Step 1. Identify and graph the boundary line. • If the inequality is $\leq$ or $\geq$, the boundary line is solid. • If the inequality is $<$ or $>$, the boundary line is dashed.	Replace the inequality sign with an equal sign to find the boundary line. Graph the boundary line $y = \frac{3}{4}x - 2$. The inequality sign is $\geq$, so we draw a solid line.	
Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?	We'll test (0, 0). Is it a solution of the inequality?	At (0, 0), is $y \geq \frac{3}{4}x - 2$? $0 \stackrel{?}{\geq} \frac{3}{4}(0) - 2$ $0 \geq -2$ So, (0, 0) is a solution.

Step 3. Shade in one side of the boundary line. • If the test point is a solution, shade in the side that includes the point. • If the test point is not a solution, shade in the opposite side.	The test point (0, 0) is a solution to $y \geq \frac{3}{4}x - 2$. So we shade in that side.	 All points in the shaded region and on the boundary line represent the solutions to $y \geq \frac{3}{4}x - 2$.

TRY IT : : 3.73 Graph the linear inequality $y > \frac{5}{2}x - 4$.

TRY IT : : 3.74 Graph the linear inequality $y < \frac{2}{3}x - 5$.

The steps we take to graph a linear inequality are summarized here.

HOW TO : : GRAPH A LINEAR INEQUALITY IN TWO VARIABLES.

Step 1. Identify and graph the boundary line.
- If the inequality is $\leq$ or $\geq$, the boundary line is solid.
- If the inequality is $<$ or $>$, the boundary line is dashed.

Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?

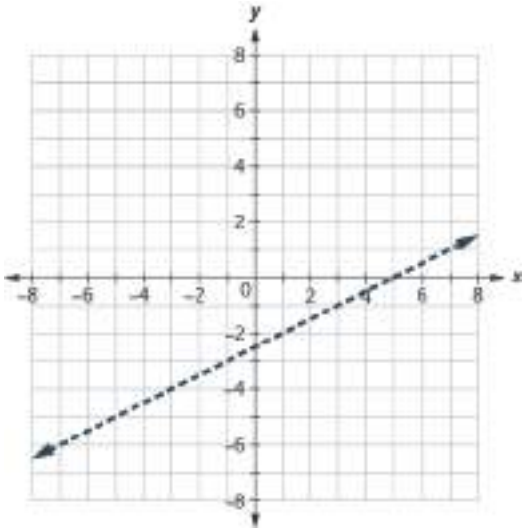
Step 3. Shade in one side of the boundary line.
- If the test point is a solution, shade in the side that includes the point.
- If the test point is not a solution, shade in the opposite side.

EXAMPLE 3.38

Graph the linear inequality $x - 2y < 5$.

Solution

First, we graph the boundary line $x - 2y = 5$. The inequality is $<$ so we draw a dashed line.

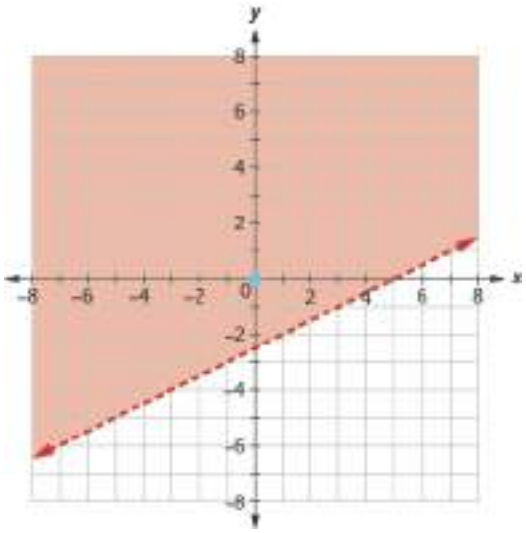


Then, we test a point. We'll use $(0, 0)$ again because it is easy to evaluate and it is not on the boundary line.

Is $(0, 0)$ a solution of $x - 2y < 5$?

$$0 - 2(0) \stackrel{?}{<} 5$$
$$0 - 0 \stackrel{?}{<} 5$$
$$0 < 5$$

The point $(0, 0)$ is a solution of $x - 2y < 5$, so we shade in that side of the boundary line.



All points in the shaded region, but not those on the boundary line, represent the solutions to $x - 2y < 5$.

TRY IT : : 3.75 Graph the linear inequality: $2x - 3y < 6$.

TRY IT : : 3.76 Graph the linear inequality: $2x - y > 3$.

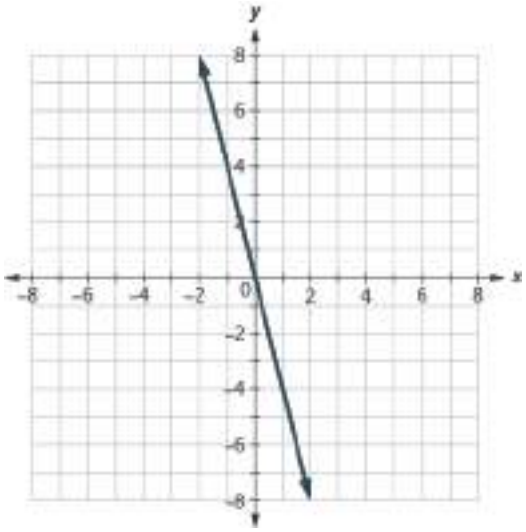
What if the boundary line goes through the origin? Then, we won't be able to use $(0, 0)$ as a test point. No problem—we'll just choose some other point that is not on the boundary line.

EXAMPLE 3.39

Graph the linear inequality: $y \leq -4x$.

Solution

First, we graph the boundary line $y = -4x$. It is in slope–intercept form, with $m = -4$ and $b = 0$. The inequality is $\leq$ so we draw a solid line.



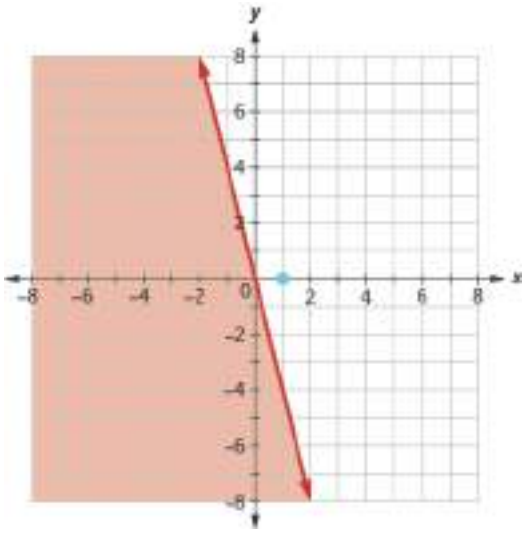
Now we need a test point. We can see that the point $(1, 0)$ is not on the boundary line.

Is $(1, 0)$ a solution of $y \leq -4x$?

$$0 \stackrel{?}{\leq} -4(1)$$

$$0 \not\leq -4$$

The point $(1, 0)$ is not a solution to $y \leq -4x$, so we shade in the opposite side of the boundary line.



All points in the shaded region and on the boundary line represent the solutions to $y \leq -4x$.

TRY IT : : 3.77 Graph the linear inequality: $y > -3x$.

TRY IT : : 3.78 Graph the linear inequality: $y \geq -2x$.

Some linear inequalities have only one variable. They may have an *x* but no *y*, or a *y* but no *x*. In these cases, the boundary

line will be either a vertical or a horizontal line.

Recall that:

$$\begin{array}{ll} x = a & \text{vertical line} \\ y = b & \text{horizontal line} \end{array}$$

EXAMPLE 3.40

Graph the linear inequality: $y > 3$.

Solution

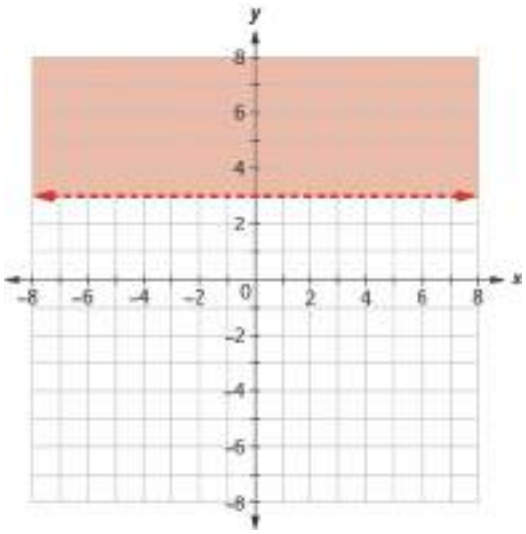
First, we graph the boundary line $y = 3$. It is a horizontal line. The inequality is $>$ so we draw a dashed line.

We test the point $(0, 0)$.

$$\begin{array}{l} y > 3 \\ 0 \not> 3 \end{array}$$

So, $(0, 0)$ is not a solution to $y > 3$.

So we shade the side that does not include $(0, 0)$ as shown in this graph.



All points in the shaded region, but not those on the boundary line, represent the solutions to $y > 3$.

TRY IT : : 3.79 Graph the linear inequality: $y < 5$.

TRY IT : : 3.80 Graph the linear inequality: $y \leq -1$.

Solve Applications using Linear Inequalities in Two Variables

Many fields use linear inequalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how thay might be used.

EXAMPLE 3.41

Hilaria works two part time jobs in order to earn enough money to meet her obligations of at least $240 a week. Her job in food service pays $10 an hour and her tutoring job on campus pays $15 an hour. How many hours does Hilaria need to work at each job to earn at least $240?

ⓐ Let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hilaria.

Solution

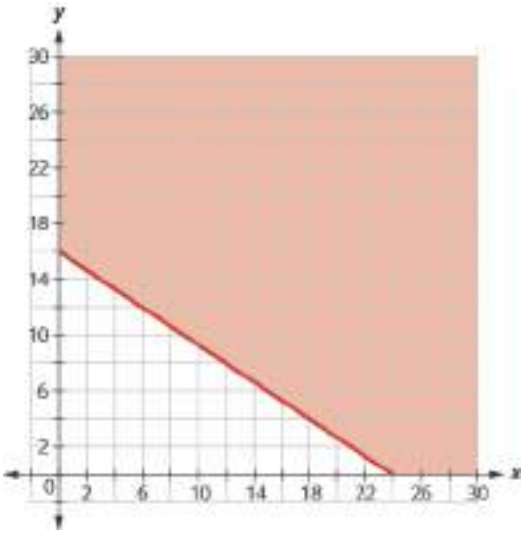
ⓐ We let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring.

She earns \$10 per hour at the job in food service and \$15 an hour tutoring. At each job, the number of hours multiplied by the hourly wage will gives the amount earned at that job.

Amount earned at the food service job	plus	the amount earned tutoring	is at least \$240
$10x$	+	$15y$	≥ 240

ⓑ To graph the inequality, we put it in slope–intercept form.

$$\begin{aligned} 10x + 15y &\geq 240 \\ 15y &\geq -10x + 240 \\ y &\geq -\frac{2}{3}x + 16 \end{aligned}$$



ⓒ From the graph, we see that the ordered pairs $(15, 10), (0, 16), (24, 0)$ represent three of infinitely many solutions. Check the values in the inequality.

(15, 10)	(0, 16)	(24, 0)
$10x + 15y \geq 240$	$10x + 15y \geq 240$	$10x + 15y \geq 240$
$10(15) + 15(10) \stackrel{?}{\geq} 240$	$10(0) + 15(16) \stackrel{?}{\geq} 240$	$10(24) + 15(0) \stackrel{?}{\geq} 240$
$300 \geq 240$ True	$240 \geq 240$ True	$240 \geq 240$ True

For Hilaria, it means that to earn at least \$240, she can work 15 hours tutoring and 10 hours at her fast-food job, earn all her money tutoring for 16 hours, or earn all her money while working 24 hours at the job in food service.

TRY IT : : 3.81

Hugh works two part time jobs. One at a grocery store that pays $10 an hour and the other is babysitting for $13 hour. Between the two jobs, Hugh wants to earn at least $260 a week. How many hours does Hugh need to work at each job to earn at least $260?

ⓐ Let x be the number of hours he works at the grocery store and let y be the number of hours he works babysitting. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hugh.

TRY IT : : 3.82

Veronica works two part time jobs in order to earn enough money to meet her obligations of at least $280 a week. Her job at the day spa pays $10 an hour and her administrative assistant job on campus pays $17.50 an hour. How many hours does Veronica need to work at each job to earn at least $280?

ⓐ Let x be the number of hours she works at the day spa and let y be the number of hours she works as administrative assistant. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Veronica

MEDIA : :

Access this online resource for additional instruction and practice with graphing linear inequalities in two variables.

- **Graphing Linear Inequalities in Two Variables (https://openstax.org/l/37lininеqgraphs)**

3.4 EXERCISES

Practice Makes Perfect

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.

237. Determine whether each ordered pair is a solution to the inequality $y > x - 1$:

ⓐ $(0, 1)$
ⓑ $(-4, -1)$
ⓒ $(4, 2)$
ⓓ $(3, 0)$
ⓔ $(-2, -3)$

238. Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

ⓐ $(0, 0)$
ⓑ $(2, 1)$
ⓒ $(-1, -5)$
ⓓ $(-6, -3)$
ⓔ $(1, 0)$

239. Determine whether each ordered pair is a solution to the inequality $y < 3x + 2$:

ⓐ $(0, 3)$
ⓑ $(-3, -2)$
ⓒ $(-2, 0)$
ⓓ $(0, 0)$
ⓔ $(-1, 4)$

240. Determine whether each ordered pair is a solution to the inequality $y < -2x + 5$:

ⓐ $(-3, 0)$
ⓑ $(1, 6)$
ⓒ $(-6, -2)$
ⓓ $(0, 1)$
ⓔ $(5, -4)$

241. Determine whether each ordered pair is a solution to the inequality $3x - 4y > 4$:

ⓐ $(5, 1)$
ⓑ $(-2, 6)$
ⓒ $(3, 2)$
ⓓ $(10, -5)$
ⓔ $(0, 0)$

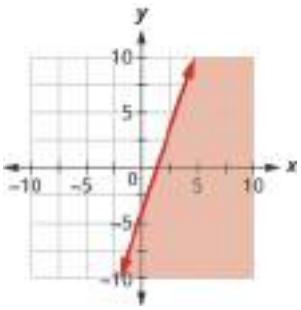
242. Determine whether each ordered pair is a solution to the inequality $2x + 3y > 2$:

ⓐ $(1, 1)$
ⓑ $(4, -3)$
ⓒ $(0, 0)$
ⓓ $(-8, 12)$
ⓔ $(3, 0)$

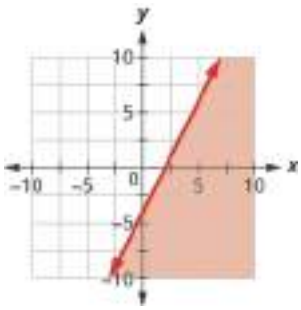
Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

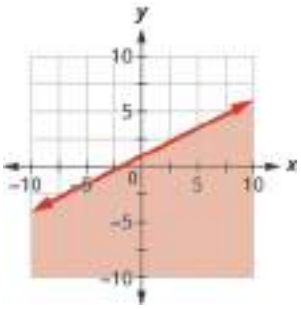
243. Write the inequality shown by the graph with the boundary line $y = 3x - 4$.



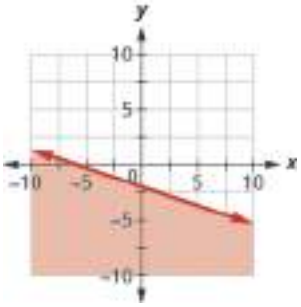
244. Write the inequality shown by the graph with the boundary line $y = 2x - 4$.



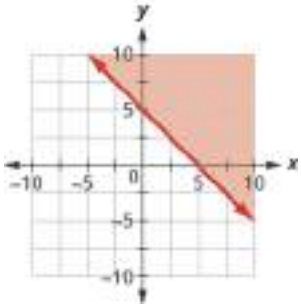
245. Write the inequality shown by the graph with the boundary line $y = -\frac{1}{2}x + 1$.



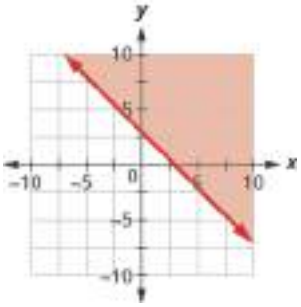
246. Write the inequality shown by the graph with the boundary line $y = -\frac{1}{3}x - 2$.



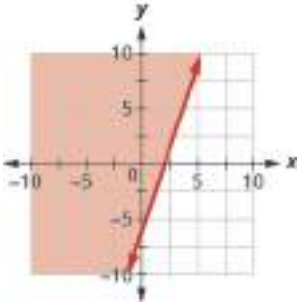
247. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 5$.



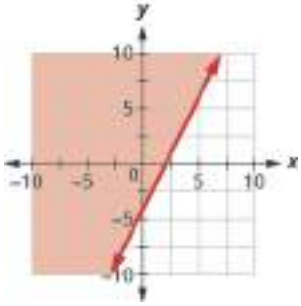
248. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 3$.



249. Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.



250. Write the inequality shown by the shaded region in the graph with the boundary line $2x - y = 4$.



Graph Linear Inequalities in Two Variables

In the following exercises, graph each linear inequality.

251. Graph the linear inequality: $y > \frac{2}{3}x - 1$.

252. Graph the linear inequality: $y < \frac{3}{5}x + 2$.

253. Graph the linear inequality: $y \leq -\frac{1}{2}x + 4$.

254. Graph the linear inequality: $y \geq -\frac{1}{3}x - 2$.

255. Graph the linear inequality: $x - y \leq 3$.

256. Graph the linear inequality: $x - y \geq -2$.

257. Graph the linear inequality: $4x + y > -4$.

258. Graph the linear inequality: $x + 5y < -5$.

259. Graph the linear inequality: $3x + 2y \geq -6$.

260. Graph the linear inequality: $4x + 2y \geq -8$.

261. Graph the linear inequality: $y > 4x$.

262. Graph the linear inequality: $y \leq -3x$.

263. Graph the linear inequality: $y < -10$.

264. Graph the linear inequality: $y \geq 2$.

265. Graph the linear inequality: $x \leq 5$.

266. Graph the linear inequality: $x \geq 0$.

267. Graph the linear inequality: $x - y < 4$.

268. Graph the linear inequality: $x - y < -3$.

269. Graph the linear inequality: $y \geq \frac{3}{2}x$.

270. Graph the linear inequality: $y \leq \frac{5}{4}x$.

271. Graph the linear inequality: $y > -2x + 1$.

272. Graph the linear inequality: $y < -3x - 4$.

273. Graph the linear inequality: $2x + y \geq -4$.

274. Graph the linear inequality: $x + 2y \leq -2$.

275. Graph the linear inequality: $2x - 5y > 10$.

276. Graph the linear inequality: $4x - 3y > 12$.

Solve Applications using Linear Inequalities in Two Variables

277. Harrison works two part time jobs. One at a gas station that pays $11 an hour and the other is IT troubleshooting for $16.50 an hour. Between the two jobs, Harrison wants to earn at least $330 a week. How many hours does Harrison need to work at each job to earn at least $330?

ⓐ Let *x* be the number of hours he works at the gas station and let *y* be the number of (hours he works troubleshooting. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Harrison.

278. Elena needs to earn at least $450 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays $9 an hour and the other as an intern in a genetics lab for $22.50 per hour. How many hours does Elena need to work at each job to earn at least $450 per week?

ⓐ Let *x* be the number of hours she works teaching swimming and let *y* be the number of hours she works as an intern. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Elena.

279. The doctor tells Laura she needs to exercise enough to burn 500 calories each day. She prefers to either run or bike and burns 15 calories per minute while running and 10 calories a minute while biking.

ⓐ If *x* is the number of minutes that Laura runs and *y* is the number minutes she bikes, find the inequality that models the situation.

ⓑ Graph the inequality.

ⓒ List three solutions to the inequality. What options do the solutions provide Laura?

280. Armando's workouts consist of kickboxing and swimming. While kickboxing, he burns 10 calories per minute and he burns 7 calories a minute while swimming. He wants to burn 600 calories each day.

ⓐ If *x* is the number of minutes that Armando will kickbox and *y* is the number minutes he will swim, find the inequality that will help Armando create a workout for today.

ⓑ Graph the inequality.

ⓒ List three solutions to the inequality. What options do the solutions provide Armando?

Writing Exercises

281. Lester thinks that the solution of any inequality with a $>$ sign is the region above the line and the solution of any inequality with a $<$ sign is the region below the line. Is Lester correct? Explain why or why not.

282. Explain why, in some graphs of linear inequalities, the boundary line is solid but in other graphs it is dashed.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
verify solutions to an inequality in two variables.			
recognize the relation between the solutions of an inequality and its graph.			
graph linear inequalities.			

ⓑ *On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

3.5 Relations and Functions

Learning Objectives

By the end of this section, you will be able to:

- Find the domain and range of a relation
- Determine if a relation is a function
- Find the value of a function

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $3x - 5$ when $x = -2$.
 If you missed this problem, review **Example 1.6**.
2. Evaluate $2x^2 - x - 3$ when $x = a$.
 If you missed this problem, review **Example 1.6**.
3. Simplify: $7x - 1 - 4x + 5$.
 If you missed this problem, review **Example 1.7**.

Find the Domain and Range of a Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number and our birthday are matched to our name. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a **relation**.

$$(\text{Student name, Student ID \#})$$

The set of all the names of the students in the class is called the **domain** of the relation and the set of all student ID numbers paired with these students is the range of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation.

Relation

A **relation** is any set of ordered pairs, (x, y). All the x-values in the ordered pairs together make up the **domain**. All the y-values in the ordered pairs together make up the **range**.

EXAMPLE 3.42

For the relation $\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

Solution

$$\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$$

ⓐ The domain is the set of all x-values of the relation. $\{1, 2, 3, 4, 5\}$

ⓑ The range is the set of all y-values of the relation. $\{1, 4, 9, 16, 25\}$

TRY IT : : 3.83 For the relation $\{(1, 1), (2, 8), (3, 27), (4, 64), (5, 125)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

TRY IT : : 3.84 For the relation $\{(1, 3), (2, 6), (3, 9), (4, 12), (5, 15)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

Mapping

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

EXAMPLE 3.43

Use the **mapping** of the relation shown to ⓐ list the ordered pairs of the relation, ⓑ find the domain of the relation, and ⓒ find the range of the relation.



Solution

ⓐ The arrow shows the matching of the person to their birthday. We create ordered pairs with the person's name as the *x*-value and their birthday as the *y*-value.

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}

ⓑ The domain is the set of all *x*-values of the relation.

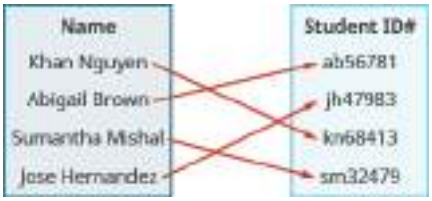
{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

ⓒ The range is the set of all *y*-values of the relation.

{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

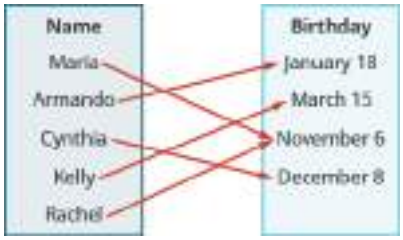
TRY IT : : 3.85

Use the mapping of the relation shown to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



TRY IT : : 3.86

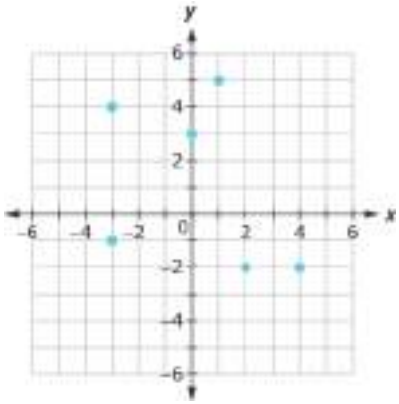
Use the mapping of the relation shown to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all *x*-coordinates is the domain of the relation and the set of all *y*-coordinates is the range. Generally we write the numbers in ascending order for both the domain and range.

EXAMPLE 3.44

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution

ⓐ The ordered pairs of the relation are: $\{(1, 5), (-3, -1), (4, -2), (0, 3), (2, -2), (-3, 4)\}$.

ⓑ The domain is the set of all *x*-values of the relation: $\{-3, 0, 1, 2, 4\}$.

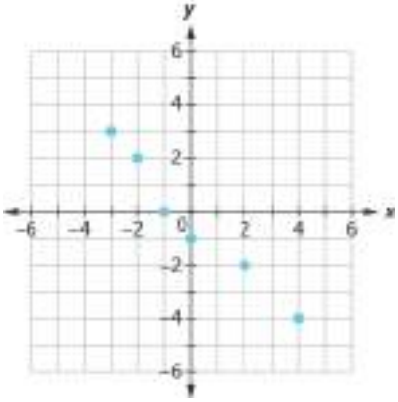
Notice that while -3 repeats, it is only listed once.

ⓒ The range is the set of all *y*-values of the relation: $\{-2, -1, 3, 4, 5\}$.

Notice that while -2 repeats, it is only listed once.

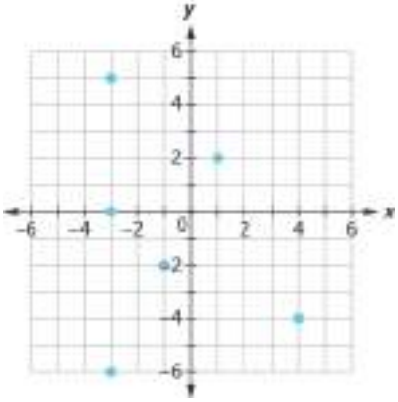
TRY IT : : 3.87

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



TRY IT : : 3.88

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Determine if a Relation is a Function

A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

Function

A **function** is a relation that assigns to each element in its domain exactly one element in the range.

The birthday example from Example 3.43 helps us understand this definition. Every person has a birthday but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24th as their birthday and that June and Liz share August 2nd. Since each person has exactly one birthday, the relation in Example 3.43 is a function.

The relation shown by the graph in Example 3.44 includes the ordered pairs $(-3, -1)$ and $(-3, 4)$. Is that okay in a function? No, as this is like one person having two different birthdays.

EXAMPLE 3.45

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

ⓑ $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

Solution

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

(i) Each x-value is matched with only one y-value. So this relation is a function.

(ii) The domain is the set of all x-values in the relation.
The domain is: $\{-3, -2, -1, 0, 1, 2, 3\}$.

(iii) The range is the set of all y-values in the relation. Notice we do not list range values twice.
The range is: $\{27, 8, 1, 0\}$.

ⓑ $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

(i) The x-value 9 is matched with two y-values, both 3 and -3. So this relation is not a function.

(ii) The domain is the set of all x-values in the relation. Notice we do not list domain values twice.
The domain is: $\{0, 1, 2, 4, 9\}$.

(iii) The range is the set of all y-values in the relation.
The range is: $\{-3, -2, -1, 0, 1, 2, 3\}$.

TRY IT : : 3.89

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the function.

ⓐ $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$

ⓑ $\{(8, -4), (4, -2), (2, -1), (0, 0), (2, 1), (4, 2), (8, 4)\}$

TRY IT : : 3.90

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

ⓐ $\{(27, -3), (8, -2), (1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$

ⓑ $\{(7, -3), (-5, -4), (8, -0), (0, 0), (-6, 4), (-2, 2), (-1, 3)\}$

EXAMPLE 3.46

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution

ⓐ Both Lydia and Marty have two phone numbers. So each x-value is not matched with only one y-value. So this relation is not a function.

ⓑ The domain is the set of all x-values in the relation. The domain is: {Lydia, Eugene, Janet, Rick, Marty}

ⓒ The range is the set of all *y*-values in the relation. The range is:

{321-549-3327, 427-658-2314, 321-964-7324, 684-358-7961, 684-369-7231, 798-367-8541}

TRY IT : : 3.91

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



TRY IT : : 3.92

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



In algebra, more often than not, functions will be represented by an equation. It is easiest to see if the equation is a function when it is solved for *y*. If each value of *x* results in only one value of *y*, then the equation defines a function.

EXAMPLE 3.47

Determine whether each equation is a function.

ⓐ $2x + y = 7$ ⓑ $y = x^2 + 1$ ⓒ $x + y^2 = 3$

Solution

ⓐ $2x + y = 7$

For each value of *x*, we multiply it by -2 and then add 7 to get the *y*-value

	$y = -2x + 7$
For example, if $x = 3$:	$y = -2 \cdot 3 + 7$
	$y = 1$

We have that when $x = 3,$ then $y = 1.$ It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓑ $y = x^2 + 1$

For each value of *x*, we square it and then add 1 to get the *y*-value.

	$y = x^2 + 1$
For example, if $x = 2$:	$y = 2^2 + 1$
	$y = 5$

We have that when $x = 2,$ then $y = 5.$ It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓒ

	$x + y^2 = 3$
Isolate the *y* term.	$y^2 = -x + 3$
Let's substitute $x = 2.$	$y^2 = -2 + 3$
	$y^2 = 1$
This give us two values for *y*.	$y = 1\ y = -1$

We have shown that when $x = 2,$ then $y = 1$ and $y = -1.$ It would work similarly for any value of *x*. Since each value of *x* does not corresponds to only one value of *y* the equation does not define a function.

TRY IT : : 3.93 Determine whether each equation is a function.

ⓐ $4x + y = -3$ ⓑ $x + y^2 = 1$ ⓒ $y - x^2 = 2$

TRY IT : : 3.94 Determine whether each equation is a function.

ⓐ $x + y^2 = 4$ ⓑ $y = x^2 - 7$ ⓒ $y = 5x - 4$

Find the Value of a Function

It is very convenient to name a function and most often we name it *f*, *g*, *h*, *F*, *G*, or *H*. In any function, for each *x*-value from the domain we get a corresponding *y*-value in the range. For the function *f*, we write this range value *y* as $f(x)$. This is called function notation and is read *f* of *x* or the value of *f* at *x*. In this case the parentheses does not indicate multiplication.

Function Notation

For the function $y = f(x)$

f is the name of the function
x is the domain value
$f(x)$ is the range value y corresponding to the value x

We read $f(x)$ as *f* of *x* or the value of *f* at *x*.

We call *x* the independent variable as it can be any value in the domain. We call *y* the dependent variable as its value depends on *x*.

Independent and Dependent Variables

For the function $y = f(x)$,

x is the independent variable as it can be any value in the domain

y the dependent variable as its value depends on x

Much as when you first encountered the variable *x*, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let's look at the equation $y = 4x - 5$. To find the value of *y* when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

	$y = 4x - 5$
Let $x = 2$.	$y = 4 \cdot 2 - 5$
	$y = 3$

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let $x = 2$.	$f(2) = 4 \cdot 2 - 5$
	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of *x* is called *evaluating the function.*

EXAMPLE 3.48

For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

ⓐ $f(3)$ ⓑ $f(-2)$ ⓒ $f(a)$

Solution

ⓐ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(3)$, substitute 3 for *x*.	$f(3) = 2(3)^2 + 3 \cdot 3 - 1$
Simplify.	$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$
	$f(3) = 18 + 9 - 1$
	$f(3) = 26$

ⓑ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(-2)$, substitute -2 for x.	$f(-2) = 2(-2)^2 + 3(-2) - 1$
Simplify.	$f(-2) = 2 \cdot 4 + (-6) - 1$
	$f(-2) = 8 + (-6) - 1$
	$f(-2) = 1$

ⓒ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(a)$, substitute a for x.	$f(a) = 2(a)^2 + 3 \cdot a - 1$
Simplify.	$f(a) = 2a^2 + 3a - 1$

TRY IT : : 3.95 For the function $f(x) = 3x^2 - 2x + 1$, evaluate the function.

ⓐ $f(3)$ ⓑ $f(-1)$ ⓒ $f(t)$

TRY IT : : 3.96 For the function $f(x) = 2x^2 + 4x - 3$, evaluate the function.

ⓐ $f(2)$ ⓑ $f(-3)$ ⓒ $f(h)$

In the last example, we found $f(x)$ for a constant value of x. In the next example, we are asked to find $g(x)$ with values of x that are variables. We still follow the same procedure and substitute the variables in for the x.

EXAMPLE 3.49

For the function $g(x) = 3x - 5$, evaluate the function.

ⓐ $g(h^2)$ ⓑ $g(x + 2)$ ⓒ $g(x) + g(2)$

Solution

ⓐ

	$g(x) = 3x - 5$
To evaluate $g(h^2)$, substitute h^2 for x.	$g(h^2) = 3h^2 - 5$
	$g(h^2) = 3h^2 - 5$

ⓑ

	$g(x) = 3x - 5$
To evaluate $g(x+2)$, substitute $x+2$ for x.	$g(x+2) = 3(x+2) - 5$
Simplify.	$g(x+2) = 3x + 6 - 5$
	$g(x+2) = 3x + 1$

ⓒ

	$g(x) = 3x - 5$
To evaluate $g(x) + g(2)$, first find $g(2)$.	$g(2) = 3 \cdot 2 - 5$
	$g(2) = 1$
Now find $g(x) + g(2)$	$g(x) + g(2) = \underbrace{3x - 5}_{g(x)} + \underbrace{1}_{g(2)}$
Simplify.	$g(x) + g(2) = 3x - 5 + 1$
	$g(x) + g(2) = 3x - 4$

Notice the difference between part ⓑ and ⓒ. We get $g(x+2) = 3x + 1$ and $g(x) + g(2) = 3x - 4$. So we see that $g(x+2) \neq g(x) + g(2)$.

TRY IT : : 3.97 For the function $g(x) = 4x - 7$, evaluate the function.

ⓐ $g(m^2)$ ⓑ $g(x-3)$ ⓒ $g(x) - g(3)$

TRY IT : : 3.98 For the function $h(x) = 2x + 1$, evaluate the function.

ⓐ $h(k^2)$ ⓑ $h(x+1)$ ⓒ $h(x) + h(1)$

Many everyday situations can be modeled using functions.

EXAMPLE 3.50

The number of unread emails in Sylvia's account is 75. This number grows by 10 unread emails a day. The function $N(t) = 75 + 10t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(5)$. Explain what this result means.

Solution

ⓐ The number of unread emails is a function of the number of days. The number of unread emails, *N*, depends on the number of days, *t*. Therefore, the variable *N*, is the dependent variable and the variable t is the independent variable.

ⓑ Find $N(5)$. Explain what this result means.

	$N(t) = 75 + 10t$
Substitute in $t = 5$.	$N(5) = 75 + 10 \cdot 5$
Simplify.	$N(5) = 75 + 50$
	$N(5) = 125$

Since 5 is the number of days, $N(5)$, is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

TRY IT : : 3.99

The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. The function $N(t) = 100 + 15t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(7)$. Explain what this result means.

TRY IT : : 3.100

The number of unread emails in Anthony's account is 110. This number grows by 25 unread emails a day. The function $N(t) = 110 + 25t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(14)$. Explain what this result means.

MEDIA : :

Access this online resource for additional instruction and practice with relations and functions.

- **Introduction to Functions (https://openstax.org/l/37introfunction)**

3.5 EXERCISES

Practice Makes Perfect

Find the Domain and Range of a Relation

In the following exercises, for each relation ⓐ find the domain of the relation ⓑ find the range of the relation.

283. $\{(1, 4), (2, 8), (3, 12), (4, 16), (5, 20)\}$

284. $\{(1, -2), (2, -4), (3, -6), (4, -8), (5, -10)\}$

285. $\{(1, 7), (5, 3), (7, 9), (-2, -3), (-2, 8)\}$

286. $\{(11, 3), (-2, -7), (4, -8), (4, 17), (-6, 9)\}$

In the following exercises, use the mapping of the relation to ⓐ list the ordered pairs of the relation, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

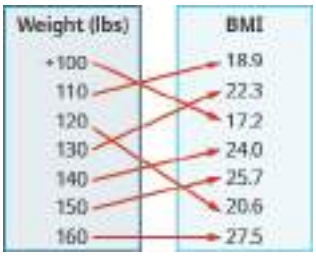
287.



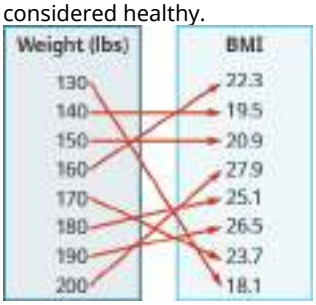
288.



289. For a woman of height $5'4''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5 - 24.9$ is considered healthy.

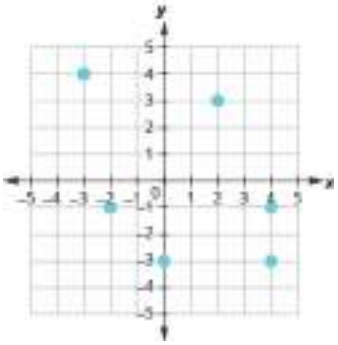


290. For a man of height $5'11''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5 - 24.9$ is considered healthy.

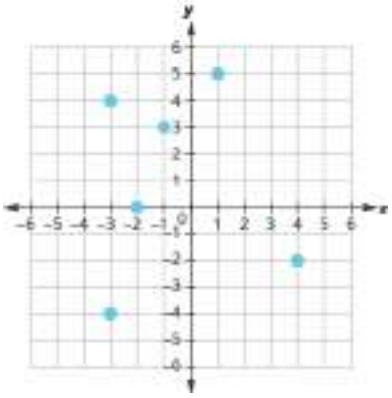


In the following exercises, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

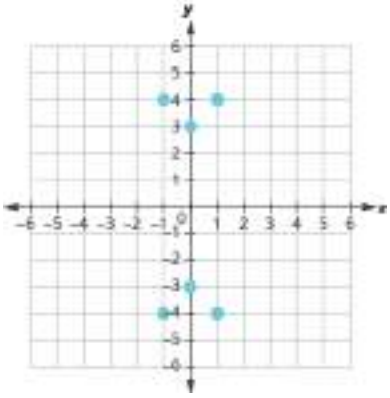
291.



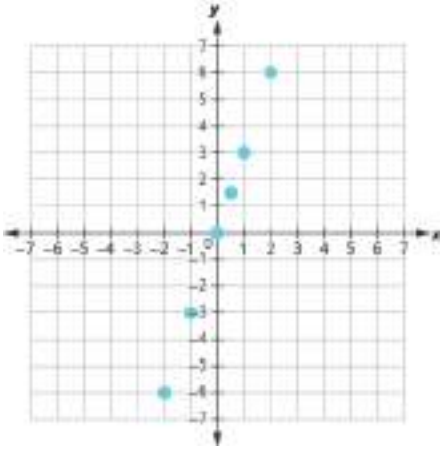
292.



293.



294.



Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

295. {(−3, 9), (−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4), (3, 9)}

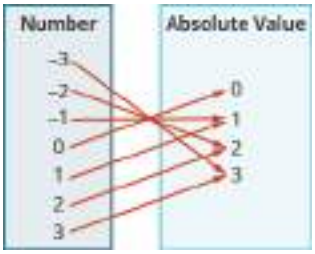
296. {(9, −3), (4, −2), (1, −1), (0, 0), (1, 1), (4, 2), (9, 3)}

297. {(−3, 27), (−2, 8), (−1, 1), (0, 0), (1, 1), (2, 8), (3, 27)}

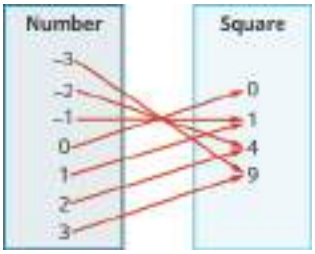
298. {(9, −3), (4, −2), (1, −1), (0, 0), (1, 1), (4, 2), (9, 3)}

In the following exercises, use the mapping to ⓐ determine whether the relation is a function, ⓑ find the domain of the function, and ⓒ find the range of the function.

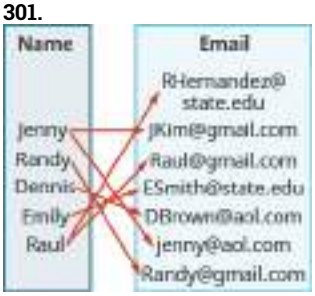
299.



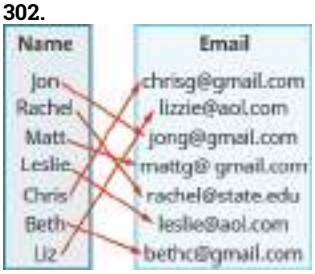
300.



301.



302.



In the following exercises, determine whether each equation is a function.

303.

ⓐ $2x + y = -3$

ⓑ $y = x^2$

ⓒ $x + y^2 = -5$

304.

ⓐ $y = 3x - 5$

ⓑ $y = x^3$

ⓒ $2x + y^2 = 4$

305.

ⓐ $y - 3x^3 = 2$

ⓑ $x + y^2 = 3$

ⓒ $3x - 2y = 6$

306.

ⓐ $2x - 4y = 8$

ⓑ $-4 = x^2 - y$

ⓒ $y^2 = -x + 5$

Find the Value of a Function

In the following exercises, evaluate the function: ⓐ $f(2)$ ⓑ $f(-1)$ ⓒ $f(a)$.

307. $f(x) = 5x - 3$

308. $f(x) = 3x + 4$

309. $f(x) = -4x + 2$

310. $f(x) = -6x - 3$

311. $f(x) = x^2 - x + 3$

312. $f(x) = x^2 + x - 2$

313. $f(x) = 2x^2 - x + 3$

314. $f(x) = 3x^2 + x - 2$

In the following exercises, evaluate the function: ⓐ $g(h^2)$ ⓑ $g(x + 2)$ ⓒ $g(x) + g(2)$.

315. $g(x) = 2x + 1$

316. $g(x) = 5x - 8$

317. $g(x) = -3x - 2$

318. $g(x) = -8x + 2$

319. $g(x) = 3 - x$

320. $g(x) = 7 - 5x$

In the following exercises, evaluate the function.

321. $f(x) = 3x^2 - 5x;\ \ f(2)$

322. $g(x) = 4x^2 - 3x;\ \ g(3)$

323. $F(x) = 2x^2 - 3x + 1;\ \ F(-1)$

324. $G(x) = 3x^2 - 5x + 2;\ \ G(-2)$

325. $h(t) = 2|t - 5| + 4;\ \ f(-4)$

326. $h(y) = 3|y - 1| - 3;\ \ h(-4)$

327. $f(x) = \frac{x+2}{x-1};\ \ f(2)$

328. $g(x) = \frac{x-2}{x+2};\ \ g(4)$

In the following exercises, solve.

329. The number of unwatched shows in Sylvia's DVR is 85. This number grows by 20 unwatched shows per week. The function $N(t) = 85 + 20t$ represents the relation between the number of unwatched shows, *N*, and the time, *t*, measured in weeks.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(4)$. Explain what this result means

330. Every day a new puzzle is downloaded into Ken's account. Right now he has 43 puzzles in his account. The function $N(t) = 43 + t$ represents the relation between the number of puzzles, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(30)$. Explain what this result means.

331. The daily cost to the printing company to print a book is modeled by the function $C(x) = 3.25x + 1500$ where *C* is the total daily cost and *x* is the number of books printed.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(0)$. Explain what this result means.

ⓒ Find $N(1000)$. Explain what this result means.

332. The daily cost to the manufacturing company is modeled by the function $C(x) = 7.25x + 2500$ where $C(x)$ is the total daily cost and *x* is the number of items manufactured.

ⓐ Determine the independent and dependent variable.

ⓑ Find $C(0)$. Explain what this result means.

ⓒ Find $C(1000)$. Explain what this result means.

Writing Exercises

333. In your own words, explain the difference between a relation and a function.

334. In your own words, explain what is meant by domain and range.

335. Is every relation a function? Is every function a relation?

336. How do you find the value of a function?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find the domain and range of a relation.			
determine if a relation is a function.			
find the value of a function.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

3.6 Graphs of Functions

Learning Objectives

By the end of this section, you will be able to:

- Use the vertical line test
- Identify graphs of basic functions
- Read information from a graph of a function

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate: ⓐ 2^3 ⓑ 3^2.
 If you missed this problem, review **Example 1.5**.
2. Evaluate: ⓐ $|7|$ ⓑ $|-3|$.
 If you missed this problem, review **Example 1.14**.
3. Evaluate: ⓐ $\sqrt{4}$ ⓑ $\sqrt{16}$.
 If you missed this problem, review **Example 1.41**.

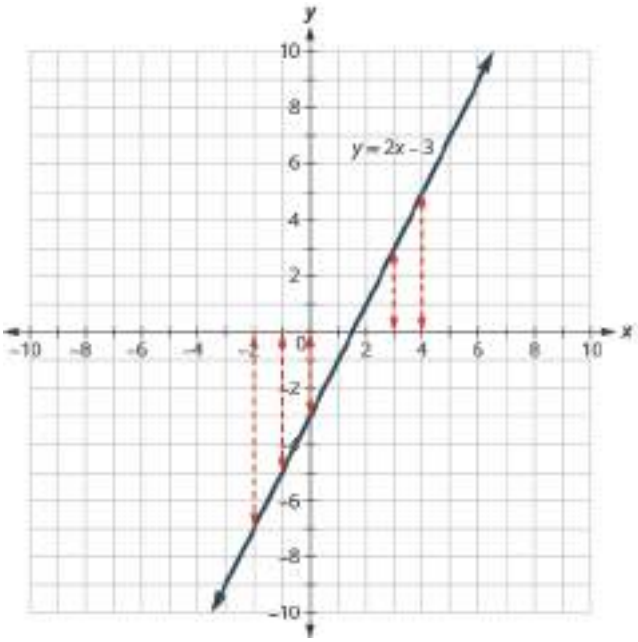
Use the Vertical Line Test

In the last section we learned how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping or an equation. We will now look at how to tell if a graph is that of a function.

An ordered pair (x, y) is a solution of a linear equation, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

The graph of a linear equation is a straight line where every point on the line is a solution of the equation and every solution of this equation is a point on this line.

In **Figure 3.14**, we can see that, in graph of the equation $y = 2x - 3,$ for every *x*-value there is only one *y*-value, as shown in the accompanying table.



y = 2x − 3		
x	y	(x, y)
−2	−7	(−2, −7)
−1	−5	(−1, −5)
0	−3	(0, −3)
3	3	(3, 3)
4	5	(4, 5)

Figure 3.14

A relation is a function if every element of the domain has exactly one value in the range. So the relation defined by the equation $y = 2x - 3$ is a function.

If we look at the graph, each vertical dashed line only intersects the line at one point. This makes sense as in a function, for every x-value there is only one y-value.

If the vertical line hit the graph twice, the x-value would be mapped to two y-values, and so the graph would not represent a function.

This leads us to the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

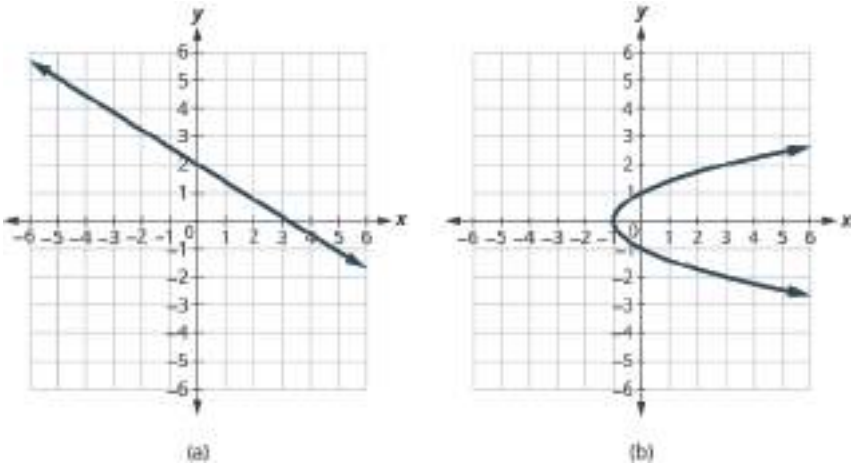
Vertical Line Test

A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.

If any vertical line intersects the graph in more than one point, the graph does not represent a function.

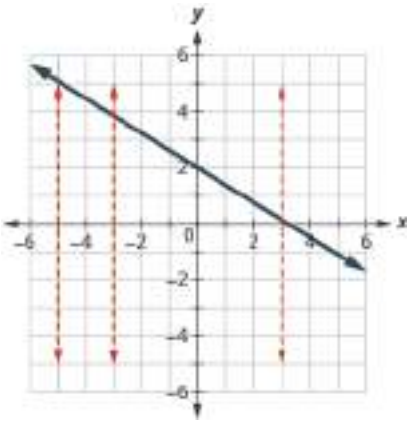
EXAMPLE 3.51

Determine whether each graph is the graph of a function.

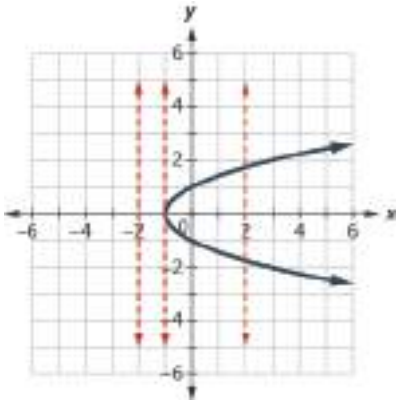


Solution

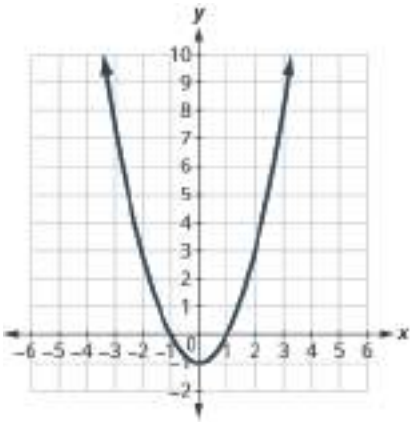
ⓐ Since any vertical line intersects the graph in at most one point, the graph is the graph of a function.



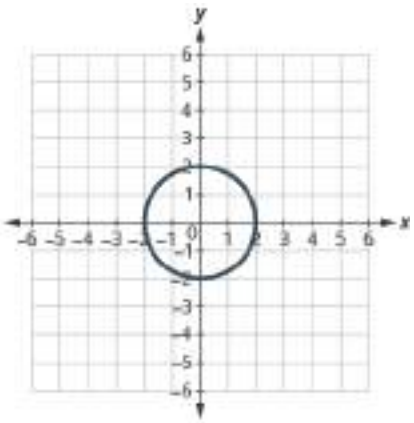
ⓑ One of the vertical lines shown on the graph, intersects it in two points. This graph does not represent a function.



TRY IT : : 3.101 Determine whether each graph is the graph of a function.

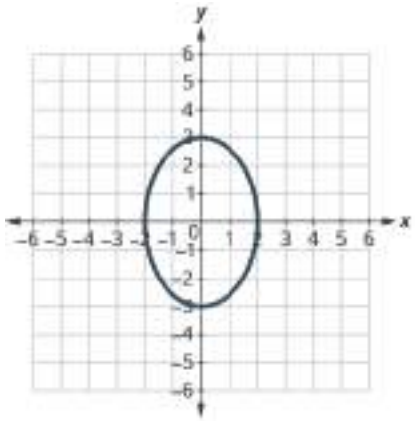


(a)

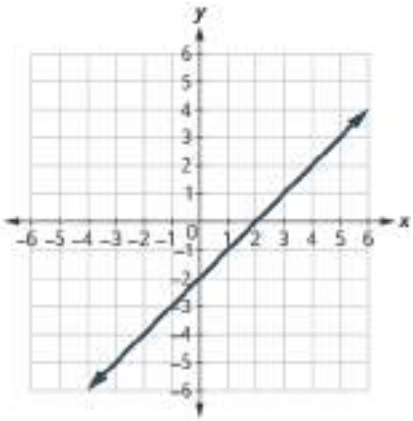


(b)

TRY IT : : 3.102 Determine whether each graph is the graph of a function.



(a)



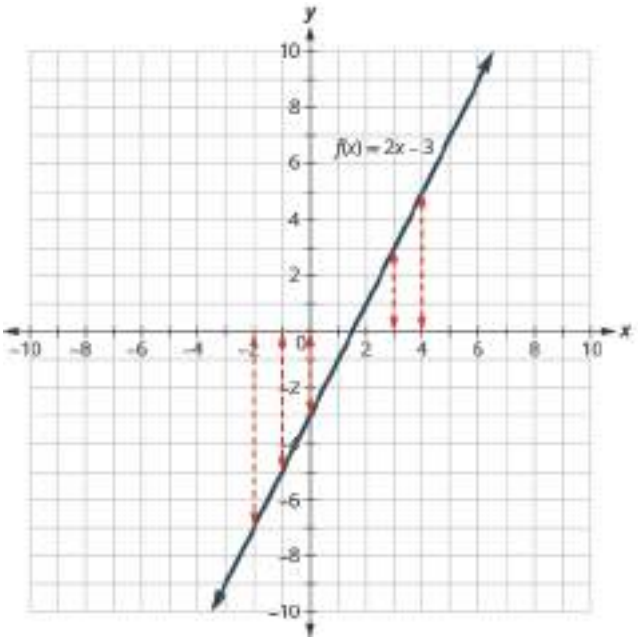
(b)

Identify Graphs of Basic Functions

We used the equation $y = 2x - 3$ and its graph as we developed the vertical line test. We said that the relation defined by the equation $y = 2x - 3$ is a function.

We can write this as in function notation as $f(x) = 2x - 3$. It still means the same thing. The graph of the function is the graph of all ordered pairs (x, y) where $y = f(x)$. So we can write the ordered pairs as $(x, f(x))$. It looks different but the graph will be the same.

Compare the graph of $y = 2x - 3$ previously shown in Figure 3.14 with the graph of $f(x) = 2x - 3$ shown in Figure 3.15. Nothing has changed but the notation.



f(x) = 2x – 3		
x	f(x)	(x, f(x))
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Figure 3.15

Graph of a Function

The graph of a function is the graph of all its ordered pairs, (x, y) or using function notation, $(x, f(x))$ where $y = f(x)$.

f	name of function
x	x-coordinate of the ordered pair
$f(x)$	y-coordinate of the ordered pair

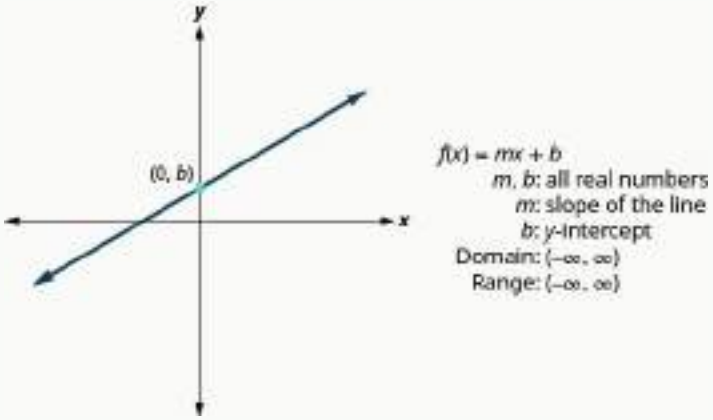
As we move forward in our study, it is helpful to be familiar with the graphs of several basic functions and be able to identify them.

Through our earlier work, we are familiar with the graphs of linear equations. The process we used to decide if $y = 2x - 3$ is a function would apply to all linear equations. All non-vertical linear equations are functions. Vertical lines are not functions as the *x*-value has infinitely many *y*-values.

We wrote linear equations in several forms, but it will be most helpful for us here to use the slope-intercept form of the linear equation. The slope-intercept form of a linear equation is $y = mx + b$. In function notation, this linear function becomes $f(x) = mx + b$ where *m* is the slope of the line and *b* is the *y*-intercept.

The domain is the set of all real numbers, and the range is also the set of all real numbers.

Linear Function



We will use the graphing techniques we used earlier, to graph the basic functions.

EXAMPLE 3.52

Graph: $f(x) = -2x - 4$.

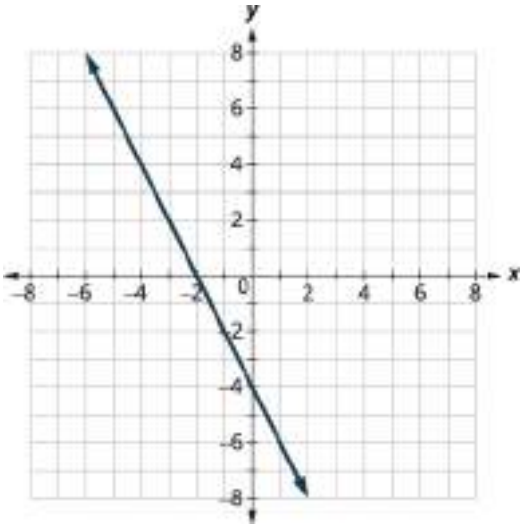
Solution

$$f(x) = -2x - 4$$

We recognize this as a linear function.

Find the slope and *y*-intercept. $m = -2$, $b = -4$

Graph using the slope intercept.



TRY IT : : 3.103 Graph: $f(x) = -3x - 1$

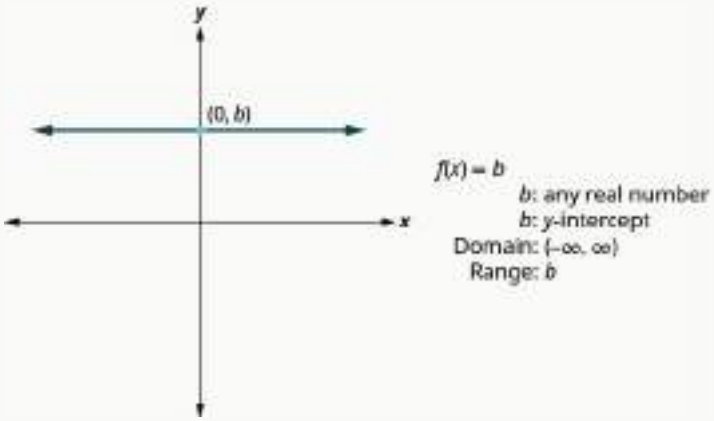
TRY IT : : 3.104 Graph: $f(x) = -4x - 5$

The next function whose graph we will look at is called the constant function and its equation is of the form $f(x) = b$,

where *b* is any real number. If we replace the $f(x)$ with y, we get $y = b$. We recognize this as the horizontal line whose *y*-intercept is *b*. The graph of the function $f(x) = b$, is also the horizontal line whose *y*-intercept is *b*.

Notice that for any real number we put in the function, the function value will be *b*. This tells us the range has only one value, *b*.

Constant Function



EXAMPLE 3.53

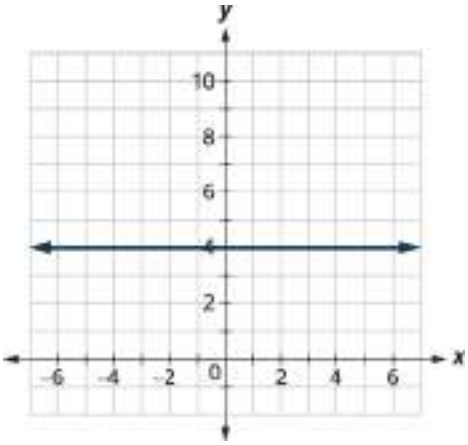
Graph: $f(x) = 4$.

Solution

$$f(x) = 4$$

We recognize this as a constant function.

The graph will be a horizontal line through $(0, 4)$.

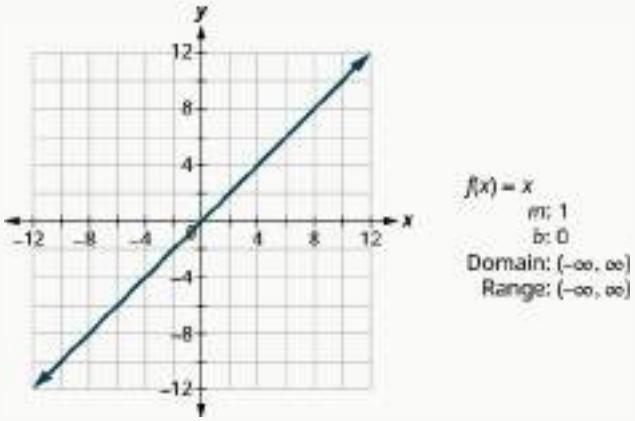


TRY IT : : 3.105 Graph: $f(x) = -2$.

TRY IT : : 3.106 Graph: $f(x) = 3$.

The identity function, $f(x) = x$ is a special case of the linear function. If we write it in linear function form, $f(x) = 1x + 0$, we see the slope is 1 and the *y*-intercept is 0.

Identity Function



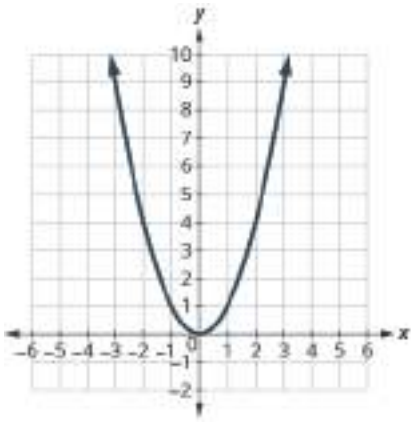
The next function we will look at is not a linear function. So the graph will not be a line. The only method we have to graph this function is point plotting. Because this is an unfamiliar function, we make sure to choose several positive and negative values as well as 0 for our x-values.

EXAMPLE 3.54

Graph: $f(x) = x^2$.

Solution

We choose x-values. We substitute them in and then create a chart as shown.



x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

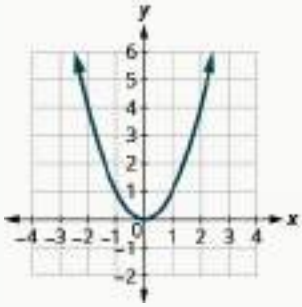
TRY IT : : 3.107 Graph: $f(x) = x^2$.

TRY IT : : 3.108 $f(x) = -x^2$

Looking at the result in **Example 3.54**, we can summarize the features of the square function. We call this graph a parabola. As we consider the domain, notice any real number can be used as an x-value. The domain is all real numbers.

The range is not all real numbers. Notice the graph consists of values of y never go below zero. This makes sense as the square of any number cannot be negative. So, the range of the square function is all non-negative real numbers.

Square Function



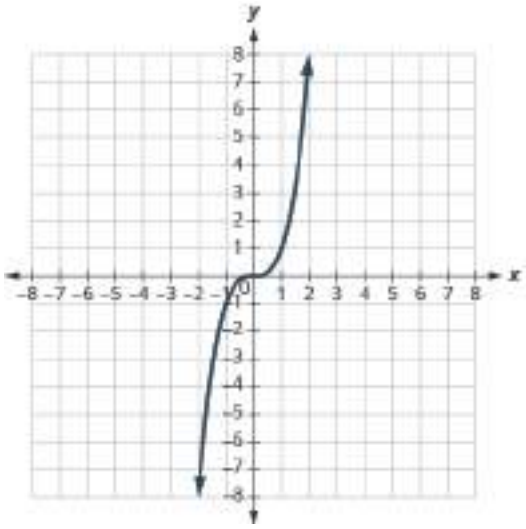
The next function we will look at is also not a linear function so the graph will not be a line. Again we will use point plotting, and make sure to choose several positive and negative values as well as 0 for our *x*-values.

EXAMPLE 3.55

Graph: $f(x) = x^3$.

Solution

We choose *x*-values. We substitute them in and then create a chart.



x	$f(x) = x^3$	$(x, f(x))$
−2	−8	(−2, −8)
−1	−1	(−1, −1)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)

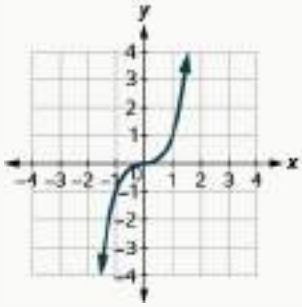
TRY IT : : 3.109 Graph: $f(x) = x^3$.

TRY IT : : 3.110 Graph: $f(x) = -x^3$.

Looking at the result in **Example 3.55**, we can summarize the features of the cube function. As we consider the domain, notice any real number can be used as an *x*-value. The domain is all real numbers.

The range is all real numbers. This makes sense as the cube of any non-zero number can be positive or negative. So, the range of the cube function is all real numbers.

Cube Function



The next function we will look at does not square or cube the input values, but rather takes the square root of those values.

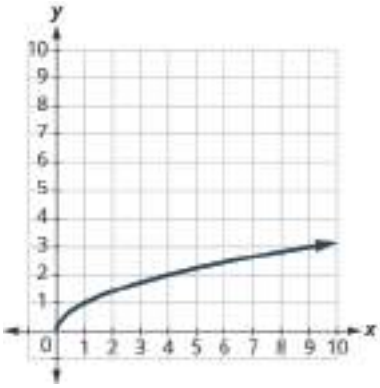
Let's graph the function $f(x) = \sqrt{x}$ and then summarize the features of the function. Remember, we can only take the square root of non-negative real numbers, so our domain will be the non-negative real numbers.

EXAMPLE 3.56

$f(x) = \sqrt{x}$

Solution

We choose x-values. Since we will be taking the square root, we choose numbers that are perfect squares, to make our work easier. We substitute them in and then create a chart.

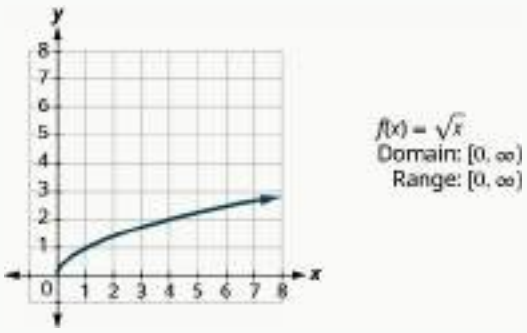


x	$f(x) = \sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

TRY IT : : 3.111 Graph: $f(x) = \sqrt{x}$.

TRY IT : : 3.112 Graph: $f(x) = -\sqrt{x}$.

Square Root Function



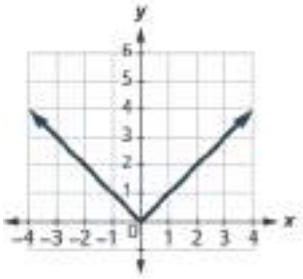
Our last basic function is the absolute value function, $f(x) = |x|$. Keep in mind that the absolute value of a number is its distance from zero. Since we never measure distance as a negative number, we will never get a negative number in the range.

EXAMPLE 3.57

Graph: $f(x) = |x|$.

Solution

We choose x-values. We substitute them in and then create a chart.

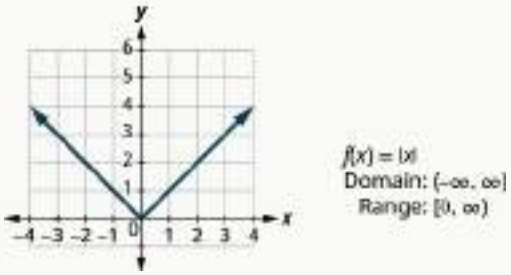


x	$f(x) = \lvert x \rvert$	$(x, f(x))$
−3	3	(−3, 3)
−2	2	(−2, 2)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

TRY IT : : 3.113 Graph: $f(x) = |x|$.

TRY IT : : 3.114 Graph: $f(x) = -|x|$.

Absolute Value Function



Read Information from a Graph of a Function

In the sciences and business, data is often collected and then graphed. The graph is analyzed, information is obtained from the graph and then often predictions are made from the data.

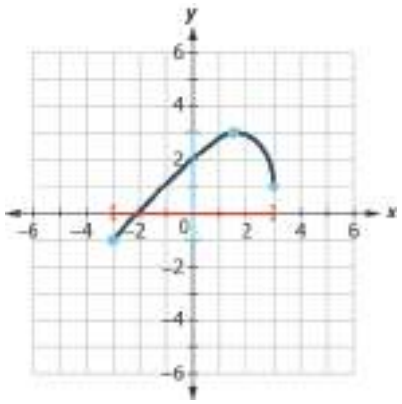
We will start by reading the domain and range of a function from its graph.

Remember the domain is the set of all the *x*-values in the ordered pairs in the function. To find the domain we look at the graph and find all the values of *x* that have a corresponding value on the graph. Follow the value *x* up or down vertically. If you hit the graph of the function then *x* is in the domain.

Remember the range is the set of all the *y*-values in the ordered pairs in the function. To find the range we look at the graph and find all the values of *y* that have a corresponding value on the graph. Follow the value *y* left or right horizontally. If you hit the graph of the function then *y* is in the range.

EXAMPLE 3.58

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



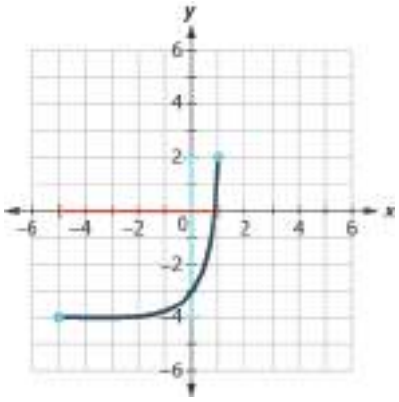
Solution

To find the domain we look at the graph and find all the values of *x* that correspond to a point on the graph. The domain is highlighted in red on the graph. The domain is $[-3, 3]$.

To find the range we look at the graph and find all the values of *y* that correspond to a point on the graph. The range is highlighted in blue on the graph. The range is $[-1, 3]$.

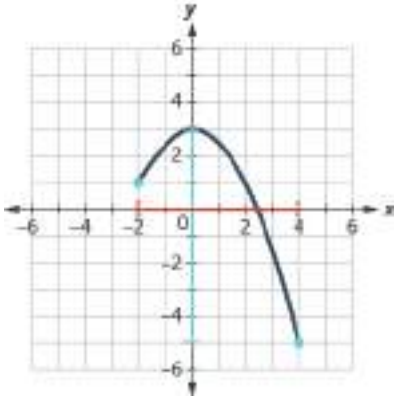
TRY IT : : 3.115

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



TRY IT : : 3.116

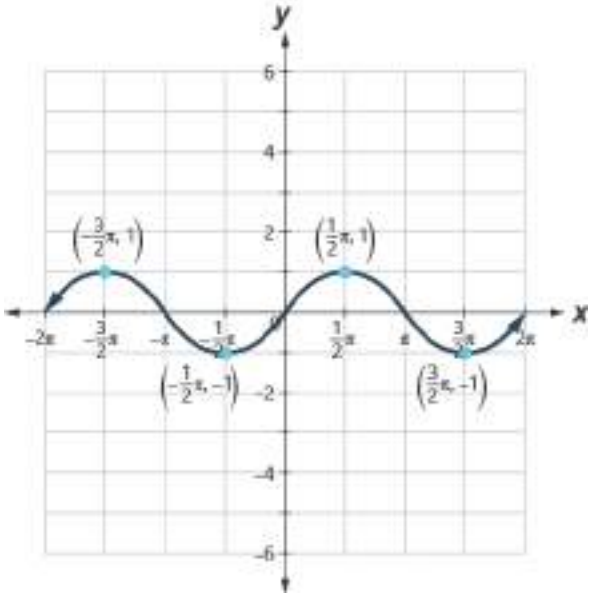
Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



We are now going to read information from the graph that you may see in future math classes.

EXAMPLE 3.59

Use the graph of the function to find the indicated values.



ⓐ Find: $f(0)$.

ⓑ Find: $f\left(\frac{3}{2}\pi\right)$.

ⓒ Find: $f\left(-\frac{1}{2}\pi\right)$.

ⓓ Find the values for *x* when $f(x) = 0$.

ⓔ Find the *x*-intercepts.

ⓕ Find the *y*-intercepts.

ⓖ Find the domain. Write it in interval notation.

ⓗ Find the range. Write it in interval notation.

Solution

ⓐ When $x = 0$, the function crosses the *y*-axis at 0. So, $f(0) = 0$.

ⓑ When $x = \frac{3}{2}\pi$, the *y*-value of the function is -1. So, $f\left(\frac{3}{2}\pi\right) = -1$.

ⓒ When $x = -\frac{1}{2}\pi$, the *y*-value of the function is -1. So, $f\left(-\frac{1}{2}\pi\right) = -1$.

ⓓ The function is 0 at the points, $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$. The *x*-values when $f(x) = 0$ are $-2\pi, -\pi, 0, \pi, 2\pi$.

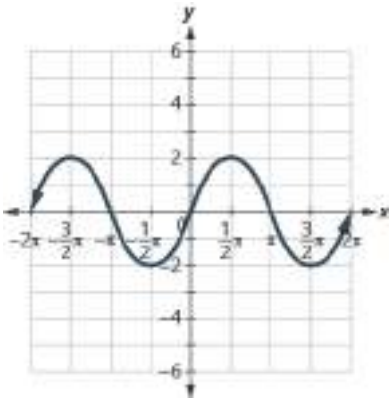
ⓔ The *x*-intercepts occur when $y = 0$. So the *x*-intercepts occur when $f(x) = 0$. The *x*-intercepts are $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$.

ⓕ The *y*-intercepts occur when $x = 0$. So the *y*-intercepts occur at $f(0)$. The *y*-intercept is $(0, 0)$.

ⓖ This function has a value when *x* is from -2π to 2π. Therefore, the domain in interval notation is $[-2\pi, 2\pi]$.

ⓗ This function values, or *y*-values go from -1 to 1. Therefore, the range, in interval notation, is $[-1, 1]$.

TRY IT : : 3.117 Use the graph of the function to find the indicated values.



ⓐ Find: $f(0)$.

ⓑ Find: $f\left(\frac{1}{2}\pi\right)$.

ⓒ Find: $f\left(-\frac{3}{2}\pi\right)$.

ⓓ Find the values for *x* when $f(x) = 0$.

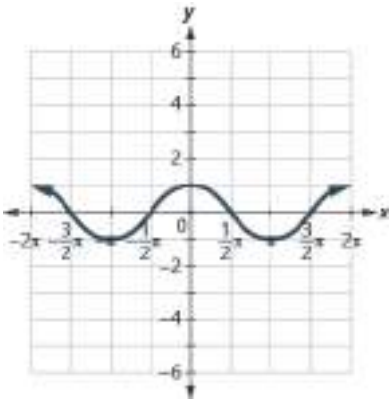
ⓔ Find the *x*-intercepts.

ⓕ Find the *y*-intercepts.

ⓖ Find the domain. Write it in interval notation.

ⓗ Find the range. Write it in interval notation.

TRY IT : : 3.118 Use the graph of the function to find the indicated values.



ⓐ Find: $f(0)$.

ⓑ Find: $f(\pi)$.

ⓒ Find: $f(-\pi)$.

ⓓ Find the values for x when $f(x) = 0$.

ⓔ Find the x-intercepts.

ⓕ Find the y-intercepts.

ⓖ Find the domain. Write it in interval notation.

ⓗ Find the range. Write it in interval notation.

MEDIA : :

Access this online resource for additional instruction and practice with graphs of functions.

- **Find Domain and Range (https://openstax.org/l/37domainrange)**

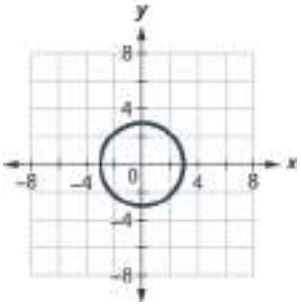
3.6 EXERCISES

Practice Makes Perfect

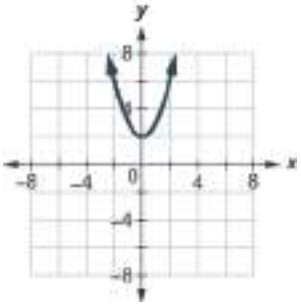
Use the Vertical Line Test

In the following exercises, determine whether each graph is the graph of a function.

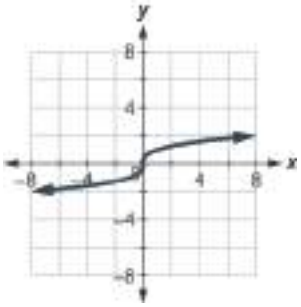
337. ⓐ



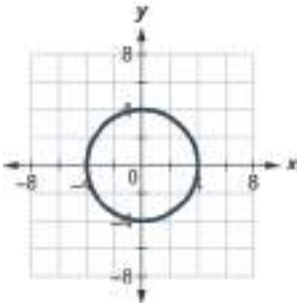
ⓑ



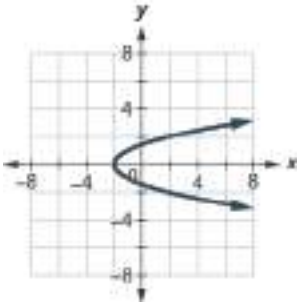
338. ⓐ



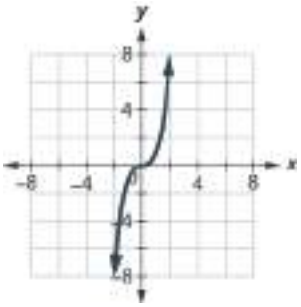
ⓑ



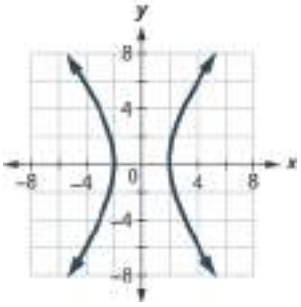
339. ⓐ



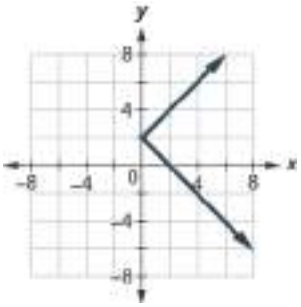
ⓑ



340. ⓐ



ⓑ



Identify Graphs of Basic Functions

In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

341. $f(x) = 3x + 4$

342. $f(x) = 2x + 5$

343. $f(x) = -x - 2$

344. $f(x) = -4x - 3$

345. $f(x) = -2x + 2$

346. $f(x) = -3x + 3$

347. $f(x) = \frac{1}{2}x + 1$

348. $f(x) = \frac{2}{3}x - 2$

349. $f(x) = 5$

350. $f(x) = 2$

351. $f(x) = -3$

352. $f(x) = -1$

353. $f(x) = 2x$

354. $f(x) = 3x$

355. $f(x) = -2x$

356. $f(x) = -3x$

357. $f(x) = 3x^2$

358. $f(x) = 2x^2$

359. $f(x) = -3x^2$

360. $f(x) = -2x^2$

361. $f(x) = \frac{1}{2}x^2$

362. $f(x) = \frac{1}{3}x^2$

363. $f(x) = x^2 - 1$

364. $f(x) = x^2 + 1$

365. $f(x) = -2x^3$

366. $f(x) = 2x^3$

367. $f(x) = x^3 + 2$

368. $f(x) = x^3 - 2$

369. $f(x) = 2\sqrt{x}$

370. $f(x) = -2\sqrt{x}$

371. $f(x) = \sqrt{x - 1}$

372. $f(x) = \sqrt{x + 1}$

373. $f(x) = 3|x|$

374. $f(x) = -2|x|$

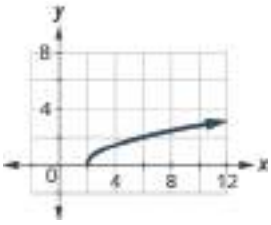
375. $f(x) = |x| + 1$

376. $f(x) = |x| - 1$

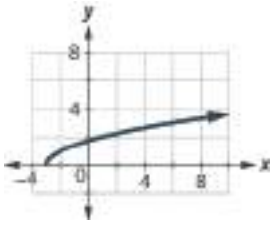
Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation.

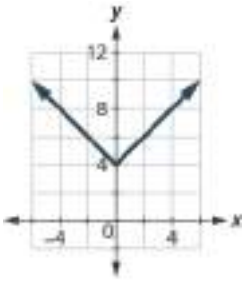
377.



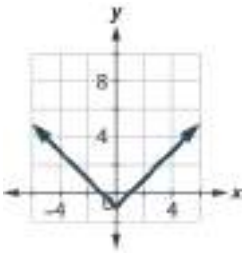
378.



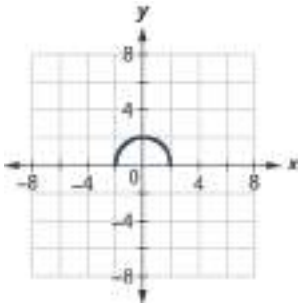
379.



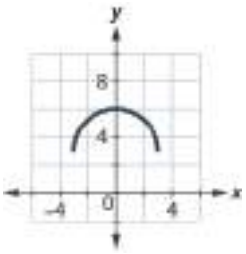
380.



381.

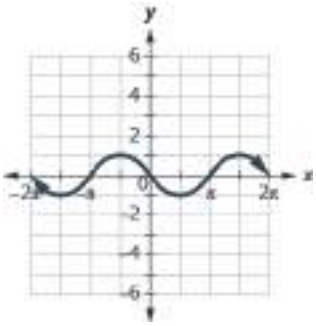


382.



In the following exercises, use the graph of the function to find the indicated values.

383.



ⓐ Find: $f(0)$.

ⓑ Find: $f\left(\frac{1}{2}\pi\right)$.

ⓒ Find: $f\left(-\frac{3}{2}\pi\right)$.

ⓓ Find the values for *x* when $f(x) = 0$.

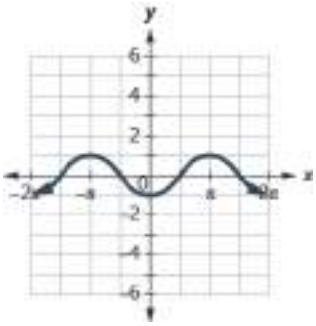
ⓔ Find the *x*-intercepts.

ⓕ Find the *y*-intercepts.

ⓖ Find the domain. Write it in interval notation.

ⓗ Find the range. Write it in interval notation.

384.



ⓐ Find: $f(0)$.

ⓑ Find: $f(\pi)$.

ⓒ Find: $f(-\pi)$.

ⓓ Find the values for *x* when $f(x) = 0$.

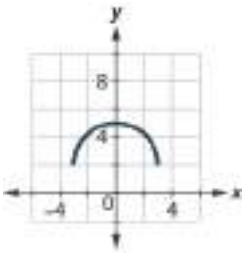
ⓔ Find the *x*-intercepts.

ⓕ Find the *y*-intercepts.

ⓖ Find the domain. Write it in interval notation.

ⓗ Find the range. Write it in interval notation

385.



ⓐ Find: $f(0)$.

ⓑ Find: $f(-3)$.

ⓒ Find: $f(3)$.

ⓓ Find the values for *x* when $f(x) = 0$.

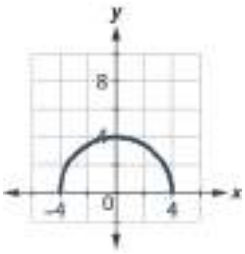
ⓔ Find the *x*-intercepts.

ⓕ Find the *y*-intercepts.

ⓖ Find the domain. Write it in interval notation.

ⓗ Find the range. Write it in interval notation.

386.



ⓐ Find: $f(0)$.

ⓑ Find the values for *x* when $f(x) = 0$.

ⓒ Find the *x*-intercepts.

ⓓ Find the *y*-intercepts.

ⓔ Find the domain. Write it in interval notation.

ⓕ Find the range. Write it in interval notation

Writing Exercises

387. Explain in your own words how to find the domain from a graph.

388. Explain in your own words how to find the range from a graph.

389. Explain in your own words how to use the vertical line test.

390. Draw a sketch of the square and cube functions. What are the similarities and differences in the graphs?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
use the vertical line test.			
identify graphs of basic functions.			
read information from a graph.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

CHAPTER 3 REVIEW

KEY TERMS

boundary line The line with equation $Ax + By = C$ is the boundary line that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

domain of a relation The domain of a relation is all the *x*-values in the ordered pairs of the relation.

function A function is a relation that assigns to each element in its domain exactly one element in the range.

horizontal line A horizontal line is the graph of an equation of the form $y = b$. The line passes through the *y*-axis at $(0, b)$.

intercepts of a line The points where a line crosses the *x*-axis and the *y*-axis are called the intercepts of the line.

linear equation An equation of the form $Ax + By = C,$ where *A* and *B* are not both zero, is called a linear equation in two variables.

linear inequality A linear inequality is an inequality that can be written in one of the following forms: $Ax + By > C,$ $Ax + By \geq C,$ $Ax + By < C,$ or $Ax + By \leq C,$ where *A* and *B* are not both zero.

mapping A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

ordered pair An ordered pair, (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the *x*-coordinate. The second number is the *y*-coordinate.

origin The point $(0, 0)$ is called the origin. It is the point where the *x*-axis and *y*-axis intersect.

parallel lines Parallel lines are lines in the same plane that do not intersect.

perpendicular lines Perpendicular lines are lines in the same plane that form a right angle.

point-slope form The point-slope form of an equation of a line with slope *m* and containing the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

range of a relation The range of a relation is all the *y*-values in the ordered pairs of the relation.

relation A relation is any set of ordered pairs, (x, y). All the *x*-values in the ordered pairs together make up the domain. All the *y*-values in the ordered pairs together make up the range.

solution of a linear equation in two variables An ordered pair (x, y) is a solution of the linear equation $Ax + By = C,$ if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

solution to a linear inequality An ordered pair (x, y) is a solution to a linear inequality if the inequality is true when we substitute the values of *x* and *y*.

standard form of a linear equation A linear equation is in standard form when it is written $Ax + By = C$.

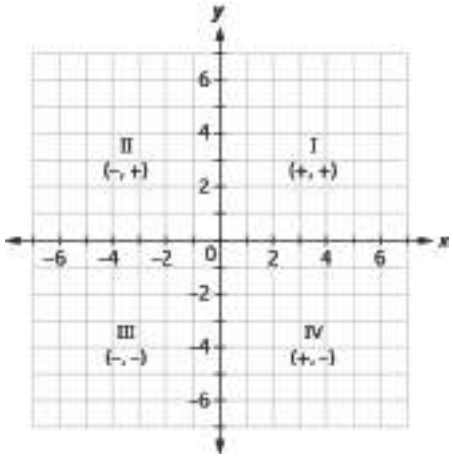
vertical line A vertical line is the graph of an equation of the form $x = a$. The line passes through the *x*-axis at $(a, 0)$.

KEY CONCEPTS

3.1 Graph Linear Equations in Two Variables

- **Points on the Axes**
 - Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates $(a, 0)$.
 - Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates $(0, b)$.
- **Quadrant**

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$



- **Graph of a Linear Equation:** The graph of a linear equation $Ax + By = C$ is a straight line.

 Every point on the line is a solution of the equation.
 Every solution of this equation is a point on this line.

- **How to graph a linear equation by plotting points.**

 Step 1. Find three points whose coordinates are solutions to the equation. Organize them in a table.

 Step 2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.

 Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

- ***x*-intercept and *y*-intercept of a Line**
 - The *x*-intercept is the point $(a, 0)$ where the line crosses the *x*-axis.
 - The *y*-intercept is the point $(0, b)$ where the line crosses the *y*-axis.

- The *x*-intercept occurs when *y* is zero.
- The *y*-intercept occurs when *x* is zero.

x	*y*
a	0
0	*b*

- **Find the *x*- and *y*-intercepts from the Equation of a Line**
 - Use the equation of the line. To find:
 the *x*-intercept of the line, let $y = 0$ and solve for *x*.
 the *y*-intercept of the line, let $x = 0$ and solve for *y*.

- **How to graph a linear equation using the intercepts.**

 Step 1. Find the *x*- and *y*-intercepts of the line.
 Let $y = 0$ and solve for *x*.
 Let $x = 0$ and solve for *y*.

 Step 2. Find a third solution to the equation.

 Step 3. Plot the three points and check that they line up.

 Step 4. Draw the line

3.2 Slope of a Line

- **Slope of a Line**
 - The slope of a line is $m = \frac{\text{rise}}{\text{run}}$.
 - The rise measures the vertical change and the run measures the horizontal change.

- **How to find the slope of a line from its graph using** $m = \frac{\text{rise}}{\text{run}}$.

 Step 1. Locate two points on the line whose coordinates are integers.

 Step 2. Starting with one point, sketch a right triangle, going from the first point to the second point.

 Step 3. Count the rise and the run on the legs of the triangle.

 Step 4. Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

- **Slope of a line between two points.**
 - The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

- **How to graph a line given a point and the slope.**

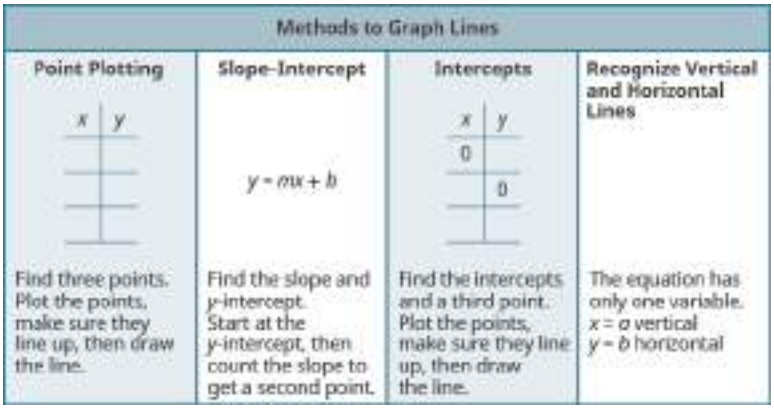
 Step 1. Plot the given point.

 Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.

 Step 3. Starting at the given point, count out the rise and run to mark the second point.

 Step 4. Connect the points with a line.

- **Slope Intercept Form of an Equation of a Line**
 - The slope–intercept form of an equation of a line with slope *m* and *y*-intercept, $(0, b)$ is $y = mx + b$



- **Parallel Lines**
 - Parallel lines are lines in the same plane that do not intersect.
 Parallel lines have the same slope and different *y*-intercepts.
 If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.

 Parallel vertical lines have different *x*-intercepts.

- **Perpendicular Lines**
 - Perpendicular lines are lines in the same plane that form a right angle.
 - If m_1 and m_2 are the slopes of two perpendicular lines, then:

 their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$.

 the product of their slopes is -1, $m_1 \cdot m_2 = -1$.

 - A vertical line and a horizontal line are always perpendicular to each other.

3.3 Find the Equation of a Line

- **How to find an equation of a line given the slope and a point.**

 Step 1. Identify the slope.

 Step 2. Identify the point.

 Step 3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.

Step 4. Write the equation in slope-intercept form.

- **How to find an equation of a line given two points.**

 Step 1. Find the slope using the given points. $m = \frac{y_2 - y_1}{x_2 - x_1}$

 Step 2. Choose one point.

 Step 3. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.

 Step 4. Write the equation in slope-intercept form.

To Write an Equation of a Line		
If given:	**Use:**	**Form:**
Slope and *y*-intercept	**slope-intercept**	$y = mx + b$
Slope and a point	**point-slope**	$y - y_1 = m(x - x_1)$
Two points	**point-slope**	$y - y_1 = m(x - x_1)$

- **How to find an equation of a line parallel to a given line.**

 Step 1. Find the slope of the given line.

 Step 2. Find the slope of the parallel line.

 Step 3. Identify the point.

 Step 4. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.

 Step 5. Write the equation in slope-intercept form

- **How to find an equation of a line perpendicular to a given line.**

 Step 1. Find the slope of the given line.

 Step 2. Find the slope of the perpendicular line.

 Step 3. Identify the point.

 Step 4. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$

 Step 5. Write the equation in slope-intercept form.

3.4 Graph Linear Inequalities in Two Variables

- **How to graph a linear inequality in two variables.**

 Step 1. Identify and graph the boundary line.
 If the inequality is $\leq$ or $\geq$, the boundary line is solid.
 If the inequality is $<$ or $>$, the boundary line is dashed.

 Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?

 Step 3. Shade in one side of the boundary line.
 If the test point is a solution, shade in the side that includes the point.
 If the test point is not a solution, shade in the opposite side.

3.5 Relations and Functions

- **Function Notation:** For the function $y = f(x)$
 - *f* is the name of the function
 - *x* is the domain value
 - $f(x)$ is the range value *y* corresponding to the value *x*

We read $f(x)$ as *f* of *x* or the value of *f* at *x*.

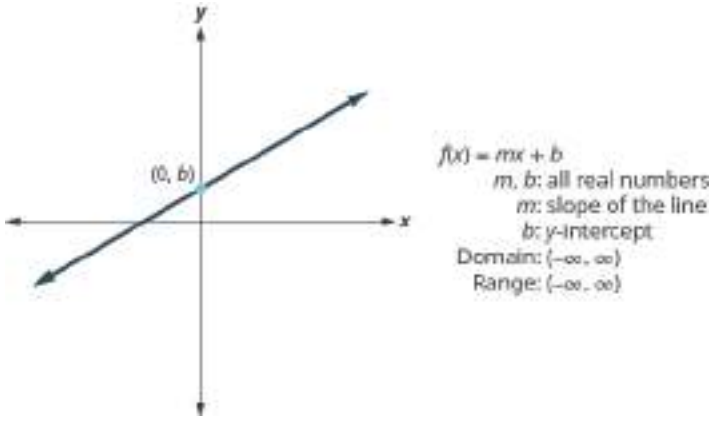
- **Independent and Dependent Variables:** For the function $y = f(x)$,
 - *x* is the independent variable as it can be any value in the domain
 - *y* is the dependent variable as its value depends on *x*

3.6 Graphs of Functions

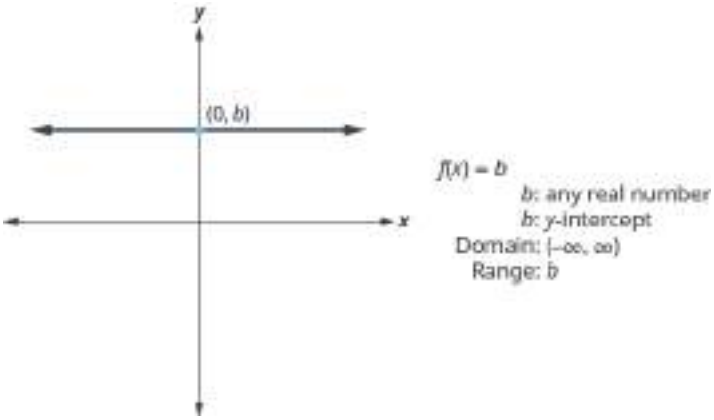
- **Vertical Line Test**
 - A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.
 - If any vertical line intersects the graph in more than one point, the graph does not represent a function.
- **Graph of a Function**
 - The graph of a function is the graph of all its ordered pairs, (x, y) or using function notation, $(x, f(x))$ where $y = f(x)$.

f	name of function
x	x-coordinate of the ordered pair
$f(x)$	y-coordinate of the ordered pair

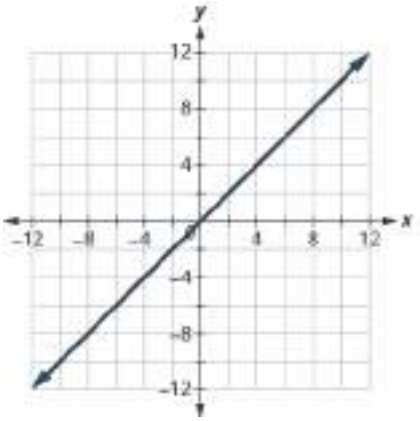
- **Linear Function**



- **Constant Function**

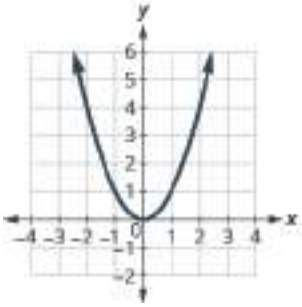


- **Identity Function**



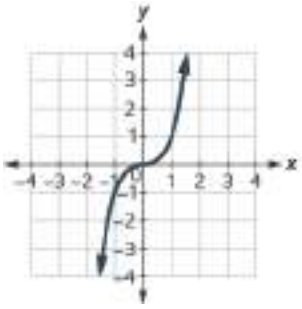
$f(x) = x$
m: 1
b: 0
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

- **Square Function**



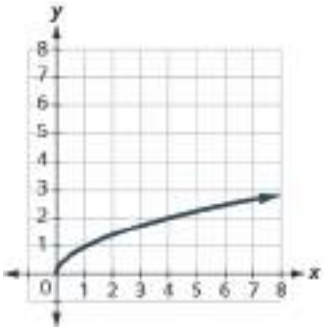
$f(x) = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

- **Cube Function**



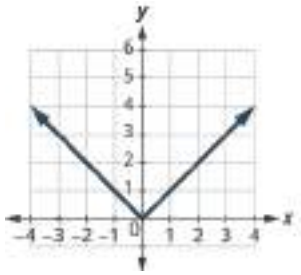
$f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

- **Square Root Function**



$f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$

- **Absolute Value Function**



REVIEW EXERCISES

3.1 Graph Linear Equations in Two Variables

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system.

391.

ⓐ $(-1, -5)$

ⓑ $(-3, 4)$

ⓒ $(2, -3)$

ⓓ $\left(1, \frac{5}{2}\right)$

392.

ⓐ $(-2, 0)$

ⓑ $(0, -4)$

ⓒ $(0, 5)$

ⓓ $(3, 0)$

In the following exercises, determine which ordered pairs are solutions to the given equations.

393. $5x + y = 10$;

ⓐ $(5, 1)$

ⓑ $(2, 0)$

ⓒ $(4, -10)$

394. $y = 6x - 2$;

ⓐ $(1, 4)$

ⓑ $\left(\frac{1}{3}, 0\right)$

ⓒ $(6, -2)$

Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

395. $y = 4x - 3$

396. $y = -3x$

397. $y = \frac{1}{2}x + 3$

398. $y = -\frac{4}{5}x - 1$

399. $x - y = 6$

400. $2x + y = 7$

401. $3x - 2y = 6$

Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

402. $y = -2$

403. $x = 3$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

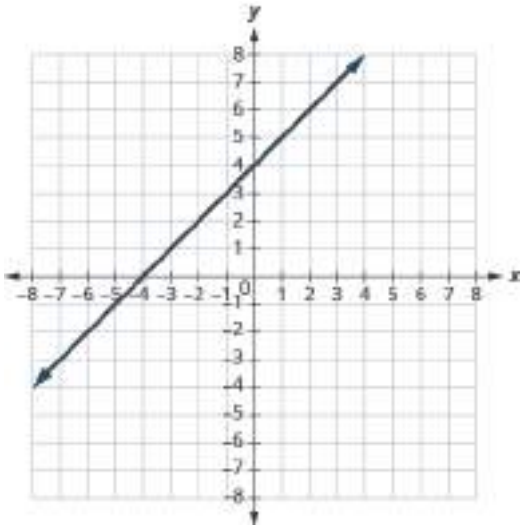
404. $y = -2x$ and $y = -2$

405. $y = \frac{4}{3}x$ and $y = \frac{4}{3}$

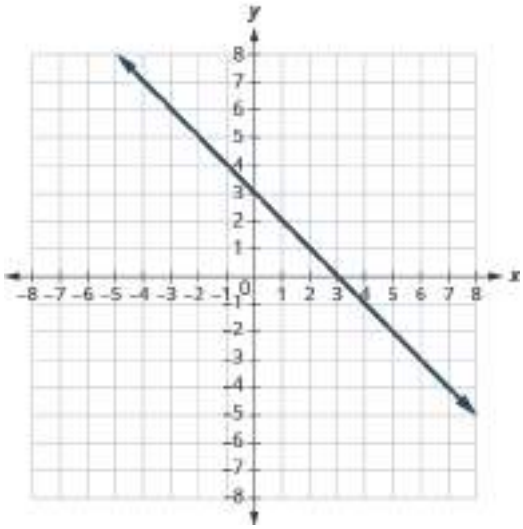
Find *x*- and *y*-Intercepts

In the following exercises, find the x- and y-intercepts.

406.



407.



In the following exercises, find the intercepts of each equation.

408. $x - y = -1$

409. $x + 2y = 6$

410. $2x + 3y = 12$

411. $y = \frac{3}{4}x - 12$

412. $y = 3x$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

413. $-x + 3y = 3$

414. $x - y = 4$

415. $2x - y = 5$

416. $2x - 4y = 8$

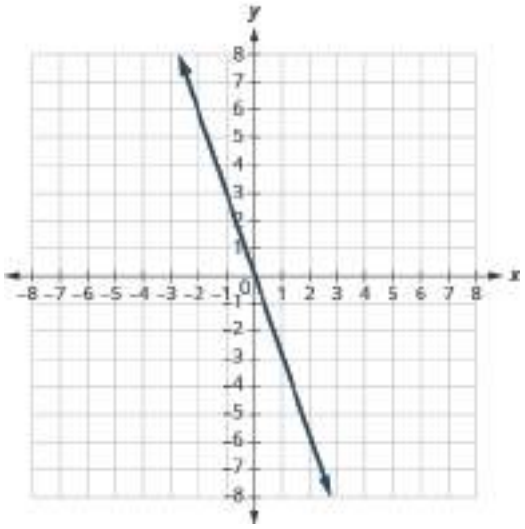
417. $y = 4x$

3.2 Slope of a Line

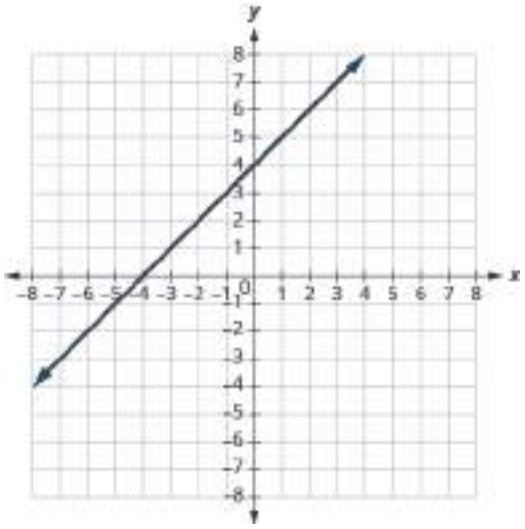
Find the Slope of a Line

In the following exercises, find the slope of each line shown.

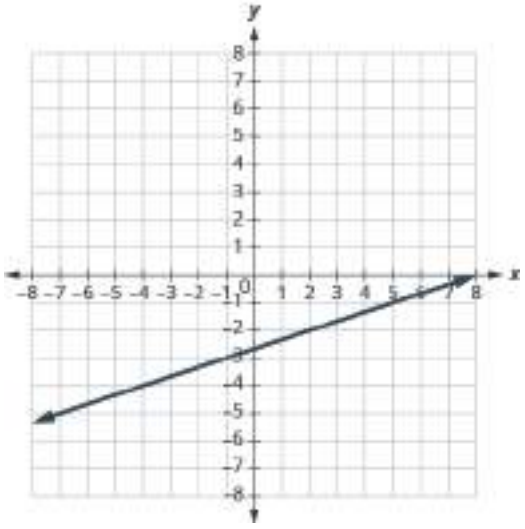
418.



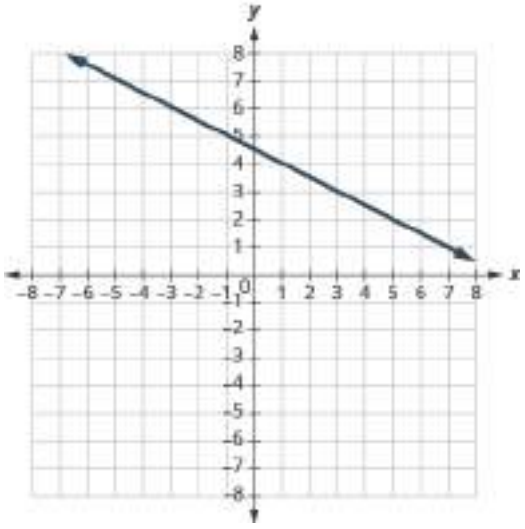
419.



420.



421.



In the following exercises, find the slope of each line.

422. $y = 2$

423. $x = 5$

424. $x = -3$

425. $y = -1$

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

426. $(-1, -1), (0, 5)$

427. $(3.5), (4, -1)$

428. $(-5, -2), (3, 2)$

429. $(2, 1), (4, 6)$

Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

430. $(2, -2);\ m = \frac{5}{2}$

431. $(-3, 4);\ m = -\frac{1}{3}$

432. x-intercept $-4;\ m = 3$

433. y-intercept 1; $m = -\frac{3}{4}$

Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and y-intercept of each line.

434. $y = -4x + 9$

435. $y = \frac{5}{3}x - 6$

436. $5x + y = 10$

437. $4x - 5y = 8$

In the following exercises, graph the line of each equation using its slope and y-intercept.

438. $y = 2x + 3$

439. $y = -x - 1$

440. $y = -\frac{2}{5}x + 3$

441. $4x - 3y = 12$

In the following exercises, determine the most convenient method to graph each line.

442. $x = 5$

443. $y = -3$

444. $2x + y = 5$

445. $x - y = 2$

446. $y = \frac{2}{2}x + 2$

447. $y = \frac{3}{4}x - 1$

Graph and Interpret Applications of Slope-Intercept

448. Katherine is a private chef. The equation $C = 6.5m + 42$ models the relation between her weekly cost, C, in dollars and the number of meals, m, that she serves.

ⓐ Find Katherine's cost for a week when she serves no meals.

ⓑ Find the cost for a week when she serves 14 meals.

ⓒ Interpret the slope and C-intercept of the equation.

ⓓ Graph the equation.

449. Marjorie teaches piano. The equation $P = 35h - 250$ models the relation between her weekly profit, P, in dollars and the number of student lessons, s, that she teaches.

ⓐ Find Marjorie's profit for a week when she teaches no student lessons.

ⓑ Find the profit for a week when she teaches 20 student lessons.

ⓒ Interpret the slope and P-intercept of the equation.

ⓓ Graph the equation.

Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

450. $4x - 3y = -1;\ y = \frac{4}{3}x - 3$

451. $y = 5x - 1;\ 10x + 2y = 0$

452. $3x - 2y = 5;\ 2x + 3y = 6$

453. $2x - y = 8;\ x - 2y = 4$

3.3 Find the Equation of a Line

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

454. slope $\frac{1}{3}$ and y-intercept $(0, -6)$

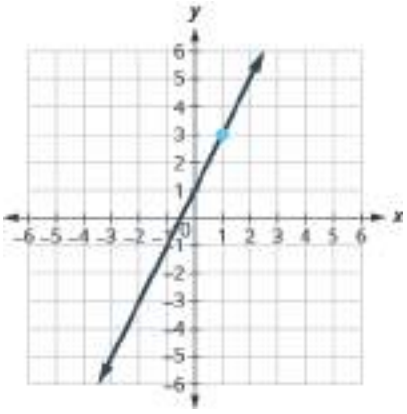
455. slope -5 and y-intercept $(0, -3)$

456. slope 0 and y-intercept $(0, 4)$

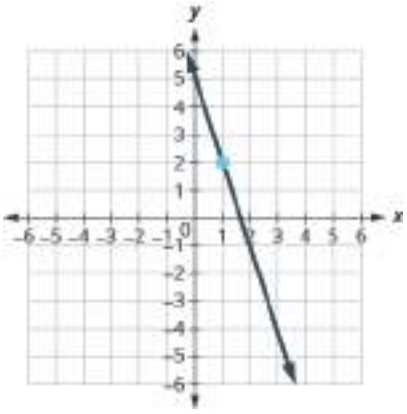
457. slope -2 and y-intercept $(0, 0)$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

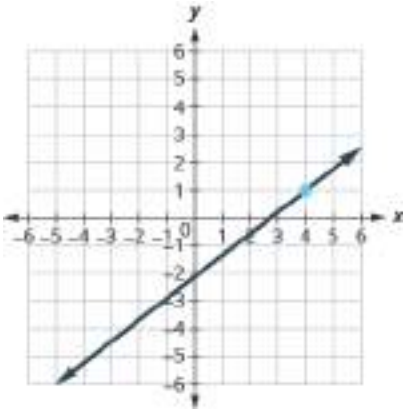
458.



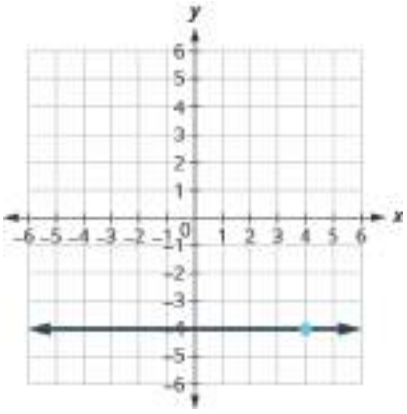
459.



460.



461.



Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

462. $m = -\frac{1}{4}$, point $(-8, 3)$

463. $m = \frac{3}{5}$, point $(10, 6)$

464. Horizontal line containing $(-2, 7)$

465. $m = -2$, point $(-1, -3)$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

466. $(2, 10)$ and $(-2, -2)$

467. $(7, 1)$ and $(5, 0)$

468. $(3, 8)$ and $(3, -4)$

469. $(5, 2)$ and $(-1, 2)$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

470. line $y = -3x + 6$, point $(1, -5)$

471. line $2x + 5y = -10$, point $(10, 4)$

472. line $x = 4$, point $(-2, -1)$

473. line $y = -5$, point $(-4, 3)$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

474. line $y = -\frac{4}{5}x + 2$, point $(8, 9)$

475. line $2x - 3y = 9$, point $(-4, 0)$

476. line $y = 3$, point $(-1, -3)$

477. line $x = -5$ point $(2, 1)$

3.4 Graph Linear Inequalities in Two Variables

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.

478. Determine whether each ordered pair is a solution to the inequality $y < x - 3$:

ⓐ $(0, 1)$ ⓑ $(-2, -4)$ ⓒ $(5, 2)$ ⓓ $(3, -1)$

ⓔ $(-1, -5)$

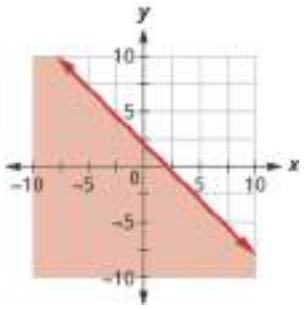
479. Determine whether each ordered pair is a solution to the inequality $x + y > 4$:

ⓐ $(6, 1)$ ⓑ $(-3, 6)$ ⓒ $(3, 2)$ ⓓ $(-5, 10)$ ⓔ $(0, 0)$

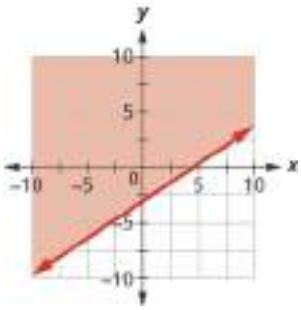
Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

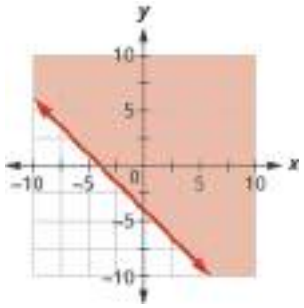
480. Write the inequality shown by the graph with the boundary line $y = -x + 2$.



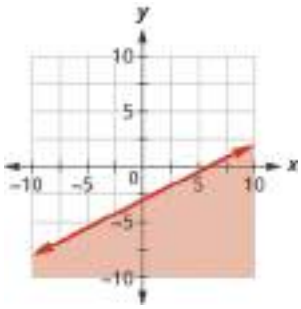
481. Write the inequality shown by the graph with the boundary line $y = \frac{2}{3}x - 3$.



482. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = -4$.



483. Write the inequality shown by the shaded region in the graph with the boundary line $x - 2y = 6$.



Graph Linear Inequalities in Two Variables

In the following exercises, graph each linear inequality.

484. Graph the linear inequality $y > \frac{2}{5}x - 4$.

485. Graph the linear inequality $y \leq -\frac{1}{4}x + 3$.

486. Graph the linear inequality $x - y \leq 5$.

487. Graph the linear inequality $3x + 2y > 10$.

488. Graph the linear inequality $y \leq -3x$.

489. Graph the linear inequality $y < 6$.

Solve Applications using Linear Inequalities in Two Variables

490. Shanthie needs to earn at least $500 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays $10 an hour and the other as an intern in a law office for $25 hour. How many hours does Shanthie need to work at each job to earn at least $500 per week?

ⓐ Let x be the number of hours she works teaching swimming and let y be the number of hours she works as an intern. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Shanthie.

491. Atsushi he needs to exercise enough to burn 600 calories each day. He prefers to either run or bike and burns 20 calories per minute while running and 15 calories a minute while biking.

ⓐ If x is the number of minutes that Atsushi runs and y is the number minutes he bikes, find the inequality that models the situation.

ⓑ Graph the inequality.

ⓒ List three solutions to the inequality. What options do the solutions provide Atsushi?

3.5 Relations and Functions

Find the Domain and Range of a Relation

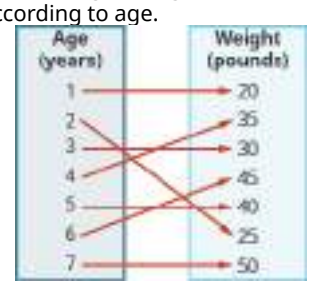
In the following exercises, for each relation, ⓐ find the domain of the relation ⓑ find the range of the relation.

492. $\{(5, -2), (5, -4), (7, -6), (8, -8), (9, -10)\}$

493. $\{(-3, 7), (-2, 3), (-1, 9), (0, -3), (-1, 8)\}$

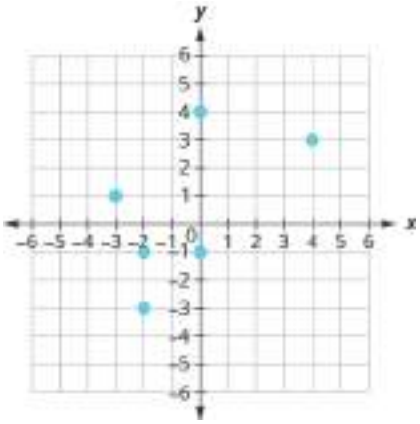
In the following exercise, use the mapping of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

494. The mapping below shows the average weight of a child according to age.



In the following exercise, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

495.



Determine if a Relation is a Function

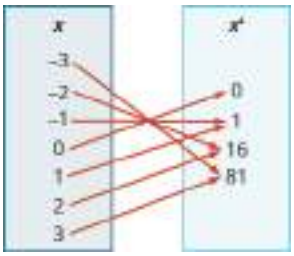
In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.

496. {(9, −5), (4, −3), (1, −1), (0, 0), (1, 1), (4, 3), (9, 5)}

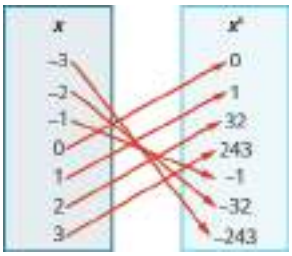
497. {(−3, 27), (−2, 8), (−1, 1), (0, 0), (1, 1), (2, 8), (3, 27)}

In the following exercises, use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the function ⓒ find the range of the function.

498.



499.



In the following exercises, determine whether each equation is a function.

500. $2x + y = -3$

501. $y = x^2$

502. $y = 3x - 5$

503. $y = x^3$

504. $2x + y^2 = 4$

Find the Value of a Function

In the following exercises, evaluate the function:

ⓐ $f(-2)$ ⓑ $f(3)$ ⓒ $f(a)$.

505. $f(x) = 3x - 4$

506. $f(x) = -2x + 5$

507. $f(x) = x^2 - 5x + 6$

508. $f(x) = 3x^2 - 2x + 1$

In the following exercises, evaluate the function.

509. $g(x) = 3x^2 - 5x;\ g(2)$

510. $F(x) = 2x^2 - 3x + 1;\ F(-1)$

511. $h(t) = 4|t - 1| + 2;\ h(-3)$

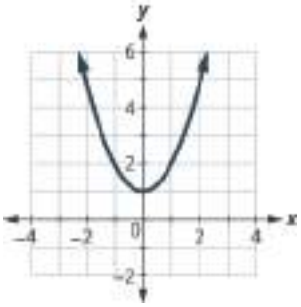
512. $f(x) = \frac{x+2}{x-1};\ f(3)$

3.6 Graphs of Functions

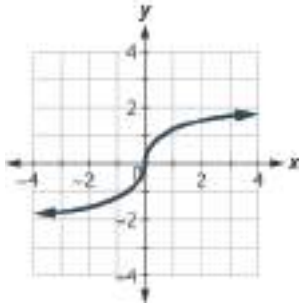
Use the Vertical line Test

In the following exercises, determine whether each graph is the graph of a function.

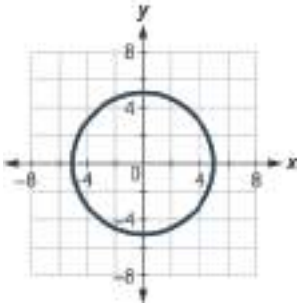
513.



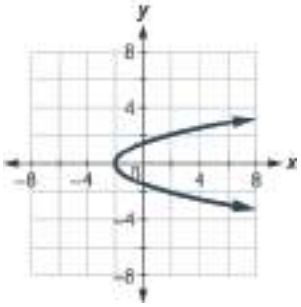
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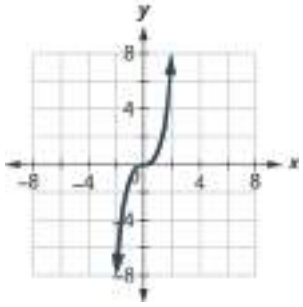
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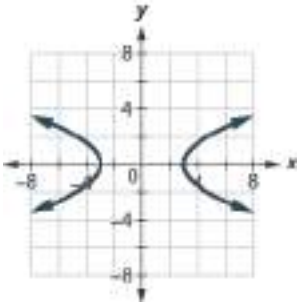
516.



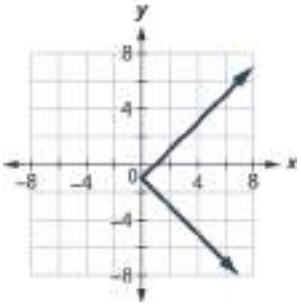
517.



518.



519.



Identify Graphs of Basic Functions

In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

520. $f(x) = 5x + 1$

521. $f(x) = -4x - 2$

522. $f(x) = \frac{2}{3}x - 1$

523. $f(x) = -6$

524. $f(x) = 2x$

525. $f(x) = 3x^2$

526. $f(x) = -\frac{1}{2}x^2$

527. $f(x) = x^2 + 2$

528. $f(x) = x^3 - 2$

529. $f(x) = \sqrt{x+2}$

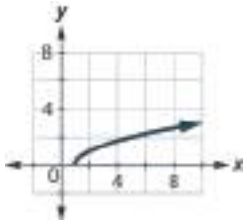
530. $f(x) = -|x|$

531. $f(x) = |x| + 1$

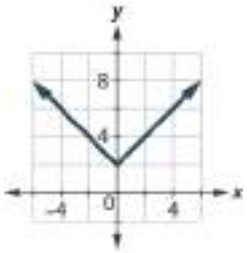
Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation

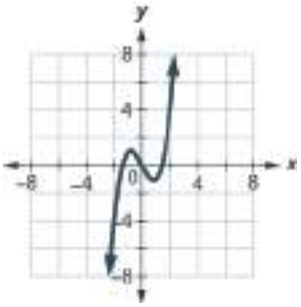
532.



533.

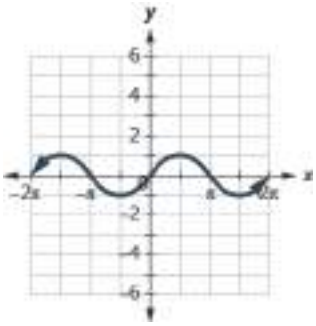


534.



In the following exercises, use the graph of the function to find the indicated values.

535.



ⓐ Find $f(0)$.

ⓑ Find $f\left(\frac{1}{2}\pi\right)$.

ⓒ Find $f\left(-\frac{3}{2}\pi\right)$.

ⓓ Find the values for x when $f(x) = 0$.

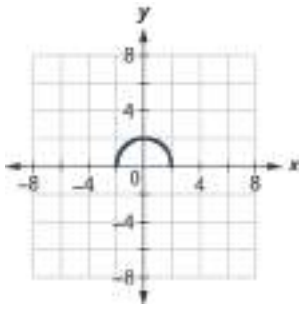
ⓔ Find the x-intercepts.

ⓕ Find the y-intercepts.

ⓖ Find the domain. Write it in interval notation.

ⓗ Find the range. Write it in interval notation.

536.



ⓐ Find $f(0)$.

ⓑ Find the values for x when $f(x) = 0$.

ⓒ Find the x-intercepts.

ⓓ Find the y-intercepts.

ⓔ Find the domain. Write it in interval notation.

ⓕ Find the range. Write it in interval notation.

PRACTICE TEST

537. Plot each point in a rectangular coordinate system.

ⓐ $(2, 5)$

ⓑ $(-1, -3)$

ⓒ $(0, 2)$

ⓓ $\left(-4, \frac{3}{2}\right)$

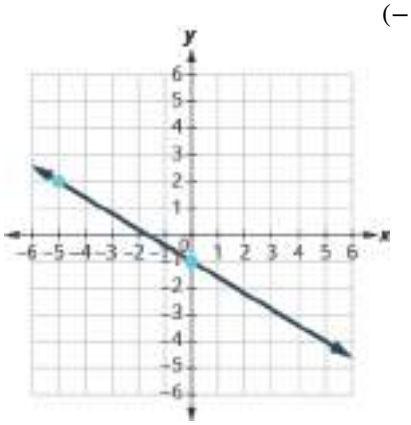
ⓔ $(5, 0)$

538. Which of the given ordered pairs are solutions to the equation $3x - y = 6$?

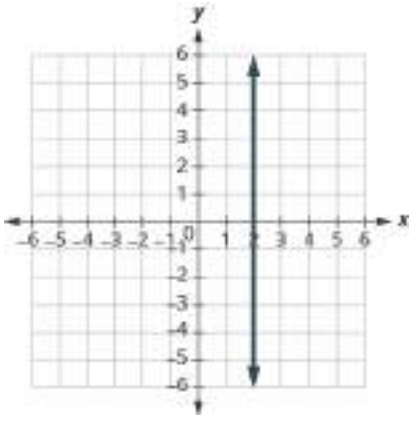
ⓐ $(3, 3)$ ⓑ $(2, 0)$ ⓒ $(4, -6)$

Find the slope of each line shown.

539. ⓐ



ⓑ



540. Find the slope of the line between the points $(5, 2)$ and $(-1, -4)$.

541. Graph the line with slope $\frac{1}{2}$ containing the point $(-3, -4)$.

542. Find the intercepts of $4x + 2y = -8$ and graph.

Graph the line for each of the following equations.

543. $y = \frac{5}{3}x - 1$

544. $y = -x$

545. $y = 2$

Find the equation of each line. Write the equation in slope-intercept form.

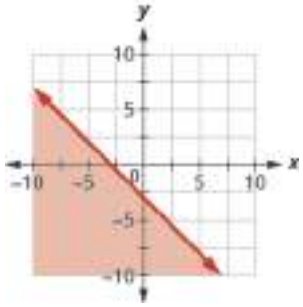
546. slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

547. $m = 2$, point $(-3, -1)$

548. containing $(10, 1)$ and $(6, -1)$

549. perpendicular to the line $y = \frac{5}{4}x + 2$, containing the point $(-10, 3)$

550. Write the inequality shown by the graph with the boundary line $y = -x - 3$.



Graph each linear inequality.

551. $y > \frac{3}{2}x + 5$

552. $x - y \geq -4$

553. $y \leq -5x$

554. Hiro works two part time jobs in order to earn enough money to meet her obligations of at least \$450 a week. Her job at the mall pays \$10 an hour and her administrative assistant job on campus pays \$15 an hour. How many hours does Hiro need to work at each job to earn at least \$450?

ⓐ Let x be the number of hours she works at the mall and let y be the number of hours she works as administrative assistant. Write an inequality that would model this situation.

ⓑ Graph the inequality .

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then explain what that means for Hiro.

555. Use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

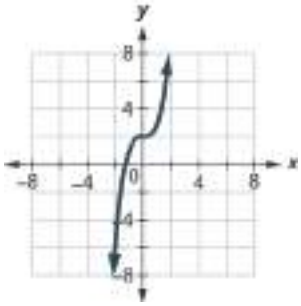
$\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

556. Evaluate the function: ⓐ $f(-1)$ ⓑ $f(2)$ ⓒ $f(c)$.

$f(x) = 4x^2 - 2x - 3$

557. For $h(y) = 3|y - 1| - 3$, evaluate $h(-4)$.

558. Determine whether the graph is the graph of a function. Explain your answer.

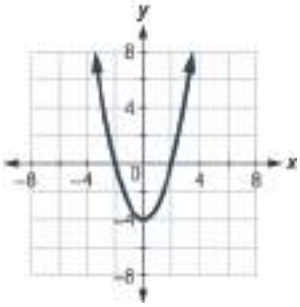


In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

559. $f(x) = x^2 + 1$

560. $f(x) = \sqrt{x + 1}$

561.



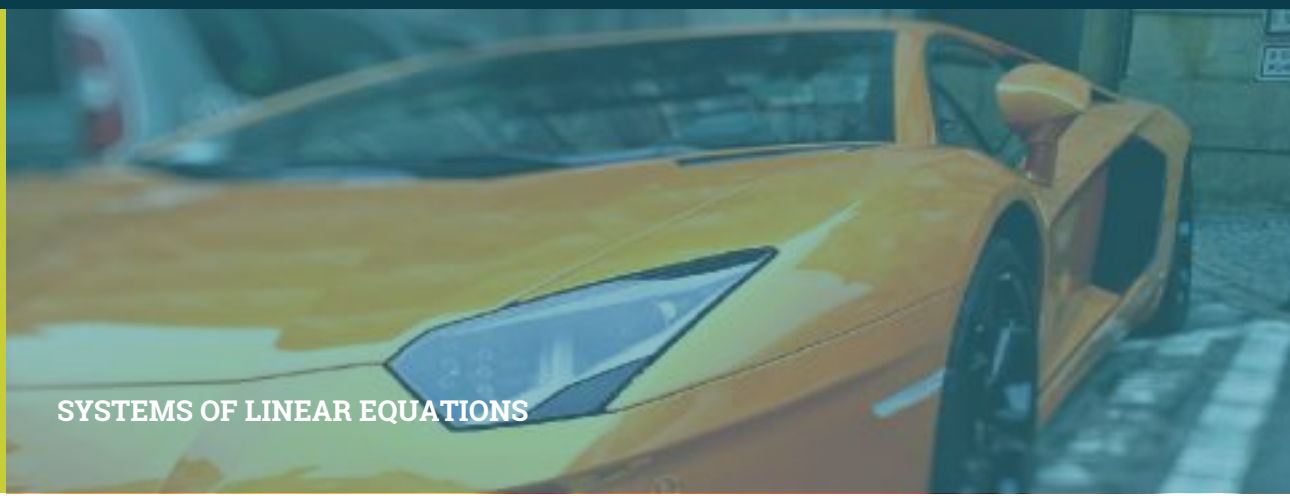
ⓑ Find the y-intercepts.

ⓒ Find $f(-1)$.

ⓓ Find $f(1)$.

ⓔ Find the domain. Write it in interval notation.

ⓕ Find the range. Write it in interval notation.



4 SYSTEMS OF LINEAR EQUATIONS

Figure 4.1 In the future, car drivers may become passengers because cars will be able to drive themselves. (credit: jingoba/Pixabay)

Chapter Outline

4.1 Solve Systems of Linear Equations with Two Variables
4.2 Solve Applications with Systems of Equations
4.3 Solve Mixture Applications with Systems of Equations
4.4 Solve Systems of Equations with Three Variables
4.5 Solve Systems of Equations Using Matrices
4.6 Solve Systems of Equations Using Determinants
4.7 Graphing Systems of Linear Inequalities

Introduction

Climb into your car. Put on your seatbelt. Choose your destination and then...relax. That's right. You don't have to do anything else because you are in an autonomous car, or one that navigates its way to your destination! No cars are fully autonomous at the moment and so you theoretically still need to have your hands on the wheel. Self-driving cars may help ease traffic congestion, prevent accidents, and lower pollution. The technology is thanks to computer programmers who are developing software to control the navigation of the car. These programmers rely on their understanding of mathematics, including relationships between equations. In this chapter, you will learn how to solve systems of linear equations in different ways and use them to analyze real-world situations.

4.1 Solve Systems of Linear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Solve a system of equations by substitution
- Solve a system of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Be Prepared!

Before you get started, take this readiness quiz.

1. For the equation $y = \frac{2}{3}x - 4$,

 ⓐ Is $(6, 0)$ a solution? ⓑ Is $(-3, -2)$ a solution?

If you missed this problem, review **Example 3.2**.

2. Find the slope and *y*-intercept of the line $3x - y = 12$.
 If you missed this problem, review **Example 3.16**.
3. Find the *x*- and *y*-intercepts of the line $2x - 3y = 12$.
 If you missed this problem, review **Example 3.8**.

Determine Whether an Ordered Pair is a Solution of a System of Equations

In **Solving Linear Equations**, we learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a **system of linear equations**.

System of Linear Equations

When two or more linear equations are grouped together, they form a **system of linear equations**.

In this section, we will focus our work on systems of two linear equations in two unknowns. We will solve larger systems of equations later in this chapter.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, such as $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to *both* equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the **solutions of a system of equations**.

Solutions of a System of Equations

The **solutions of a system of equations** are the values of the variables that make *all* the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

EXAMPLE 4.1

Determine whether the ordered pair is a solution to the system $\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$.

ⓐ $(-2, -1)$ ⓑ $(-4, -3)$

Solution

ⓐ

$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

We substitute $x = -2$ and $y = -1$ into both equations.

$x - y = -1$	$2x - y = -5$
$-2 - (-1) \stackrel{?}{=} -1$	$2 \cdot -2 - (-1) \stackrel{?}{=} -5$
$-1 = -1$ ✓	$5 \neq -5$
$(-2, -1)$ does not make both equations true.	$(-2, -1)$ is not a solution.

ⓑ

We substitute $x = -4$ and $y = -3$ into both equations.

$x - y = -1$	$2x - y = -5$
$-4 - (-3) \stackrel{?}{=} -1$	$2 \cdot (-4) - (-3) \stackrel{?}{=} -5$
$-1 = -1$ ✓	$-5 = -5$ ✓
$(-4, -3)$ does make both equations true.	$(-4, -3)$ is a solution.

TRY IT : : 4.1 Determine whether the ordered pair is a solution to the system $\begin{cases} 3x + y = 0 \\ x + 2y = -5 \end{cases}$.

ⓐ $(1, -3)$ ⓑ $(0, 0)$

TRY IT : : 4.2 Determine whether the ordered pair is a solution to the system $\begin{cases} x - 3y = -8 \\ -3x - y = 4 \end{cases}$.

ⓐ $(2, -2)$ ⓑ $(-2, 2)$

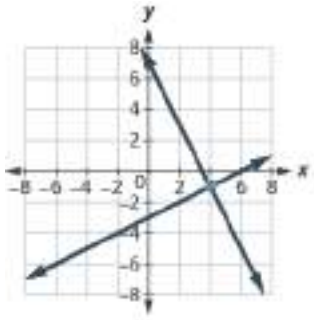
Solve a System of Linear Equations by Graphing

In this section, we will use three methods to solve a system of linear equations. The first method we'll use is graphing.

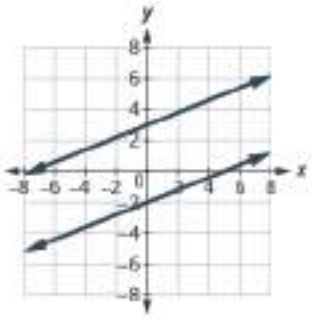
The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

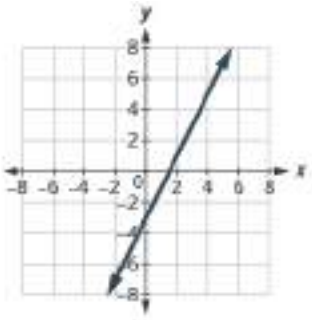
Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.



The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.



Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

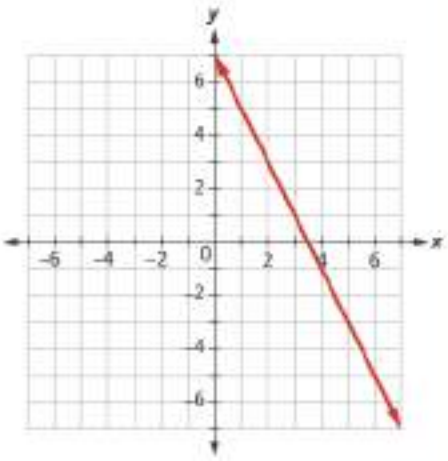
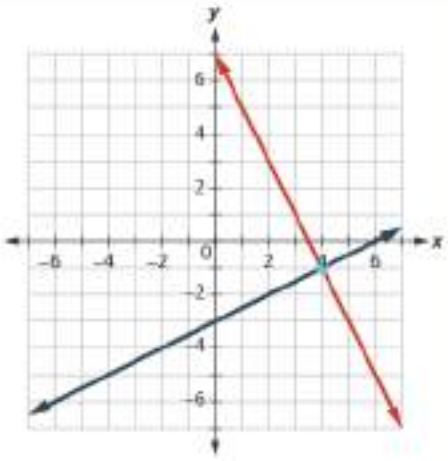
Figure 4.2

Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end of the section you'll decide which method was the most convenient way to solve this system.

EXAMPLE 4.2 HOW TO SOLVE A SYSTEM OF EQUATIONS BY GRAPHING

Solve the system by graphing $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$.

Solution

Step 1. Graph the first equation.	To graph the first line, write the equation in slope–intercept form. $2x + y = 7$ $y = -2x + 7$ $m = -2 \qquad b = 7$	$2x + y = 7$ $x - 2y = 6$ 
Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x - 2y = 6$ $(0, -3) \quad (6, 0)$	
Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4. Identify the solution to the system. • If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. • If the lines are parallel, the system has no solution. • If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at $(4, -1)$. $2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $8 - 1 \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is $(4, -1)$.

TRY IT : : 4.3 Solve the system by graphing: $\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$.

TRY IT : : 4.4 Solve the system by graphing: $\begin{cases} -x+y=1 \\ 3x+2y=12 \end{cases}$.

The steps to use to solve a system of linear equations by graphing are shown here.

HOW TO : : SOLVE A SYSTEM OF LINEAR EQUATIONS BY GRAPHING.

Step 1. Graph the first equation.

Step 2. Graph the second equation on the same rectangular coordinate system.

Step 3. Determine whether the lines intersect, are parallel, or are the same line.

Step 4. Identify the solution to the system.

- If the lines intersect, identify the point of intersection. This is the solution to the system.
- If the lines are parallel, the system has no solution.
- If the lines are the same, the system has an infinite number of solutions.

Step 5. Check the solution in both equations.

In the next example, we'll first re-write the equations into slope–intercept form as this will make it easy for us to quickly graph the lines.

EXAMPLE 4.3

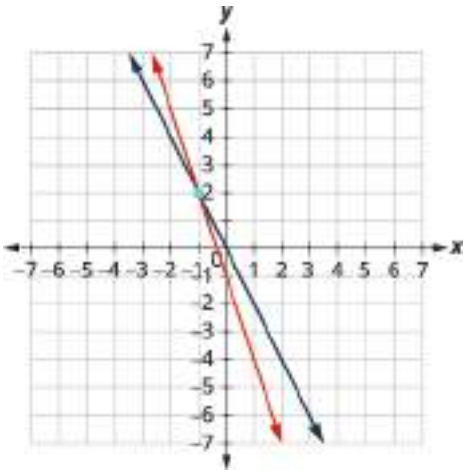
Solve the system by graphing: $\begin{cases} 3x+y=-1 \\ 2x+y=0 \end{cases}$.

Solution

We'll solve both of these equations for y so that we can easily graph them using their slopes and *y*-intercepts.

	$\begin{array}{l} 3x+y=-1 \\ 2x+y=0 \end{array}$
Solve the first equation for *y*.	$\begin{array}{l} 3x+y=-1 \\ y=-3x-1 \end{array}$
Find the slope and *y*-intercept.	$\begin{array}{l} m=-3 \\ b=-1 \end{array}$
Solve the second equation for *y*.	$\begin{array}{l} 2x+y=0 \\ y=-2x \end{array}$
Find the slope and *y*-intercept.	$\begin{array}{l} m=-2 \\ b=0 \end{array}$

Graph the lines.



Determine the point of intersection.	The lines intersect at $(-1, 2)$.
Check the solution in both equations.	

$$\begin{aligned} 3x+y&=-1 \\ 3(-1)+2&\stackrel{?}{=}-1 \\ -1&=-1\ \checkmark \end{aligned} \qquad \begin{aligned} 2x+y&=0 \\ 2(-1)+2&\stackrel{?}{=}0 \\ 0&=0\ \checkmark \end{aligned}$$

	The solution is $(-1, 2)$.

TRY IT : : 4.5 Solve the system by graphing: $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$.

TRY IT : : 4.6 Solve the system by graphing: $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$.

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

EXAMPLE 4.4

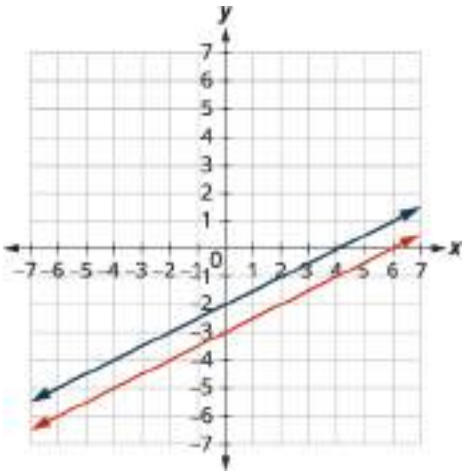
Solve the system by graphing: $\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$.

Solution

$$\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$$

To graph the first equation, we will use its slope and *y*-intercept.	$y=\frac{1}{2}x-3$ $m=\frac{1}{2}$ $b=-3$
To graph the second equation, we use the intercepts.	$x-2y=4$

x	y
0	−2
4	0

Graph the lines.	
Determine the points of intersection.	The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

TRY IT : : 4.7

Solve the system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = -8 \end{cases}$.

TRY IT : : 4.8

Solve the system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$.

Sometimes the equations in a system represent the same line. Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to the system.

EXAMPLE 4.5

Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$.

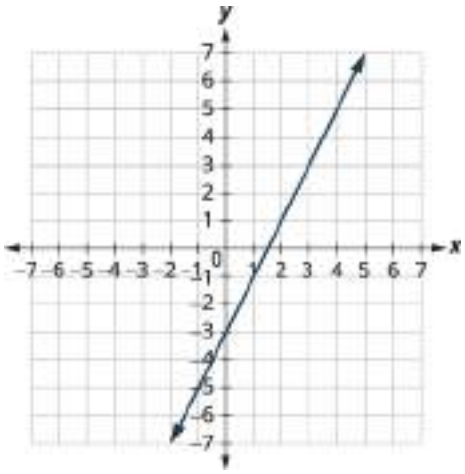
Solution

$$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$$

Find the slope and *y*-intercept of the first equation.	$y = 2x - 3$ $m = 2$ $b = -3$
Find the intercepts of the second equation.	$-6x + 3y = -9$

x	y
0	−3
$\frac{3}{2}$	0

Graph the lines.



The lines are the same!
Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true.
There are infinitely many solutions to this system.

If you write the second equation in slope-intercept form, you may recognize that the equations have the same slope and same *y*-intercept.

TRY IT : : 4.9 Solve the system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$.

TRY IT : : 4.10 Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are **coincident**. Coincident lines have the same slope and same *y*-intercept.

Coincident Lines

Coincident lines have the same slope and same *y*-intercept.

The systems of equations in Example 4.2 and Example 4.3 each had two intersecting lines. Each system had one solution.

In Example 4.5, the equations gave coincident lines, and so the system had infinitely many solutions.

The systems in those three examples had at least one solution. A system of equations that has at least one solution is called a *consistent* system.

A system with parallel lines, like Example 4.4, has no solution. We call a system of equations like this *inconsistent.* It has no solution.

Consistent and Inconsistent Systems

A **consistent system of equations** is a system of equations with at least one solution.

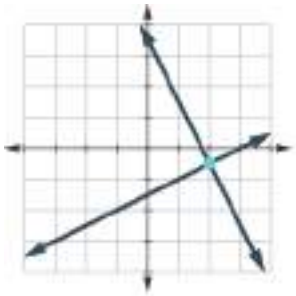
An **inconsistent system of equations** is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two

equations are independent, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines.

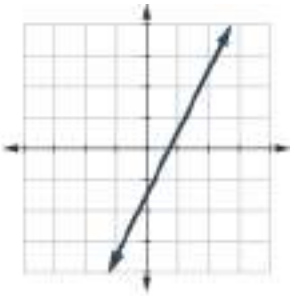
Let's sum this up by looking at the graphs of the three types of systems. See below and Table 4.4.



Intersecting



Parallel



Coincident

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/ independent	Independent	Independent	Dependent

Table 4.4

EXAMPLE 4.6

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$ ⓑ $\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$

Solution

ⓐ We will compare the slopes and intercepts of the two lines.

The first equation is already in slope-intercept form.

$$\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$$

$$y = 3x - 1$$

Write the second equation in slope-intercept form.

$$\begin{aligned} 6x - 2y &= 12 \\ -2y &= -6x + 12 \\ \frac{-2y}{-2} &= \frac{-6x + 12}{-2} \\ y &= 3x - 6 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= 3x - 1 \\ m &= 3 \\ b &= -1 \end{aligned} \qquad \begin{aligned} y &= 3x - 6 \\ m &= 3 \\ b &= -6 \end{aligned}$$

Since the slopes are the same and y-intercepts are different, the lines are parallel.

A system of equations whose graphs are parallel lines has no solution and is inconsistent and independent.

ⓑ We will compare the slope and intercepts of the two lines.

$$\begin{cases} 2x+y &= -3 \\ x-5y &= 5 \end{cases}$$

Write both equations in slope–intercept form.

$$\begin{aligned} 2x+y &= -3 \\ y &= -2x-3 \end{aligned} \qquad \begin{aligned} x-5y &= 5 \\ -5y &= -x+5 \\ \frac{-5y}{-5} &= \frac{-x+5}{-5} \\ y &= \frac{1}{5}x-1 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= -2x-3 \\ m &= -2 \\ b &= -3 \end{aligned} \qquad \begin{aligned} y &= \frac{1}{5}x-1 \\ m &= \frac{1}{5} \\ b &= -1 \end{aligned}$$

Since the slopes are diffe ent, the lines intersect.

A system of equations whose graphs are intersect has 1 solution and is consistent and independent.

TRY IT : : 4.11

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y=-2x-4 \\ 4x+2y=9 \end{cases}$ ⓑ $\begin{cases} 3x+2y=2 \\ 2x+y=1 \end{cases}$

TRY IT : : 4.12

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y=\frac{1}{3}x-5 \\ x-3y=6 \end{cases}$ ⓑ $\begin{cases} x+4y=12 \\ -x+y=3 \end{cases}$

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between -10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

Solve a System of Equations by Substitution

We will now solve systems of linear equations by the substitution method.

We will use the same system we used first for graphing.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

EXAMPLE 4.7 HOW TO SOLVE A SYSTEM OF EQUATIONS BY SUBSTITUTION

Solve the system by substitution: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$.

Solution

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for y.	$2x + y = 7$ $x - 2y = 6$ $2x + y = 7$ $y = 7 - 2x$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$
Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x - 2(7 - 2x) = 6$ $x - 14 + 4x = 6$ $5x = 20$ $x = 4$
Step 4. Substitute the solution from Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x + y = 7$ $2(4) + y = 7$ $8 + y = 7$ $y = -1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = 4, y = -1$ into both equations and make sure they are both true.	$2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

TRY IT : : 4.13 Solve the system by substitution: $\begin{cases} -2x + y = -11 \\ x + 3y = 9 \end{cases}$.

TRY IT : : 4.14 Solve the system by substitution: $\begin{cases} 2x + y = -1 \\ 4x + 3y = 3 \end{cases}$.

HOW TO : : SOLVE A SYSTEM OF EQUATIONS BY SUBSTITUTION.

Step 1. Solve one of the equations for either variable.
Step 2. Substitute the expression from Step 1 into the other equation.
Step 3. Solve the resulting equation.
Step 4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
Step 5. Write the solution as an ordered pair.
Step 6. Check that the ordered pair is a solution to **both** original equations.

Be very careful with the signs in the next example.

EXAMPLE 4.8

Solve the system by substitution: $\begin{cases} 4x + 2y = 4 \\ 6x - y = 8 \end{cases}$.

Solution

We need to solve one equation for one variable. We will solve the first equation for y.

	$4x + 2y = 4$ $6x - y = 8$
Solve the first equation for y. Substitute $-2x + 2$ for y in the second equation.	$4x + 2y = 4$ $2y = -4x + 4$ $y = -2x + 2$ $6x - y = 8$
Replace the y with $-2x + 2$.	$6x - (-2x + 2) = 8$
Solve the equation for x.	$6x + 2x - 2 = 8$ $8x - 2 = 8$ $8x = 10$ $x = \frac{5}{4}$
Substitute $x = \frac{5}{4}$ into $4x + 2y = 4$ to find y.	$4x + 2y = 4$ $4\left(\frac{5}{4}\right) + 2y = 4$ $5 + 2y = 4$ $2y = -1$ $y = -\frac{1}{2}$
	The ordered pair is $\left(\frac{5}{4}, -\frac{1}{2}\right)$.
Check the ordered pair in both equations.	

$$4x + 2y = 4 \qquad 6x - y = 8$$
$$4\left(\frac{5}{4}\right) + 2\left(-\frac{1}{2}\right) \stackrel{?}{=} 4 \qquad 6\left(\frac{5}{4}\right) - \left(-\frac{1}{2}\right) \stackrel{?}{=} 8$$
$$5 - 1 \stackrel{?}{=} 4 \qquad \frac{15}{4} - \left(-\frac{1}{2}\right) \stackrel{?}{=} 8$$
$$4 = 4 \checkmark \qquad \frac{16}{2} \stackrel{?}{=} 8$$
$$8 = 8 \checkmark$$

	The solution is $\left(\frac{5}{4}, -\frac{1}{2}\right)$.

TRY IT : : 4.15 Solve the system by substitution: $\begin{cases} x - 4y = -4 \\ -3x + 4y = 0 \end{cases}$.

TRY IT : : 4.16 Solve the system by substitution: $\begin{cases} 4x - y = 0 \\ 2x - 3y = 5 \end{cases}$.

Solve a System of Equations by Elimination

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation

for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a, b, c,* and *d*.

$$\begin{array}{lrcl} \text{if} & a & = & b \\ \text{and} & c & = & d \\ \text{then} & a+c & = & b+d. \end{array}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{cases} 3x+y=5 \\ 2x-y=0 \end{cases}$$
$$5x = 5$$

The *y*'s add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{cases} x+4y=2 \\ 2x+5y=-2 \end{cases}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by $-2,$ we will make the coefficients of *x* opposites. We must multiply every term on both sides of the equation by $-2.$

$$\begin{cases} -2(x+4y)=-2(2) \\ 2x+5y=-2 \end{cases}$$

Then rewrite the system of equations.

$$\begin{cases} -2x-8y=-4 \\ 2x+5y=-2 \end{cases}$$

Now we see that the coefficients of the *x* terms are opposites, so *x* will be eliminated when we add these two equations.

$$\begin{array}{r} -2x-8y=-4 \\ 2x+5y=-2 \\ \hline -3y=-6 \end{array}$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

EXAMPLE 4.9 HOW TO SOLVE A SYSTEM OF EQUATIONS BY ELIMINATION

Solve the system by elimination: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}.$

Solution

Step 1. Write both equations in standard form. • If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax+By=C.$ There are no fractions.	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Step 2. Make the coefficients of one variable opposites. • Decide which variable you will eliminate. • Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{cases} 4x + 2y = 14 \\ x - 2y = 6 \end{cases}$ $5x \quad = 20$
Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = 4, y = -1$ into $2x + y = 7$ and $x - 2y = 6$. Do they make both equations true? Yes!	$2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is $(4, -1)$.

TRY IT : : 4.17 Solve the system by elimination: $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$.

TRY IT : : 4.18 Solve the system by elimination: $\begin{cases} 4x + y = -5 \\ -2x - 2y = -2 \end{cases}$.

The steps are listed here for easy reference.

HOW TO : : SOLVE A SYSTEM OF EQUATIONS BY ELIMINATION.

Step 1. Write both equations in standard form. If any coefficients are fractions, clear them.

Step 2. Make the coefficients of one variable opposites.
- Decide which variable you will eliminate.
- Multiply one or both equations so that the coefficients of that variable are opposites.

Step 3. Add the equations resulting from Step 2 to eliminate one variable.

Step 4. Solve for the remaining variable.

Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.

Step 6. Write the solution as an ordered pair.

Step 7. Check that the ordered pair is a solution to **both** original equations.

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

EXAMPLE 4.10

Solve the system by elimination: $\begin{cases} 4x - 3y = 9 \\ 7x + 2y = -6 \end{cases}$.

Solution

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by different constants to get the opposites.

	$\begin{aligned} 4x - 3y &= 9 \\ 7x + 2y &= -6 \end{aligned}$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$\begin{aligned} 2(4x - 3y) &= 2(9) \\ 3(7x + 2y) &= 3(-6) \end{aligned}$
Simplify.	$\begin{aligned} 8x - 6y &= 18 \\ 21x + 6y &= -18 \end{aligned}$
Add the two equations to eliminate y.	$\begin{aligned} 8x - 6y &= 18 \\ 21x + 6y &= -18 \\ \hline 29x &= 0 \end{aligned}$
Solve for x.	$\begin{aligned} x &= 0 \\ 7x + 2y &= -6 \end{aligned}$
Substitute $x = 0$ into one of the original equations.	$7 \cdot 0 + 2y = -6$
Solve for y.	$\begin{aligned} 2y &= -6 \\ y &= -3 \end{aligned}$
Write the solution as an ordered pair.	The ordered pair is $(0, -3)$.
Check that the ordered pair is a solution to **both** original equations.	

$$\begin{aligned} 4x - 3y &= 9 \\ 4(0) - 3(-3) &\stackrel{?}{=} 9 \\ 9 &= 9 \checkmark \end{aligned} \qquad \begin{aligned} 7x + 2y &= -6 \\ 7(0) + 2(-3) &\stackrel{?}{=} -6 \\ -6 &= -6 \checkmark \end{aligned}$$

	The solution is $(0, -3)$.

TRY IT : : 4.19 Solve the system by elimination: $\begin{cases} 3x - 4y = -9 \\ 5x + 3y = 14 \end{cases}$.

TRY IT : : 4.20 Solve each system by elimination: $\begin{cases} 7x + 8y = 4 \\ 3x - 5y = 27 \end{cases}$.

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by the LCD of all the fractions in the equation.

EXAMPLE 4.11

Solve the system by elimination: $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$.

Solution

In this example, both equations have fractions. Our first step will be to multiply each equation by the LCD of all the fractions in the equation to clear the fractions.

	$x+\frac{1}{2}y=6$ $\frac{3}{2}x+\frac{2}{3}y=\frac{17}{2}$
To clear the fractions, multiply each equation by its LCD.	$2\left(x+\frac{1}{2}y\right)=2(6)$ $6\left(\frac{3}{2}x+\frac{2}{3}y\right)=6\left(\frac{17}{2}\right)$
Simplify.	$2x+y=12$ $9x+4y=51$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.	
We can eliminate y by multiplying the top equation by -4.	$-4(2x+y)=-4(12)$ $9x+4y=51$
Simplify and add. Substitute $x=3$ into one of the original equations.	$-8x-4y=-48$ $9x+4y=51$ $x=3$ $x+\frac{1}{2}y=6$
Solve for y.	$3+\frac{1}{2}y=6$ $\frac{1}{2}y=3$ $y=6$
Write the solution as an ordered pair.	The ordered pair is $(3, 6)$.
Check that the ordered pair is a solution to both original equations.	$x+\frac{1}{2}y=6$ $3+\frac{1}{2}(6)\stackrel{?}{=}6$ $3+3\stackrel{?}{=}6$ $6=6$ ✓ $\frac{3}{2}x+\frac{2}{3}y=\frac{17}{2}$ $\frac{3}{2}(3)+\frac{2}{3}(6)\stackrel{?}{=}\frac{17}{2}$ $\frac{9}{2}+4\stackrel{?}{=}\frac{17}{2}$ $\frac{9}{2}+\frac{8}{2}\stackrel{?}{=}\frac{17}{2}$ $\frac{17}{2}=\frac{17}{2}$ ✓
	The solution is $(3, 6)$.

TRY IT : : 4.21

Solve each system by elimination: $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$.

TRY IT : : 4.22

Solve each system by elimination: $\begin{cases} x+\frac{3}{5}y=-\frac{1}{5} \\ -\frac{1}{2}x-\frac{2}{3}y=\frac{5}{6} \end{cases}$.

When we solved the system by graphing, we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

The same is true using substitution or elimination. If the equation at the end of substitution or elimination is a true statement, we have a consistent but dependent system and the system of equations has infinitely many solutions. If the equation at the end of substitution or elimination is a false statement, we have an inconsistent system and the system of equations has no solution.

EXAMPLE 4.12

Solve the system by elimination: $\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$.

Solution

$$\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$$

Write the second equation in standard form. $\begin{cases} 3x+4y=12 \\ \frac{3}{4}x+y=3 \end{cases}$

Clear the fractions by multiplying the second equation by 4. $\begin{cases} 3x+4y=12 \\ 4\left(\frac{3}{4}x+y\right)=4(3) \end{cases}$

Simplify. $\begin{cases} 3x+4y=12 \\ 3x+4y=12 \end{cases}$

To eliminate a variable, we multiply the second equation by -1. Simplify and add.

$$\begin{array}{l} 3x+4y=12 \\ \underline{-3x-4y=-12} \\ 0=0 \end{array}$$

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

TRY IT : : 4.23

Solve the system by elimination: $\begin{cases} 5x-3y=15 \\ y=-5+\frac{5}{3}x \end{cases}$.

TRY IT : : 4.24

Solve the system by elimination: $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$.

Choose the Most Convenient Method to Solve a System of Linear Equations

When you solve a system of linear equations in in an application, you will not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations are in standard form.

EXAMPLE 4.13

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$ ⓑ $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Solution

ⓐ

$$\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$$

Since both equations are in standard form, using elimination will be most convenient.

ⓑ

$$\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$$

Since one equation is already solved for y, using substitution will be most convenient.

TRY IT : : 4.25

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$ ⓑ $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

TRY IT : : 4.26

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} y=2x-1 \\ 3x-4y=-6 \end{cases}$ ⓑ $\begin{cases} 6x-2y=12 \\ 3x+7y=-13 \end{cases}$

4.1 EXERCISES

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Equations

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} 2x-6y=0 \\ 3x-4y=5 \end{cases}$

ⓐ $(3,\ 1)$

ⓑ $(-3,\ 4)$

2. $\begin{cases} -3x+y=8 \\ -x+2y=-9 \end{cases}$

ⓐ $(-5,\ -7)$

ⓑ $(-5,\ 7)$

3. $\begin{cases} x+y=2 \\ y=\frac{3}{4}x \end{cases}$

ⓐ $\left(\frac{8}{7},\ \frac{6}{7}\right)$

ⓑ $\left(1,\ \frac{3}{4}\right)$

4. $\begin{cases} 2x+3y=6 \\ y=\frac{2}{3}x+2 \end{cases}$

ⓐ $(-6,\ 2)$

ⓑ $(-3,\ 4)$

Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

5. $\begin{cases} 3x+y=-3 \\ 2x+3y=5 \end{cases}$

6. $\begin{cases} -x+y=2 \\ 2x+y=-4 \end{cases}$

7. $\begin{cases} y=x+2 \\ y=-2x+2 \end{cases}$

8. $\begin{cases} y=x-2 \\ y=-3x+2 \end{cases}$

9. $\begin{cases} y=\frac{3}{2}x+1 \\ y=-\frac{1}{2}x+5 \end{cases}$

10. $\begin{cases} y=\frac{2}{3}x-2 \\ y=-\frac{1}{3}x-5 \end{cases}$

11. $\begin{cases} x+y=-4 \\ -x+2y=-2 \end{cases}$

12. $\begin{cases} -x+3y=3 \\ x+3y=3 \end{cases}$

13. $\begin{cases} -2x+3y=3 \\ x+3y=12 \end{cases}$

14. $\begin{cases} 2x-y=4 \\ 2x+3y=12 \end{cases}$

15. $\begin{cases} x+3y=-6 \\ y=-\frac{4}{3}x+4 \end{cases}$

16. $\begin{cases} -x+2y=-6 \\ y=-\frac{1}{2}x-1 \end{cases}$

17. $\begin{cases} -2x+4y=4 \\ y=\frac{1}{2}x \end{cases}$

18. $\begin{cases} 3x+5y=10 \\ y=-\frac{3}{5}x+1 \end{cases}$

19. $\begin{cases} 4x-3y=8 \\ 8x-6y=14 \end{cases}$

20. $\begin{cases} x+3y=4 \\ -2x-6y=3 \end{cases}$

21. $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$

22. $\begin{cases} 4x=3y+7 \\ 8x-6y=14 \end{cases}$

23. $\begin{cases} 2x+y=6 \\ -8x-4y=-24 \end{cases}$

24. $\begin{cases} 5x+2y=7 \\ -10x-4y=-14 \end{cases}$

Without graphing, determine the number of solutions and then classify the system of equations.

25. $\begin{cases} y = \frac{2}{3}x + 1 \\ -2x + 3y = 5 \end{cases}$

26. $\begin{cases} y = \frac{3}{2}x + 1 \\ 2x - 3y = 7 \end{cases}$

27. $\begin{cases} 5x + 3y = 4 \\ 2x - 3y = 5 \end{cases}$

28. $\begin{cases} y = -\frac{1}{2}x + 5 \\ x + 2y = 10 \end{cases}$

29. $\begin{cases} 5x - 2y = 10 \\ y = \frac{5}{2}x - 5 \end{cases}$

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

30. $\begin{cases} 2x + y = -4 \\ 3x - 2y = -6 \end{cases}$

31. $\begin{cases} 2x + y = -2 \\ 3x - y = 7 \end{cases}$

32. $\begin{cases} x - 2y = -5 \\ 2x - 3y = -4 \end{cases}$

33. $\begin{cases} x - 3y = -9 \\ 2x + 5y = 4 \end{cases}$

34. $\begin{cases} 5x - 2y = -6 \\ y = 3x + 3 \end{cases}$

35. $\begin{cases} -2x + 2y = 6 \\ y = -3x + 1 \end{cases}$

36. $\begin{cases} 2x + 5y = 1 \\ y = \frac{1}{3}x - 2 \end{cases}$

37. $\begin{cases} 3x + 4y = 1 \\ y = -\frac{2}{5}x + 2 \end{cases}$

38. $\begin{cases} 2x + y = 5 \\ x - 2y = -15 \end{cases}$

39. $\begin{cases} 4x + y = 10 \\ x - 2y = -20 \end{cases}$

40. $\begin{cases} y = -2x - 1 \\ y = -\frac{1}{3}x + 4 \end{cases}$

41. $\begin{cases} y = x - 6 \\ y = -\frac{3}{2}x + 4 \end{cases}$

42. $\begin{cases} x = 2y \\ 4x - 8y = 0 \end{cases}$

43. $\begin{cases} 2x - 16y = 8 \\ -x - 8y = -4 \end{cases}$

44. $\begin{cases} y = \frac{7}{8}x + 4 \\ -7x + 8y = 6 \end{cases}$

45. $\begin{cases} y = -\frac{2}{3}x + 5 \\ 2x + 3y = 11 \end{cases}$

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination.

46. $\begin{cases} 5x + 2y = 2 \\ -3x - y = 0 \end{cases}$

47. $\begin{cases} 6x - 5y = -1 \\ 2x + y = 13 \end{cases}$

48. $\begin{cases} 2x - 5y = 7 \\ 3x - y = 17 \end{cases}$

49. $\begin{cases} 5x - 3y = -1 \\ 2x - y = 2 \end{cases}$

50. $\begin{cases} 3x - 5y = -9 \\ 5x + 2y = 16 \end{cases}$

51. $\begin{cases} 4x - 3y = 3 \\ 2x + 5y = -31 \end{cases}$

52. $\begin{cases} 3x + 8y = -3 \\ 2x + 5y = -3 \end{cases}$

53. $\begin{cases} 11x + 9y = -5 \\ 7x + 5y = -1 \end{cases}$

54. $\begin{cases} 3x + 8y = 67 \\ 5x + 3y = 60 \end{cases}$

55. $\begin{cases} 2x + 9y = -4 \\ 3x + 13y = -7 \end{cases}$

56. $\begin{cases} \frac{1}{3}x - y = -3 \\ x + \frac{5}{2}y = 2 \end{cases}$

57. $\begin{cases} x + \frac{1}{2}y = \frac{3}{2} \\ \frac{1}{5}x - \frac{1}{5}y = 3 \end{cases}$

58. $\begin{cases} x+\frac{1}{3}y=-1 \\ \frac{1}{3}x+\frac{1}{2}y=1 \end{cases}$

59. $\begin{cases} \frac{1}{3}x-y=-3 \\ \frac{2}{3}x+\frac{5}{2}y=3 \end{cases}$

60. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$

61. $\begin{cases} x-4y=-1 \\ -3x+12y=3 \end{cases}$

62. $\begin{cases} -3x-y=8 \\ 6x+2y=-16 \end{cases}$

63. $\begin{cases} 4x+3y=2 \\ 20x+15y=10 \end{cases}$

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

64.

ⓐ $\begin{cases} 8x-15y=-32 \\ 6x+3y=-5 \end{cases}$

ⓑ $\begin{cases} x=4y-3 \\ 4x-2y=-6 \end{cases}$

65.

ⓐ $\begin{cases} y=7x-5 \\ 3x-2y=16 \end{cases}$

ⓑ $\begin{cases} 12x-5y=-42 \\ 3x+7y=-15 \end{cases}$

66.

ⓐ $\begin{cases} y=4x+9 \\ 5x-2y=-21 \end{cases}$

ⓑ $\begin{cases} 9x-4y=24 \\ 3x+5y=-14 \end{cases}$

67.

ⓐ $\begin{cases} 14x-15y=-30 \\ 7x+2y=10 \end{cases}$

ⓑ $\begin{cases} x=9y-11 \\ 2x-7y=-27 \end{cases}$

Writing Exercises

68. In a system of linear equations, the two equations have the same intercepts. Describe the possible solutions to the system.

69. Solve the system of equations by substitution and explain all your steps in words: $\begin{cases} 3x+y=12 \\ x=y-8 \end{cases}$.

70. Solve the system of equations by elimination and explain all your steps in words: $\begin{cases} 5x+4y=10 \\ 2x=3y+27 \end{cases}$.

71. Solve the system of equations $\begin{cases} x+y=10 \\ x-y=6 \end{cases}$

ⓐ by graphing ⓑ by substitution

ⓒ Which method do you prefer? Why?

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered pair is a solution of a system of equations.			
solve a system of linear equations by graphing.			
solve a system of equations by substitution.			
solve a system of equations by elimination.			
choose the most convenient method to solve a system of linear equations.			

If most of your checks were:

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

4.2 Solve Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve direct translation applications
- Solve geometry applications
- Solve uniform motion applications

Be Prepared!

Before you get started, take this readiness quiz.

1. The sum of twice a number and nine is 31. Find the number.
 If you missed this problem, review Example 2.15.
2. Twins Jon and Ron together earned $96,000 last year. Ron earned $8000 more than three times what Jon earned. How much did each of the twins earn?
 If you missed this problem, review Example 2.19.
3. An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in four hours and the local train takes five hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.
 If you missed this problem, review Example 2.43.

Solve Direct Translation Applications

Systems of linear equations are very useful for solving applications. Some people find setting up word problems with two variables easier than setting them up with just one variable. To solve an application, we'll first translate the words into a system of linear equations. Then we will decide the most convenient method to use, and then solve the system.



HOW TO : : SOLVE APPLICATIONS WITH SYSTEMS OF EQUATIONS.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what we are looking for.
Step 3. **Name** what we are looking for. Choose variables to represent those quantities.
Step 4. **Translate** into a system of equations.
Step 5. **Solve** the system of equations using good algebra techniques.
Step 6. **Check** the answer in the problem and make sure it makes sense.
Step 7. **Answer** the question with a complete sentence.

We solved number problems with one variable earlier. Let's see how differently it works using two variables.

EXAMPLE 4.14

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let $n =$ the first number. $m =$ the second number
Step 4. Translate into a system of equations.	The sum of two numbers is zero.

	$n+m=0$
	One number is nine less than the other.
	$n=m-9$
The system is:	$\begin{cases} n+m=0 \\ n=m-9 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for n.	
Substitute $m-9$ for n in the first equation.	$n=m-9$ $n+m=0$
Solve for m.	$m-9+m=0$
	$2m-9=0$
	$2m=9$
Substitute $m=\frac{9}{2}$ into the second equation and then solve for n.	$m=\frac{9}{2}$ $n=m-9$
	$n=\frac{9}{2}-9$
	$n=\frac{9}{2}-\frac{18}{2}$
	$n=-\frac{9}{2}$
Step 6. Check the answer in the problem.	Do these numbers make sense in the problem? We will leave this to you!
Step 7. Answer the question.	The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$.

TRY IT : : 4.27 The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

TRY IT : : 4.28 The sum of two numbers is -6. One number is 10 less than the other. Find the numbers.

EXAMPLE 4.15

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her $25,000 plus $15 for each training session. Option B would pay her $\$10{,}000+\$40$ for each training session. How many training sessions would make the salary options equal?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of training sessions that would make the pay equal.
Step 3. Name what we are looking for.	Let $s =$ Heather's salary. $n =$ the number of training sessions
Step 4. Translate into a system of equations.	Option A would pay her \$25,000 plus \$15 for each training session.
	$s = 25{,}000 + 15n$
	Option B would pay her \$10,000 + \$40 for each training session.
	$s = 10{,}000 + 40n$
The system is shown.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Step 5. Solve the system of equations. We will use substitution.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Substitute $25{,}000 + 15n$ for s in the second equation.	$25{,}000 + 15n = 10{,}000 + 40n$
Solve for n.	$25{,}000 = 10{,}000 + 25n$ $15{,}000 = 25n$ $600 = n$
Step 6. Check the answer.	Are 600 training sessions a year reasonable? Are the two options equal when $n = 600$?
Step 7. Answer the question.	The salary options would be equal for 600 training sessions.

TRY IT : : 4.29

Geraldine has been offered positions by two insurance companies. The first company pays a salary of \$12,000 plus a commission of \$100 for each policy sold. The second pays a salary of \$20,000 plus a commission of \$50 for each policy sold. How many policies would need to be sold to make the total pay the same?

TRY IT : : 4.30

Kenneth currently sells suits for company A at a salary of \$22,000 plus a \$10 commission for each suit sold. Company B offers him a position with a salary of \$28,000 plus a \$4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

As you solve each application, remember to analyze which method of solving the system of equations would be most convenient.

EXAMPLE 4.16

Translate to a system of equations and then solve:

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she

burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories for each minute of circuit training?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories burned each minute on the elliptical trainer and each minute of circuit training.
Step 3. Name what we are looking for.	Let $e =$ number of calories burned per minute on the elliptical trainer. $c =$ number of calories burned per minute while circuit training
Step 4. Translate into a system of equations.	10 minutes on the elliptical and circuit training for 20 minutes, burned 278 calories
	$10e + 20c = 278$
	20 minutes on the elliptical and 30 minutes of circuit training burned 473 calories
	$20e + 30c = 473$
The system is:	$\begin{aligned} 10e + 20c &= 278 \\ 20e + 30c &= 473 \end{aligned}$
Step 5. Solve the system of equations.	
Multiply the first equation by −2 to get opposite coefficients of e.	$\begin{aligned} -2(10e + 20c) &= -2(278) \\ 20e + 30c &= 473 \end{aligned}$
Simplify and add the equations. Solve for c.	$\begin{aligned} -20e - 40c &= -556 \\ 20e + 30c &= 473 \\ \hline -10c &= -83 \\ c &= 8.3 \end{aligned}$
Substitute $c = 8.3$ into one of the original equations to solve for e.	$\begin{aligned} 10e + 20c &= 278 \\ 10e + 20(8.3) &= 278 \\ 10e + 166 &= 278 \\ 10e &= 112 \\ e &= 11.2 \end{aligned}$
Step 6. Check the answer in the problem.	Check the math on your own.
	$\begin{aligned} 10(11.2) + 20(8.3) &\stackrel{?}{=} 278 \\ 20(11.2) + 30(8.3) &\stackrel{?}{=} 473 \end{aligned}$
Step 7. Answer the question.	Jenna burns 8.3 calories per minute circuit training and 11.2 calories per minute while on the elliptical trainer.

TRY IT : : 4.31

Translate to a system of equations and then solve:

Mark went to the gym and did 40 minutes of Bikram hot yoga and 10 minutes of jumping jacks. He burned 510 calories. The next time he went to the gym, he did 30 minutes of Bikram hot yoga and 20 minutes of jumping jacks burning 470 calories. How many calories were burned for each minute of yoga? How many calories were burned for each minute of jumping jacks?

TRY IT : : 4.32

Translate to a system of equations and then solve:

Erin spent 30 minutes on the rowing machine and 20 minutes lifting weights at the gym and burned 430 calories. During her next visit to the gym she spent 50 minutes on the rowing machine and 10 minutes lifting weights and burned 600 calories. How many calories did she burn for each minutes on the rowing machine? How many calories did she burn for each minute of weight lifting?

Solve Geometry Applications

We will now solve geometry applications using systems of linear equations. We will need to add complementary angles and supplementary angles to our list some properties of angles.

The measures of two **complementary angles** add to 90 degrees. The measures of two **supplementary angles** add to 180 degrees.

Complementary and Supplementary Angles

Two angles are complementary if the sum of the measures of their angles is 90 degrees.

Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

If two angles are complementary, we say that *one angle is the complement of the other.*

If two angles are supplementary, we say that *one angle is the supplement of the other.*

EXAMPLE 4.17

Translate to a system of equations and then solve.

The difference of two complementary angles is 26 degrees. Find the measures of the angles.

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for.

We are looking for the measure of each angle.

Step 3. Name what we are looking for.

Let $x =$ the measure of the first angle.

$y =$ the measure of the second angle

Step 4. Translate into a system of equations.

The angles are complementary.

$$x + y = 90$$

The difference of the two angles is 26 degrees.

$$x - y = 26$$

The system is shown.

$$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$$

Step 5. Solve the system of equations by elimination.

$$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$$

$$\begin{aligned} 2x &= 116 \\ x &= 58 \end{aligned}$$

Substitute $x = 58$ into the first equation.

$$\begin{aligned} x + y &= 90 \\ 58 + y &= 90 \\ y &= 32 \end{aligned}$$

Step 6. Check the answer in the problem.

$58 + 32 = 90$✓

$58 - 32 = 26$✓

Step 7. Answer the question.

The angle measures are 58 and 32 degrees.

TRY IT : : 4.33

Translate to a system of equations and then solve:

The difference of two complementary angles is 20 degrees. Find the measures of the angles.

TRY IT : : 4.34

Translate to a system of equations and then solve:

The difference of two complementary angles is 80 degrees. Find the measures of the angles.

In the next example, we remember that the measures of supplementary angles add to 180.

EXAMPLE 4.18

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is twelve degrees less than five times the measure of the smaller angle. Find the measures of both angles.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for measure of each angle.

Step 3. Name what we are looking for.	Let $x =$ the measure of the first angle. $y =$ the measure of the second angle
Step 4. Translate into a system of equations.	The angles are supplementary.
	$x+y=180$
	The larger angle is twelve less than five times the smaller angle.
	$y=5x-12$
The system is shown. **Step 5. Solve** the system of equations substitution.	$x+y=180$ $y=5x-12$ $x+y=180$
Substitute $5x-12$ for y in the first equation. Solve for x.	$x+5x-12=180$ $6x-12=180$
Substitute 32 for x in the second equation, then solve for y.	$6x=192$ $x=32$ $y=5x-12$ $y=5\cdot32-12$ $y=160-12$ $y=148$
Step 6. Check the answer in the problem.	$32+148=180$ ✓ $5\cdot32-12=148$ ✓
Step 7. Answer the question.	The angle measures are 148 and 32 degrees.

TRY IT : : 4.35

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 12 degrees more than three times the smaller angle. Find the measures of the angles.

TRY IT : : 4.36

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 18 less than twice the measure of the smaller angle. Find the measures of the angles.

Recall that the angles of a triangle add up to 180 degrees. A right triangle has one angle that is 90 degrees. What does that tell us about the other two angles? In the next example we will be finding the measures of the other two angles.

EXAMPLE 4.19

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Solution

We will draw and label a figure.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the measures of the angles.
Step 3. Name what we are looking for.	Let $a =$ the measure of the first angle. $b =$ the measure of the second angle
Step 4. Translate into a system of equations.	The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle.
	$a = 3b + 10$
	The sum of the measures of the angles of a triangle is 180.
	$a + b + 90 = 180$
The system is shown.	$a = 3b + 10$ $a + b + 90 = 180$
Step 5. Solve the system of equations. We will use substitution since the first equation is solved for a.	$a = 3b + 10$ $a + b + 90 = 180$
Substitute $3b + 10$ for a in the second equation.	$(3b + 10) + b + 90 = 180$
Solve for b.	$4b + 100 = 180$ $4b = 80$ $b = 20$ $a = 3b + 10$
Substitute $b = 20$ into the first equation and then solve for a.	$a = 3 \cdot 20 + 10$ $a = 70$
Step 6. Check the answer in the problem.	We will leave this to you!
Step 7. Answer the question.	The measures of the small angles are 20 and 70 degrees.

TRY IT : : 4.37

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

TRY IT : : 4.38

The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Often it is helpful when solving geometry applications to draw a picture to visualize the situation.

EXAMPLE 4.20

Translate to a system of equations and then solve:

Randall has 125 feet of fencing to enclose the part of his backyard adjacent to his house. He will only need to fence around three sides, because the fourth side will be the wall of the house. He wants the length of the fenced yard (parallel to the house wall) to be 5 feet more than four times as long as the width. Find the length and the width.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
	
Step 3. Name what we are looking for.	Let $L =$ the length of the fenced yard. $W =$ the width of the fenced yard
Step 4. Translate into a system of equations.	One lenth and two widths equal 125.
	$L + 2W = 125$
	The length will be 5 feet more than four times the width.
	$L = 4W + 5$
The system is shown. **Step 5. Solve** The system of equations by substitution.	$L + 2W = 125$ $L = 4W + 5$ $L + 2W = 125$
Substitute $L = 4W + 5$ into the first equation, then solve for W.	$4W + 5 + 2W = 125$
	$6W + 5 = 125$ $6W = 120$ $W = 20$
Substitute 20 for W in the second equation, then solve for L.	$L = 4W + 5$ $L = 4 \cdot 20 + 5$ $L = 80 + 5$ $L = 85$
Step 6. Check the answer in the problem.	$20 + 85 + 20 = 125$ $85 = 4 \cdot 20 + 5$ $85 = 85$ ✓
Step 7. Answer the equation.	The length is 85 feet and the width is 20 feet.

TRY IT : : 4.39

Translate to a system of equations and then solve:

Mario wants to put a fence around the pool in his backyard. Since one side is adjacent to the house, he will only need to fence three sides. There are two long sides and the one shorter side is parallel to the house. He needs 155 feet of fencing to enclose the pool. The length of the long side is 10 feet less than twice the width. Find the length and width of the pool area to be enclosed.

TRY IT : : 4.40

Translate to a system of equations and then solve:

Alexis wants to build a rectangular dog run in her yard adjacent to her neighbor's fence. She will use 136 feet of fencing to completely enclose the rectangular dog run. The length of the dog run along the neighbor's fence will be 16 feet less than twice the width. Find the length and width of the dog run.

Solve uniform motion applications

We used a table to organize the information in uniform motion problems when we introduced them earlier. We'll continue using the table here. The basic equation was $D = rt$ where D is the distance traveled, r is the rate, and t is the time.

Our first example of a uniform motion application will be for a situation similar to some we have already seen, but now we can use two variables and two equations.

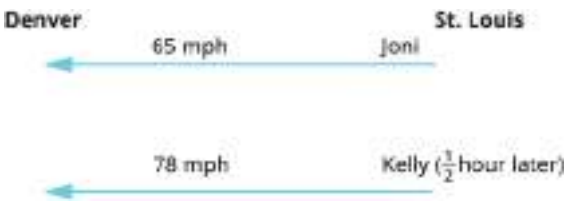
EXAMPLE 4.21

Translate to a system of equations and then solve:

Joni left St. Louis on the interstate, driving west towards Denver at a speed of 65 miles per hour. Half an hour later, Kelly left St. Louis on the same route as Joni, driving 78 miles per hour. How long will it take Kelly to catch up to Joni?

Solution

A diagram is useful in helping us visualize the situation.



Identify and name what we are looking for. A chart will help us organize the data. We know the rates of both Joni and Kelly, and so we enter them in the chart. We are looking for the length of time Kelly, k, and Joni, j, will each drive.

	Rate · Time = Distance		
Joni	65	j	$65j$
Kelly	78	k	$78k$

Since $D = r \cdot t$ we can fill in the Distance column.

Translate into a system of equations.

To make the system of equations, we must recognize that Kelly and Joni will drive the same distance. So,

$$65j = 78k$$

Also, since Kelly left later, her time will be $\frac{1}{2}$ hour less than Joni's time. So,

$$k = j - \frac{1}{2}$$

Now we have the system.

$$\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$$

Solve the system of equations by substitution.

Substitute $k = j - \frac{1}{2}$ into the second equation,

then solve for j.

$$\begin{aligned} 65j &= 78k \\ 65j &= 78\left(j - \tfrac{1}{2}\right) \\ 65j &= 78j - 39 \\ -13j &= -39 \\ j &= 3 \end{aligned}$$

To find elly's time, substitute $j = 3$ into the fir t equation, then solve for k.

$$k = j - \frac{1}{2}$$

$$k = 3 - \frac{1}{2}$$

$$k = \frac{5}{2} \text{ or } k = 2\frac{1}{2}$$

Check the answer in the problem.

Joni $3 \text{ hours} \, (65 \text{ mph}) = 195 \text{ miles}$

Kelly $2\frac{1}{2} \text{ hours} \left(78 \text{ mph}\right) = 195 \text{ miles}$

Yes, they will have traveled the same distance when they meet.

Answer the question.

Kelly will catch up to Joni in $2\frac{1}{2}$ hours. By then, Joni will have traveled 3 hours.

TRY IT : : 4.41

Translate to a system of equations and then solve:

Mitchell left Detroit on the interstate driving south towards Orlando at a speed of 60 miles per hour. Clark left Detroit 1 hour later traveling at a speed of 75 miles per hour, following the same route as Mitchell. How long will it take Clark to catch Mitchell?

TRY IT : : 4.42

Translate to a system of equations and then solve:

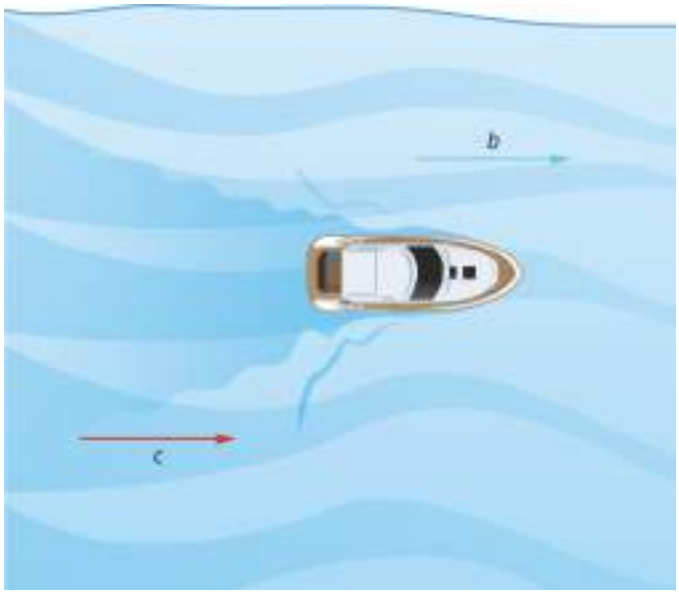
Charlie left his mother's house traveling at an average speed of 36 miles per hour. His sister Sally left 15 minutes $(\frac{1}{4} \text{ hour})$ later traveling the same route at an average speed of 42 miles per hour. How long before Sally catches up to Charlie?

Many real-world applications of uniform motion arise because of the effects of currents—of water or air—on the actual speed of a vehicle. Cross-country airplane flights in the United States generally take longer going west than going east because of the prevailing wind currents.

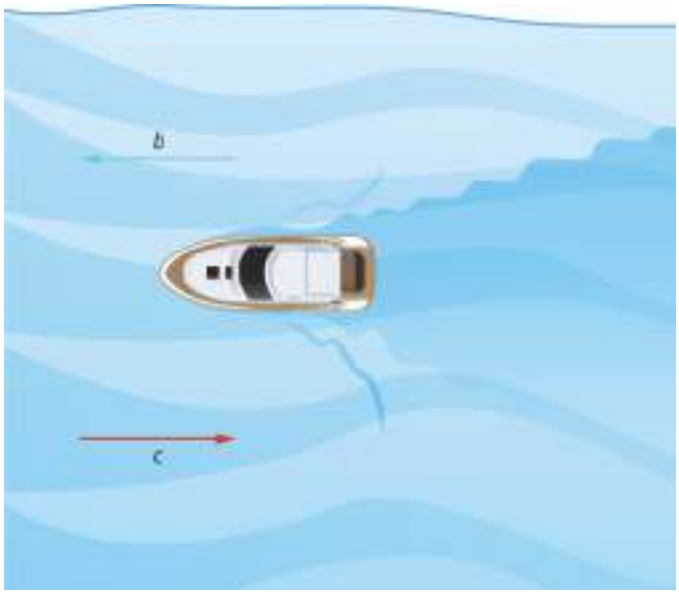
Let's take a look at a boat travelling on a river. Depending on which way the boat is going, the current of the water is either slowing it down or speeding it up.

The images below show how a river current affects the speed at which a boat is actually travelling. We'll call the speed of the boat in still water *b* and the speed of the river current *c*.

The boat is going downstream, in the same direction as the river current. The current helps push the boat, so the boat's actual speed is faster than its speed in still water. The actual speed at which the boat is moving is $b + c$.



Now, the boat is going upstream, opposite to the river current. The current is going against the boat, so the boat's actual speed is slower than its speed in still water. The actual speed of the boat is $b - c$.



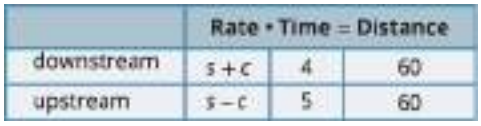
We'll put some numbers to this situation in the next example.

EXAMPLE 4.22

Translate to a system of equations and then solve.

A river cruise ship sailed 60 miles downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Solution

Read the problem.	This is a uniform motion problem and a picture will help us visualize the situation.
	
Identify what we are looking for.	We are looking for the speed of the ship in still water and the speed of the current.
Name what we are looking for.	Let $s =$ the rate of the ship in still water. $c =$ the rate of the current
A chart will help us organize the information. The ship goes downstream and then upstream. Going downstream, the current helps the ship and so the ship's actual rate is $s + c$. Going upstream, the current slows the ship and so the actual rate is $s - c$.	
Downstream it takes 4 hours. Upstream it takes 5 hours. Each way the distance is 60 miles.	
Translate into a system of equations. Since rate times time is distance, we can write the system of equations.	$4(s + c) = 60$ $5(s - c) = 60$
Solve the system of equations. Distribute to put both equations in standard form, then solve by elimination.	$4s + 4c = 60$ $5s - 5c = 60$
Multiply the top equation by 5 and the bottom equation by 4. Add the equations, then solve for s.	$20s + 20c = 300$ $20s - 20c = 240$ $40s = 540$ $s = 13.5$ $4(s + c) = 60$
Substitute $s = 13.5$ into of the original equations.	$4(13.5 + c) = 60$ $54 + 4c = 60$ $4c = 6$ $c = 1.5$
Check the answer in the problem. The downstream rate would be $13.5 + 1.5 = 15$ mph. In 4 hours the ship would travel $15 \cdot 4 = 60$ miles. The upstream rate would be $13.5 - 1.5 = 12$ mph. In 5 hours the ship would travel $12 \cdot 5 = 60$ miles.	
Answer the question.	The rate of the ship is 13.5 mph and the rate of the current is 1.5 mph.

TRY IT : : 4.43

Translate to a system of equations and then solve:

A Mississippi river boat cruise sailed 120 miles upstream for 12 hours and then took 10 hours to return to the dock. Find the speed of the river boat in still water and the speed of the river current.

TRY IT : : 4.44

Translate to a system of equations and then solve:

Jason paddled his canoe 24 miles upstream for 4 hours. It took him 3 hours to paddle back. Find the speed of the canoe in still water and the speed of the river current.

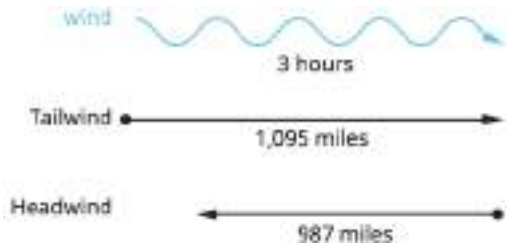
Wind currents affect airplane speeds in the same way as water currents affect boat speeds. We'll see this in the next example. A wind current in the same direction as the plane is flying is called a *tailwind*. A wind current blowing against the direction of the plane is called a *headwind*.

EXAMPLE 4.23

Translate to a system of equations and then solve:

A private jet can fly 1,095 miles in three hours with a tailwind but only 987 miles in three hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution

Read the problem.	This is a uniform motion problem and a picture will help us visualize.
	
Identify what we are looking for.	We are looking for the speed of the jet in still air and the speed of the wind.
Name what we are looking for.	Let $j =$ the speed of the jet in still air. $w =$ the speed of the wind.
A chart will help us organize the information. The jet makes two trips—one in a tailwind and one in a headwind. In a tailwind, the wind helps the jet and so the rate is $j + w$. In a headwind, the wind slows the jet and so the rate is $j - w$.	

	Rate • Time = Distance		
tailwind	$j + w$	3	1095
headwind	$j - w$	3	987

Each trip takes 3 hours. In a tailwind the jet flies 1,095 miles. In a headwind the jet flies 987 miles.	
Translate into a system of equations. Since rate times time is distance, we get the system of equations.	$\begin{aligned} 3(j + w) &= 1095 \\ 3(j - w) &= 987 \end{aligned}$

Solve the system of equations. Distribute, then solve by elimination. Add, and solve for j.	$3j + 3w = 1095$ $3j - 3w = 987$ $6j = 2082$ $j = 347$ $4(j + w) = 1095$
Substitute $j = 347$ into one of the original equations, then solve for w.	$3(347 + w) = 1095$ $1041 + 3w = 1095$ $3w = 54$ $w = 18$
Check the answer in the problem. With the tailwind, the actual rate of the jet would be $347 + 18 = 365$ mph. In 3 hours the jet would travel $365 \cdot 3 = 1{,}095$ miles Going into the headwind, the jet's actual rate would be $347 - 18 = 329$ mph. In 3 hours the jet would travel $329 \cdot 3 = 987$ miles.	
Answer the question.	The rate of the jet is 347 mph and the rate of the wind is 18 mph.

TRY IT : : 4.45

Translate to a system of equations and then solve:

A small jet can fly 1,325 miles in 5 hours with a tailwind but only 1,035 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

TRY IT : : 4.46

Translate to a system of equations and then solve:

A commercial jet can fly 1,728 miles in 4 hours with a tailwind but only 1,536 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

MEDIA : :

Access this online resource for additional instruction and practice with systems of equations.

- **Systems of Equations (https://openstax.org/l/37sysequations)**

4.2 EXERCISES

Practice Makes Perfect

Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

72. The sum of two number is 15. One number is 3 less than the other. Find the numbers.

73. The sum of two number is 30. One number is 4 less than the other. Find the numbers.

74. The sum of two number is −16. One number is 20 less than the other. Find the numbers.

75. The sum of two number is -26. One number is 12 less than the other. Find the numbers.

76. The sum of two numbers is 65. Their difference is 25. Find the numbers.

77. The sum of two numbers is 37. Their difference is 9. Find the numbers.

78. The sum of two numbers is -27. Their difference is -59. Find the numbers.

79. The sum of two numbers is -45. Their difference is -89. Find the numbers.

80. Maxim has been offered positions by two car companies. The first company pays a salary of $10,000 plus a commission of $1000 for each car sold. The second pays a salary of $20,000 plus a commission of $500 for each car sold. How many cars would need to be sold to make the total pay the same?

81. Jackie has been offered positions by two cable companies. The first company pays a salary of $14,000 plus a commission of $100 for each cable package sold. The second pays a salary of $20,000 plus a commission of $25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?

82. Amara currently sells televisions for company A at a salary of $17,000 plus a $100 commission for each television she sells. Company B offers her a position with a salary of $29,000 plus a $20 commission for each television she sells. How televisions would Amara need to sell for the options to be equal?

83. Mitchell currently sells stoves for company A at a salary of $12,000 plus a $150 commission for each stove he sells. Company B offers him a position with a salary of $24,000 plus a $50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to be equal?

84. Two containers of gasoline hold a total of fifty gallons. The big container can hold ten gallons less than twice the small container. How many gallons does each container hold?

85. June needs 48 gallons of punch for a party and has two different coolers to carry it in. The bigger cooler is five times as large as the smaller cooler. How many gallons can each cooler hold?

86. Shelly spent 10 minutes jogging and 20 minutes cycling and burned 300 calories. The next day, Shelly swapped times, doing 20 minutes of jogging and 10 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

87. Drew burned 1800 calories Friday playing one hour of basketball and canoeing for two hours. Saturday he spent two hours playing basketball and three hours canoeing and burned 3200 calories. How many calories did he burn per hour when playing basketball? How many calories did he burn per hour when canoeing?

88. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and thumb drive. Troy bought four notebooks and five thumb drives for $116. Lisa bought two notebooks and three thumb dives for $68. Find the cost of each notebook and each thumb drive.

89. Nancy bought seven pounds of oranges and three pounds of bananas for $17. Her husband later bought three pounds of oranges and six pounds of bananas for $12. What was the cost per pound of the oranges and the bananas?

90. Andrea is buying some new shirts and sweaters. She is able to buy 3 shirts and 2 sweaters for $114 or she is able to buy 2 shirts and 4 sweaters for $164. How much does a shirt cost? How much does a sweater cost?

91. Peter is buying office supplies. He is able to buy 3 packages of paper and 4 staplers for $40 or he is able to buy 5 packages of paper and 6 staplers for $62. How much does a package of paper cost? How much does a stapler cost?

92. The total amount of sodium in 2 hot dogs and 3 cups of cottage cheese is 4720 mg. The total amount of sodium in 5 hot dogs and 2 cups of cottage cheese is 6300 mg. How much sodium is in a hot dog? How much sodium is in a cup of cottage cheese?

93. The total number of calories in 2 hot dogs and 3 cups of cottage cheese is 960 calories. The total number of calories in 5 hot dogs and 2 cups of cottage cheese is 1190 calories. How many calories are in a hot dog? How many calories are in a cup of cottage cheese?

94. Molly is making strawberry infused water. For each ounce of strawberry juice, she uses three times as many ounces of water as juice. How many ounces of strawberry juice and how many ounces of water does she need to make 64 ounces of strawberry infused water?

95. Owen is making lemonade from concentrate. The number of quarts of water he needs is 4 times the number of quarts of concentrate. How many quarts of water and how many quarts of concentrate does Owen need to make 100 quarts of lemonade?

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

96. The difference of two complementary angles is 55 degrees. Find the measures of the angles.

97. The difference of two complementary angles is 17 degrees. Find the measures of the angles.

98. Two angles are complementary. The measure of the larger angle is twelve less than twice the measure of the smaller angle. Find the measures of both angles.

99. Two angles are complementary. The measure of the larger angle is ten more than four times the measure of the smaller angle. Find the measures of both angles.

100. The difference of two supplementary angles is 8 degrees. Find the measures of the angles.

101. The difference of two supplementary angles is 88 degrees. Find the measures of the angles.

102. Two angles are supplementary. The measure of the larger angle is four more than three times the measure of the smaller angle. Find the measures of both angles.

103. Two angles are supplementary. The measure of the larger angle is five less than four times the measure of the smaller angle. Find the measures of both angles.

104. The measure of one of the small angles of a right triangle is 14 more than 3 times the measure of the other small angle. Find the measure of both angles.

105. The measure of one of the small angles of a right triangle is 26 more than 3 times the measure of the other small angle. Find the measure of both angles.

106. The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

107. The measure of one of the small angles of a right triangle is 45 less than twice the measure of the other small angle. Find the measure of both angles.

108. Wayne is hanging a string of lights 45 feet long around the three sides of his patio, which is adjacent to his house. The length of his patio, the side along the house, is five feet longer than twice its width. Find the length and width of the patio.

109. Darrin is hanging 200 feet of Christmas garland on the three sides of fencing that enclose his front yard. The length is five feet less than three times the width. Find the length and width of the fencing.

110. A frame around a family portrait has a perimeter of 90 inches. The length is fifteen less than twice the width. Find the length and width of the frame.

111. The perimeter of a toddler play area is 100 feet. The length is ten more than three times the width. Find the length and width of the play area.

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

112. Sarah left Minneapolis heading east on the interstate at a speed of 60 mph. Her sister followed her on the same route, leaving two hours later and driving at a rate of 70 mph. How long will it take for Sarah's sister to catch up to Sarah?

113. College roommates John and David were driving home to the same town for the holidays. John drove 55 mph, and David, who left an hour later, drove 60 mph. How long will it take for David to catch up to John?

114. At the end of spring break, Lucy left the beach and drove back towards home, driving at a rate of 40 mph. Lucy's friend left the beach for home 30 minutes (half an hour) later, and drove 50 mph. How long did it take Lucy's friend to catch up to Lucy?

115. Felecia left her home to visit her daughter driving 45 mph. Her husband waited for the dog sitter to arrive and left home twenty minutes (1/3 hour) later. He drove 55 mph to catch up to Felecia. How long before he reaches her?

116. The Jones family took a 12-mile canoe ride down the Indian River in two hours. After lunch, the return trip back up the river took three hours. Find the rate of the canoe in still water and the rate of the current.

117. A motor boat travels 60 miles down a river in three hours but takes five hours to return upstream. Find the rate of the boat in still water and the rate of the current.

118. A motor boat traveled 18 miles down a river in two hours but going back upstream, it took 4.5 hours due to the current. Find the rate of the motor boat in still water and the rate of the current. (Round to the nearest hundredth.)

119. A river cruise boat sailed 80 miles down the Mississippi River for four hours. It took five hours to return. Find the rate of the cruise boat in still water and the rate of the current.

120. A small jet can fly 1072 miles in 4 hours with a tailwind but only 848 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

121. A small jet can fly 1435 miles in 5 hours with a tailwind but only 1,215 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

122. A commercial jet can fly 868 miles in 2 hours with a tailwind but only 792 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

123. A commercial jet can fly 1,320 miles in 3 hours with a tailwind but only 1170 miles in 3 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Writing Exercises

124. Write an application problem similar to **Example 4.14**. Then translate to a system of equations and solve it.

125. Write a uniform motion problem similar to **Example 4.15** that relates to where you live with your friends or family members. Then translate to a system of equations and solve it.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve direct translation applications.			
solve geometry applications.			
solve uniform motion applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

4.3 Solve Mixture Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve mixture applications
- Solve interest applications
- Solve applications of cost and revenue functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Multiply: $4.025(1{,}562)$.
 If you missed this problem, review **Example 1.36**.
2. Write 8.2% as a decimal.
 If you missed this problem, review **Example 1.40**.
3. Earl's dinner bill came to $32.50 and he wanted to leave an 18% tip. How much should the tip be?
 If you missed this problem, review **Example 2.20**.

Solve Mixture Applications

Mixture application involve combining two or more quantities. When we solved mixture applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:

	Number •	Value =	Total Value
nickels			
dimes			

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let n be the number of nickels and then write the number of dimes in terms of n, or if we would let d be the number of dimes and write the number of nickels in terms of d.

Now that we know how to solve systems of equations with two variables, we'll just let n be the number of nickels and d be the number of dimes. We'll write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we'll do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

EXAMPLE 4.24

Translate to a system of equations and solve:

A science center sold 1,363 tickets on a busy weekend. The receipts totaled $12,146. How many $12 adult tickets and how many $7 child tickets were sold?

Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of adult tickets and the number of child tickets sold.
Step 3. Name what we are looking for.	Let $a =$ the number of adult tickets. $c =$ the number of child tickets
A table will help us organize the data. We have two types of tickets, adult and child.	Write in a and c for the number of tickets.

Write the total number of tickets sold at the bottom of the Number column.	Altogether 1,363 were sold.
Write the value of each type of ticket in the Value column.	The value of each adult ticket is \$12. The value of each child tickets is \$7.
The number times the value gives the total value, so the total value of adult tickets is $a \cdot 12 = 12a$, and the total value of child tickets is $c \cdot 7 = 7c$.	Fill in the Total Value column.
Altogether the total value of the tickets was \$12,146.	

Type	Number	• Value	= Total Value
adult	a	12	$11a$
child	c	7	$8c$
	1,363		12,146

Step 4. Translate into a system of equations.	
The Number column and the Total value column give us the system of equations.	$\begin{aligned} a + c &= 1{,}363 \\ 12a + 7c &= 12{,}146 \end{aligned}$
We will use the elimination method to solve this system. Multiply the first equation by -7.	$\begin{aligned} -7(a + c) &= -7(1{,}363) \\ 12a + 7c &= 12{,}146 \end{aligned}$
Simplify and add, then solve for a.	$\begin{aligned} -7a - 7c &= -9{,}541 \\ 12a + 7c &= 12{,}146 \\ 5a &= 2{,}605 \\ a &= 521 \end{aligned}$
Substitute $a = 521$ into the first equation, then solve for c.	$\begin{aligned} a + c &= 1{,}363 \\ 521 + c &= 1{,}363 \\ c &= 842 \end{aligned}$
Step 6. Check the answer in the problem. 521 adult at \$12 per ticket makes \$ 6,252 842 child at \$7 per ticket makes \$58,994 The total receipts are \$12,146✓	
Step 7. Answer the question.	The science center sold 521 adult tickets and 842 child tickets.

TRY IT : : 4.47

Translate to a system of equations and solve:

The ticket office at the zoo sold 553 tickets one day. The receipts totaled \$3,936. How many \$9 adult tickets and how many \$6 child tickets were sold?

TRY IT : : 4.48

Translate to a system of equations and solve:

The box office at a movie theater sold 147 tickets for the evening show, and receipts totaled \$1,302. How many \$11 adult and how many \$8 child tickets were sold?

In the next example, we'll solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the 'number' column will be easy.

EXAMPLE 4.25

Translate to a system of equations and solve:

Juan has a pocketful of nickels and dimes. The total value of the coins is $8.10. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Solution

Step 1. Read the problem.
We will create a table to organize the information.

Step 2. Identify what we are looking for.

We are looking for the number of nickels and the number of dimes.

Step 3. Name what we are looking for.

Let $n =$ the number of nickels.
$d =$ the number of dimes

A table will help us organize the data.
We have two types of coins, nickels and dimes.

Write *n* and *d* for the number of each type of coin.

Fill in the Value column with the value of each type of coin.

The value of each nickel is $0.05.
The value of each dime is $0.10.

The number times the value gives the total value, so, the total value of the nickels is $n(0.05) = 0.05n$ and the total value of dimes is $d(0.10) = 0.10d$.
Altogether the total value of the coins is $8.10.

Type	Number ·	Value	= Total Value
nickels	n	0.05	$0.05n$
dimes	d	0.10	$0.10d$
			8.10

Step 4. Translate into a system of equations.

The Total Value column gives one equation.

$$0.05n + 0.10d = 8.10$$

We also know the number of dimes is 9 less than twice the number of nickels.

Translate to get the second equation.

$$d = 2n - 9$$

Now we have the system to solve.

$$\begin{aligned} 0.05n + 0.10d &= 8.10 \\ d &= 2n - 9 \end{aligned}$$

Step 5. Solve the system of equations
We will use the substitution method.

Substitute $d = 2n - 9$ into the first equation.

$$\begin{aligned} 0.05n + 0.10d &= 8.10 \\ 0.05n + 0.10(2n - 9) &= 8.10 \end{aligned}$$

Simplify and solve for *n*.

$$\begin{aligned} 0.05n + 0.2n - 0.90 &= 8.10 \\ 0.25n - 0.90 &= 8.10 \\ 0.25n &= 9.00 \\ n &= 36 \end{aligned}$$

To find the number of dimes, substitute $n = 36$ into the second equation.

$$\begin{aligned} d &= 2n - 9 \\ d &= 2 \cdot 36 - 9 \\ d &= 63 \end{aligned}$$

Step 6. Check the answer in the problem

63 dimes at $\$0.10 = \$6.30$
36 nickels at $\$0.05 = \$1.80$
Total $= \$8.10$ ✓

Step 7. Answer the question.

Juan has 36 nickels and 63 dimes.

TRY IT : : 4.49

Translate to a system of equations and solve:

Matilda has a handful of quarters and dimes, with a total value of $8.55. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

TRY IT : : 4.50

Translate to a system of equations and solve:

Priam has a collection of nickels and quarters, with a total value of $7.30. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

EXAMPLE 4.26

Translate to a system of equations and solve:

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $7.60. per pound. Nuts cost $9.00 per pound and chocolate chips cost $2.00 per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Solution

Step 1. Read the problem.
We will create a table to organize the information.

Step 2. Identify what we are looking for.

We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.

Step 3. Name what we are looking for.

Let $n =$ the number of pound of nuts.
$c =$ the number of pounds of chips

Carson will mix nuts and chocolate chips to get trail mix.
Write in n and c for the number of pounds of nuts and chocolate chips.

Type	Number of pounds	· Value	= Total Value
Nuts	n	9.00	$9n$
Chocolate chips	c	2.00	$2c$
Trail mix	20	7.60	$7.60(20) = 152$

There will be 20 pounds of trail mix.
Put the price per pound of each item in the Value column.
Fill in the last column using
Number • Value = Total Value

Step 4. Translate into a system of equations.
We get the equations from the Number and Total Value columns.

$$\begin{cases} n + c = 20 \\ 9n + 2c = 152 \end{cases}$$

Step 5. Solve the system of equations
We will use elimination to solve the system.
Multiply the first equation by -2 to eliminate c.

$$\begin{cases} -2(n + c) = -2(20) \\ 9n + 2c = 152 \end{cases}$$

Simplify and add.
Solve for n.

$$\begin{cases} -2n - 2c = -40 \\ 9n + 2c = 152 \end{cases}$$
$$7n = 112$$
$$n = 16$$

To find the number of pounds of chocolate chips, substitute $n = 16$ into the first equation, then solve for c.	$n + c = 20$ $16 + c = 20$ $c = 4$
Step 6. Check the answer in the problem. $16 + 4 = 20$ ✓ $9 \cdot 16 + 2 \cdot 4 = 152$ ✓	
Step 7. Answer the question.	Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

TRY IT : : 4.51

Translate to a system of equations and solve:

Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her $6 per pound. Peanuts are $4 per pound and cashews are $9 per pound. How many pounds of peanuts and how many pounds of cashews should she use?

TRY IT : : 4.52

Translate to a system of equations and solve:

Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of $3 per pound. The price of beans is $1 per pound and the price of ground beef is $5 per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20% of the total amount is cleanser, and the rest is water. To make 35 ounces of a 20% concentration, you mix 7 ounces (20% of 35) of the cleanser with 28 ounces of water.

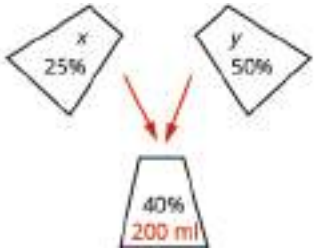
For these kinds of mixture problems, we'll use "percent" instead of "value" for one of the columns in our table.

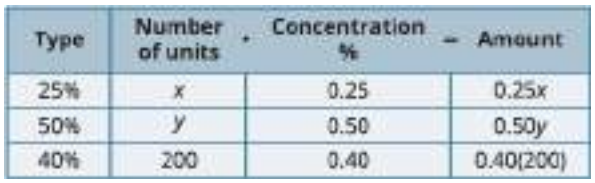
EXAMPLE 4.27

Translate to a system of equations and solve:

Sasheena is lab assistant at her community college. She needs to make 200 milliliters of a 40% solution of sulfuric acid for a lab experiment. The lab has only 25% and 50% solutions in the storeroom. How much should she mix of the 25% and the 50% solutions to make the 40% solution?

Solution

Step 1. Read the problem. A figure may help us visualize the situation, then we will create a table to organize the information.	Sasheena must mix some of the 25% solution and some of the 50% solution together to get 200 ml of the 40% solution.
	

Step 2. Identify what we are looking for.	We are looking for how much of each solution she needs.
Step 3. Name what we are looking for.	Let $x =$ number of ml of 25% solution. $y =$ number of ml of 50% solution
A table will help us organize the data. She will mix *x* ml of 25% with *y* ml of 50% to get 200 ml of 40% solution. We write the percents as decimals in the chart. We multiply the number of units times the concentration to get the total amount of sulfuric acid in each solution.	
Step 4. Translate into a system of equations. We get the equations from the Number column and the Amount column. Now we have the system.	$x + y = 200$ $0.25x + 0.50y = 0.40(200)$
Step 5. Solve the system of equations We will solve the system by elimination. Multiply the first equation by -0.5 to eliminate *y*.	$-0.5(x + y) = -0.5(200)$ $0.25x + 0.50y = 80$
Simplify and add to solve for *x*.	$-0.5x - 0.5y = -100$ $0.25x + 0.5y = 80$ $-0.25x = -20$ $x = 80$
To solve for *y*, substitute $x = 80$ into the first equation.	$x + y = 200$ $80 + y = 200$ $y = 120$
Step 6. Check the answer in the problem. $80 + 120 = 200$ ✓ $0.25(80) + 0.50(120) = 200$ ✓ Yes!	
Step 7. Answer the question.	Sasheena should mix 80 ml of the 25% solution with 120 ml of the 50% solution to get the 200 ml of the 40% solution.

TRY IT : : 4.53

Translate to a system of equations and solve:

LeBron needs 150 milliliters of a 30% solution of sulfuric acid for a lab experiment but only has access to a 25% and a 50% solution. How much of the 25% and how much of the 50% solution should he mix to make the 30% solution?

TRY IT : : 4.54

Translate to a system of equations and solve:

Anatole needs to make 250 milliliters of a 25% solution of hydrochloric acid for a lab experiment. The lab only has a 10% solution and a 40% solution in the storeroom. How much of the 10% and how much of the 40% solutions should he mix to make the 25% solution?

Solve Interest Applications

The formula to model simple interest applications is $I = Prt$. Interest, *I*, is the product of the principal, *P*, the rate, *r*, and

the time, *t*. In our work here, we will calculate the interest earned in one year, so *t* will be 1.

We modify the column titles in the mixture table to show the formula for interest, as you'll see in the next example.

EXAMPLE 4.28

Translate to a system of equations and solve:

Adnan has $40,000 to invest and hopes to earn 7.1% interest per year. He will put some of the money into a stock fund that earns 8% per year and the rest into bonds that earns 3% per year. How much money should he put into each fund?

Solution

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the amount to invest in each fund.
Step 3. Name what we are looking for.	Let $s =$ the amount invested in stocks. $b =$ the amount invested in stocks
Write the interest rate as a decimal for each fund. Multiply: Principal · Rate · Time	

Account	Principal ·	Rate ·	Time =	Interest
Stock fund	s	0.08	1	0.08s
Bonds	b	0.03	1	0.03b
Total	40,000	0.071		0.071(40,000)

Step 4. Translate into a system of equations. We get our system of equations from the Principal column and the Interest column.	$s + b = 40{,}000$ $0.08s + 0.03b = 0.071(40{,}000)$
Step 5. Solve the system of equations by elimination. Multiply the top equation by −0.03.	$-0.03(s + b) = -0.03(40{,}000)$ $0.08s + 0.03b = 2{,}840$
Simplify and add to solve for *s*.	$-0.03s - 0.03b = -1{,}200$ $0.08s + 0.03b = 2{,}840$ $0.05s = 1{,}640$ $s = 32{,}800$
To find *b*, substitute *s* = 32,800 into the first equation.	$s + b = 40{,}000$ $32{,}800 + b = 40{,}000$ $b = 7{,}200$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	Adnan should invest $32,000 in stock and $7,200 in bonds.

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s + b = 40{,}000$, represents the total amount of money invested and the second equation, $0.08s + 0.03b = 0.071(40{,}000)$, represents the interest earned.

TRY IT : : 4.55

Translate to a system of equations and solve:

Leon had $50,000 to invest and hopes to earn 6.2% interest per year. He will put some of the money into a stock fund that earns 7% per year and the rest in to a savings account that earns 2% per year. How much money should he put into each fund?

TRY IT : : 4.56

Translate to a system of equations and solve:

Julius invested $7000 into two stock investments. One stock paid 11% interest and the other stock paid 13% interest. He earned 12.5% interest on the total investment. How much money did he put in each stock?

The next example requires that we find the principal given the amount of interest earned.

EXAMPLE 4.29

Translate to a system of equations and solve:

Rosie owes $21,540 on her two student loans. The interest rate on her bank loan is 10.5% and the interest rate on the federal loan is 5.9%. The total amount of interest she paid last year was $\$1{,}669.68$. What was the principal for each loan?

Solution

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the principal of each loan.
Step 3. Name what we are looking for.	Let $b =$ the principal for the bank loan. $f =$ the principal on the federal loan
The total loans are $21,540.	
Record the interest rates as decimals in the chart. Multiply using the formula $I = Prt$ to get the Interest.	

Account	Principal	· Rate	· Time	= Interest
Bank	b	0.105	1	$0.105b$
Federal	f	0.059	1	$0.059f$
Total	21,540			1669.68

Step 4. Translate into a system of equations. The system of equations comes from the Principal column and the Interest column.	$b + f = 21{,}540$ $0.105b + 0.059f = 1669.68$
Step 5. Solve the system of equations We will use substitution to solve. Solve the first equation for b.	$b + f = 21{,}540$ $b = -f + 21{,}540$
	$0.105b + 0.059f = 1669.68$
Substitute $b = -f + 21.540$ into the second equation.	$0.105(-f + 21{,}540) + 0.059f = 1669.68$ $-0.105f + 2261.70 + 0.059f = 1669.68$
Simplify and solve for f.	$-0.046f + 2261.70 = 1669.68$ $-0.046f = -592.02$ $f = 12{,}870$
To find b, substitute $f = 12{,}870$ into the first equation.	$b + f = 21{,}540$ $b + 12{,}870 = 21{,}540$ $b = 8{,}670$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	The principal of the federal loan is $12,870 and the principal for the bank loan is $8,670.

TRY IT : : 4.57

Translate to a system of equations and solve:

Laura owes $18,000 on her student loans. The interest rate on the bank loan is 2.5% and the interest rate on the federal loan is 6.9%. The total amount of interest she paid last year was $1,066. What was the principal for each loan?

TRY IT : : 4.58

Translate to a system of equations and solve:

Jill's Sandwich Shoppe owes $65,200 on two business loans, one at 4.5% interest and the other at 7.2% interest. The total amount of interest owed last year was $3,582. What was the principal for each loan?

Solve applications of cost and revenue functions

Suppose a company makes and sells x units of a product. The cost to the company is the total costs to produce x units. This is the cost to manufacture for each unit times x, the number of units manufactured, plus the fixed costs.

The **revenue** is the money the company brings in as a result of selling x units. This is the selling price of each unit times the number of units sold.

When the costs equal the revenue we say the business has reached the **break-even point**.

Cost and Revenue Functions

The **cost function** is the cost to manufacture each unit times x, the number of units manufactured, plus the fixed costs.

$$C(x) = (\text{cost per unit}) \cdot x + \text{fi ed costs}$$

The **revenue** function is the selling price of each unit times x, the number of units sold.

$$R(x) = (\text{selling price per unit}) \cdot x$$

The **break-even point** is when the revenue equals the costs.

$$C(x) = R(x)$$

EXAMPLE 4.30

The manufacturer of a weight training bench spends $105 to build each bench and sells them for $245. The manufacturer also has fixed costs each month of $7,000.

ⓐ Find the cost function C when x benches are manufactured.

ⓑ Find the revenue function R when x benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

Solution

ⓐ The manufacturer has $7,000 of fixed costs no matter how many weight training benches it produces. In addition to the fixed costs, the manufacturer also spends $105 to produce each bench. Suppose x benches are sold.

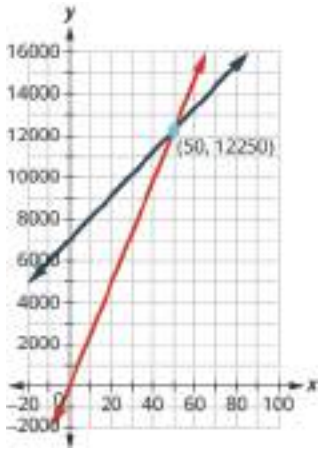
Write the general Cost function formula.	$C(x) = (\text{cost per unit}) \cdot x + \text{fi ed costs}$
Substitute in the cost values.	$C(x) = 105x + 7000$

ⓑ The manufacturer sells each weight training bench for $245. We get the total revenue by multiplying the revenue per unit times the number of units sold.

Write the general Revenue function.	$R(x) = (\text{selling price per unit}) \cdot x$
Substitute in the revenue per unit.	$R(x) = 245x$

ⓒ Essentially we have a system of linear equations. We will show the graph of the system as this helps make the idea of a break-even point more visual.

$$\begin{cases} C(x) = 105x + 7000 \\ R(x) = 245x \end{cases} \quad \text{or} \quad \begin{cases} y = 105x + 7000 \\ y = 245x \end{cases}$$



ⓓ To find the actual value, we remember the break-even point occurs when costs equal revenue.

Write the break-even formula.

$$C(x) = R(x)$$

Solve.

$$\begin{aligned} 105x + 7000 &= 245x \\ 7000 &= 140x \\ 50 &= x \end{aligned}$$

When 50 benches are sold, the costs equal the revenue.

$C(x) = 105x + 7000$ $\quad$ $R(x) = 245x$

$C(50) = 105(50) + 7000$ $\quad$ $R(50) = 245 \cdot 50$

$C(50) = 12{,}250$ $\quad$ $R(50) = 12{,}250$

When 50 benches are sold, the revenue and costs are both $12,250. Notice this corresponds to the ordered pair (50, 12,250).

TRY IT : : 4.59

The manufacturer of a weight training bench spends $15 to build each bench and sells them for $32. The manufacturer also has fixed costs each month of $25,500.

ⓐ Find the cost function C when x benches are manufactured.

ⓑ Find the revenue function R when x benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

TRY IT : : 4.60

The manufacturer of a weight training bench spends $120 to build each bench and sells them for $170. The manufacturer also has fixed costs each month of $150,000.

ⓐ Find the cost function C when x benches are manufactured.

ⓑ Find the revenue function R when x benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

MEDIA : :

Access this online resource for additional instruction and practice with interest and mixtures.

- **Interest and Mixtures (https://openstax.org/l/37intmixure)**

4.3 EXERCISES

Practice Makes Perfect

Solve Mixture Applications

In the following exercises, translate to a system of equations and solve.

126. Tickets to a Broadway show cost $35 for adults and $15 for children. The total receipts for 1650 tickets at one performance were $47,150. How many adult and how many child tickets were sold?

127. Tickets for the Cirque du Soleil show are $70 for adults and $50 for children. One evening performance had a total of 300 tickets sold and the receipts totaled $17,200. How many adult and how many child tickets were sold?

128. Tickets for an Amtrak train cost $10 for children and $22 for adults. Josie paid $1200 for a total of 72 tickets. How many children tickets and how many adult tickets did Josie buy?

129. Tickets for a Minnesota Twins baseball game are $69 for Main Level seats and $39 for Terrace Level seats. A group of sixteen friends went to the game and spent a total of $804 for the tickets. How many of Main Level and how many Terrace Level tickets did they buy?

130. Tickets for a dance recital cost $15 for adults and $7 dollars for children. The dance company sold 253 tickets and the total receipts were $2771. How many adult tickets and how many child tickets were sold?

131. Tickets for the community fair cost $12 for adults and $5 dollars for children. On the first day of the fair, 312 tickets were sold for a total of $2204. How many adult tickets and how many child tickets were sold?

132. Brandon has a cup of quarters and dimes with a total value of $\$3.80$. The number of quarters is four less than twice the number of quarters. How many quarters and how many dimes does Brandon have?

133. Sherri saves nickels and dimes in a coin purse for her daughter. The total value of the coins in the purse is $\$0.95$. The number of nickels is two less than five times the number of dimes. How many nickels and how many dimes are in the coin purse?

134. Peter has been saving his loose change for several days. When he counted his quarters and nickels, he found they had a total value $\$13.10$. The number of quarters was fifteen more than three times the number of dimes. How many quarters and how many dimes did Peter have?

135. Lucinda had a pocketful of dimes and quarters with a value of $\$6.20$. The number of dimes is eighteen more than three times the number of quarters. How many dimes and how many quarters does Lucinda have?

136. A cashier has 30 bills, all of which are $10 or $20 bills. The total value of the money is $460. How many of each type of bill does the cashier have?

137. A cashier has 54 bills, all of which are $10 or $20 bills. The total value of the money is $910. How many of each type of bill does the cashier have?

138. Marissa wants to blend candy selling for $\$1.80$ per pound with candy costing $\$1.20$ per pound to get a mixture that costs her $\$1.40$ per pound to make. She wants to make 90 pounds of the candy blend. How many pounds of each type of candy should she use?

139. How many pounds of nuts selling for $6 per pound and raisins selling for $3 per pound should Kurt combine to obtain 120 pounds of trail mix that cost him $5 per pound?

140. Hannah has to make twenty-five gallons of punch for a potluck. The punch is made of soda and fruit drink. The cost of the soda is $\$1.79$ per gallon and the cost of the fruit drink is $\$2.49$ per gallon. Hannah's budget requires that the punch cost $\$2.21$ per gallon. How many gallons of soda and how many gallons of fruit drink does she need?

141. Joseph would like to make twelve pounds of a coffee blend at a cost of $6 per pound. He blends Ground Chicory at $5 a pound with Jamaican Blue Mountain at $9 per pound. How much of each type of coffee should he use?

142. Julia and her husband own a coffee shop. They experimented with mixing a City Roast Columbian coffee that cost $7.80 per pound with French Roast Columbian coffee that cost $8.10 per pound to make a twenty-pound blend. Their blend should cost them $7.92 per pound. How much of each type of coffee should they buy?

143. Twelve-year old Melody wants to sell bags of mixed candy at her lemonade stand. She will mix M&M's that cost $4.89 per bag and Reese's Pieces that cost $3.79 per bag to get a total of twenty-five bags of mixed candy. Melody wants the bags of mixed candy to cost her $4.23 a bag to make. How many bags of M&M's and how many bags of Reese's Pieces should she use?

144. Jotham needs 70 liters of a 50% solution of an alcohol solution. He has a 30% and an 80% solution available. How many liters of the 30% and how many liters of the 80% solutions should he mix to make the 50% solution?

145. Joy is preparing 15 liters of a 25% saline solution. She only has 40% and 10% solution in her lab. How many liters of the 40% and how many liters of the 10% should she mix to make the 25% solution?

146. A scientist needs 65 liters of a 15% alcohol solution. She has available a 25% and a 12% solution. How many liters of the 25% and how many liters of the 12% solutions should she mix to make the 15% solution?

147. A scientist needs 120 milliliters of a 20% acid solution for an experiment. The lab has available a 25% and a 10% solution. How many liters of the 25% and how many liters of the 10% solutions should the scientist mix to make the 20% solution?

148. A 40% antifreeze solution is to be mixed with a 70% antifreeze solution to get 240 liters of a 50% solution. How many liters of the 40% and how many liters of the 70% solutions will be used?

149. A 90% antifreeze solution is to be mixed with a 75% antifreeze solution to get 360 liters of an 85% solution. How many liters of the 90% and how many liters of the 75% solutions will be used?

Solve Interest Applications

In the following exercises, translate to a system of equations and solve.

150. Hattie had $3000 to invest and wants to earn 10.6% interest per year. She will put some of the money into an account that earns 12% per year and the rest into an account that earns 10% per year. How much money should she put into each account?

151. Carol invested $2560 into two accounts. One account paid 8% interest and the other paid 6% interest. She earned 7.25% interest on the total investment. How much money did she put in each account?

152. Sam invested $48,000, some at 6% interest and the rest at 10%. How much did he invest at each rate if he received $4000 in interest in one year?

153. Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4500 in interest in one year?

154. After four years in college, Josie owes $65, 800 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $\$2878.50$. What is the amount of each loan?

155. Mark wants to invest $10,000 to pay for his daughter's wedding next year. He will invest some of the money in a short term CD that pays 12% interest and the rest in a money market savings account that pays 5% interest. How much should he invest at each rate if he wants to earn $1095 in interest in one year?

156. A trust fund worth $25,000 is invested in two different portfolios. This year, one portfolio is expected to earn 5.25% interest and the other is expected to earn 4%. Plans are for the total interest on the fund to be $1150 in one year. How much money should be invested at each rate?

157. A business has two loans totaling $85,000. One loan has a rate of 6% and the other has a rate of 4.5% This year, the business expects to pay $4,650 in interest on the two loans. How much is each loan?

Solve Applications of Cost and Revenue Functions

158. The manufacturer of an energy drink spends $1.20 to make each drink and sells them for $2. The manufacturer also has fixed costs each month of $8,000.

ⓐ Find the cost function C when x energy drinks are manufactured.

ⓑ Find the revenue function R when x drinks are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

159. The manufacturer of a water bottle spends $5 to build each bottle and sells them for $10. The manufacturer also has fixed costs each month of $6500. ⓐ Find the cost function C when x bottles are manufactured. ⓑ Find the revenue function R when x bottles are sold. ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid. ⓓ Find the break-even point. Interpret what the break-even point means.

Writing Exercises

160. Take a handful of two types of coins, and write a problem similar to **Example 4.25** relating the total number of coins and their total value. Set up a system of equations to describe your situation and then solve it.

161. In **Example 4.28**, we used elimination to solve the system of equations

$$\begin{cases} s + b = 40{,}000 \\ 0.08s + 0.03b = 0.071(40{,}000) \end{cases}.$$

Could you have used substitution or elimination to solve this system? Why?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve mixture applications.			
solve interest applications.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

4.4 Solve Systems of Equations with Three Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered triple is a solution of a system of three linear equations with three variables
- Solve a system of linear equations with three variables
- Solve applications using systems of linear equations with three variables

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $5x - 2y + 3z$ when $x = -2, \quad y = -4, \quad$ and $z = 3$.
 If you missed this problem, review **Example 1.21**.
2. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution.
 $\begin{cases} -2x + y = -11 \\ x + 3y = 9 \end{cases}$.
 If you missed this problem, review **Example 2.6**.
3. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution.
 $\begin{cases} 7x + 8y = 4 \\ 3x - 5y = 27 \end{cases}$.
 If you missed this problem, review **Example 2.8**.

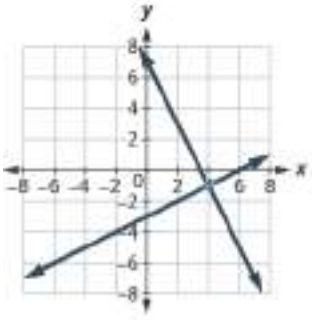
Determine Whether an Ordered Triple is a Solution of a System of Three Linear Equations with Three Variables

In this section, we will extend our work of solving a system of linear equations. So far we have worked with systems of equations with two equations and two variables. Now we will work with systems of three equations with three variables. But first let's review what we already know about solving equations and systems involving up to two variables.

We learned earlier that the graph of a linear equation, $ax + by = c, \quad$ is a line. Each point on the line, an ordered pair $(x, y), \quad$ is a solution to the equation. For a system of two equations with two variables, we graph two lines. Then we can see that all the points that are solutions to each equation form a line. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions

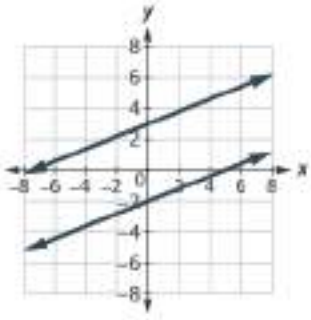
We know when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



One solution

The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.

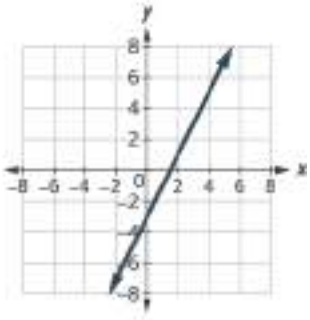
Consistent
Independent



No solution

The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.

Inconsistent



Infinitely many solutions

Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

Consistent
Dependent

Similarly, for a linear equation with three variables $ax + by + cz = d$, every solution to the equation is an ordered triple, (x, y, z) that makes the equation true.

Linear Equation in Three Variables

A linear equation with three variables, where *a, b, c,* and *d* are real numbers and *a, b*, and *c* are not all 0, is of the form

$$ax + by + cz = d$$

Every solution to the equation is an ordered triple, (x, y, z) that makes the equation true.

All the points that are solutions to one equation form a plane in three-dimensional space. And, by finding what the planes have in common, we'll find the solution to the system.

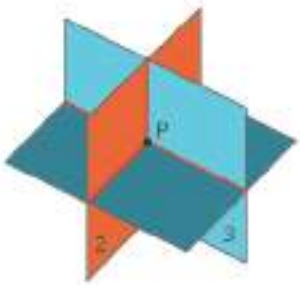
When we solve a system of three linear equations represented by a graph of three planes in space, there are three possible cases.

One solution

Consistent system and Independent equations

The 3 planes intersect.

The three intersecting planes have one point in common.



No solution

Inconsistent system

The planes are parallel.

Parallel planes have no points in common.



Two planes are coincident and parallel to the third plane.

The planes have no points in common.



Two planes are parallel and each intersect the third plane.

The planes have no points in common.



Each plane intersects the other two, but all three share no points.

The planes have no points in common.



Infinitely many solutions
Consistent system and dependent equations

Three planes intersect in one line.

There is just one line, so there are infinitely many solutions.



Two planes are coincident and intersect the third plane in a line.

There is just one line, so there are infinitely many solutions.



Three planes are coincident.

There is just one plane, so there are infinitely many solutions.



To solve a system of three linear equations, we want to find the values of the variables that are solutions to all three equations. In other words, we are looking for the ordered triple (x, y, z) that makes all three equations true. These are called the **solutions of the system of three linear equations with three variables**.

Solutions of a System of Linear Equations with Three Variables

Solutions of a system of equations are the values of the variables that make all the equations true. A solution is represented by an **ordered triple** (x, y, z).

To determine if an ordered triple is a solution to a system of three equations, we substitute the values of the variables into each equation. If the ordered triple makes all three equations true, it is a solution to the system.

EXAMPLE 4.31

Determine whether the ordered triple is a solution to the system: $\begin{cases} x - y + z = 2 \\ 2x - y - z = -6 \\ 2x + 2y + z = -3 \end{cases}$.

ⓐ $(-2, -1, 3)$ ⓑ $(-4, -3, 4)$

Solution

ⓐ

$$\begin{cases} x - y + z = 2 \\ 2x - y - z = -6 \\ 2x + 2y + z = -3 \end{cases}$$

We substitute $x = -2$ and $y = -1$ and $z = 3$ into all three equations.

$x - y + z = 2$	$2x - y - z = -6$	$2x + 2y + z = -3$
$-2 - (-1) + 3 \stackrel{?}{=} 2$	$2 \cdot (-2) - (-1) - 3 \stackrel{?}{=} -6$	$2 \cdot (-2) + 2(-1) + 3 \stackrel{?}{=} -3$
$2 = 2$ ✓	$-6 = -6$ ✓	$-3 = -3$ ✓
$(-2, -1, 3)$ does make all three equations true.		$(-2, -1, 3)$ is a solution.

ⓑ

$$\begin{cases} x - y + z = 2 \\ 2x - y - z = -6 \\ 2x + 2y + z = -3 \end{cases}$$

We substitute $x = -4$ and $y = -3$ and $z = 4$ into all three equations.

$x - y + z = 2$	$2x - y - z = -6$	$2x + 2y + z = -3$
$-4 - (-3) + 4 \stackrel{?}{=} 2$	$2 \cdot (-4) - (-3) - 4 \stackrel{?}{=} -6$	$2 \cdot (-4) + 2(-3) + 4 \stackrel{?}{=} -3$
$3 \neq 2$	$-9 \neq -6$	$-10 \neq -3$
$(-4, -3, 4)$ does not make all three equations true.		$(-4, -3, 4)$ is not a solution.

TRY IT : : 4.61

Determine whether the ordered triple is a solution to the system: $\begin{cases} 3x+y+z=2 \\ x+2y+z=-3. \\ 3x+y+2z=4 \end{cases}$

ⓐ $(1, -3, 2)$ ⓑ $(4, -1, -5)$

TRY IT : : 4.62

Determine whether the ordered triple is a solution to the system: $\begin{cases} x-3y+z=-5 \\ -3x-y-z=1 \\ 2x-2y+3z=1 \end{cases}$.

ⓐ $(2, -2, 3)$ ⓑ $(-2, 2, 3)$

Solve a System of Linear Equations with Three Variables

To solve a system of linear equations with three variables, we basically use the same techniques we used with systems that had two variables. We start with two pairs of equations and in each pair we eliminate the same variable. This will then give us a system of equations with only two variables and then we know how to solve that system!

Next, we use the values of the two variables we just found to go back to the original equation and find the third variable. We write our answer as an ordered triple and then check our results.

EXAMPLE 4.32 HOW TO SOLVE A SYSTEM OF EQUATIONS WITH THREE VARIABLES BY ELIMINATION

Solve the system by elimination: $\begin{cases} x-2y+z=3 \\ 2x+y+z=4 \\ 3x+4y+3z=-1 \end{cases}$.

Solution

Step 1. Write the equations in standard form. • If any coefficients are fractions, clear them.	The equations are in standard form. There are no fractions.	$x-2y+z=3$ (1) $2x+y+z=4$ (2) $3x+4y+3z=-1$ (3)
Step 2. Eliminate the same variable from two equations. • Decide which variable you will eliminate. • Work with a pair of equations to eliminate the chosen variable. • Multiply one or both equations so that the coefficients of that variable are opposites. • Add the equations to eliminate one variable.	We can eliminate the y's from equations (1) and (2) by multiplying equation (2) by 2. Multiply. We add the x's, y's, and constants. The new equation, (4), has only x and z.	$x-2y+z=3$ (1) $2(2x+y+z)=(4)2$ (2) $x-2y+z=3$ (1) $4x+2y+2z=8$ (2) $x-2y+z=3$ (1) $4x+2y+2z=8$ (2) $5x+3z=11$ (4)
Step 3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.	We can again eliminate the y's using the equations (1) and (3) by multiplying equation (1) by 2. Add the new equations and the result will be equation (5).	$2(x-2y+z)=(3)2$ (1) $3x+4y+3z=-1$ (3) $2x-4y+2z=6$ (1) $3x+4y+3z=-1$ (3) $5x+5z=5$ (5)

Step 4. The two new equations form a system of two equations with two variables. Solve this system.	Use equations (4) and (5) to eliminate either x or z.	$\begin{cases} 5x+3z=11 & (4) \\ 5x+5z=5 & (5) \end{cases}$
	Multiply (5) by –1.	$\begin{aligned} 5x+3z&=11 && (4)\\ -1(5x+5z)&=5(-1) && (5) \end{aligned}$
		$\begin{aligned} 5x+3z&=11 && (4)\\ -5x-5z&=-5 && (5)\\ \hline -2z&=6\\ z&=-3 \end{aligned}$
	Use either (4) or (5) to solve for x using substitution.	$\begin{aligned} 5x+3z&=11 && (4)\\ 5x+3(-3)&=11\\ 5x-9&=11\\ 5x&=20\\ x&=4 \end{aligned}$
Step 5. Use the values of the two variables found in Step 4 to find the third variable.	Use an original equation to find y.	$\begin{aligned} 2x+y+z&=4 && (2)\\ 2\cdot 4+y+z&=4\\ 8+y-3&=4\\ y&=-1 \end{aligned}$
Step 6. Write the solution as an ordered triple.	Write it as (x, y, z).	$(4, -1, -3)$
Step 7. Check that the ordered triple is a solution to **all three** original equations.	Substitute $x=4$, $y=-1$ and $z=-3$ into all three equations. Do they make the equations true?	We'll leave the checks to you. The solution is $(4, -1, -3)$.

TRY IT : : 4.63

Solve the system by elimination: $\begin{cases} 3x+y-z=2 \\ 2x-3y-2z=1. \\ 4x-y-3z=0 \end{cases}$

TRY IT : : 4.64

Solve the system by elimination: $\begin{cases} 4x+y+z=-1 \\ -2x-2y+z=2. \\ 2x+3y-z=1 \end{cases}$

The steps are summarized here.

HOW TO : : SOLVE A SYSTEM OF LINEAR EQUATIONS WITH THREE VARIABLES.

Step 1. Write the equations in standard form
- If any coefficients are fractions, clear them.

Step 2. Eliminate the same variable from two equations.
- Decide which variable you will eliminate.
- Work with a pair of equations to eliminate the chosen variable.
- Multiply one or both equations so that the coefficients of that variable are opposites.
- Add the equations resulting from Step 2 to eliminate one variable

Step 3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.

Step 4. The two new equations form a system of two equations with two variables. Solve this system.

Step 5. Use the values of the two variables found in Step 4 to find the third variable.

Step 6. Write the solution as an ordered triple.

Step 7. Check that the ordered triple is a solution to **all three** original equations.

EXAMPLE 4.33

Solve: $\begin{cases} 3x-4z=0 \\ 3y+2z=-3 \\ 2x+3y=-5 \end{cases}$.

Solution

$$\begin{cases} 3x-4z=0 \quad (1) \\ 3y+2z=-3 \ (2) \\ 2x+3y=-5 \ (3) \end{cases}$$

We can eliminate z from equations (1) and (2) by multiplying equation (2) by 2 and then adding the resulting equations.

$$\begin{array}{rr} 3x & -4z=0 \quad (1) \\ 3y+2z=-3 & (2) \end{array} \longrightarrow \begin{array}{rr} 3x & -4z=0 \\ 2(3y+2z)=(-3)2 & \end{array} \longrightarrow \begin{array}{rr} 3x & -4z=0 \\ 6y+4z=-6 & \\ \hline 3x+6y & =-6 \quad (4) \end{array}$$

Notice that equations (3) and (4) both have the variables x and y. We will solve this new system for x and y.

$$\begin{array}{rr} 2x+3y=-5 & (3) \\ 3x+6y=-6 & (4) \end{array} \longrightarrow \begin{array}{rr} -2(2x+3y)=-2(-5) & \\ 3x+6y=-6 & \end{array} \longrightarrow \begin{array}{rr} -4x-6y=10 & \\ 3x+6y=-6 & \\ \hline -x & =4 \\ x=-4 & \end{array}$$

To solve for *y*, we substitute $x=-4$ into equation (3).

$$\begin{array}{rr} 2x+3y=-5 & (3) \\ 2(-4)+3y=-5 & \\ -8+3y=-5 & \\ 3y=3 & \\ y=1 & \end{array}$$

We now have $x=-4$ and $y=1$. We need to solve for *z*. We can substitute $x=-4$ into equation (1) to find *z*.

$$\begin{array}{rr} 3x-4z=0 & (1) \\ 3(-4)-4z=0 & \\ -12-4z=0 & \\ -4z=12 & \\ z=-3 & \end{array}$$

We write the solution as an ordered triple. $(-4,\ 1,\ -3)$

We check that the solution makes all three equations true.

$$\begin{aligned} 3x-4z &= 0 \quad (1) \\ 3(-4)-4(-3) &\stackrel{?}{=} 0 \\ 0 &= 0\ \checkmark \end{aligned}$$

$$\begin{aligned} 3y+2z &= -3 \quad (2) \\ 3(1)+2(-3) &\stackrel{?}{=} -3 \\ -3 &= -3\ \checkmark \end{aligned}$$

$$\begin{aligned} 2x+3y &= -5 \quad (3) \\ 2(-4)+3(1) &\stackrel{?}{=} -5 \\ -5 &= -5\ \checkmark \end{aligned}$$

The solution is $(-4,\ 1,\ -3)$.

TRY IT : : 4.65

Solve: $\begin{cases} 3x-4z=-1 \\ 2y+3z=2 \\ 2x+3y=6 \end{cases}$.

TRY IT : : 4.66

Solve: $\begin{cases} 4x-3z=-5 \\ 3y+2z=7 \\ 3x+4y=6 \end{cases}$.

When we solve a system and end up with no variables and a false statement, we know there are no solutions and that the system is inconsistent. The next example shows a system of equations that is inconsistent.

EXAMPLE 4.34

Solve the system of equations: $\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$.

Solution

$$\begin{cases} x+2y-3z=-1 & (1) \\ x-3y+z=1 & (2) \\ 2x-y-2z=2 & (3) \end{cases}$$

Use equation (1) and (2) to eliminate z.

$$\begin{aligned} x+2y-3z &= -1 \quad (1) \\ x-3y+z &= 1 \quad (2) \end{aligned} \xrightarrow{\text{multiply by 3}} \begin{aligned} x+2y-3z &= -1 \\ 3x-9y+3z &= 3 \\ \hline 4x-7y &= 2 \quad (4) \end{aligned}$$

Use (2) and (3) to eliminate z again.

$$\begin{aligned} x-3y+z &= 1 \quad (2) \\ 2x-y-2z &= 2 \quad (3) \end{aligned} \xrightarrow{\text{multiply by 2}} \begin{aligned} 2x-6y+2z &= 2 \\ 2x-y-2z &= 2 \\ \hline 4x-7y &= 4 \quad (5) \end{aligned}$$

Use (4) and (5) to eliminate a variable.

$$\begin{aligned} 4x-7y &= 2 \quad (4) \\ 4x-7y &= 4 \quad (5) \end{aligned} \xrightarrow{\text{multiply by } -1} \begin{aligned} 4x-7y &= 2 \\ -4x+7y &= -4 \\ \hline 0 &= -2 \ \textit{false} \end{aligned}$$

There is no solution.

We are left with a false statement and this tells us the system is inconsistent and has no solution.

TRY IT : : 4.67

Solve the system of equations: $\begin{cases} x+2y+6z=5 \\ -x+y-2z=3. \\ x-4y-2z=1 \end{cases}$

TRY IT : : 4.68

Solve the system of equations: $\begin{cases} 2x-2y+3z=6 \\ 4x-3y+2z=0 \\ -2x+3y-7z=1 \end{cases}$.

When we solve a system and end up with no variables but a true statement, we know there are infinitely many solutions. The system is consistent with dependent equations. Our solution will show how two of the variables depend on the third.

EXAMPLE 4.35

Solve the system of equations: $\begin{cases} x+2y-z=1 \\ 2x+7y+4z=11. \\ x+3y+z=4 \end{cases}$

Solution

$$\begin{cases} x+2y-z=1 & (1) \\ 2x+7y+4z=11 & (2) \\ x+3y+z=4 & (3) \end{cases}$$

Use equation (1) and (3) to eliminate *x*.

$$\begin{array}{l} x+2y-z=1 \quad (1) \\ x+3y+z=4 \quad (3) \end{array} \xrightarrow{\text{multiply by } -1} \begin{array}{r} -x-2y+z=-1 \quad (1) \\ x+3y+z=4 \quad (3) \\ \hline y+2z=3 \quad (4) \end{array}$$

Use equation (1) and (2) to eliminate *x* again.

$$\begin{array}{l} x+2y-z=1 \quad (1) \\ 2x+7y+4z=11 \quad (2) \end{array} \xrightarrow{\text{multiply by } -2} \begin{array}{r} -2x-4y+2z=-2 \quad (1) \\ 2x+7y+4z=11 \quad (2) \\ \hline 3y+6z=9 \quad (5) \end{array}$$

Use equation (4) and (5) to eliminate y.

$$\begin{array}{l} y+2z=3 \quad (4) \\ 3y+6z=9 \quad (5) \end{array} \xrightarrow{\text{multiply by } -3} \begin{array}{r} -3y-6z=-9 \quad (4) \\ 3y+6z=9 \quad (5) \\ \hline 0=0 \end{array}$$

	There are infinitely many solutions.
Solve equation (4) for *y*.	Represent the solution showing how *x* and *y* are dependent on *z*. $y+2z = 3$ $y = -2z+3$
Use equation (1) to solve for *x*.	$x+2y-z=1$
Substitute $y=-2z+3$.	$x+2(-2z+3)-z = 1$ $x-4z+6-z = 1$ $x-5z+6 = 1$ $x = 5z-5$

The true statement $0=0$ tells us that this is a dependent system that has infinitely many solutions. The solutions are of the form (x, y, z) where $x=5z-5$; $y=-2z+3$ and *z* is any real number.

TRY IT : : 4.69

Solve the system by equations: $\begin{cases} x+y-z=0 \\ 2x+4y-2z=6. \\ 3x+6y-3z=9 \end{cases}$

TRY IT : : 4.70

Solve the system by equations: $\begin{cases} x-y-z=1 \\ -x+2y-3z=-4. \\ 3x-2y-7z=0 \end{cases}$

Solve Applications using Systems of Linear Equations with Three Variables

Applications that are modeled by a systems of equations can be solved using the same techniques we used to solve the systems. Many of the application are just extensions to three variables of the types we have solved earlier.

EXAMPLE 4.36

The community college theater department sold three kinds of tickets to its latest play production. The adult tickets sold for $15, the student tickets for $10 and the child tickets for $8. The theater department was thrilled to have sold 250 tickets and brought in $2,825 in one night. The number of student tickets sold is twice the number of adult tickets sold. How many of each type did the department sell?

Solution

We will use a chart to organize the information.

Type	Number	· Value	= Total Value
adult	x	15	$15x$
student	y	10	$10y$
child	z	8	$8z$
	250		2825

Number of students is twice number of adults.

Rewrite the equation in standard form.

$$\begin{aligned} y &= 2x \\ 2x-y &= 0 \end{aligned}$$

Write the system of equations.

$$\begin{aligned} x+y+z &= 250 \quad (1) \\ 15x+10y+8z &= 2825 \quad (2) \\ -2x+y &= 0 \quad (3) \end{aligned}$$

Use equations (1) and (2) to eliminate z.

$$\begin{aligned} x+y+z &= 250 \quad (1) \\ 15x+10y+8z &= 2825 \quad (2) \end{aligned} \xrightarrow{\text{Multiply by } -8} \begin{aligned} -8x-8y-8z &= 250 \quad (1) \\ 15x+10y+8z &= 2825 \quad (2) \\ 7x+2y &= 825 \quad (4) \end{aligned}$$

Use (3) and (4) to eliminate y.

$$\begin{aligned} -2x+y &= 0 \quad (3) \\ 7x+2y &= 825 \quad (4) \end{aligned} \xrightarrow{\text{Multiply by } -2} \begin{aligned} 4x-2y &= 0 \quad (3) \\ 7x+2y &= 825 \quad (4) \\ 11x &= 825 \end{aligned}$$

Solve for x. $x = 75$ adult tickets

Use equation (3) to find y. $-2x+y=0$

Substitute $x = 75$.	$\begin{aligned} -2(75) + y &= 0 \\ -150 + y &= 0 \\ y &= 150 \text{ student tickets} \end{aligned}$
Use equation (1) to find z.	$x + y + z = 250$
Substitute in the values $x = 75,\ y = 150$.	$\begin{aligned} 75 + 150 + z &= 250 \\ 225 + z &= 250 \\ z &= 25 \text{ child tickets} \end{aligned}$
Write the solution.	The theater department sold 75 adult tickets, 150 student tickets, and 25 child tickets.

TRY IT : : 4.71

The community college fine arts department sold three kinds of tickets to its latest dance presentation. The adult tickets sold for $20, the student tickets for $12 and the child tickets for $10.The fine arts department was thrilled to have sold 350 tickets and brought in $4,650 in one night. The number of child tickets sold is the same as the number of adult tickets sold. How many of each type did the department sell?

TRY IT : : 4.72

The community college soccer team sold three kinds of tickets to its latest game. The adult tickets sold for $10, the student tickets for $8 and the child tickets for $5. The soccer team was thrilled to have sold 600 tickets and brought in $4,900 for one game. The number of adult tickets is twice the number of child tickets. How many of each type did the soccer team sell?

MEDIA : :

Access this online resource for additional instruction and practice with solving a linear system in three variables with no or infinite solutions.

- **Solving a Linear System in Three Variables with No or Infinite Solutions (https://openstax.org/l/37linsys3var)**
- **3 Variable Application (https://openstax.org/l/37variableapp)**

4.4 EXERCISES

Practice Makes Perfect

Determine Whether an Ordered Triple is a Solution of a System of Three Linear Equations with Three Variables

In the following exercises, determine whether the ordered triple is a solution to the system.

162. $\begin{cases} 2x-6y+z=3 \\ 3x-4y-3z=2 \\ 2x+3y-2z=3 \end{cases}$

ⓐ $(3,\ 1,\ 3)$

ⓑ $(4,\ 3,\ 7)$

163. $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

ⓐ $(-5,\ -7,\ 4)$

ⓑ $(5,\ 7,\ 4)$

164. $\begin{cases} y-10z=-8 \\ 2x-y=2 \\ x-5z=3 \end{cases}$

ⓐ $(7,\ 12,\ 2)$

ⓑ $(2,\ 2,\ 1)$

165. $\begin{cases} x+3y-z=15 \\ y=\frac{2}{3}x-2 \\ x-3y+z=-2 \end{cases}$

ⓐ $\left(-6,\ 5,\ \frac{1}{2}\right)$

ⓑ $\left(5,\ \frac{4}{3},\ -3\right)$

Solve a System of Linear Equations with Three Variables

In the following exercises, solve the system of equations.

166. $\begin{cases} 5x+2y+z=5 \\ -3x-y+2z=6 \\ 2x+3y-3z=5 \end{cases}$

167. $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

168. $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

169. $\begin{cases} 5x-3y+2z=-5 \\ 2x-y-z=4 \\ 3x-2y+2z=-7 \end{cases}$

170. $\begin{cases} 3x-5y+4z=5 \\ 5x+2y+z=0 \\ 2x+3y-2z=3 \end{cases}$

171. $\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$

172. $\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$

173. $\begin{cases} 11x+9y+2z=-9 \\ 7x+5y+3z=-7 \\ 4x+3y+z=-3 \end{cases}$

174. $\begin{cases} \frac{1}{3}x-y-z=1 \\ x+\frac{5}{2}y+z=-2 \\ 2x+2y+\frac{1}{2}z=-4 \end{cases}$

175. $\begin{cases} x+\frac{1}{2}y+\frac{1}{2}z=0 \\ \frac{1}{5}x-\frac{1}{5}y+z=0 \\ \frac{1}{3}x-\frac{1}{3}y+2z=-1 \end{cases}$

176. $\begin{cases} x+\frac{1}{3}y-2z=-1 \\ \frac{1}{3}x+y+\frac{1}{2}z=0 \\ \frac{1}{2}x+\frac{1}{3}y-\frac{1}{2}z=-1 \end{cases}$

177. $\begin{cases} \frac{1}{3}x-y+\frac{1}{2}z=4 \\ \frac{2}{3}x+\frac{5}{2}y-4z=0 \\ x-\frac{1}{2}y+\frac{3}{2}z=2 \end{cases}$

178. $\begin{cases} x+2z=0 \\ 4y+3z=-2 \\ 2x-5y=3 \end{cases}$

179. $\begin{cases} 2x+5y=4 \\ 3y-z=3 \\ 4x+3z=-3 \end{cases}$

180. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

181. $\begin{cases} 3x - z = -3 \\ 5y + 2z = -6 \\ 4x + 3y = -8 \end{cases}$

182. $\begin{cases} 4x - 3y + 2z = 0 \\ -2x + 3y - 7z = 1 \\ 2x - 2y + 3z = 6 \end{cases}$

183. $\begin{cases} x - 2y + 2z = 1 \\ -2x + y - z = 2 \\ x - y + z = 5 \end{cases}$

184. $\begin{cases} 2x + 3y + z = 12 \\ x + y + z = 9 \\ 3x + 4y + 2z = 20 \end{cases}$

185. $\begin{cases} x + 4y + z = -8 \\ 4x - y + 3z = 9 \\ 2x + 7y + z = 0 \end{cases}$

186. $\begin{cases} x + 2y + z = 4 \\ x + y - 2z = 3 \\ -2x - 3y + z = -7 \end{cases}$

187. $\begin{cases} x + y - 2z = 3 \\ -2x - 3y + z = -7 \\ x + 2y + z = 4 \end{cases}$

188. $\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ -x + 2y = 1 \end{cases}$

189. $\begin{cases} x - 2y + 3z = 1 \\ x + y - 3z = 7 \\ 3x - 4y + 5z = 7 \end{cases}$

Solve Applications using Systems of Linear Equations with Three Variables

In the following exercises, solve the given problem.

190. The sum of the measures of the angles of a triangle is 180. The sum of the measures of the second and third angles is twice the measure if the first angle. The third angle is twelve more than the second. Find the measures of the three angles.

191. The sum of the measures of the angles of a triangle is 180. The sum of the measures of the second and third angles is three the measure if the first angle. The third angle is fifteen more than the second. Find the measures of the three angles.

192. After watching a major musical production at the theater, the patrons can purchase souvenirs. If a family purchases 4 t-shirts, the video and 1 stuffed animal their total is $135.

A couple buys 2 t-shirts, the video and 3 stuffed animal for their nieces and spends $115. Another couple buys 2 t-shirts, the video and 1 stuffed animal and their total is $85. What is the cost of each item?

193. The church youth group is selling snacks to raise money to attend their convention. Amy sold 2 pounds of candy, 3 boxes of cookies and 1 can of popcorn for a total sales of $65. Brian sold 4 pounds of candy, 6 boxes of cookies and 3 cans of popcorn for a total sales of $140. Paulina sold 8 pounds of candy, 8 boxes of cookies and 5 can of popcorn for a total sales of $250. What is the cost of each item?

Writing Exercises

194. In your own words explain the steps to solve a system of linear equations with three variables by elimination.

195. How can you tell when a system of three linear equations with three variables has no solution? Infinitely many solutions?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered triple is a solution of a system of three linear equations with three variables.			
solve a system of linear equations with three variables.			
solve applications using systems of linear equations with three variables.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

4.5 Solve Systems of Equations Using Matrices

Learning Objectives

By the end of this section, you will be able to:

- Write the augmented matrix for a system of equations
- Use row operations on a matrix
- Solve systems of equations using matrices

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $3(x+2)+4=4(2x-1)+9$.
 If you missed this problem, review **Example 2.2**.
2. Solve: $0.25p+0.25(x+4)=5.20$.
 If you missed this problem, review **Example 2.13**.
3. Evaluate when $x=-2$ and $y=3$: $2x^2-xy+3y^2$.
 If you missed this problem, review **Example 1.21**.

Write the Augmented Matrix for a System of Equations

Solving a system of equations can be a tedious operation where a simple mistake can wreak havoc on finding the solution. An alternative method which uses the basic procedures of elimination but with notation that is simpler is available. The method involves using a **matrix**. A matrix is a rectangular array of numbers arranged in rows and columns.

Matrix

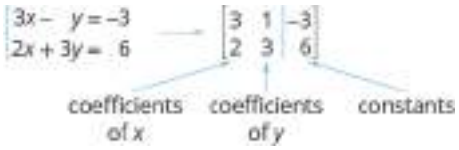
A **matrix** is a rectangular array of numbers arranged in rows and columns.

A matrix with *m* rows and *n* columns has order $m\times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.



Each number in the matrix is called an element or entry in the matrix.

We will use a matrix to represent a system of linear equations. We write each equation in standard form and the coefficients of the variables and the constant of each equation becomes a row in the matrix. Each column then would be the coefficients of one of the variables in the system or the constants. A vertical line replaces the equal signs. We call the resulting matrix the augmented matrix for the system of equations.



Notice the first column is made up of all the coefficients of *x*, the second column is the all the coefficients of *y*, and the third column is all the constants.

EXAMPLE 4.37

Write each system of linear equations as an augmented matrix:

ⓐ $\begin{cases} 5x-3y=-1 \\ y=2x-2 \end{cases}$ ⓑ $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

Solution

ⓐ The second equation is not in standard form. We rewrite the second equation in standard form.

$$\begin{aligned} y &= 2x-2 \\ -2x+y &= -2 \end{aligned}$$

We replace the second equation with its standard form. In the augmented matrix, the first equation gives us the first row and the second equation gives us the second row. The vertical line replaces the equal signs.

$$\begin{bmatrix} 5x-3y=-1 \\ 2x-y=2 \end{bmatrix} \qquad \begin{array}{c} x \quad y \\ \left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \end{array}$$

ⓑ All three equations are in standard form. In the augmented matrix the first equation gives us the first row, the second equation gives us the second row, and the third equation gives us the third row. The vertical line replaces the equal signs.

$$\begin{bmatrix} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{bmatrix} \qquad \begin{array}{c} x \quad y \quad z \\ \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \end{array}$$

TRY IT : : 4.73 Write each system of linear equations as an augmented matrix:

ⓐ $\begin{cases} 3x+8y=-3 \\ 2x=-5y-3 \end{cases}$ ⓑ $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

TRY IT : : 4.74 Write each system of linear equations as an augmented matrix:

ⓐ $\begin{cases} 11x=-9y-5 \\ 7x+5y=-1 \end{cases}$ ⓑ $\begin{cases} 5x-3y+2z=-5 \\ 2x-y-z=4 \\ 3x-2y+2z=-7 \end{cases}$

It is important as we solve systems of equations using matrices to be able to go back and forth between the system and the matrix. The next example asks us to take the information in the matrix and write the system of equations.

EXAMPLE 4.38

Write the system of equations that corresponds to the augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & -3 & 3 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & -1 & 3 & -4 \end{array}\right].$$

Solution

We remember that each row corresponds to an equation and that each entry is a coefficient of a variable or the constant. The vertical line replaces the equal sign. Since this matrix is a 4×3, we know it will translate into a system of three equations with three variables.

$$\begin{array}{c} x \quad y \quad z \\ \left[\begin{array}{ccc|c} 4 & -3 & 3 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & -1 & 3 & -4 \end{array}\right] \end{array} \longrightarrow \begin{bmatrix} 4x-3y+3z=-1 \\ x+2y-z=2 \\ -2x-y+3z=-4 \end{bmatrix}$$

TRY IT : : 4.75

Write the system of equations that corresponds to the augmented matrix: $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & -2 & 1 \\ 4 & -1 & 2 & 0 \end{bmatrix}$.

TRY IT : : 4.76

Write the system of equations that corresponds to the augmented matrix: $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & -1 & 8 \\ 1 & 1 & -1 & 3 \end{bmatrix}$.

Use Row Operations on a Matrix

Once a system of equations is in its augmented matrix form, we will perform operations on the rows that will lead us to the solution.

To solve by elimination, it doesn't matter which order we place the equations in the system. Similarly, in the matrix we can interchange the rows.

When we solve by elimination, we often multiply one of the equations by a constant. Since each row represents an equation, and we can multiply each side of an equation by a constant, similarly we can multiply each entry in a row by any real number except 0.

In elimination, we often add a multiple of one row to another row. In the matrix we can replace a row with its sum with a multiple of another row.

These actions are called row operations and will help us use the matrix to solve a system of equations.

Row Operations

In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.

1. Interchange any two rows.
2. Multiply a row by any real number except 0.
3. Add a nonzero multiple of one row to another row.

Performing these operations is easy to do but all the arithmetic can result in a mistake. If we use a system to record the row operation in each step, it is much easier to go back and check our work.

We use capital letters with subscripts to represent each row. We then show the operation to the left of the new matrix. To show interchanging a row:

$$\left[\begin{array}{ccc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \begin{matrix} \\ \end{matrix} \begin{matrix} \\ \end{matrix} \begin{matrix} R_2 \\ R_1 \end{matrix} \left[\begin{array}{ccc|c} 2 & -1 & 2 \\ 5 & -3 & -1 \end{array}\right]$$

To multiply row 2 by -3 :

$$\left[\begin{array}{ccc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \begin{matrix} \\ \end{matrix} \begin{matrix} \\ -3R_2 \end{matrix} \left[\begin{array}{ccc|c} 5 & -3 & -1 \\ -6 & 3 & -6 \end{array}\right]$$

To multiply row 2 by -3 and add it to row 1:

$$\left[\begin{array}{ccc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \begin{matrix} \\ \end{matrix} \begin{matrix} \\ -3R_2 + R_1 \end{matrix} \left[\begin{array}{ccc|c} -1 & 0 & -7 \\ 2 & -1 & 2 \end{array}\right]$$

EXAMPLE 4.39

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 2 and 3.

ⓑ Multiply row 2 by 5.

ⓒ Multiply row 3 by -2 and add to row 1.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

Solution

ⓐ We interchange rows 2 and 3.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{\substack{R_2 \\ R_3}} \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 3 & -3 & 1 & -1 \\ 2 & 1 & -4 & 5 \end{array}\right]$$

ⓑ We multiply row 2 by 5.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{5R_2} \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 10 & 5 & -20 & 25 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

ⓒ We multiply row 3 by -2 and add to row 1.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{-2R_3+R_1} \left[\begin{array}{ccc|c} (-2\cdot 3)+6 & (-2\cdot(-3))+(-5) & (-2\cdot 1)+2 & (-2\cdot(-1))+3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-2R_3+R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 5 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

TRY IT : : 4.77 Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 3.

ⓑ Multiply row 3 by 3.

ⓒ Multiply row 3 by 2 and add to row 2.

$$\left[\begin{array}{ccc|c} 5 & -2 & -2 & -2 \\ 4 & -1 & -4 & 4 \\ -2 & 3 & 0 & -1 \end{array}\right]$$

TRY IT : : 4.78 Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 2,

ⓑ Multiply row 1 by 2,

ⓒ Multiply row 2 by 3 and add to row 1.

$$\left[\begin{array}{ccc|c} 2 & -3 & -2 & -4 \\ 4 & 1 & -3 & 2 \\ 5 & 0 & 4 & -1 \end{array}\right]$$

Now that we have practiced the row operations, we will look at an augmented matrix and figure out what operation we will use to reach a goal. This is exactly what we did when we did elimination. We decided what number to multiply a row by in order that a variable would be eliminated when we added the rows together.

Given this system, what would you do to eliminate x?

$$\begin{array}{l} x - y = 2 \\ 4x - 8y = 0 \end{array} \xrightarrow{\text{multiply the first equation by } -4} \begin{array}{l} -4x + 4y = -8 \\ 4x - 8y = 0 \end{array} \xrightarrow{\text{then add}} \begin{array}{r} -4x + 4y = -8 \\ 4x - 8y = 0 \\ \hline -4y = -8 \end{array}$$

This next example essentially does the same thing, but to the matrix.

EXAMPLE 4.40

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & -8 & 0 \end{array}\right]$.

Solution

To make the 4 a 0, we could multiply row 1 by -4 and then add it to row 2.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & -8 & 0 \end{array}\right] \xrightarrow{-4R_1 + R_2} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -4 & -8 \end{array}\right]$$

TRY IT : : 4.79

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 3 & -6 & 2 \end{array}\right]$.

TRY IT : : 4.80

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 3 \\ -2 & -3 & 2 \end{array}\right]$.

Solve Systems of Equations Using Matrices

To solve a system of equations using matrices, we transform the augmented matrix into a matrix in **row-echelon form** using row operations. For a consistent and independent system of equations, its augmented matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

Row-Echelon Form

For a consistent and independent system of equations, its augmented matrix is in **row-echelon form** when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right] \quad a, b, c, d, e, f \text{ are real numbers}$$

Once we get the augmented matrix into row-echelon form, we can write the equivalent system of equations and read the value of at least one variable. We then substitute this value in another equation to continue to solve for the other variables. This process is illustrated in the next example.

EXAMPLE 4.41 HOW TO SOLVE A SYSTEM OF EQUATIONS USING A MATRIX

Solve the system of equations using a matrix: $\begin{cases} 3x + 4y = 5 \\ x + 2y = 1 \end{cases}$.

Solution

Step 1. Write the augmented matrix for the system of equations.		$3x + 4y = 5$ $x + 2y = 1$ $\left[\begin{array}{cc|c} 3 & 4 & 5 \\ 1 & 2 & 1 \end{array}\right]$
Step 2. Using row operations get the entry in row 1, column 1 to be 1.	Interchange the rows, so 1 will be in row 1, column 1.	$\begin{matrix} R_2 \\ R_1 \end{matrix} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 5 \end{array}\right]$

Step 3. Using row operations, get zeros in column 1 below the 1.	Multiply row 1 by –3 and add it to row 2.	$-3R_1 + R_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \end{bmatrix}$
Step 4. Using row operations, get the entry in row 2, column 2 to be 1.	Multiply row 2 by $-\frac{1}{2}$.	$-\frac{1}{2}R_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
Step 5. Continue the process until the matrix is in row-echelon form.	The matrix is now in row-echelon form.	$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
Step 6. Write the corresponding system of equations.	$\begin{matrix} x & y & \\ \end{matrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$	$\begin{cases} x + 2y = 1 \\ y = -1 \end{cases}$
Step 7. Use substitution to find the remaining variables.	Substitute $y = -1$ into $x + 2y = 1$.	$y = -1$ $x + 2y = 1$ $x + 2(-1) = 1$ $x - 2 = 1$ $x = 3$
Step 8. Write the solution as an ordered pair or triple.		$(3, -1)$
Step 9. Check that the solution makes the original equations true.		We leave the check to you.

TRY IT : : 4.81 Solve the system of equations using a matrix: $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$.

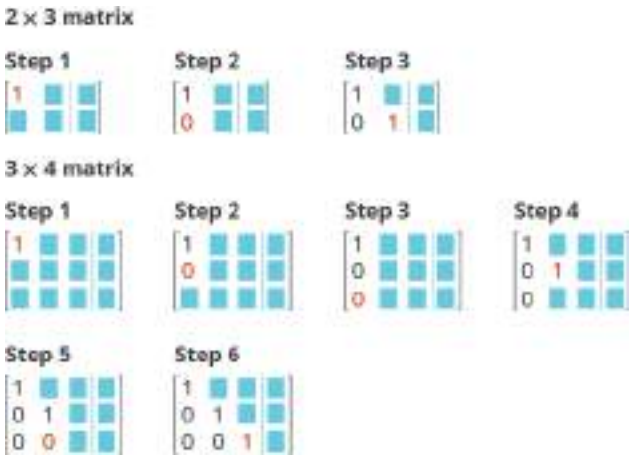
TRY IT : : 4.82 Solve the system of equations using a matrix: $\begin{cases} 2x + y = -4 \\ x - y = -2 \end{cases}$.

The steps are summarized here.

HOW TO : : SOLVE A SYSTEM OF EQUATIONS USING MATRICES.

Step 1. Write the augmented matrix for the system of equations.
Step 2. Using row operations get the entry in row 1, column 1 to be 1.
Step 3. Using row operations, get zeros in column 1 below the 1.
Step 4. Using row operations, get the entry in row 2, column 2 to be 1.
Step 5. Continue the process until the matrix is in row-echelon form.
Step 6. Write the corresponding system of equations.
Step 7. Use substitution to find the remaining variables.
Step 8. Write the solution as an ordered pair or triple.
Step 9. Check that the solution makes the original equations true.

Here is a visual to show the order for getting the 1's and 0's in the proper position for row-echelon form.



We use the same procedure when the system of equations has three equations.

EXAMPLE 4.42

Solve the system of equations using a matrix: $\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$.

Solution

$$\begin{aligned} 3x+8y+2z&=-5\\ 2x+5y-3z&=0\\ x+2y-2z&=-1 \end{aligned}$$

Write the augmented matrix for the equations.

$$\begin{bmatrix} 3 & 8 & 2 & -5 \\ 2 & 5 & -3 & 0 \\ 1 & 2 & -2 & -1 \end{bmatrix}$$

Interchange row 1 and 3 to get the entry in row 1, column 1 to be 1.

$$\begin{matrix} R_1 \\ \\ R_3 \end{matrix}\begin{bmatrix} 1 & 2 & -2 & -1 \\ 2 & 5 & -3 & 0 \\ 3 & 8 & 2 & -5 \end{bmatrix}$$

Using row operations, get zeros in column 1 below the 1.

$$\begin{matrix} \\ -2R_1+R_2 \\ \end{matrix}\begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 3 & 8 & 2 & -5 \end{bmatrix}$$

$$\begin{matrix} \\ \\ -3R_1+R_3 \end{matrix}\begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 8 & -2 \end{bmatrix}$$

The entry in row 2, column 2 is now 1.

Continue the process until the matrix is in row-echelon form.

$$\begin{matrix} \\ \\ -2R_2+R_3 \end{matrix}\begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & -6 \end{bmatrix}$$

$$\begin{matrix} \\ \\ \frac{1}{6}R_3 \end{matrix}\begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The matrix is now in row-echelon form.	$\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$
Write the corresponding system of equations.	$\begin{cases} x+2y-2z=-1 \\ y+z=2 \\ z=-1 \end{cases}$
Use substitution to find the remaining variables.	$y+z=2$ $y+(-1)=2$ $y=3$
	$x+2y-2z=-1$ $x+2(3)-2(-1)=-1$ $x+6+2=-1$ $x=-9$
Write the solution as an ordered pair or triple.	$(-9, 3, -1)$
Check that the solution makes the original equations true.	We leave the check for you.

TRY IT : : 4.83 Solve the system of equations using a matrix: $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$.

TRY IT : : 4.84 Solve the system of equations using a matrix: $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$.

So far our work with matrices has only been with systems that are consistent and independent, which means they have exactly one solution. Let's now look at what happens when we use a matrix for a dependent or inconsistent system.

EXAMPLE 4.43

Solve the system of equations using a matrix: $\begin{cases} x+y+3z=0 \\ x+3y+5z=0 \\ 2x+4z=1 \end{cases}$.

Solution

	$\begin{cases} x+y+3z=0 \\ x+3y+5z=0 \\ 2x+4z=1 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 1 & 3 & 5 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$
The entry in row 1, column 1 is 1.	
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -1R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$

$-2R_1 + R_3$ $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 1 \end{bmatrix}$

Continue the process until the matrix is in row-echelon form.

$\frac{1}{2}R_2$ $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{bmatrix}$

Multiply row 2 by 2 and add it to row 3.

$2R_2 + R_3$ $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

At this point, we have all zeros on the left of row 3.

Write the corresponding system of equations.

$$\begin{aligned} x + y + 3z &= 0 \\ y + z &= 0 \\ 0 &\neq 1 \end{aligned}$$

Since $0 \neq 1$ we have a false statement. Just as when we solved a system using other methods, this tells us we have an inconsistent system. There is no solution.

TRY IT : : 4.85

Solve the system of equations using a matrix: $\begin{cases} x - 2y + 2z = 1 \\ -2x + y - z = 2. \\ x - y + z = 5 \end{cases}$

TRY IT : : 4.86

Solve the system of equations using a matrix: $\begin{cases} 3x + 4y - 3z = -2 \\ 2x + 3y - z = -12. \\ x + y - 2z = 6 \end{cases}$

The last system was inconsistent and so had no solutions. The next example is dependent and has infinitely many solutions.

EXAMPLE 4.44

Solve the system of equations using a matrix: $\begin{cases} x - 2y + 3z = 1 \\ x + y - 3z = 7 \\ 3x - 4y + 5z = 7 \end{cases}$.

Solution

$$\begin{aligned} x - 2y + 3z &= 1 \\ x + y - 3z &= 7 \\ 3x - 4y + 5z &= 7 \end{aligned}$$

Write the augmented matrix for the equations.

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 1 & 1 & -3 & 7 \\ 3 & -4 & 5 & 7 \end{bmatrix}$$

The entry in row 1, column 1 is 1.

Using row operations, get zeros in column 1 below the 1.

$$\begin{bmatrix}1 & -2 & 3 & 1\\ 0 & 3 & -6 & 6\\ 3 & -4 & 5 & 7\end{bmatrix} \quad -1R_1 + R_2$$

$$\begin{bmatrix}1 & -2 & 3 & 1\\ 0 & 3 & -6 & 6\\ 0 & 2 & -4 & 4\end{bmatrix} \quad -3R_1 + R_3$$

Continue the process until the matrix is in row-echelon form.

$$\begin{bmatrix}1 & -2 & 3 & 1\\ 0 & 1 & -2 & 2\\ 0 & 2 & -4 & 4\end{bmatrix} \quad \frac{1}{3}R_2$$

Multiply row 2 by -2 and add it to row 3.

$$\begin{bmatrix}1 & -2 & 3 & 1\\ 0 & 1 & -2 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} \quad -2R_2 + R_3$$

At this point, we have all zeros in the bottom row.

Write the corresponding system of equations.

$$\begin{cases} x - 2y + 3z = 1\\ y - 2z = 2\\ 0 = 0 \end{cases}$$

Since $0 = 0$ we have a true statement. Just as when we solved by substitution, this tells us we have a dependent system. There are infinitely many solutions.

Solve for y in terms of z in the second equation.

$$\begin{aligned} y - 2z &= 2\\ y &= 2z + 2 \end{aligned}$$

Solve the first equation for x in terms of z.

$$x - 2y + 3z = 1$$

Substitute $y = 2z + 2$.

$$x - 2(2z + 2) + 3z = 1$$

Simplify.

$$x - 4z - 4 + 3z = 1$$

Simplify.

$$x - z - 4 = 1$$

Simplify.

$$x = z + 5$$

The system has infinitely many solutions (x, y, z), where $x = z + 5$; $y = 2z + 2$; z is any real number.

TRY IT : : 4.87

Solve the system of equations using a matrix: $\begin{cases} x + y - z = 0\\ 2x + 4y - 2z = 6.\\ 3x + 6y - 3z = 9 \end{cases}$

TRY IT : : 4.88

Solve the system of equations using a matrix: $\begin{cases} x - y - z = 1\\ -x + 2y - 3z = -4.\\ 3x - 2y - 7z = 0 \end{cases}$

MEDIA : :

Access this online resource for additional instruction and practice with Gaussian Elimination.

- **Gaussian Elimination (https://openstax.org/l/37GaussElim)**

4.5 EXERCISES

Practice Makes Perfect

Write the Augmented Matrix for a System of Equations

In the following exercises, write each system of linear equations as an augmented matrix.

196.

ⓐ $\begin{cases} 3x - y = -1 \\ 2y = 2x + 5 \end{cases}$

ⓑ $\begin{cases} 4x + 3y = -2 \\ x - 2y - 3z = 7 \\ 2x - y + 2z = -6 \end{cases}$

197.

ⓐ $\begin{cases} 2x + 4y = -5 \\ 3x - 2y = 2 \end{cases}$

ⓑ $\begin{cases} 3x - 2y - z = -2 \\ -2x + y = 5 \\ 5x + 4y + z = -1 \end{cases}$

198.

ⓐ $\begin{cases} 3x - y = -4 \\ 2x = y + 2 \end{cases}$

ⓑ $\begin{cases} x - 3y - 4z = -2 \\ 4x + 2y + 2z = 5 \\ 2x - 5y + 7z = -8 \end{cases}$

199.

ⓐ $\begin{cases} 2x - 5y = -3 \\ 4x = 3y - 1 \end{cases}$

ⓑ $\begin{cases} 4x + 3y - 2z = -3 \\ -2x + y - 3z = 4 \\ -x - 4y + 5z = -2 \end{cases}$

Write the system of equations that that corresponds to the augmented matrix.

200. $\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & -3 & 2 \end{array}\right]$

201. $\left[\begin{array}{cc|c} 2 & -4 & -2 \\ 3 & -3 & -1 \end{array}\right]$

202. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 1 & -2 & 0 & -2 \\ 0 & -1 & 2 & 3 \end{array}\right]$

203. $\left[\begin{array}{ccc|c} 2 & -2 & 0 & -1 \\ 0 & 2 & -1 & 2 \\ 3 & 0 & -1 & -2 \end{array}\right]$

Use Row Operations on a Matrix

In the following exercises, perform the indicated operations on the augmented matrices.

204. $\left[\begin{array}{cc|c} 6 & -4 & 3 \\ 3 & -2 & 1 \end{array}\right]$

ⓐ Interchange rows 1 and 2

ⓑ Multiply row 2 by 3

ⓒ Multiply row 2 by -2 and add to row 1.

205. $\left[\begin{array}{cc|c} 4 & -6 & -3 \\ 3 & 2 & 1 \end{array}\right]$

ⓐ Interchange rows 1 and 2

ⓑ Multiply row 1 by 4

ⓒ Multiply row 2 by 3 and add to row 1.

206. $\left[\begin{array}{ccc|c} 1 & -3 & -2 & 4 \\ 2 & 2 & -1 & -3 \\ 4 & -2 & -3 & -1 \end{array}\right]$

ⓐ Interchange rows 2 and 3

ⓑ Multiply row 1 by 4

ⓒ Multiply row 2 by -2 and add to row 3.

207. $\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$

ⓐ Interchange rows 2 and 3

ⓑ Multiply row 2 by 5

ⓒ Multiply row 3 by -2 and add to row 1.

208. Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ -3 & -4 & -1 \end{array}\right]$.

209. Perform the needed row operations that will get the first entry in both row 2 and row 3 to be zero in the augmented matrix: $\left[\begin{array}{ccc|c} 1 & -2 & 3 & -4 \\ 3 & -1 & -2 & 5 \\ 2 & -3 & -4 & -1 \end{array}\right]$.

Solve Systems of Equations Using Matrices

In the following exercises, solve each system of equations using a matrix.

210. $\begin{cases} 2x+y=2 \\ x-y=-2 \end{cases}$

211. $\begin{cases} 3x+y=2 \\ x-y=2 \end{cases}$

212. $\begin{cases} -x+2y=-2 \\ x+y=-4 \end{cases}$

213. $\begin{cases} -2x+3y=3 \\ x+3y=12 \end{cases}$

In the following exercises, solve each system of equations using a matrix.

214. $\begin{cases} 2x-3y+z=19 \\ -3x+y-2z=-15 \\ x+y+z=0 \end{cases}$

215. $\begin{cases} 2x-y+3z=-3 \\ -x+2y-z=10 \\ x+y+z=5 \end{cases}$

216. $\begin{cases} 2x-6y+z=3 \\ 3x+2y-3z=2 \\ 2x+3y-2z=3 \end{cases}$

217. $\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$

218. $\begin{cases} x+2z=0 \\ 4y+3z=-2 \\ 2x-5y=3 \end{cases}$

219. $\begin{cases} 2x+5y=4 \\ 3y-z=3 \\ 4x+3z=-3 \end{cases}$

220. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

221. $\begin{cases} 3x-z=-3 \\ 5y+2z=-6 \\ 4x+3y=-8 \end{cases}$

222. $\begin{cases} 2x+3y+z=12 \\ x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

223. $\begin{cases} x+2y+6z=5 \\ -x+y-2z=3 \\ x-4y-2z=1 \end{cases}$

224. $\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$

225. $\begin{cases} 4x-3y+2z=0 \\ -2x+3y-7z=1 \\ 2x-2y+3z=6 \end{cases}$

226. $\begin{cases} x-y+2z=-4 \\ 2x+y+3z=2 \\ -3x+3y-6z=12 \end{cases}$

227. $\begin{cases} -x-3y+2z=14 \\ -x+2y-3z=-4 \\ 3x+y-2z=6 \end{cases}$

228. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

229. $\begin{cases} x+2y+z=4 \\ x+y-2z=3 \\ -2x-3y+z=-7 \end{cases}$

Writing Exercises

230. Solve the system of equations $\begin{cases} x+y=10 \\ x-y=6 \end{cases}$ ⓐ by graphing and ⓑ by substitution. ⓒ Which method do you prefer? Why?

231. Solve the system of equations $\begin{cases} 3x+y=12 \\ x=y-8 \end{cases}$ by substitution and explain all your steps in words.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
write the augmented matrix for a system of equations.			
use row operations on a matrix.			
solve systems of equations using matrices.			
write the augmented matrix for a system of equations.			
use row operations on a matrix.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

4.6 Solve Systems of Equations Using Determinants

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the determinant of a 2×2 matrix
- Evaluate the determinant of a 3×3 matrix
- Use Cramer's Rule to solve systems of equations
- Solve applications using determinants

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $5(-2) - (-4)(1)$.
 If you missed this problem, review **Example 1.20**.
2. Simplify: $-3(8 - 10) + (-2)(6 - 3) - 4(-3 - (-4))$.
 If you missed this problem, review **Example 1.19**.
3. Simplify: $\frac{-12}{-8}$.
 If you missed this problem, review **Example 1.18**.

In this section we will learn of another method to solve systems of linear equations called Cramer's rule. Before we can begin to use the rule, we need to learn some new definitions and notation.

Evaluate the Determinant of a 2×2 Matrix

If a matrix has the same number of rows and columns, we call it a **square matrix**. Each square matrix has a real number associated with it called its **determinant**. To find the determinant of the square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we first write it as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. To get the real number value of the determinate we subtract the products of the diagonals, as shown.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinant

The determinant of any square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where *a*, *b*, *c*, and *d* are real numbers, is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE 4.45

Evaluate the determinate of ⓐ $\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$ ⓑ $\begin{bmatrix} -3 & -4 \\ -2 & 0 \end{bmatrix}$.

Solution

ⓐ

	$\begin{vmatrix} 4 & -2 \\ 3 & -1 \end{vmatrix}$
Write the determinant.	$\begin{vmatrix} 4 & -2 \\ 3 & -1 \end{vmatrix}$
Subtract the products of the diagonals.	$4(-1)-3(-2)$
Simplify.	$-4+6$
Simplify.	2

ⓑ

	$\begin{vmatrix} -3 & -4 \\ -2 & 0 \end{vmatrix}$
Write the determinant.	$\begin{vmatrix} -3 & -4 \\ -2 & 0 \end{vmatrix}$
Subtract the products of the diagonals.	$-3(0)-(-2)(-4)$
Simplify.	$0-8$
Simplify.	-8

TRY IT : : 4.89 Evaluate the determinate of ⓐ $\begin{bmatrix} 5 & -3 \\ 2 & -4 \end{bmatrix}$ ⓑ $\begin{bmatrix} -4 & -6 \\ 0 & 7 \end{bmatrix}$.

TRY IT : : 4.90 Evaluate the determinate of ⓐ $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ ⓑ $\begin{bmatrix} -7 & -3 \\ -5 & 0 \end{bmatrix}$.

Evaluate the Determinant of a 3×3 Matrix

To evaluate the determinant of a 3×3 matrix, we have to be able to evaluate the **minor of an entry** in the determinant. The minor of an entry is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

Minor of an entry in 3×3 a Determinant

The **minor of an entry** in a 3×3 determinant is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

To find the minor of entry a_1, we eliminate the row and column which contain it. So we eliminate the first row and first column. Then we write the 2×2 determinant that remains.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{minor of } a_1 \quad \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

To find the minor of entry b_2, we eliminate the row and column that contain it. So we eliminate the 2nd row and 2nd column. Then we write the 2×2 determinant that remains.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{minor of } b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

EXAMPLE 4.46

For the determinant $\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$, find and then evaluate the minor of ⓐ a_1 ⓑ b_3 ⓒ c_2.

Solution

ⓐ

	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Eliminate the row and column that contains a_1.	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Write the 2×2 determinant that remains.	minor of $a_1 \begin{vmatrix} 0 & -3 \\ -4 & 2 \end{vmatrix}$
Evaluate.	$0(2) - (-3)(-4)$
Simplify.	-12

ⓑ

Eliminate the row and column that contains b_3.	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Write the 2×2 determinant that remains.	minor of $b_3 \begin{vmatrix} 4 & 3 \\ 1 & -3 \end{vmatrix}$
Evaluate.	$4(-3) - (1)(3)$
Simplify.	-15

ⓒ

	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Eliminate the row and column that contains c_2.	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$

Write the 2×2 determinant that remains.	minor of c_1 $\begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix}$
Evaluate.	$4(4)-(-2)(-2)$
Simplify.	12

TRY IT : : 4.91

For the determinant $\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & -1 \\ -2 & -3 & 3 \end{vmatrix}$, find and then evaluate the minor of ⓐ a_1 ⓑ b_2 ⓒ c_3.

TRY IT : : 4.92

For the determinant $\begin{vmatrix} -2 & -1 & 0 \\ 3 & 0 & -1 \\ -1 & -2 & 3 \end{vmatrix}$, find and then evaluate the minor of ⓐ a_2 ⓑ b_3 ⓒ c_2.

We are now ready to evaluate a 3×3 determinant. To do this we expand by minors, which allows us to evaluate the 3×3 determinant using 2×2 determinants—which we already know how to evaluate!

To evaluate a 3×3 determinant by expanding by minors along the first row, we use the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

Remember, to find the minor of an entry we eliminate the row and column that contains the entry.

Expanding by Minors along the First Row to Evaluate a 3×3 Determinant

To evaluate a 3×3 determinant by **expanding by minors along the first row**, the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

EXAMPLE 4.47

Evaluate the determinant $\begin{vmatrix} 2 & -3 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & -2 \end{vmatrix}$ by expanding by minors along the first row.

Solution

$$\begin{vmatrix} 2 & -3 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & -2 \end{vmatrix}$$

Expand by minors along the first row	$2\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & 0 \\ -1 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 2 \\ -1 & -1 \end{vmatrix}$ (minor of 2, minor of −3, minor of −1)
Evaluate each determinant.	$2(-4-0)+3(-6-0)-1(-3-(-2))$
Simplify.	$2(-4)+3(-6)-1(-1)$
Simplify.	$-8-18+1$
Simplify.	-25

TRY IT : : 4.93

Evaluate the determinant $\begin{vmatrix} 3 & -2 & 4 \\ 0 & -1 & -2 \\ 2 & 3 & -1 \end{vmatrix}$, by expanding by minors along the first row.

TRY IT : : 4.94

Evaluate the determinant $\begin{vmatrix} 3 & -2 & -2 \\ 2 & -1 & 4 \\ -1 & 0 & -3 \end{vmatrix}$, by expanding by minors along the first row.

To evaluate a 3×3 determinant we can expand by minors using any row or column. Choosing a row or column other than the first row sometimes makes the work easier.

When we expand by any row or column, we must be careful about the sign of the terms in the expansion. To determine the sign of the terms, we use the following sign pattern chart.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Sign Pattern

When expanding by minors using a row or column, the sign of the terms in the expansion follow the following pattern.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Notice that the sign pattern in the first row matches the signs between the terms in the expansion by the first row.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

(minor of a_1, minor of b_1, minor of c_1)

Since we can expand by any row or column, how do we decide which row or column to use? Usually we try to pick a row or column that will make our calculation easier. If the determinant contains a 0, using the row or column that contains the 0 will make the calculations easier.

EXAMPLE 4.48

Evaluate the determinant $\begin{vmatrix} 4 & -1 & -3 \\ 3 & 0 & 2 \\ 5 & -4 & -3 \end{vmatrix}$ by expanding by minors.

Solution

To expand by minors, we look for a row or column that will make our calculations easier. Since 0 is in the second row and second column, expanding by either of those is a good choice. Since the second row has fewer negatives than the second column, we will expand by the second row.

$$\begin{vmatrix} 4 & -1 & -3 \\ 3 & 0 & 2 \\ 5 & -4 & -3 \end{vmatrix}$$

Expand using the second row.	
Be careful of the signs.	$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ $-3\underbrace{\begin{vmatrix} -1 & -3 \\ -4 & -3 \end{vmatrix}}_{\text{minor of 3}} + (0)\underbrace{\begin{vmatrix} 4 & -3 \\ 5 & -3 \end{vmatrix}}_{\text{minor of 0}} - 2\underbrace{\begin{vmatrix} 4 & -1 \\ 5 & -4 \end{vmatrix}}_{\text{minor of 2}}$
Evaluate each determinant.	$-3(3-12)+0(-12-(-15))-2(-16-(-5))$
Simplify.	$-3(-9)+0-2(-11)$
Simplify.	$27+0+22$
Add.	49

TRY IT : : 4.95 Evaluate the determinant $\begin{vmatrix} 2 & -1 & -3 \\ 0 & 3 & -4 \\ 3 & -4 & -3 \end{vmatrix}$ by expanding by minors.

TRY IT : : 4.96 Evaluate the determinant $\begin{vmatrix} -2 & -1 & -3 \\ -1 & 2 & 2 \\ 4 & -4 & 0 \end{vmatrix}$ by expanding by minors.

Use Cramer's Rule to Solve Systems of Equations

Cramer's Rule is a method of solving systems of equations using determinants. It can be derived by solving the general form of the systems of equations by elimination. Here we will demonstrate the rule for both systems of two equations with two variables and for systems of three equations with three variables.

Let's start with the systems of two equations with two variables.

Cramer's Rule for Solving a System of Two Equations

For the system of equations $\begin{cases} a_1x+b_1y=k_1 \\ a_2x+b_2y=k_2 \end{cases}$, the solution $(x,\ y)$ can be determined by

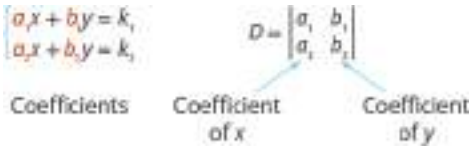
$$x = \frac{D_x}{D} \text{ and } y = \frac{D_y}{D}$$

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ use the coefficients of the variables.

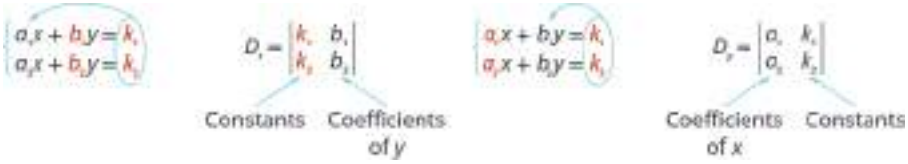
$D_x = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$ replace the y coefficients with the constants.

Notice that to form the determinant D, we use take the coefficients of the variables.



Notice that to form the determinant D_x and D_y, we substitute the constants for the coefficients of the variable we are finding.



EXAMPLE 4.49 HOW TO SOLVE A SYSTEM OF EQUATIONS USING CRAMER'S RULE

Solve using Cramer's Rule: $\begin{cases} 2x + y = -4 \\ 3x - 2y = -6 \end{cases}$.

Solution

Step 1. Evaluate the determinant D, using the coefficients of the variables.		$2x + y = -4$ $3x - 2y = -6$ $D = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$ $D = -4 - 3$ $D = -7$
Step 2. Evaluate the determinant D_x. Use the constants in place of the x coefficients.	We replace the coefficients of x, 2 and 3, with the constants, –4 and –6.	$D_x = \begin{vmatrix} -4 & 1 \\ -6 & -2 \end{vmatrix}$ $D_x = 8 - (-6)$ $D_x = 14$
Step 3. Evaluate the determinant D_y. Use the constants in place of the y coefficients.	We replace the coefficients of y, 1 and 2, with the constants, –4 and –6.	$D_y = \begin{vmatrix} 2 & -4 \\ 3 & -6 \end{vmatrix}$ $D_y = -12 - (-12)$ $D_y = 0$
Step 4. Find x and y.	Substitute in the values of D, D_x, and D_y.	$x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$ $x = \frac{14}{-7}$ and $y = \frac{0}{-7}$ $x = -2$ and $y = 0$

Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(-2, 0)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = -2, y = 0$ into both equations and make sure they are both true.	$(-2, 0)$ is the solution to the system.

TRY IT : : 4.97 Solve using Cramer's rule: $\begin{cases} 3x + y = -3 \\ 2x + 3y = 6 \end{cases}$.

TRY IT : : 4.98 Solve using Cramer's rule: $\begin{cases} -x + y = 2 \\ 2x + y = -4 \end{cases}$.

HOW TO : : SOLVE A SYSTEM OF TWO EQUATIONS USING CRAMER'S RULE.

Step 1. Evaluate the determinant D, using the coefficients of the variables.

Step 2. Evaluate the determinant D_x. Use the constants in place of the x coefficients.

Step 3. Evaluate the determinant D_y. Use the constants in place of the y coefficients.

Step 4. Find x and y. $x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$

Step 5. Write the solution as an ordered pair.

Step 6. Check that the ordered pair is a solution to both original equations.

To solve a system of three equations with three variables with Cramer's Rule, we basically do what we did for a system of two equations. However, we now have to solve for three variables to get the solution. The determinants are also going to be 3×3 which will make our work more interesting!

Cramer's Rule for Solving a System of Three Equations

For the system of equations $\begin{cases} a_1 x + b_1 y + c_1 z = k_1 \\ a_2 x + b_2 y + c_2 z = k_2, \\ a_3 x + b_3 y + c_3 z = k_3 \end{cases}$ the solution (x, y, z) can be determined by

$$x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ use the coefficients of the variables.

$D_x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}$ replace the y coefficients with the constants.

$D_z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$ replace the z coefficients with the constants.

EXAMPLE 4.50

Solve the system of equations using Cramer's Rule: $\begin{cases} 3x-5y+4z=5 \\ 5x+2y+z=0 \\ 2x+3y-2z=3 \end{cases}$.

Solution

Evaluate the determinant D.	$D=\begin{vmatrix} 3 & -5 & 4 \\ 5 & 2 & 1 \\ 2 & 3 & -2 \end{vmatrix}$
Expand by minors using column 1.	
Be careful of the signs. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$	$D=3\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}-5\begin{vmatrix} -5 & 4 \\ 3 & -2 \end{vmatrix}+2\begin{vmatrix} -5 & 4 \\ 2 & 1 \end{vmatrix}$
Evaluate the determinants.	$D=3(-4-3)-5(10-12)+2(-5-8)$
Simplify.	$D=3(-7)-5(-2)+2(-13)$
Simplify.	$D=-21+10-26$
Simplify.	$D=-37$
Evaluate the determinant D_x. Use the constants to replace the coefficients of x.	$D_x=\begin{vmatrix} 5 & -5 & 4 \\ 0 & 2 & 1 \\ 3 & 3 & -2 \end{vmatrix}$
Expand by minors using column 1.	$D_x=5\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}-0\begin{vmatrix} -5 & 4 \\ 3 & -2 \end{vmatrix}+3\begin{vmatrix} -5 & 4 \\ 2 & 1 \end{vmatrix}$
Evaluate the determinants.	$D_x=5(-4-3)-0(10-12)+3(-5-8)$
Simplify.	$D_x=5(-7)-0+3(-13)$
Simplify.	$D_x=-74$
Evaluate the determinant D_y. Use the constants to replace the coefficients of y.	$D_y=\begin{vmatrix} 3 & 5 & 4 \\ 5 & 0 & 1 \\ 2 & 3 & -2 \end{vmatrix}$
Expand by minors using column 2. Be careful of the signs. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$	$D_y=-5\begin{vmatrix} 5 & 1 \\ 2 & -2 \end{vmatrix}+0\begin{vmatrix} 3 & 4 \\ 3 & -2 \end{vmatrix}-3\begin{vmatrix} 3 & 4 \\ 5 & 1 \end{vmatrix}$
Evaluate the determinants.	$D_y=-5(-10-2)+0(-10-12)-3(3-20)$
Simplify.	$D_y=-5(-12)+0-3(-17)$
Simplify.	$D_y=60+0+51$
Simplify.	$D_y=111$

Evaluate the determinant D_z. Use the constants to replace the coefficients of z.	$D_z=\begin{vmatrix}3&-5&5\\5&2&0\\2&3&3\end{vmatrix}$
Expand by minors using column 3. Be careful of the signs. $\begin{vmatrix}+&-&+\\-&+&-\\+&-&+\end{vmatrix}$	$D_z=5\begin{vmatrix}5&2\\2&3\end{vmatrix}-0\begin{vmatrix}3&-5\\2&3\end{vmatrix}+3\begin{vmatrix}3&-5\\5&2\end{vmatrix}$
Evaluate the determinants.	$D_z=5(15-4)-0(9-(-10))+3(6-(-25))$
Simplify.	$D_z=5(11)-0+3(31)$
Simplify.	$D_z=55-0+93$
Simplify.	$D_z=148$
Find x, y, and z.	$x=\frac{D_x}{D}, y=\frac{D_y}{D}$ and $z=\frac{D_z}{D}$
Substitute in the values.	$x=\frac{-74}{-37}, y=\frac{111}{-37}$ and $z=\frac{148}{-37}$
Simplify.	$x=2, y=-3$ and $z=-4$
Write the solution as an ordered triple.	$(2,-3,-4)$
Check that the ordered triple is a solution to **all three** original equations.	We leave the check to you.
	The solution is $(2,\ -3,\ -4)$.

TRY IT : : 4.99

Solve the system of equations using Cramer's Rule: $\begin{cases}3x+8y+2z=-5\\2x+5y-3z=0\\x+2y-2z=-1\end{cases}$.

TRY IT : : 4.100

Solve the system of equations using Cramer's Rule: $\begin{cases}3x+y-6z=-3\\2x+6y+3z=0\\3x+2y-3z=-6\end{cases}$.

Cramer's rule does not work when the value of the D determinant is 0, as this would mean we would be dividing by 0. But when $D=0,$ the system is either inconsistent or dependent.

When the value of $D=0$ and D_x, D_y and D_z are all zero, the system is consistent and dependent and there are infinitely many solutions.

When the value of $D=0$ and D_x, D_y and D_z are not all zero, the system is inconsistent and there is no solution.

Dependent and Inconsistent Systems of Equations

For any system of equations, where the **value of the determinant** $D=0$,

Value of determinants	Type of system	Solution
$D=0$ and D_x, D_y and D_z are all zero	consistent and dependent	infinite y many solutions
$D=0$ and D_x, D_y and D_z are not all zero	inconsistent	no solution

In the next example, we will use the values of the determinants to find the solution of the system.

EXAMPLE 4.51

Solve the system of equations using Cramer's rule : $\begin{cases} x+3y=4 \\ -2x-6y=3 \end{cases}$.

Solution

$$\begin{cases} x+3y=4 \\ -2x-6y=3 \end{cases}$$

Evaluate the determinant D, using the coefficients of he variables.

$$D=\begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix}$$
$$D=-6-(-6)$$
$$D=0$$

We cannot use Cramer's Rule to solve this system. But by looking at the value of the determinants D_x and D_y, we can determine whether the system is dependent or inconsistent.

Evaluate the determinant D_x.

$$D_x=\begin{vmatrix} 4 & 3 \\ 3 & -6 \end{vmatrix}$$
$$D_x=-24-9$$
$$D_x=15$$

Since all the determinants are not zero, the system is inconsistent. There is no solution.

TRY IT : : 4.101 Solve the system of equations using Cramer's rule: $\begin{cases} 4x-3y=8 \\ 8x-6y=14 \end{cases}$.

TRY IT : : 4.102 Solve the system of equations using Cramer's rule: $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$.

Solve Applications using Determinants

An interesting application of determinants allows us to test if points are collinear. Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if the determinant below is zero.

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

Test for Collinear Points

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

We will use this property in the next example.

EXAMPLE 4.52

Determine whether the points $(5, -5)$, $(4, -3)$, and $(3, -1)$ are collinear.

Solution

	$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$
Substitute the values into the determinant. $(5, -5)$, $(4, -3)$, and $(3, -1)$	$\begin{vmatrix} 5 & -5 & 1 \\ 4 & -3 & 1 \\ 3 & -1 & 1 \end{vmatrix}$
Evaluate the determinant by expanding by minors using column 3.	$1\begin{vmatrix} 4 & -3 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 5 & -5 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 5 & -5 \\ 4 & -3 \end{vmatrix}$
Evaluate the determinants.	$1(-4-(-9))-1(-5-(-15))+1(-15-(-20))$
Simplify.	$1(5)-1(10)+1(5)$
Simplify.	0
	The value of the determinate is 0, so the points are collinear.

TRY IT : : 4.103 Determine whether the points $(3, -2)$, $(5, -3)$, and $(1, -1)$ are collinear.

TRY IT : : 4.104 Determine whether the points $(-4, -1)$, $(-6, 2)$, and $(-2, -4)$ are collinear.

MEDIA : :

Access these online resources for additional instruction and practice with solving systems of linear inequalities by graphing.

- **Solving Systems of Linear Inequalities by Graphing (https://www.openstax.org/l/37syslineqgph)**
- **Systems of Linear Inequalities (https://www.openstax.org/l/37syslineqs)**

4.6 EXERCISES

Practice Makes Perfect

Evaluate the Determinant of a 2 × 2 Matrix

In the following exercises, evaluate the determinate of each square matrix.

232. $\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}$

233. $\begin{bmatrix} -4 & 8 \\ -3 & 5 \end{bmatrix}$

234. $\begin{bmatrix} -3 & 5 \\ 0 & -4 \end{bmatrix}$

235. $\begin{bmatrix} -2 & 0 \\ 7 & -5 \end{bmatrix}$

Evaluate the Determinant of a 3 × 3 Matrix

In the following exercises, find and then evaluate the indicated minors.

236. $\begin{vmatrix} 3 & -1 & 4 \\ -1 & 0 & -2 \\ -4 & 1 & 5 \end{vmatrix}$

Find the minor ⓐ a_1 ⓑ b_2 ⓒ c_3

237. $\begin{vmatrix} -1 & -3 & 2 \\ 4 & -2 & -1 \\ -2 & 0 & -3 \end{vmatrix}$

Find the minor ⓐ a_1 ⓑ b_1 ⓒ c_2

238. $\begin{vmatrix} 2 & -3 & -4 \\ -1 & 2 & -3 \\ 0 & -1 & -2 \end{vmatrix}$

Find the minor ⓐ a_2 ⓑ b_2 ⓒ c_2

239. $\begin{vmatrix} -2 & -2 & 3 \\ 1 & -3 & 0 \\ -2 & 3 & -2 \end{vmatrix}$

Find the minor ⓐ a_3 ⓑ b_3 ⓒ c_3

In the following exercises, evaluate each determinant by expanding by minors along the first row.

240. $\begin{vmatrix} -2 & 3 & -1 \\ -1 & 2 & -2 \\ 3 & 1 & -3 \end{vmatrix}$

241. $\begin{vmatrix} 4 & -1 & -2 \\ -3 & -2 & 1 \\ -2 & -5 & 7 \end{vmatrix}$

242. $\begin{vmatrix} -2 & -3 & -4 \\ 5 & -6 & 7 \\ -1 & 2 & 0 \end{vmatrix}$

243. $\begin{vmatrix} 1 & 3 & -2 \\ 5 & -6 & 4 \\ 0 & -2 & -1 \end{vmatrix}$

In the following exercises, evaluate each determinant by expanding by minors.

244. $\begin{vmatrix} -5 & -1 & -4 \\ 4 & 0 & -3 \\ 2 & -2 & 6 \end{vmatrix}$

245. $\begin{vmatrix} 4 & -1 & 3 \\ 3 & -2 & 2 \\ -1 & 0 & 4 \end{vmatrix}$

246. $\begin{vmatrix} 3 & 5 & 4 \\ -1 & 3 & 0 \\ -2 & 6 & 1 \end{vmatrix}$

247. $\begin{vmatrix} 2 & -4 & -3 \\ 5 & -1 & -4 \\ 3 & 2 & 0 \end{vmatrix}$

Use Cramer's Rule to Solve Systems of Equations

In the following exercises, solve each system of equations using Cramer's Rule.

248. $\begin{cases} -2x + 3y = 3 \\ x + 3y = 12 \end{cases}$

249. $\begin{cases} x - 2y = -5 \\ 2x - 3y = -4 \end{cases}$

250. $\begin{cases} x - 3y = -9 \\ 2x + 5y = 4 \end{cases}$

251. $\begin{cases} 2x + y = -4 \\ 3x - 2y = -6 \end{cases}$

252. $\begin{cases} x - 2y = -5 \\ 2x - 3y = -4 \end{cases}$

253. $\begin{cases} x - 3y = -9 \\ 2x + 5y = 4 \end{cases}$

254. $\begin{cases} 5x-3y=-1 \\ 2x-y=2 \end{cases}$

255. $\begin{cases} 3x+8y=-3 \\ 2x+5y=-3 \end{cases}$

256. $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

257. $\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$

258. $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

259. $\begin{cases} 11x+9y+2z=-9 \\ 7x+5y+3z=-7 \\ 4x+3y+z=-3 \end{cases}$

260. $\begin{cases} x+2z=0 \\ 4y+3z=-2 \\ 2x-5y=3 \end{cases}$

261. $\begin{cases} 2x+5y=4 \\ 3y-z=3 \\ 4x+3z=-3 \end{cases}$

262. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

263. $\begin{cases} 3x-z=-3 \\ 5y+2z=-6 \\ 4x+3y=-8 \end{cases}$

264. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$

265. $\begin{cases} x-4y=-1 \\ -3x+12y=3 \end{cases}$

266. $\begin{cases} -3x-y=4 \\ 6x+2y=-16 \end{cases}$

267. $\begin{cases} 4x+3y=2 \\ 20x+15y=5 \end{cases}$

268. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

269. $\begin{cases} 2x+3y+z=12 \\ x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

270. $\begin{cases} 3x+4y-3z=-2 \\ 2x+3y-z=-12 \\ x+y-2z=6 \end{cases}$

271. $\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$

Solve Applications Using Determinants

In the following exercises, determine whether the given points are collinear.

272. $(0, 1)$, $(2, 0)$, and $(-2, 2)$.

273. $(0, -5)$, $(-2, -2)$, and $(2, -8)$.

274. $(4, -3)$, $(6, -4)$, and $(2, -2)$.

275. $(-2, 1)$, $(-4, 4)$, and $(0, -2)$.

Writing Exercises

276. Explain the difference between a square matrix and its determinant. Give an example of each.

277. Explain what is meant by the minor of an entry in a square matrix.

278. Explain how to decide which row or column you will use to expand a 3×3 determinant.

279. Explain the steps for solving a system of equations using Cramer's rule.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
evaluate the determinant of a 2×2 matrix.			
evaluate the determinant of a 3×3 matrix.			
use Cramer's rule to solve systems of equations.			
solve applications using determinants.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

4.7 Graphing Systems of Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of linear inequalities
- Solve a system of linear inequalities by graphing
- Solve applications of systems of inequalities

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve the inequality $2a < 5a + 12$.
 If you missed this problem, review Example 2.52.
2. Determine whether the ordered pair $\left(3, \frac{1}{2}\right)$ is a solution to the system $y > 2x + 3$.
 If you missed this problem, review Example 3.34.

Determine whether an ordered pair is a solution of a system of linear inequalities

The definition of a **system of linear inequalities** is very similar to the definition of a system of linear equations.

System of Linear Inequalities

Two or more linear inequalities grouped together form a system of linear inequalities.

A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true.

Solutions of a System of Linear Inequalities

Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true.

The solution of a system of linear inequalities is shown as a shaded region in the *x, y* coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, we substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

EXAMPLE 4.53

Determine whether the ordered pair is a solution to the system $\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$.

ⓐ $(-2, 4)$ ⓑ $(3, 1)$

Solution

ⓐ Is the ordered pair $(-2, 4)$ a solution?

We substitute $x = -2$ and $y = 4$ into both inequalities.

$$x + 4y \geq 10 \qquad 3x - 2y < 12$$

$$-2 + 4(4) \stackrel{?}{\geq} 10 \qquad 3(-2) - 2(4) \stackrel{?}{<} 12$$

$$14 \geq 10 \text{ true} \qquad -14 < 12 \text{ true}$$

The ordered pair $(-2, 4)$ made both inequalities true. Therefore $(-2, 4)$ is a solution to this system.

ⓑ Is the ordered pair $(3, 1)$ a solution?

We substitute $x = 3$ and $y = 1$ into both inequalities.

$x + 4y \geq 10$	$3x - 2y < 12$
$3 + 4(1) \stackrel{?}{\geq} 10$	$3(3) - 2(1) \stackrel{?}{<} 12$
$7 \geq 10$ false	$7 < 12$ true

The ordered pair $(3, 1)$ made one inequality true, but the other one false. Therefore $(3, 1)$ is not a solution to this system.

TRY IT : : 4.105 Determine whether the ordered pair is a solution to the system: $\begin{cases} x - 5y > 10 \\ 2x + 3y > -2 \end{cases}$.

ⓐ $(3, -1)$ ⓑ $(6, -3)$

TRY IT : : 4.106 Determine whether the ordered pair is a solution to the system: $\begin{cases} y > 4x - 2 \\ 4x - y < 20 \end{cases}$.

ⓐ $(-2, 1)$ ⓑ $(4, -1)$

Solve a System of Linear Inequalities by Graphing

The solution to a single linear inequality is the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

EXAMPLE 4.54 HOW TO SOLVE A SYSTEM OF LINEAR INEQUALITIES BY GRAPHING

Solve the system by graphing: $\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$.

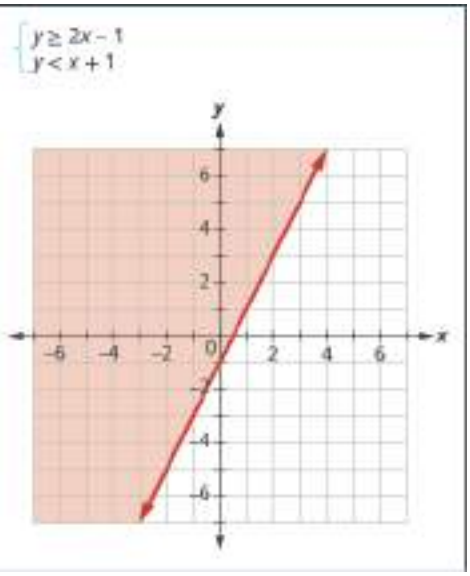
Solution

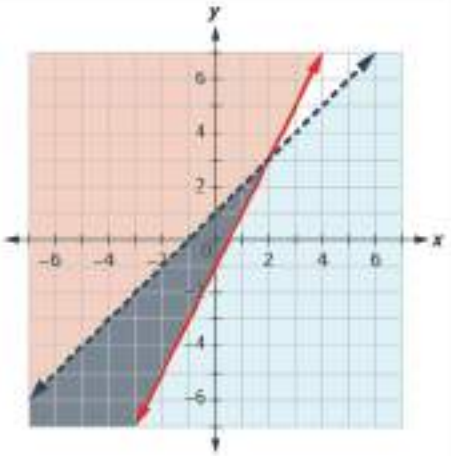
Step 1. Graph the first inequality.
- Graph the boundary line.
- Shade in the side of the boundary line where the inequality is true.

We will graph $y \geq 2x - 1$.

We graph the line $y = 2x - 1$. It is a solid line because the inequality sign is $\geq$.

We choose $(0, 0)$ as a test point. It is a solution to $y \geq 2x - 1$, so we shade in above the boundary line.



Step 2. On the same grid, graph the second inequality. • Graph the boundary line. • Shade in the side of that boundary line where the inequality is true.	We will graph $y < x + 1$ on the same grid. We graph the line $y = x + 1$. It is a dashed line because the inequality sign is $<$. Again, we use (0, 0) as a test point. It is a solution so we shade in that side of the line $y = x + 1$.	
Step 3. The solution is the region where the shading overlaps.	The point where the boundary lines intersect is not a solution because it is not a solution to $y < x + 1$.	The solution is all points in the area shaded twice—which appears as the darkest shaded region.
Step 4. Check by choosing a test point.	We'll use $(-1, -1)$ as a test point.	Is $(-1, -1)$ a solution to $y \geq 2x - 1$? $-1 \stackrel{?}{\geq} 2(-1) - 1$ $-1 \geq -3$ true Is $(-1, -1)$ a solution to $y < x + 1$? $-1 \stackrel{?}{<} -1 + 1$ $-1 < 0$ true The region containing $(-1, -1)$ is the solution to this system.

TRY IT : : 4.107 Solve the system by graphing: $\begin{cases} y < 3x + 2 \\ y > -x - 1 \end{cases}$.

TRY IT : : 4.108 Solve the system by graphing: $\begin{cases} y < -\frac{1}{2}x + 3 \\ y < 3x - 4 \end{cases}$.

HOW TO : : SOLVE A SYSTEM OF LINEAR INEQUALITIES BY GRAPHING.

Step 1. Graph the first inequality.
- Graph the boundary line.
- Shade in the side of the boundary line where the inequality is true.

Step 2. On the same grid, graph the second inequality.
- Graph the boundary line.
- Shade in the side of that boundary line where the inequality is true.

Step 3. The solution is the region where the shading overlaps.

Step 4. Check by choosing a test point.

EXAMPLE 4.55

Solve the system by graphing: $\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$.

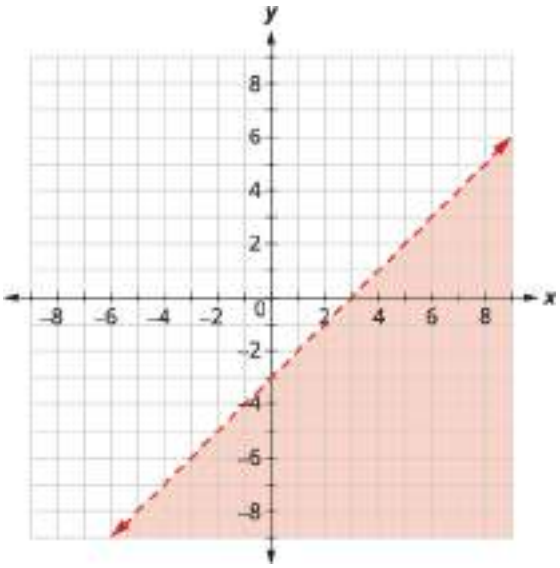
Solution

$$\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$$

Graph $x - y > 3$, by graphing $x - y = 3$ and testing a point.

The intercepts are $x = 3$ and $y = -3$ and the boundary line will be dashed.

Test (0, 0) which makes the inequality false so shade (red) the side that does not contain (0, 0).

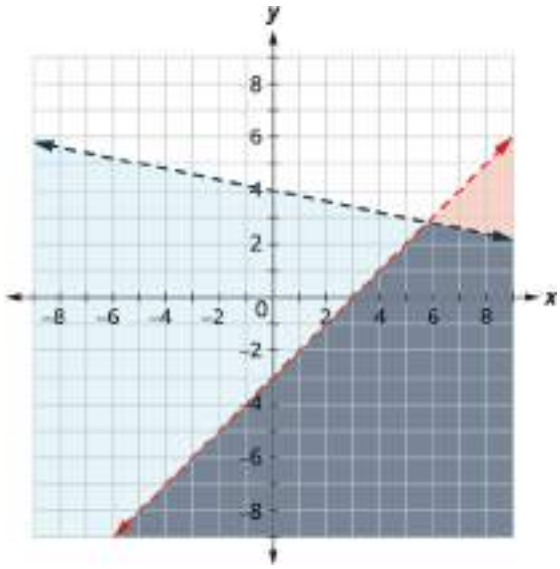


Graph $y < -\frac{1}{5}x + 4$ by graphing $y = -\frac{1}{5}x + 4$ using the slope $m = -\frac{1}{5}$ and *y*-intercept *b* = 4.

The boundary line will be dashed

Test (0, 0) which makes the inequality true, so shade (blue) the side that contains (0, 0).

Choose a test point in the solution and verify that it is a solution to both inequalties.



The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice—which appears as the darkest shaded region.

TRY IT : : 4.109

Solve the system by graphing: $\begin{cases} x + y \leq 2 \\ y \geq \frac{2}{3}x - 1 \end{cases}$.

TRY IT : : 4.110

Solve the system by graphing: $\begin{cases} 3x - 2y \leq 6 \\ y > -\frac{1}{4}x + 5 \end{cases}$.

EXAMPLE 4.56

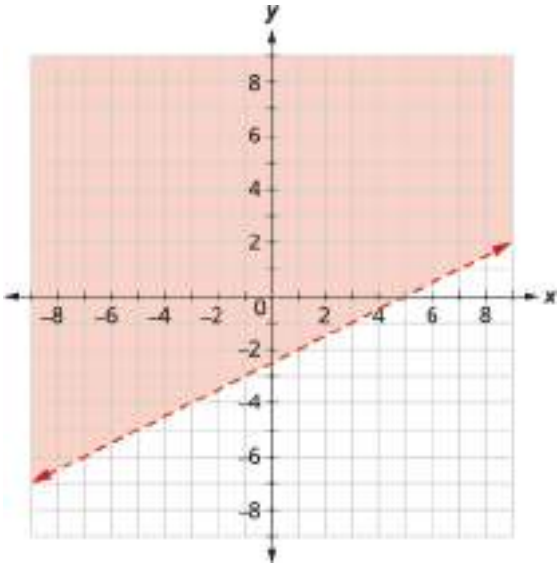
Solve the system by graphing: $\begin{cases} x - 2y < 5 \\ y > -4 \end{cases}$.

Solution

$$\begin{cases} x - 2y < 5 \\ y > -4 \end{cases}$$

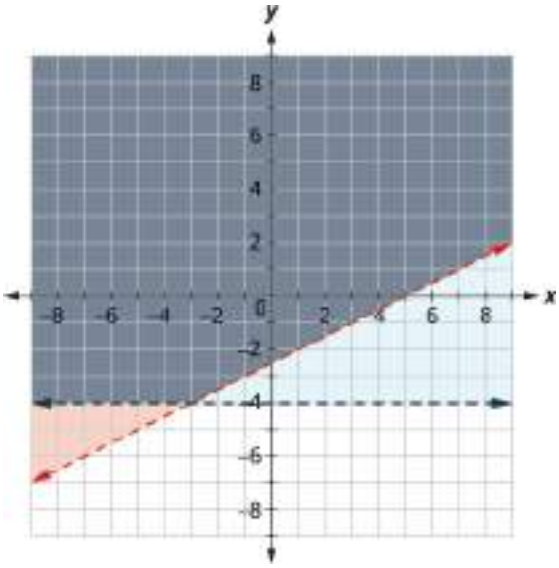
Graph $x - 2y < 5$, by graphing $x - 2y = 5$ and testing a point. The intercepts are $x = 5$ and $y = -2.5$ and the boundary line will be dashed.

Test (0, 0) which makes the inequality true, so shade (red) the side that contains (0, 0).



Graph $y > -4$, by graphing $y = -4$ and recognizing that it is a horizontal line through $y = -4$. The boundary line will be dashed.

Test (0, 0) which makes the inequality true so shade (blue) the side that contains (0, 0).



The point $(0, 0)$ is in the solution and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed.

The solution is the area shaded twice—which appears as the darkest shaded region.

TRY IT : : 4.111 Solve the system by graphing: $\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$.

TRY IT : : 4.112 Solve the system by graphing: $\begin{cases} x > -4 \\ x - 2y \geq -4 \end{cases}$.

Systems of linear inequalities where the boundary lines are parallel might have no solution. We'll see this in the next example.

EXAMPLE 4.57

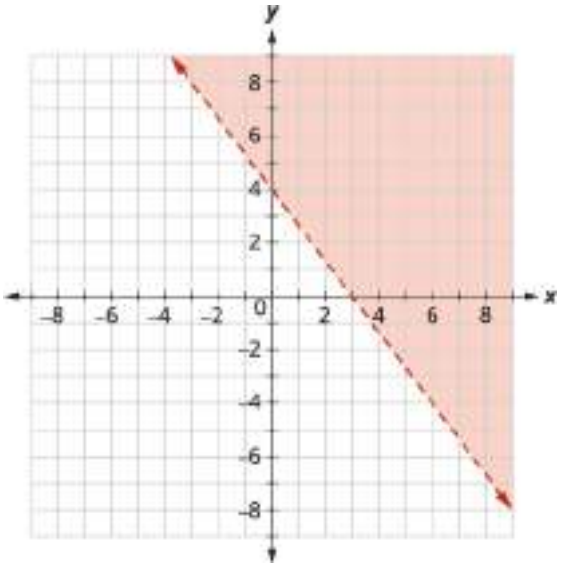
Solve the system by graphing: $\begin{cases} 4x+3y \geq 12 \\ y < -\frac{4}{3}x+1 \end{cases}$.

Solution

$$\begin{cases} 4x+3y \geq 12 \\ y < -\frac{4}{3}x+1 \end{cases}$$

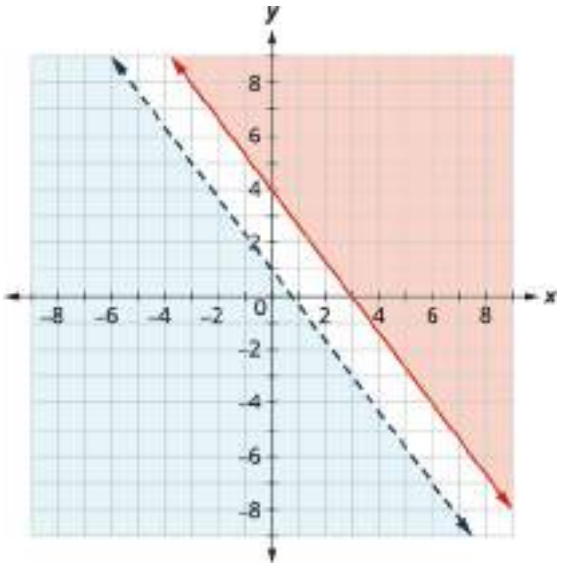
Graph $4x+3y \geq 12$, by graphing $4x+3y=12$ and testing a point. The intercepts are $x = 3$ and $y = 4$ and the boundary line will be solid.

Test (0, 0) which makes the inequality false, so shade (red) the side that does not contain (0, 0).



Graph $y < -\frac{4}{3}x+1$ by graphing $y = -\frac{4}{3}x+1$ using the slope $m = -\frac{4}{3}$ and y-intercept $b = 1$. The boundary line will be dashed.

Test (0, 0) which makes the inequality true, so shade (blue) the side that contains (0, 0).



There is no point in both shaded regions, so the system has no solution.

TRY IT : : 4.113 Solve the system by graphing: $\begin{cases} 3x - 2y \geq 12 \\ y \geq \frac{3}{2}x + 1 \end{cases}$.

TRY IT : : 4.114 Solve the system by graphing: $\begin{cases} x + 3y > 8 \\ y < -\frac{1}{3}x - 2 \end{cases}$.

Some systems of linear inequalities where the boundary lines are parallel will have a solution. We'll see this in the next example.

EXAMPLE 4.58

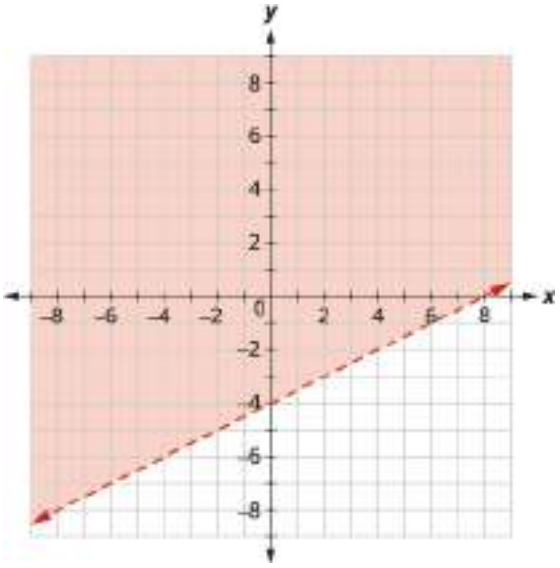
Solve the system by graphing: $\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$.

Solution

$$\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$$

Graph $y > \frac{1}{2}x - 4$ by graphing $y = \frac{1}{2}x - 4$ using the slope $m = \frac{1}{2}$ and the intercept $b = -4$. The boundary line will be dashed.

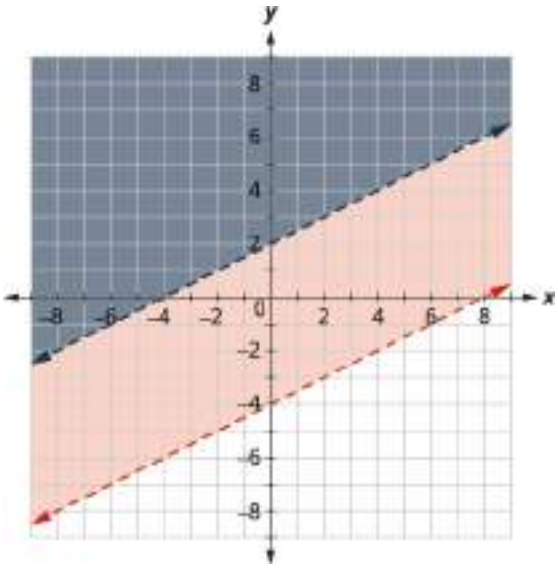
Test (0, 0) which makes the inequality true, so shade (red) the side that contains (0, 0).



Graph $x - 2y < -4$ by graphing $x - 2y = -4$ and testing a point. The intercepts are $x = -4$ and $y = 2$ and the boundary line will be dashed.

Choose a test point in the solution and verify that it is a solution to both inequalties.

Test (0, 0) which makes the inequality false, so shade (blue) the side that does not contain (0, 0).



No point on the boundary lines is included in the solution as both lines are dashed.

The solution is the region that is shaded twice which is also the solution to $x - 2y < -4$.

TRY IT : : 4.115 Solve the system by graphing: $\begin{cases} y \geq 3x + 1 \\ -3x + y \geq -4 \end{cases}$.

TRY IT : : 4.116 Solve the system by graphing: $\begin{cases} y \leq -\frac{1}{4}x + 2 \\ x + 4y \leq 4 \end{cases}$.

Solve Applications of Systems of Inequalities

The first thing we'll need to do to solve applications of systems of inequalities is to translate each condition into an inequality. Then we graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so we add inequalities to the system as additional requirements.

EXAMPLE 4.59

Christy sells her photographs at a booth at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her $4 and each large photo costs her $10. She doesn't want to spend more than $200 on photos to display.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Could she display 10 small and 20 large photos?

ⓓ Could she display 20 large and 10 small photos?

Solution

ⓐ

Let $x =$ the number of small photos.

$y =$ the number of large photos

To find the system of equations translate the information.

She wants to have at least 25 photos.
The number of small plus the number of large should be at least 25.

$$x + y \geq 25$$

$4 for each small and $10 for each large must be no more than $200

$$4x + 10y \leq 200$$

The number of small photos must be greater than or equal to 0.

$$x \geq 0$$

The number of large photos must be greater than or equal to 0.

$$y \geq 0$$

We have our system of equations.

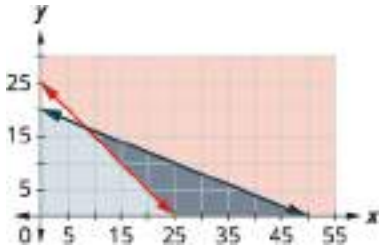
$$\begin{cases} x + y \geq 25 \\ 4x + 10y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

ⓑ

Since $x \geq 0$ and $y \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $x + y \geq 25$, graph $x + y = 25$ as a solid line.
Choose (0, 0) as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point (0, 0).

To graph $4x + 10y \leq 200$, graph $4x + 10y = 200$ as a solid line.
Choose (0, 0) as a test point. Since it does make the inequality true, shade (blue) the side that include the point (0, 0).



The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly-shaded section are included in the solution as are the points on the *x*-axis from (25, 0) to (55, 0).

ⓒ To determine if 10 small and 20 large photos would work, we look at the graph to see if the point (10, 20) is in the solution region. We could also test the point to see if it is a solution of both equations.
It is not, Christy would not display 10 small and 20 large photos.

ⓓ To determine if 20 small and 10 large photos would work, we look at the graph to see if the point (20, 10) is in the solution region. We could also test the point to see if it is a solution of both equations.
It is, so Christy could choose to display 20 small and 10 large photos.
Notice that we could also test the possible solutions by substituting the values into each inequality.

TRY IT : : 4.117

A trailer can carry a maximum weight of 160 pounds and a maximum volume of 15 cubic feet. A microwave oven weighs 30 pounds and has 2 cubic feet of volume, while a printer weighs 20 pounds and has 3 cubic feet of space.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could 4 microwaves and 2 printers be carried on this trailer?
ⓓ Could 7 microwaves and 3 printers be carried on this trailer?

TRY IT : : 4.118

Mary needs to purchase supplies of answer sheets and pencils for a standardized test to be given to the juniors at her high school. The number of the answer sheets needed is at least 5 more than the number of pencils. The pencils cost $2 and the answer sheets cost $1. Mary's budget for these supplies allows for a maximum cost of $400.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Could Mary purchase 100 pencils and 100 answer sheets?

ⓓ Could Mary purchase 150 pencils and 150 answer sheets?

When we use variables other than *x* and *y* to define an unknown quantity, we must change the names of the axes of the graph as well.

EXAMPLE 4.60

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than $5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs $1.40. Each cookie has 160 calories and costs $0.50.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Could he eat 3 hamburgers and 1 cookie?

ⓓ Could he eat 2 hamburgers and 4 cookies?

Solution

ⓐ

Let $h =$ the number of hamburgers.
$c =$ the number of cookies

To find the system of equations translate the information.

The calories from hamburgers at 240 calories each, plus the calories from cookies at 160 calories each must be more that 800.

$$240h + 160c \geq 800$$

The amount spent on hamburgers at $1.40 each, plus the amount spent on cookies at $0.50 each must be no more than $5.00.

$$1.40h + 0.50c \leq 5$$

The number of hamburgers must be greater than or equal to 0.

$$h \geq 0$$

The number of cookies must be greater than or equal to 0.

$$c \geq 0$$

We have our system of equations.

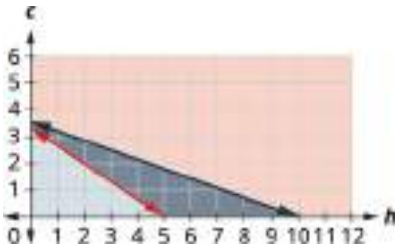
$$\begin{cases} 240h + 160c \geq 800 \\ 1.40h + 0.50c \leq 5 \\ h \geq 0 \\ c \geq 0 \end{cases}$$

ⓑ

Since $h >= 0$ and $c >= 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $240h + 160c \geq 800$, graph $240h + 160c = 800$ as a solid line.

Choose (0, 0) as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point (0, 0).



Graph $1.40h + 0.50c \leq 5$. The boundary line is $1.40h + 0.50c = 5$. We test (0, 0) and it makes the inequality true. We shade the side of the line that includes (0, 0).

The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the *x*-axis from (5, 0) to (10, 0).

ⓒ To determine if 3 hamburgers and 2 cookies would meet Omar's criteria, we see if the point (3, 2) is in the solution region. It is, so Omar might choose to eat 3 hamburgers and 2 cookies.

ⓓ To determine if 2 hamburgers and 4 cookies would meet Omar's criteria, we see if the point (2, 4) is in the solution region. It is, Omar might choose to eat 2 hamburgers and 4 cookies.

We could also test the possible solutions by substituting the values into each inequality.

TRY IT : : 4.119

Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only $25 to spend on the extra food he needs and will spend it on $0.75 donuts which have 360 calories each and $2 energy drinks which have 110 calories.

ⓐ Write a system of inequalities that models this situation.
ⓑ Graph the system.
ⓒ Can he buy 8 donuts and 4 energy drinks and satisfy his caloric needs?
ⓓ Can he buy 1 donut and 3 energy drinks and satisfy his caloric needs?

TRY IT : : 4.120

Philip's doctor tells him he should add at least 1,000 more calories per day to his usual diet. Philip wants to buy protein bars that cost $1.80 each and have 140 calories and juice that costs $1.25 per bottle and have 125 calories. He doesn't want to spend more than $12.

ⓐ Write a system of inequalities that models this situation.
ⓑ Graph the system.
ⓒ Can he buy 3 protein bars and 5 bottles of juice?
ⓓ Can he buy 5 protein bars and 3 bottles of juice?

MEDIA : :

Access these online resources for additional instruction and practice with solving systems of linear inequalities by graphing.

- **Solving Systems of Linear Inequalities by Graphing (https://openstax.org/l/37sysineqgraph)**
- **Systems of Linear Inequalities (https://openstax.org/l/37sysineqgraph2)**

4.7 EXERCISES

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.

280. $\begin{cases} 3x+y>5 \\ 2x-y\leq 10 \end{cases}$

ⓐ $(3,-3)$

ⓑ $(7,1)$

281. $\begin{cases} 4x-y<10 \\ -2x+2y>-8 \end{cases}$

ⓐ $(5,-2)$

ⓑ $(-1,3)$

282. $\begin{cases} y>\frac{2}{3}x-5 \\ x+\frac{1}{2}y\leq 4 \end{cases}$

ⓐ $(6,-4)$

ⓑ $(3,0)$

283. $\begin{cases} y<\frac{3}{2}x+3 \\ \frac{3}{4}x-2y<5 \end{cases}$

ⓐ $(-4,-1)$

ⓑ $(8,3)$

284. $\begin{cases} 7x+2y>14 \\ 5x-y\leq 8 \end{cases}$

ⓐ $(2,3)$

ⓑ $(7,-1)$

285. $\begin{cases} 6x-5y<20 \\ -2x+7y>-8 \end{cases}$

ⓐ $(1,-3)$

ⓑ $(-4,4)$

Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

286. $\begin{cases} y\leq 3x+2 \\ y>x-1 \end{cases}$

287. $\begin{cases} y<-2x+2 \\ y\geq -x-1 \end{cases}$

288. $\begin{cases} y<2x-1 \\ y\leq -\frac{1}{2}x+4 \end{cases}$

289. $\begin{cases} y\geq -\frac{2}{3}x+2 \\ y>2x-3 \end{cases}$

290. $\begin{cases} x-y>1 \\ y<-\frac{1}{4}x+3 \end{cases}$

291. $\begin{cases} x+2y<4 \\ y<x-2 \end{cases}$

292. $\begin{cases} 3x-y\geq 6 \\ y\geq -\frac{1}{2}x \end{cases}$

293. $\begin{cases} 2x+4y\geq 8 \\ y\leq \frac{3}{4}x \end{cases}$

294. $\begin{cases} 2x-5y<10 \\ 3x+4y\geq 12 \end{cases}$

295. $\begin{cases} 3x-2y\leq 6 \\ -4x-2y>8 \end{cases}$

296. $\begin{cases} 2x+2y>-4 \\ -x+3y\geq 9 \end{cases}$

297. $\begin{cases} 2x+y>-6 \\ -x+2y\geq -4 \end{cases}$

298. $\begin{cases} x-2y<3 \\ y\leq 1 \end{cases}$

299. $\begin{cases} x-3y>4 \\ y\leq -1 \end{cases}$

300. $\begin{cases} y\geq -\frac{1}{2}x-3 \\ x\leq 2 \end{cases}$

301. $\begin{cases} y\leq -\frac{2}{3}x+5 \\ x\geq 3 \end{cases}$

302. $\begin{cases} y\geq \frac{3}{4}x-2 \\ y<2 \end{cases}$

303. $\begin{cases} y\leq -\frac{1}{2}x+3 \\ y<1 \end{cases}$

304. $\begin{cases} 3x-4y<8 \\ x<1 \end{cases}$

305. $\begin{cases} -3x+5y>10 \\ x>-1 \end{cases}$

306. $\begin{cases} x\geq 3 \\ y\leq 2 \end{cases}$

307. $\begin{cases} x \le -1 \\ y \ge 3 \end{cases}$

308. $\begin{cases} 2x + 4y > 4 \\ y \le -\frac{1}{2}x - 2 \end{cases}$

309. $\begin{cases} x - 3y \ge 6 \\ y > \frac{1}{3}x + 1 \end{cases}$

310. $\begin{cases} -2x + 6y < 0 \\ 6y > 2x + 4 \end{cases}$

311. $\begin{cases} -3x + 6y > 12 \\ 4y \le 2x - 4 \end{cases}$

312. $\begin{cases} y \ge -3x + 2 \\ 3x + y > 5 \end{cases}$

313. $\begin{cases} y \ge \frac{1}{2}x - 1 \\ -2x + 4y \ge 4 \end{cases}$

314. $\begin{cases} y \le -\frac{1}{4}x - 2 \\ x + 4y < 6 \end{cases}$

315. $\begin{cases} y \ge 3x - 1 \\ -3x + y > -4 \end{cases}$

316. $\begin{cases} 3y > x + 2 \\ -2x + 6y > 8 \end{cases}$

317. $\begin{cases} y < \frac{3}{4}x - 2 \\ -3x + 4y < 7 \end{cases}$

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.

318. Caitlyn sells her drawings at the county fair. She wants to sell at least 60 drawings and has portraits and landscapes. She sells the portraits for $15 and the landscapes for $10. She needs to sell at least $800 worth of drawings in order to earn a profit.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Will she make a profit if she sells 20 portraits and 35 landscapes?

ⓓ Will she make a profit if she sells 50 portraits and 20 landscapes?

319. Jake does not want to spend more than $50 on bags of fertilizer and peat moss for his garden. Fertilizer costs $2 a bag and peat moss costs $5 a bag. Jake's van can hold at most 20 bags.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Can he buy 15 bags of fertilizer and 4 bags of peat moss?

ⓓ Can he buy 10 bags of fertilizer and 10 bags of peat moss?

320. Reiko needs to mail her Christmas cards and packages and wants to keep her mailing costs to no more than $500. The number of cards is at least 4 more than twice the number of packages. The cost of mailing a card (with pictures enclosed) is $3 and for a package the cost is $7.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Can she mail 60 cards and 26 packages?

ⓓ Can she mail 90 cards and 40 packages?

321. Juan is studying for his final exams in chemistry and algebra. he knows he only has 24 hours to study, and it will take him at least three times as long to study for algebra than chemistry.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Can he spend 4 hours on chemistry and 20 hours on algebra?

ⓓ Can he spend 6 hours on chemistry and 18 hours on algebra?

322. Jocelyn is pregnant and so she needs to eat at least 500 more calories a day than usual. When buying groceries one day with a budget of $15 for the extra food, she buys bananas that have 90 calories each and chocolate granola bars that have 150 calories each. The bananas cost $0.35 each and the granola bars cost $2.50 each.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Could she buy 5 bananas and 6 granola bars?

ⓓ Could she buy 3 bananas and 4 granola bars?

323. Mark is attempting to build muscle mass and so he needs to eat more than an additional 80 grams of protein a day. A bottle of protein water costs $3.20 and a protein bar costs $1.75. The protein water supplies 27 grams of protein and the bar supplies 16 gram. If he has $10 dollars to spend

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Could he buy 3 bottles of protein water and 1 protein bar?

ⓓ Could he buy no bottles of protein water and 5 protein bars?

324. Jocelyn desires to increase both her protein consumption and caloric intake. She desires to have at least 35 more grams of protein each day and no more than an additional 200 calories daily. An ounce of cheddar cheese has 7 grams of protein and 110 calories. An ounce of parmesan cheese has 11 grams of protein and 22 calories.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Could she eat 1 ounce of cheddar cheese and 3 ounces of parmesan cheese?

ⓓ Could she eat 2 ounces of cheddar cheese and 1 ounce of parmesan cheese?

325. Mark is increasing his exercise routine by running and walking at least 4 miles each day. His goal is to burn a minimum of 1500 calories from this exercise. Walking burns 270 calories/mile and running burns 650 calories.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Could he meet his goal by walking 3 miles and running 1 mile?

ⓓ Could he his goal by walking 2 miles and running 2 mile

Writing Exercises

326. Graph the inequality $x - y \geq 3$. How do you know which side of the line $x - y = 3$ should be shaded?

327. Graph the system $\begin{cases} x + 2y \leq 6 \\ y \geq -\frac{1}{2}x - 4 \end{cases}$. What does the solution mean?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No. I don't get it!
determine whether an ordered pair is a solution of a system of linear inequalities.			
solve a system of linear inequalities by graphing.			
solve applications of systems of inequalities.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

CHAPTER 4 REVIEW

KEY TERMS

break-even point The point at which the revenue equals the costs is the break-even point; $C(x) = R(x)$.

coincident lines Coincident lines have the same slope and same *y*-intercept.

complementary angles Two angles are complementary if the sum of the measures of their angles is 90 degrees.

consistent and inconsistent systems Consistent system of equations is a system of equations with at least one solution; inconsistent system of equations is a system of equations with no solution.

cost function The cost function is the cost to manufacture each unit times x, the number of units manufactured, plus the fixed costs; $C(x)$ = (cost per unit)x + fixed costs.

determinant Each square matrix has a real number associated with it called its determinant.

matrix A matrix is a rectangular array of numbers arranged in rows and columns.

minor of an entry in a 3×3 determinant The minor of an entry in a 3×3 determinant is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

revenue The revenue is the selling price of each unit times *x*, the number of units sold; $R(x)$ = (selling price per unit)x.

row-echelon form A matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

solutions of a system of equations Solutions of a system of equations are the values of the variables that make *all* the equations true; solution is represented by an ordered pair (x, y).

solutions of a system of linear equations with three variables The solutions of a system of equations are the values of the variables that make all the equations true; a solution is represented by an ordered triple (x, y, z).

square matrix A square matrix is a matrix with the same number of rows and columns.

supplementary angles Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

system of linear equations When two or more linear equations are grouped together, they form a system of linear equations.

system of linear inequalities Two or more linear inequalities grouped together form a system of linear inequalities.

KEY CONCEPTS

4.1 Solve Systems of Linear Equations with Two Variables

- **How to solve a system of linear equations by graphing.**

 Step 1. Graph the first equation.

 Step 2. Graph the second equation on the same rectangular coordinate system.

 Step 3. Determine whether the lines intersect, are parallel, or are the same line.

 Step 4. Identify the solution to the system.
 If the lines intersect, identify the point of intersection. This is the solution to the system.
 If the lines are parallel, the system has no solution.
 If the lines are the same, the system has an infinite number of solutions.

 Step 5. Check the solution in both equations.

- **How to solve a system of equations by substitution.**

 Step 1. Solve one of the equations for either variable.

 Step 2. Substitute the expression from Step 1 into the other equation.

 Step 3. Solve the resulting equation.

 Step 4. Substitute the solution in Step 3 into either of the original equations to find the other variable.

 Step 5. Write the solution as an ordered pair.

 Step 6. Check that the ordered pair is a solution to **both** original equations.

- **How to solve a system of equations by elimination.**

 Step 1. Write both equations in standard form. If any coefficients are fractions, clear them.

Step 2. Make the coefficients of one variable opposites.
Decide which variable you will eliminate.
Multiply one or both equations so that the coefficients of that variable are opposites.

Step 3. Add the equations resulting from Step 2 to eliminate one variable.

Step 4. Solve for the remaining variable.

Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.

Step 6. Write the solution as an ordered pair.

Step 7. Check that the ordered pair is a solution to **both** original equations.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations are in standard form.

4.2 Solve Applications with Systems of Equations

- **How To Solve Applications with Systems of Equations**

 Step 1. **Read** the problem. Make sure all the words and ideas are understood.

 Step 2. **Identify** what we are looking for.

 Step 3. **Name** what we are looking for. Choose variables to represent those quantities.

 Step 4. **Translate** into a system of equations.

 Step 5. **Solve** the system of equations using good algebra techniques.

 Step 6. **Check** the answer in the problem and make sure it makes sense.

 Step 7. **Answer** the question with a complete sentence.

4.3 Solve Mixture Applications with Systems of Equations

- **Cost function:** The cost function is the cost to manufacture each unit times *x*, the number of units manufactured, plus the fixed costs.

$$C(x) = (\text{cost per unit}) \cdot x + \text{fi ed costs}$$

- **Revenue:** The revenue function is the selling price of each unit times *x*, the number of units sold.

$$R(x) = (\text{sellingpriceperunit}) \cdot x$$

- **Break-even point:** The break-even point is when the revenue equals the costs.

$$C(x) = R(x)$$

4.4 Solve Systems of Equations with Three Variables

- **Linear Equation in Three Variables:** A linear equation with three variables, where *a, b, c*, and *d* are real numbers and *a, b,* and *c* are not all 0, is of the form

$$ax + by + cz = d$$

 Every solution to the equation is an ordered triple, (x, y, z) that makes the equation true.

- **How to solve a system of linear equations with three variables.**

 Step 1. Write the equations in standard form
 If any coefficients are fractions, clear them.

 Step 2. Eliminate the same variable from two equations.
 Decide which variable you will eliminate.
 Work with a pair of equations to eliminate the chosen variable.
 Multiply one or both equations so that the coefficients of that variable are opposites.
 Add the equations resulting from Step 2 to eliminate one variable

 Step 3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.

Step 4. The two new equations form a system of two equations with two variables. Solve this system.

Step 5. Use the values of the two variables found in Step 4 to find the third variable.

Step 6. Write the solution as an ordered triple.

Step 7. Check that the ordered triple is a solution to **all three** original equations.

4.5 Solve Systems of Equations Using Matrices

- **Matrix:** A matrix is a rectangular array of numbers arranged in rows and columns. A matrix with *m* rows and *n* columns has *order* $m \times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.



 Each number in the matrix is called an *element* or *entry* in the matrix.

- **Row Operations:** In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.
 - Interchange any two rows
 - Multiply a row by any real number except 0
 - Add a nonzero multiple of one row to another row

- **Row-Echelon Form:** For a consistent and independent system of equations, its augmented matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix} \quad \begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{bmatrix} \quad a, b, c, d, e, f \text{ are real numbers}$$

- **How to solve a system of equations using matrices.**

 Step 1. Write the augmented matrix for the system of equations.

 Step 2. Using row operations get the entry in row 1, column 1 to be 1.

 Step 3. Using row operations, get zeros in column 1 below the 1.

 Step 4. Using row operations, get the entry in row 2, column 2 to be 1.

 Step 5. Continue the process until the matrix is in row-echelon form.

 Step 6. Write the corresponding system of equations.

 Step 7. Use substitution to find the remaining variables.

 Step 8. Write the solution as an ordered pair or triple.

 Step 9. Check that the solution makes the original equations true.

4.6 Solve Systems of Equations Using Determinants

- **Determinant:** The determinant of any square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where *a, b, c,* and *d* are real numbers, is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- **Expanding by Minors along the First Row to Evaluate a 3 × 3 Determinant:** To evaluate a 3×3 determinant by expanding by minors along the first row, the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

- **Sign Pattern:** When expanding by minors using a row or column, the sign of the terms in the expansion follow the following pattern.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

- **Cramer's Rule:** For the system of equations $\begin{cases} a_1 x + b_1 y = k_1 \\ a_2 x + b_2 y = k_2 \end{cases}$, the solution $(x,\ y)$ can be determined by

$$x = \frac{D_x}{D} \text{ and } y = \frac{D_y}{D}$$

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ use the coefficients of the variables.

$D_x = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$ replace the y coefficients with the constants.

Notice that to form the determinant D, we use take the coefficients of the variables.

- **How to solve a system of two equations using Cramer's rule.**

Step 1. Evaluate the determinant D, using the coefficients of the variables.

Step 2. Evaluate the determinant D_x. Use the constants in place of the x coefficients.

Step 3. Evaluate the determinant D_y. Use the constants in place of the y coefficients.

Step 4. Find x and y. $x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$.

Step 5. Write the solution as an ordered pair.

Step 6. Check that the ordered pair is a solution to **both** original equations.

Step 7. **Dependent and Inconsistent Systems of Equations:** For any system of equations, where the **value of the determinant** $D = 0$,

Value of determinants	Type of system	Solution
$D = 0$ and D_x, D_y and D_z are all zero	consistent and dependent	infinite y many solutions
$D = 0$ and D_x, D_y and D_z are not all zero	inconsistent	no solution

Step 8. **Test for Collinear Points:** Three points $(x_1,\ y_1), \quad (x_2,\ y_2), \quad$ and $(x_3,\ y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

4.7 Graphing Systems of Linear Inequalities

- **Solutions of a System of Linear Inequalities:** Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true. The solution of a system of linear inequalities is shown as a shaded region in the x, y coordinate system that includes all the points whose ordered pairs make the inequalities true.
- **How to solve a system of linear inequalities by graphing.**

Step 1. Graph the first inequality.
Graph the boundary line.
Shade in the side of the boundary line where the inequality is true.

Step 2. On the same grid, graph the second inequality.
Graph the boundary line.
Shade in the side of that boundary line where the inequality is true.

Step 3. The solution is the region where the shading overlaps.

Step 4. Check by choosing a test point.

REVIEW EXERCISES

4.1 Solve Systems of Linear Equations with Two Variables

Determine Whether an Ordered Pair is a Solution of a System of Equations.

In the following exercises, determine if the following points are solutions to the given system of equations.

328. $\begin{cases} x+3y=-9 \\ 2x-4y=12 \end{cases}$

ⓐ $(-3, -2)$

ⓑ $(0, -3)$

329. $\begin{cases} x+y=8 \\ y=x-4 \end{cases}$

ⓐ $(6, 2)$

ⓑ $(9, -1)$

Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

330. $\begin{cases} 3x+y=6 \\ x+3y=-6 \end{cases}$

331. $\begin{cases} x+4y=-1 \\ x=3 \end{cases}$

332. $\begin{cases} 2x-y=5 \\ 4x-2y=10 \end{cases}$

333. $\begin{cases} -x+2y=4 \\ y=\frac{1}{2}x-3 \end{cases}$

In the following exercises, without graphing determine the number of solutions and then classify the system of equations.

334. $\begin{cases} y=\frac{2}{5}x+2 \\ -2x+5y=10 \end{cases}$

335. $\begin{cases} 3x+2y=6 \\ y=-3x+4 \end{cases}$

336. $\begin{cases} 5x-4y=0 \\ y=\frac{5}{4}x-5 \end{cases}$

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

337. $\begin{cases} 3x-2y=2 \\ y=\frac{1}{2}x+3 \end{cases}$

338. $\begin{cases} x-y=0 \\ 2x+5y=-14 \end{cases}$

339. $\begin{cases} y=-2x+7 \\ y=\frac{2}{3}x-1 \end{cases}$

340. $\begin{cases} y=-5x \\ 5x+y=6 \end{cases}$

341. $\begin{cases} y=-\frac{1}{3}x+2 \\ x+3y=6 \end{cases}$

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination

342. $\begin{cases} x+y=12 \\ x-y=-10 \end{cases}$

343. $\begin{cases} 3x-8y=20 \\ x+3y=1 \end{cases}$

344. $\begin{cases} 9x+4y=2 \\ 5x+3y=5 \end{cases}$

345. $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$

346. $\begin{cases} -x+3y=8 \\ 2x-6y=-20 \end{cases}$

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

347. $\begin{cases} 6x-5y=27 \\ 3x+10y=-24 \end{cases}$

348. $\begin{cases} y=3x-9 \\ 4x-5y=23 \end{cases}$

4.2 Solve Applications with Systems of Equations

Solve Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

349. Mollie wants to plant 200 bulbs in her garden, all irises and tulips. She wants to plant three times as many tulips as irises. How many irises and how many tulips should she plant?

350. Ashanti has been offered positions by two phone companies. The first company pays a salary of $22,000 plus a commission of $100 for each contract sold. The second pays a salary of $28,000 plus a commission of $25 for each contract sold. How many contract would need to be sold to make the total pay the same?

351. Leroy spent 20 minutes jogging and 40 minutes cycling and burned 600 calories. The next day, Leroy swapped times, doing 40 minutes of jogging and 20 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

352. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and calculator. Troy bought four notebooks and five calculators for $116. Lisa bought two notebooks and three calculators for $68. Find the cost of each notebook and each thumb drive.

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

353. The difference of two supplementary angles is 58 degrees. Find the measures of the angles.

354. Two angles are complementary. The measure of the larger angle is five more than four times the measure of the smaller angle. Find the measures of both angles.

355. The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

356. Becca is hanging a 28 foot floral garland on the two sides and top of a pergola to prepare for a wedding. The height is four feet less than the width. Find the height and width of the pergola.

357. The perimeter of a city rectangular park is 1428 feet. The length is 78 feet more than twice the width. Find the length and width of the park.

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

358. Sheila and Lenore were driving to their grandmother's house. Lenore left one hour after Sheila. Sheila drove at a rate of 45 mph, and Lenore drove at a rate of 60 mph. How long will it take for Lenore to catch up to Sheila?

359. Bob left home, riding his bike at a rate of 10 miles per hour to go to the lake. Cheryl, his wife, left 45 minutes ($\frac{3}{4}$ hour) later, driving her car at a rate of 25 miles per hour. How long will it take Cheryl to catch up to Bob?

360. Marcus can drive his boat 36 miles down the river in three hours but takes four hours to return upstream. Find the rate of the boat in still water and the rate of the current.

361. A passenger jet can fly 804 miles in 2 hours with a tailwind but only 776 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

4.3 Solve Mixture Applications with Systems of Equations

Solve Mixture Applications with Systems of Equations

For the following exercises, translate to a system of equations and solve.

362. Lynn paid a total of $2,780 for 261 tickets to the theater. Student tickets cost $10 and adult tickets cost $15. How many student tickets and how many adult tickets did Lynn buy?

363. Priam has dimes and pennies in a cup holder in his car. The total value of the coins is $4.21. The number of dimes is three less than four times the number of pennies. How many dimes and how many pennies are in the cup?

364. Yumi wants to make 12 cups of party mix using candies and nuts. Her budget requires the party mix to cost her $1.29 per cup. The candies are $2.49 per cup and the nuts are $0.69 per cup. How many cups of candies and how many cups of nuts should she use?

365. A scientist needs 70 liters of a 40% solution of alcohol. He has a 30% and a 60% solution available. How many liters of the 30% and how many liters of the 60% solutions should he mix to make the 40% solution?

Solve Interest Applications

For the following exercises, translate to a system of equations and solve.

366. Jack has $12,000 to invest and wants to earn 7.5% interest per year. He will put some of the money into a savings account that earns 4% per year and the rest into CD account that earns 9% per year. How much money should he put into each account?

367. When she graduates college, Linda will owe $43,000 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $1,585. What is the amount of each loan?

4.4 Solve Systems of Equations with Three Variables

Solve Systems of Equations with Three Variables

In the following exercises, determine whether the ordered triple is a solution to the system.

368. $\begin{cases} 3x-4y-3z=2 \\ 2x-6y+z=3 \\ 2x+3y-2z=3 \end{cases}$

ⓐ $(2,\ 3,\ -1)$

ⓑ $(3,\ 1,\ 3)$

369. $\begin{cases} y=\frac{2}{3}x-2 \\ x+3y-z=15 \\ x-3y+z=-2 \end{cases}$

ⓐ $\left(-6,\ 5,\ \frac{1}{2}\right)$

ⓑ $\left(5,\ \frac{4}{3},\ -3\right)$

Solve a System of Linear Equations with Three Variables

In the following exercises, solve the system of equations.

370. $\begin{cases} 3x-5y+4z=5 \\ 5x+2y+z=0 \\ 2x+3y-2z=3 \end{cases}$

371. $\begin{cases} x+\frac{5}{2}y+z=-2 \\ 2x+2y+\frac{1}{2}z=-4 \\ \frac{1}{3}x-y-z=1 \end{cases}$

372. $\begin{cases} 5x+3y=-6 \\ 2y+3z=-1 \\ 7x+z=1 \end{cases}$

373. $\begin{cases} 2x+3y+z=12 \\ x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

374. $\begin{cases} -x-3y+2z=14 \\ -x+2y-3z=-4 \\ 3x+y-2z=6 \end{cases}$

Solve Applications using Systems of Linear Equations with Three Variables

375. After attending a major league baseball game, the patrons often purchase souvenirs. If a family purchases 4 t-shirts, a cap and 1 stuffed animal their total is $135. A couple buys 2 t-shirts, a cap and 3 stuffed animals for their nieces and spends $115. Another couple buys 2 t-shirts, a cap and 1 stuffed animal and their total is $85. What is the cost of each item?

4.5 Solve Systems of Equations Using Matrices

Write the Augmented Matrix for a System of Equations.

Write each system of linear equations as an augmented matrix.

376. $\begin{cases} 3x - y = -1 \\ -2x + 2y = 5 \end{cases}$

377. $\begin{cases} 4x + 3y = -2 \\ x - 2y - 3z = 7 \\ 2x - y + 2z = -6 \end{cases}$

Write the system of equations that that corresponds to the augmented matrix.

378. $\left[\begin{array}{cc|c} 2 & -4 & -2 \\ 3 & -3 & -1 \end{array}\right]$

379. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 1 & -2 & 0 & -2 \\ 0 & -1 & 2 & 3 \end{array}\right]$

In the following exercises, perform the indicated operations on the augmented matrices.

380. $\left[\begin{array}{cc|c} 4 & -6 & -3 \\ 3 & 2 & 1 \end{array}\right]$

ⓐ Interchange rows 2 and 1.

ⓑ Multiply row 1 by 4.

ⓒ Multiply row 2 by 3 and add to row 1.

381. $\left[\begin{array}{ccc|c} 1 & -3 & -2 & 4 \\ 2 & 2 & -1 & -3 \\ 4 & -2 & -3 & -1 \end{array}\right]$

ⓐ Interchange rows 2 and 3.

ⓑ Multiply row 1 by 2.

ⓒ Multiply row 3 by -2 and add to row 2.

Solve Systems of Equations Using Matrices

In the following exercises, solve each system of equations using a matrix.

382. $\begin{cases} 4x + y = 6 \\ x - y = 4 \end{cases}$

383. $\begin{cases} 2x - y + 3z = -3 \\ -x + 2y - z = 10 \\ x + y + z = 5 \end{cases}$

384. $\begin{cases} 2y + 3z = -1 \\ 5x + 3y = -6 \\ 7x + z = 1 \end{cases}$

385. $\begin{cases} x + 2y - 3z = -1 \\ x - 3y + z = 1 \\ 2x - y - 2z = 2 \end{cases}$

386. $\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ -x + 2y = 1 \end{cases}$

4.6 Solve Systems of Equations Using Determinants

Evaluate the Determinant of a 2 × 2 Matrix

In the following exercise, evaluate the determinate of the square matrix.

387. $\left[\begin{array}{cc} 8 & -4 \\ 5 & -3 \end{array}\right]$

Evaluate the Determinant of a 3 × 3 Matrix

In the following exercise, find and then evaluate the indicated minors.

388. $\begin{vmatrix} -1 & -3 & 2 \\ 4 & -2 & -1 \\ -2 & 0 & -3 \end{vmatrix}$; Find the minor ⓐ a_1 ⓑ b_1 ⓒ c_2

In the following exercise, evaluate each determinant by expanding by minors along the first row.

389. $\begin{vmatrix} -2 & -3 & -4 \\ 5 & -6 & 7 \\ -1 & 2 & 0 \end{vmatrix}$

In the following exercise, evaluate each determinant by expanding by minors.

390. $\begin{vmatrix} 3 & 5 & 4 \\ -1 & 3 & 0 \\ -2 & 6 & 1 \end{vmatrix}$

Use Cramer's Rule to Solve Systems of Equations

In the following exercises, solve each system of equations using Cramer's rule

391. $\begin{cases} x - 3y = -9 \\ 2x + 5y = 4 \end{cases}$

392. $\begin{cases} 4x - 3y + z = 7 \\ 2x - 5y - 4z = 3 \\ 3x - 2y - 2z = -7 \end{cases}$

393. $\begin{cases} 2x + 5y = 4 \\ 3y - z = 3 \\ 4x + 3z = -3 \end{cases}$

394. $\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ -x + 2y = 1 \end{cases}$

395. $\begin{cases} 3x + 4y - 3z = -2 \\ 2x + 3y - z = -12 \\ x + y - 2z = 6 \end{cases}$

Solve Applications Using Determinants

In the following exercises, determine whether the given points are collinear.

396. $(0, 2)$, $(-1, -1)$, and $(-2, 4)$

4.7 Graphing Systems of Linear Inequalities

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.

397. $\begin{cases} 4x + y > 6 \\ 3x - y \leq 12 \end{cases}$

ⓐ $(2, -1)$

ⓑ $(3, -2)$

398. $\begin{cases} y > \frac{1}{3}x + 2 \\ x - \frac{1}{4}y \leq 10 \end{cases}$

ⓐ $(6, 5)$

ⓑ $(15, 8)$

Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

399. $\begin{cases} y < 3x + 1 \\ y \geq -x - 2 \end{cases}$

400. $\begin{cases} x - y > -1 \\ y < \frac{1}{3}x - 2 \end{cases}$

401. $\begin{cases} 2x - 3y < 6 \\ 3x + 4y \geq 12 \end{cases}$

402. $\begin{cases} y \leq -\frac{3}{4}x + 1 \\ x \geq -5 \end{cases}$

403. $\begin{cases} x + 3y < 5 \\ y \geq -\frac{1}{3}x + 6 \end{cases}$

404. $\begin{cases} y \geq 2x - 5 \\ -6x + 3y > -4 \end{cases}$

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.

405. Roxana makes bracelets and necklaces and sells them at the farmers' market. She sells the bracelets for $12 each and the necklaces for $18 each. At the market next weekend she will have room to display no more than 40 pieces, and she needs to sell at least $500 worth in order to earn a profit.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Should she display 26 bracelets and 14 necklaces?

ⓓ Should she display 39 bracelets and 1 necklace?

406. Annie has a budget of $600 to purchase paperback books and hardcover books for her classroom. She wants the number of hardcover to be at least 5 more than three times the number of paperback books. Paperback books cost $4 each and hardcover books cost $15 each.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Can she buy 8 paperback books and 40 hardcover books?

ⓓ Can she buy 10 paperback books and 37 hardcover books?

PRACTICE TEST

In the following exercises, solve the following systems by graphing.

407. $\begin{cases} x - y = 5 \\ x + 2y = -4 \end{cases}$

408. $\begin{cases} x - y > -2 \\ y \leq 3x + 1 \end{cases}$

In the following exercises, solve each system of equations. Use either substitution or elimination.

409. $\begin{cases} x + 4y = 6 \\ -2x + y = -3 \end{cases}$

410. $\begin{cases} -3x + 4y = 25 \\ x - 5y = -23 \end{cases}$

411. $\begin{cases} x + y - z = -1 \\ 2x - y + 2z = 8 \\ -3x + 2y + z = -9 \end{cases}$

Solve the system of equations using a matrix.

412. $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

413. $\begin{cases} -3x + y + z = -4 \\ -x + 2y - 2z = 1 \\ 2x - y - z = -1 \end{cases}$

Solve using Cramer's rule.

414. $\begin{cases} 3x + y = -3 \\ 2x + 3y = 6 \end{cases}$

415. Evaluate the determinant by expanding by minors:

$$\begin{vmatrix} 3 & -2 & -2 \\ 2 & -1 & 4 \\ -1 & 0 & -3 \end{vmatrix}.$$

In the following exercises, translate to a system of equations and solve.

416. Greg is paddling his canoe upstream, against the current, to a fishing spot 10 miles away. If he paddles upstream for 2.5 hours and his return trip takes 1.25 hours, find the speed of the current and his paddling speed in still water.

417. A pharmacist needs 20 liters of a 2% saline solution. He has a 1% and a 5% solution available. How many liters of the 1% and how many liters of the 5% solutions should she mix to make the 2% solution?

418. Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4,500 in interest in one year?

419. The church youth group is selling snacks to raise money to attend their convention. Amy sold 2 pounds of candy, 3 boxes of cookies and 1 can of popcorn for a total sales of $65. Brian sold 4 pounds of candy, 6 boxes of cookies and 3 cans of popcorn for a total sales of $140. Paulina sold 8 pounds of candy, 8 boxes of cookies and 5 can of popcorn for a total sales of $250. What is the cost of each item?

420. The manufacturer of a granola bar spends $1.20 to make each bar and sells them for $2. The manufacturer also has fixed costs each month of $8,000.

ⓐ Find the cost function C when x granola bars are manufactured

ⓑ Find the revenue function R when x granola bars are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

421. Translate to a system of inequalities and solve.

Andi wants to spend no more than $50 on Halloween treats. She wants to buy candy bars that cost $1 each and lollipops that cost $0.50 each, and she wants the number of lollipops to be at least three times the number of candy bars.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Can she buy 20 candy bars and 40 lollipops?



5 POLYNOMIALS AND POLYNOMIAL FUNCTIONS

Figure 5.1 There are many different kinds of coins in circulation, but a new type of coin exists only in the virtual world. It is the bitcoin.

Chapter Outline

5.1 Add and Subtract Polynomials
5.2 Properties of Exponents and Scientific Notation
5.3 Multiply Polynomials
5.4 Dividing Polynomials

Introduction

You may have coins and paper money in your wallet, but you may soon want to acquire a type of currency called bitcoins. They exist only in a digital wallet on your computer. You can use bitcoins to pay for goods at some companies, or save them as an investment. Although the future of bitcoins is uncertain, investment brokers are beginning to investigate ways to make business predictions using this digital currency. Understanding how bitcoins are created and obtained requires an understanding of a type of function known as a polynomial function. In this chapter you will investigate polynomials and polynomial functions and learn how to perform mathematical operations on them.

5.1 Add and Subtract Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Determine the degree of polynomials
- Add and subtract polynomials
- Evaluate a polynomial function for a given value
- Add and subtract polynomial functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $3x^2 + 3x + 1 + 8x^2 + 5x + 5$.
 If you missed this problem, review **Example 1.7**.
2. Subtract: $(5n + 8) - (2n - 1)$.
 If you missed this problem, review **Example 1.5**.
3. Evaluate: $4xy^2$ when $x = -2$ and $y = 5$.
 If you missed this problem, review **Example 1.21**.

Determine the Degree of Polynomials

We have learned that a *term* is a constant or the product of a constant and one or more variables. A **monomial** is an algebraic expression with one term. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial in one variable. Some examples of monomial in one variable are. Monomials can also have more than one variable such as and $-4a^2b^3c^2$.

Monomial

A **monomial** is an algebraic expression with one term.

A monomial in one variable is a term of the form ax^m, where a is a constant and m is a whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a **polynomial**. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a **trinomial** has exactly three terms. There are no special names for polynomials with more than three terms.

Polynomials

polynomial—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.

monomial—A polynomial with exactly one term is called a monomial.

binomial—A polynomial with exactly two terms is called a binomial.

trinomial—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$y+1$	$4a^2-7ab+2b^2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	$-13a^3b^2c$
Binomial	$a+7b$	$4x^2-y^2$	y^2-16	$3p^3q-9p^2q$
Trinomial	$x^2-7x+12$	$9m^2+2mn-8n^2$	$6k^4-k^3+8k$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The **degree of a constant** is 0.

Degree of a Polynomial

The **degree of a term** is the sum of the exponents of its variables.

The **degree of a constant** is 0.

The **degree of a polynomial** is the highest degree of all its terms.

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms.

Let's start by looking at a monomial. The monomial $8ab^2$ has two variables a and b. To find the degree we need to find the sum of the exponents. The variable a doesn't have an exponent written, but remember that means the exponent is 1. The exponent of b is 2. The sum of the exponents, $1+2$, is 3 so the degree is 3.



Here are some additional examples.

Monomials	14	$8ab^2$	$-9x^2y^8$	$-13a$
Degree	0	3	8	1
Binomial	$h+7$	$7b^2-3b$	x^4y^4-25	$4n^3-8n^2$
Degree of each term	1 0	2 1	4 0	3 2
Degree of polynomial	1	2	4	3
Trinomial	$x^2-12x+27$	$9a^2+6ab+b^2$	$6m^4-m^5n^2+8mn^2$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4
Polynomial	$y-1$	$3y^2-2y-5$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

Working with polynomials is easier when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in **standard form of a polynomial**. Get in the habit of writing the term with the highest degree first.

EXAMPLE 5.1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

ⓐ $7y^2-5y+3$ ⓑ $-2a^4b^2$ ⓒ $3x^5-4x^3-6x^2+x-8$ ⓓ $2y-8xy^3$ ⓔ 15

Solution

	Polynomial	Number of terms	Type	Degree of terms	Degree of polynomial
ⓐ	$7y^2-5y+3$	3	Trinomial	2, 1, 0	2
ⓑ	$-2a^4b^2$	1	Monomial	4, 2	6
ⓒ	$3x^5-4x^3-6x^2+x-8$	5	Polynomial	5, 3, 2, 1, 0	5
ⓓ	$2y-8xy^3$	2	Binomial	1, 4	4
ⓔ	15	1	Monomial	0	0

TRY IT : : 5.1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

ⓐ -5 ⓑ $8y^3-7y^2-y-3$ ⓒ $-3x^2y-5xy+9xy^3$ ⓓ $81m^2-4n^2$ ⓔ $-3x^6y^3z$

TRY IT : : 5.2

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

ⓐ $64k^3 - 8$ ⓑ $9m^3 + 4m^2 - 2$ ⓒ $\frac{5}{6}$ ⓓ $8a^4 - 7a^3b - 6a^2b^2 - 4ab^3 + 7b^4$ ⓔ $-p^4q^3$

Add and Subtract Polynomials

We have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficients.

EXAMPLE 5.2

Add or subtract: ⓐ $25y^2 + 15y^2$ ⓑ $16pq^3 - \left(-7pq^3\right)$.

Solution

ⓐ

$$25y^2 + 15y^2$$

Combine like terms. $$40y^2$$

ⓑ

$$16pq^3 - \left(-7pq^3\right)$$

Combine like terms. $$23pq^3$$

TRY IT : : 5.3 Add or subtract: ⓐ $12q^2 + 9q^2$ ⓑ $8mn^3 - \left(-5mn^3\right)$.

TRY IT : : 5.4 Add or subtract: ⓐ $-15c^2 + 8c^2$ ⓑ $-15y^2z^3 - \left(-5y^2z^3\right)$.

Remember that like terms must have the same variables with the same exponents.

EXAMPLE 5.3

Simplify: ⓐ $a^2 + 7b^2 - 6a^2$ ⓑ $u^2v + 5u^2 - 3v^2$.

Solution

ⓐ

$$a^2 + 7b^2 - 6a^2$$

Combine like terms. $$-5a^2 + 7b^2$$

ⓑ

$$u^2v + 5u^2 - 3v^2$$

There are no like terms to combine.
In this case, the polynomial is unchanged. $$u^2v + 5u^2 - 3v^2$$

TRY IT : : 5.5 Add: ⓐ $8y^2 + 3z^2 - 3y^2$ ⓑ $m^2n^2 - 8m^2 + 4n^2$.

TRY IT : : 5.6 Add: ⓐ $3m^2 + n^2 - 7m^2$ ⓑ $pq^2 - 6p - 5q^2$.

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

EXAMPLE 5.4

Find the sum: $\left(7y^2-2y+9\right)+\left(4y^2-8y-7\right)$.

Solution

Identify like terms.	$\left(\underline{\underline{7y^2}}-\underline{2y}+9\right)+\left(\underline{\underline{4y^2}}-\underline{8y}-7\right)$
Rewrite without the parentheses, rearranging to get the like terms together.	$\underline{\underline{7y^2+4y^2}}-\underline{2y-8y}+9-7$
Combine like terms.	$11y^2-10y+2$

TRY IT : : 5.7 Find the sum: $\left(7x^2-4x+5\right)+\left(x^2-7x+3\right)$.

TRY IT : : 5.8 Find the sum: $\left(14y^2+6y-4\right)+\left(3y^2+8y+5\right)$.

Be careful with the signs as you distribute while subtracting the polynomials in the next example.

EXAMPLE 5.5

Find the difference: $\left(9w^2-7w+5\right)-\left(2w^2-4\right)$.

Solution

	$\left(9w^2-7w+5\right)-\left(2w^2-4\right)$
Distribute and identify like terms.	$\underline{\underline{9w^2}}-\underline{7w}+5-\underline{\underline{2w^2}}+4$
Rearrange the terms.	$\underline{\underline{9w^2-2w^2}}-\underline{7w}+5+4$
Combine like terms.	$7w^2-7w+9$

TRY IT : : 5.9 Find the difference: $\left(8x^2+3x-19\right)-\left(7x^2-14\right)$.

TRY IT : : 5.10 Find the difference: $\left(9b^2-5b-4\right)-\left(3b^2-5b-7\right)$.

To subtract a from b, we write it as $b-a$, placing the b first.

EXAMPLE 5.6

Subtract $\left(p^2+10pq-2q^2\right)$ from $\left(p^2+q^2\right)$.

Solution

	$\left(p^2+q^2\right)-\left(p^2+10pq-2q^2\right)$
Distribute.	$p^2+q^2-p^2-10pq+2q^2$
Rearrange the terms, to put like terms together.	$p^2-p^2-10pq+q^2+2q^2$
Combine like terms.	$-10pq^2+3q^2$

TRY IT : : 5.11 Subtract $\left(a^2+5ab-6b^2\right)$ from $\left(a^2+b^2\right)$.

TRY IT : : 5.12 Subtract $\left(m^2-7mn-3n^2\right)$ from $\left(m^2+n^2\right)$.

EXAMPLE 5.7

Find the sum: $\left(u^2-6uv+5v^2\right)+\left(3u^2+2uv\right)$.

Solution

$$\left(u^2-6uv+5v^2\right)+\left(3u^2+2uv\right)$$

Distribute.	$u^2-6uv+5v^2+3u^2+2uv$
Rearrange the terms to put like terms together.	$u^2+3u^2-6uv+2uv+5v^2$
Combine like terms.	$4u^2-4uv+5v^2$

TRY IT : : 5.13 Find the sum: $\left(3x^2-4xy+5y^2\right)+\left(2x^2-xy\right)$.

TRY IT : : 5.14 Find the sum: $\left(2x^2-3xy-2y^2\right)+\left(5x^2-3xy\right)$.

When we add and subtract more than two polynomials, the process is the same.

EXAMPLE 5.8

Simplify: $\left(a^3-a^2b\right)-\left(ab^2+b^3\right)+\left(a^2b+ab^2\right)$.

Solution

$$\left(a^3-a^2b\right)-\left(ab^2+b^3\right)+\left(a^2b+ab^2\right)$$

Distribute.	$a^3-a^2b-ab^2-b^3+a^2b+ab^2$
Rewrite without the parentheses, rearranging to get the like terms together.	$a^3-a^2b+a^2b-ab^2+ab^2-b^3$
Combine like terms.	a^3-b^3

TRY IT : : 5.15 Simplify: $\left(x^3-x^2y\right)-\left(xy^2+y^3\right)+\left(x^2y+xy^2\right)$.

TRY IT : : 5.16 Simplify: $\left(p^3-p^2q\right)+\left(pq^2+q^3\right)-\left(p^2q+pq^2\right)$.

Evaluate a Polynomial Function for a Given Value

A **polynomial function** is a function defined by a polynomial. For example, $f(x)=x^2+5x+6$ and $g(x)=3x-4$ are polynomial functions, because x^2+5x+6 and $3x-4$ are polynomials.

Polynomial Function

A **polynomial function** is a function whose range values are defined by a polynomial.

In Graphs and Functions, where we first introduced functions, we learned that evaluating a function means to find the value of $f(x)$ for a given value of x. To evaluate a polynomial function, we will substitute the given value for the variable and then simplify using the order of operations.

EXAMPLE 5.9

For the function $f(x) = 5x^2 - 8x + 4$ find: ⓐ $f(4)$ ⓑ $f(-2)$ ⓒ $f(0)$.

Solution

ⓐ

	$f(x) = 5x^2 - 8x + 4$
To find $f(4)$, substitute 4 for x.	$f(4) = 5(4)^2 - 8(4) + 4$
Simplify the exponents.	$f(4) = 5 \cdot 16 - 8(4) + 4$
Multiply.	$f(4) = 80 - 32 + 4$
Simplify.	$f(4) = 52$

ⓑ

	$f(x) = 5x^2 - 8x + 4$
To find $f(-2)$, substitute -2 for x.	$f(-2) = 5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$f(-2) = 5 \cdot 4 - 8(-2) + 4$
Multiply.	$f(-2) = 20 + 16 + 4$
Simplify.	$f(-2) = 40$

ⓒ

	$f(x) = 5x^2 - 8x + 4$
To find $f(0)$, substitute 0 for x.	$f(0) = 5(0)^2 - 8(0) + 4$
Simplify the exponents.	$f(0) = 5 \cdot 0 - 8(0) + 4$
Multiply.	$f(0) = 0 + 0 + 4$
Simplify.	$f(0) = 4$

TRY IT : : 5.17 For the function $f(x) = 3x^2 + 2x - 15$, find ⓐ $f(3)$ ⓑ $f(-5)$ ⓒ $f(0)$.

TRY IT : : 5.18 For the function $g(x) = 5x^2 - x - 4$, find ⓐ $g(-2)$ ⓑ $g(-1)$ ⓒ $g(0)$.

The polynomial functions similar to the one in the next example are used in many fields to determine the height of an object at some time after it is projected into the air. The polynomial in the next function is used specifically for dropping something from 250 ft.

EXAMPLE 5.10

The polynomial function $h(t) = -16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 2$ seconds.

Solution

	$h(t) = -16t^2 + 250$
To fin $h(2)$, substitute $t = 2$.	$h(2) = -16(2)^2 + 250$
Simplify.	$h(2) = -16 \cdot 4 + 250$
Simplify.	$h(2) = -64 + 250$
Simplify.	$h(2) = 186$
	After 2 seconds the height of the ball is 186 feet.

TRY IT : : 5.19

The polynomial function $h(t) = -16t^2 + 150$ gives the height of a stone t seconds after it is dropped from a 150-foot tall cliff. Find the height after $t = 0$ seconds (the initial height of the object).

TRY IT : : 5.20

The polynomial function $h(t) = -16t^2 + 175$ gives the height of a ball t seconds after it is dropped from a 175-foot tall bridge. Find the height after $t = 3$ seconds.

Add and Subtract Polynomial Functions

Just as polynomials can be added and subtracted, polynomial functions can also be added and subtracted.

Addition and Subtraction of Polynomial Functions

For functions $f(x)$ and $g(x)$,

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$

EXAMPLE 5.11

For functions $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$, find:

ⓐ $(f + g)(x)$ ⓑ $(f + g)(3)$ ⓒ $(f - g)(x)$ ⓓ $(f - g)(-2)$.

Solution

ⓐ

	$(f+g)(x) = f(x) + g(x)$
Substitute $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$.	$(f+g)(x) = (3x^2 - 5x + 7) + (x^2 - 4x - 3)$
Rewrite without the parentheses.	$(f+g)(x) = 3x^2 - 5x + 7 + x^2 - 4x - 3$
Put like terms together.	$(f+g)(x) = 3x^2 + x^2 - 5x - 4x + 7 - 3$
Combine like terms.	$(f+g)(x) = 4x^2 - 9x + 4$

ⓑ In part (a) we found $(f+g)(x)$ and now are asked to find $(f+g)(3)$.

$$(f+g)(x) = 4x^2 - 9x + 4$$

To find $(f+g)(3)$, substitute $x = 3$. $(f+g)(3) = 4(3)^2 - 9 \cdot 3 + 4$

$$(f+g)(3) = 4 \cdot 9 - 9 \cdot 3 + 4$$

$$(f+g)(3) = 36 - 27 + 4$$

Notice that we could have found $(f+g)(3)$ by first finding the values of $f(3)$ and $g(3)$ separately and then adding the results.

Find $f(3)$.	$f(x) = 3x^2 - 5x + 7$
	$f(3) = 3(3)^2 - 5(3) + 7$
	$f(3) = 19$
Find $g(3)$.	$g(x) = x^2 - 4x - 3$
	$g(3) = 3^2 - 4(3) - 3$
	$g(3) = -6$
Find $(f+g)(3)$.	$(f+g)(x) = f(x) + g(x)$
	$(f+g)(3) = f(3) + g(3)$
Substitute $f(3) = 19$ and $g(3) = -6$.	$(f+g)(3) = 19 + (-6)$
	$(f+g)(3) = 13$

ⓒ

	$(f-g)(x) = f(x) - g(x)$
Substitute $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$.	$(f-g)(x) = (3x^2 - 5x + 7) - (x^2 - 4x - 3)$
Rewrite without the parentheses.	$(f-g)(x) = 3x^2 - 5x + 7 - x^2 + 4x + 3$
Put like terms together.	$(f-g)(x) = 3x^2 - x^2 - 5x + 4x + 7 + 3$
Combine like terms.	$(f-g)(x) = 2x^2 - x + 10$

ⓓ

	$(f-g)(x) = 2x^2 - x + 10$
To find $(f-g)(-2)$, substitute $x = -2$.	$(f-g)(-2) = 2(-2)^2 - (-2) + 10$
	$(f-g)(-2) = 2 \cdot 4 - (-2) + 10$
	$(f-g)(-2) = 20$

TRY IT : : 5.21

For functions $f(x) = 2x^2 - 4x + 3$ and $g(x) = x^2 - 2x - 6$, find: ⓐ $(f + g)(x)$ ⓑ $(f + g)(3)$ ⓒ $(f - g)(x)$ ⓓ $(f - g)(-2)$.

TRY IT : : 5.22

For functions $f(x) = 5x^2 - 4x - 1$ and $g(x) = x^2 + 3x + 8$, find ⓐ $(f + g)(x)$ ⓑ $(f + g)(3)$ ⓒ $(f - g)(x)$ ⓓ $(f - g)(-2)$.

MEDIA : :

Access this online resource for additional instruction and practice with adding and subtracting polynomials.

- **Adding and Subtracting Polynomials (https://openstax.org/l/37AddSubtrPoly)**

5.1 EXERCISES

Practice Makes Perfect

Determine the Degree of Polynomials

In the following exercises, determine if the polynomial is a monomial, binomial, trinomial, or other polynomial.

1.

ⓐ $47x^5 - 17x^2y^3 + y^2$

ⓑ $5c^3 + 11c^2 - c - 8$

ⓒ $\frac{5}{9}ab + \frac{1}{3}b$

ⓓ 4

ⓔ $4pq + 17$

2.

ⓐ $x^2 - y^2$

ⓑ $-13c^4$

ⓒ $a^2 + 2ab - 7b^2$

ⓓ $4x^2y^2 - 3xy + 8$

ⓔ 19

3.

ⓐ $8y - 5x$

ⓑ $y^2 - 5yz - 6z^2$

ⓒ $y^3 - 8y^2 + 2y - 16$

ⓓ $81ab^4 - 24a^2b^2 + 3b$

ⓔ -18

4.

ⓐ $11y^2$

ⓑ -73

ⓒ $6x^2 - 3xy + 4x - 2y + y^2$

ⓓ $4y^2 + 17z^2$

ⓔ $5c^3 + 11c^2 - c - 8$

5.

ⓐ $5a^2 + 12ab - 7b^2$

ⓑ $18xy^2z$

ⓒ $5x + 2$

ⓓ $y^3 - 8y^2 + 2y - 16$

ⓔ -24

6.

ⓐ $9y^3 - 10y^2 + 2y - 6$

ⓑ $-12p^3q$

ⓒ $a^2 + 9ab + 18b^2$

ⓓ $20x^2y^2 - 10a^2b^2 + 30$

ⓔ 17

7.

ⓐ $14s - 29t$

ⓑ $z^2 - 5z - 6$

ⓒ $y^3 - 8y^2z + 2yz^2 - 16z^3$

ⓓ $23ab^2 - 14$

ⓔ -3

8.

ⓐ $15xy$

ⓑ 15

ⓒ $6x^2 - 3xy + 4x - 2y + y^2$

ⓓ $10p - 9q$

ⓔ $m^4 + 4m^3 + 6m^2 + 4m + 1$

Add and Subtract Polynomials

In the following exercises, add or subtract the monomials.

9.

ⓐ $7x^2 + 5x^2$

ⓑ $4a - 9a$

10.

ⓐ $4y^3 + 6y^3$

ⓑ $-y - 5y$

11.

ⓐ $-12w + 18w$

ⓑ $7x^2y - (-12x^2y)$

12.

ⓐ $-3m + 9m$

ⓑ $15yz^2 - (-8yz^2)$

13. $7x^2 + 5x^2 + 4a - 9a$

14. $4y^3 + 6y^3 - y - 5y$

15. $-12w + 18w + 7x^2y - (-12x^2y)$

16. $-3m + 9m + 15yz^2 - (-8yz^2)$

17.

ⓐ $-5b - 17b$

ⓑ $3xy - (-8xy) + 5xy$

18.

ⓐ $-10x - 35x$

ⓑ $17mn^2 - (-9mn^2) + 3mn^2$

19.

ⓐ $12a + 5b - 22a$

ⓑ $pq^2 - 4p - 3q^2$

20.

ⓐ $14x - 3y - 13x$

ⓑ $a^2b - 4a - 5ab^2$

21.
ⓐ $2a^2+b^2-6a^2$
ⓑ $x^2y-3x+7xy^2$

22.
ⓐ $5u^2+4v^2-6u^2$
ⓑ $12a+8b$

23.
ⓐ $xy^2-5x-5y^2$
ⓑ $19y+5z$

24. $12a+5b-22a+pq^2-4p-3q^2$

25. $14x-3y-13x+a^2b-4a-5ab^2$

26. $2a^2+b^2-6a^2+x^2y-3x+7xy^2$

27. $5u^2+4v^2-6u^2+12a+8b$

28. $xy^2-5x-5y^2+19y+5z$

29. Add: $4a, -3b, -8a$

30. Add: $4x, 3y, -3x$

31. Subtract $5x^6$ from $-12x^6$

32. Subtract $2p^4$ from $-7p^4$

In the following exercises, add the polynomials.

33. $(5y^2+12y+4)+(6y^2-8y+7)$

34. $(4y^2+10y+3)+(8y^2-6y+5)$

35. $(x^2+6x+8)+(-4x^2+11x-9)$

36. $(y^2+9y+4)+(-2y^2-5y-1)$

37. $(8x^2-5x+2)+(3x^2+3)$

38. $(7x^2-9x+2)+(6x^2-4)$

39. $(5a^2+8)+(a^2-4a-9)$

40. $(p^2-6p-18)+(2p^2+11)$

In the following exercises, subtract the polynomials.

41. $(4m^2-6m-3)-(2m^2+m-7)$

42. $(3b^2-4b+1)-(5b^2-b-2)$

43. $(a^2+8a+5)-(a^2-3a+2)$

44. $(b^2-7b+5)-(b^2-2b+9)$

45. $(12s^2-15s)-(s-9)$

46. $(10r^2-20r)-(r-8)$

In the following exercises, subtract the polynomials.

47. Subtract $(9x^2+2)$ from $(12x^2-x+6)$

48. Subtract $(5y^2-y+12)$ from $(10y^2-8y-20)$

49. Subtract $(7w^2-4w+2)$ from $(8w^2-w+6)$

50. Subtract $(5x^2-x+12)$ from $(9x^2-6x-20)$

In the following exercises, find the difference of the polynomials.

51. Find the difference of (w^2+w-42) and $(w^2-10w+24)$

52. Find the difference of $(z^2-3z-18)$ and $(z^2+5z-20)$

In the following exercises, add the polynomials.

53. $(7x^2-2xy+6y^2)+(3x^2-5xy)$

54. $(-5x^2-4xy-3y^2)+(2x^2-7xy)$

55. $(7m^2+mn-8n^2)+(3m^2+2mn)$

56. $(2r^2-3rs-2s^2)+(5r^2-3rs)$

In the following exercises, add or subtract the polynomials.

57. $\left(a^2-b^2\right)-\left(a^2+3ab-4b^2\right)$

58. $\left(m^2+2n^2\right)-\left(m^2-8mn-n^2\right)$

59. $\left(p^3-3p^2q\right)+\left(2pq^2+4q^3\right)-\left(3p^2q+pq^2\right)$

60. $\left(a^3-2a^2b\right)+\left(ab^2+b^3\right)-\left(3a^2b+4ab^2\right)$

61. $\left(x^3-x^2y\right)-\left(4xy^2-y^3\right)+\left(3x^2y-xy^2\right)$

62. $\left(x^3-2x^2y\right)-\left(xy^2-3y^3\right)-\left(x^2y-4xy^2\right)$

Evaluate a Polynomial Function for a Given Value

In the following exercises, find the function values for each polynomial function.

63. For the function $f(x)=8x^2-3x+2$, find:

ⓐ $f(5)$ ⓑ $f(-2)$ ⓒ $f(0)$

64. For the function $f(x)=5x^2-x-7$, find:

ⓐ $f(-4)$ ⓑ $f(1)$ ⓒ $f(0)$

65. For the function $g(x)=4-36x$, find:

ⓐ $g(3)$ ⓑ $g(0)$ ⓒ $g(-1)$

66. For the function $g(x)=16-36x^2$, find:

ⓐ $g(-1)$ ⓑ $g(0)$ ⓒ $g(2)$

In the following exercises, find the height for each polynomial function.

67. A painter drops a brush from a platform 75 feet high. The polynomial function $h(t)=-16t^2+75$ gives the height of the brush t seconds after it was dropped. Find the height after $t=2$ seconds.

68. A girl is throwing a ball off the cliff into the ocean. The polynomial $h(t)=-16t^2+200$ gives the height of a ball t seconds after it is dropped from a 250-foot tall cliff. Find the height after $t=3$ seconds.

69. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of p dollars each is given by the polynomial function $R(p)=-4p^2+420p$. Find the revenue received when $p=60$ dollars.

70. A manufacturer of the latest basketball shoes has found that the revenue received from selling the shoes at a cost of p dollars each is given by the polynomial $R(p)=-4p^2+420p$. Find the revenue received when $p=90$ dollars.

71. The polynomial $C(x)=6x^2+90x$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and height 6 feet. Find the cost of producing a box with $x=4$ feet.

72. The polynomial $C(x)=6x^2+90x$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and height 4 feet. Find the cost of producing a box with $x=6$ feet.

Add and Subtract Polynomial Functions

For each function, find ⓐ (f + g)(x) ⓑ (f + g)(2) ⓒ (f − g)(x) ⓓ (f − g)(−3).

73. $f(x)=2x^2-4x+1$ and $g(x)=5x^2+8x+3$

74. $f(x)=4x^2-7x+3$ and $g(x)=4x^2+2x-1$

75. $f(x)=3x^3-x^2-2x+3$ and $g(x)=3x^3-7x$

76. $f(x)=5x^3-x^2+3x+4$ and $g(x)=8x^3-1$

Writing Exercises

77. Using your own words, explain the difference between a monomial, a binomial, and a trinomial.

78. Using your own words, explain the difference between a polynomial with five terms and a polynomial with a degree of 5.

79. Ariana thinks the sum $6y^2+5y^4$ is $11y^6$. What is wrong with her reasoning?

80. Is every trinomial a second degree polynomial? If not, give an example.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No. I don't get it!
identify polynomials, monomials, binomials, and trinomials.			
determine the degree of polynomials.			
add and subtract monomials.			
add and subtract polynomials.			
evaluate a polynomial for a given value.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

5.2 Properties of Exponents and Scientific Notation

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions using the properties for exponents
- Use the definition of a negative exponent
- Use scientific notation

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(-2)(-2)(-2)$.
 If you missed this problem, review **Example 1.19**.
2. Simplify: $\frac{8x}{24y}$.
 If you missed this problem, review **Example 1.24**.
3. Name the decimal $(-2.6)(4.21)$.
 If you missed this problem, review **Example 1.36**.

Simplify Expressions Using the Properties for Exponents

Remember that an exponent indicates repeated multiplication of the same quantity. For example, in the expression a^m, the *exponent m* tells us how many times we use the *base a* as a factor.

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}} \qquad (-9)^5 = \underbrace{(-9)(-9)(-9)(-9)(-9)}_{5 \text{ factors}}$$

Let's review the vocabulary for expressions with exponents.

Exponential Notation

a^m ← exponent, base

a^m means multiply a, m times

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}}$$

This is read a to the m^{th} power.

In the expression a^m, the *exponent m* tells us how many times we use the *base a* as a factor.

When we combine like terms by adding and subtracting, we need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

First, we will look at an example that leads to the **Product Property**.

$x^2 \cdot x^3$

What does this mean?

$$\underbrace{x \cdot x}_{2 \text{ factors}} \cdot \underbrace{x \cdot x \cdot x}_{3 \text{ factors}}$$
$$\underbrace{\qquad\qquad}_{5 \text{ factors}}$$

x^5

Notice that 5 is the sum of the exponents, 2 and 3. We see $x^2 \cdot x^3$ is x^{2+3} or x^5.

The base stayed the same and we added the exponents. This leads to the Product Property for Exponents.

Product Property for Exponents

If *a* is a real number and *m* and *n* are integers, then

$$a^m \cdot a^n = a^{m+n}$$

To multiply with like bases, add the exponents.

EXAMPLE 5.12

Simplify each expression: ⓐ $y^5 \cdot y^6$ ⓑ $2^x \cdot 2^{3x}$ ⓒ $2a^7 \cdot 3a$.

Solution

ⓐ

	$y^5 \cdot y^6$
Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	y^{5+6}
Simplify.	y^{11}

ⓑ

	$2^x \cdot 2^{3x}$
Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	2^{x+3x}
Simplify.	2^{4x}

ⓒ

	$2a^7 \cdot 3a$
Rewrite, $a = a^1$.	$2a^7 \cdot 3a^1$
Use the Commutative Property and use the Product Property, $a^m \cdot a^n = a^{m+n}$.	$2 \cdot 3 \cdot a^{7+1}$
Simplify.	$6a^8$

ⓓ

	$a^5 \cdot a^3 \cdot a^1$
Add the exponents, since bases are the same.	a^{5+3+1}
Simplify.	a^9

TRY IT : : 5.23 Simplify each expression:

ⓐ $b^9 \cdot b^8$ ⓑ $4^{2x} \cdot 4^x$ ⓒ $3p^5 \cdot 4p$ ⓓ $x^6 \cdot x^4 \cdot x^8$.

TRY IT : : 5.24 Simplify each expression:

ⓐ $x^{12} \cdot x^4$ ⓑ $10 \cdot 10^x$ ⓒ $2z \cdot 6z^7$ ⓓ $b^5 \cdot b^9 \cdot b^5$.

Now we will look at an exponent property for division. As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$		$\frac{x \cdot x}{x \cdot x \cdot x}$
Use the Equivalent Fractions Property.	$\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$		$\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$
Simplify.	x^3		$\frac{1}{x}$

Notice, in each case the bases were the same and we subtracted exponents. We see $\frac{x^5}{x^2}$ is x^{5-2} or x^3. We see $\frac{x^2}{x^3}$ is or $\frac{1}{x}$. When the larger exponent was in the numerator, we were left with factors in the numerator. When the larger exponent was in the denominator, we were left with factors in the denominator--notice the numerator of 1. When all the factors in the numerator have been removed, remember this is really dividing the factors to one, and so we need a 1 in the numerator. $\frac{\cancel{x}}{\cancel{x}} = 1$. This leads to the **Quotient Property** for Exponents.

Quotient Property for Exponents

If *a* is a real number, $a \neq 0$, and *m* and *n* are integers, then

$$\frac{a^m}{a^n} = a^{m-n}, \quad m > n \quad \text{and} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad n > m$$

EXAMPLE 5.13

Simplify each expression: ⓐ $\frac{x^9}{x^7}$ ⓑ $\frac{3^{10}}{3^2}$ ⓒ $\frac{b^8}{b^{12}}$ ⓓ $\frac{7^3}{7^5}$.

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since $9 > 7$, there are more factors of x in the numerator.	$\frac{x^9}{x^7}$
Use Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	x^{9-7}
Simplify.	x^2

ⓑ

Since $10 > 2,$ there are more factors of 3 in the numerator.	
Use Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	
Simplify.	

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

ⓒ

Since $12 > 8,$ there are more factors of b in the denominator.	
Use Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	
Simplify.	

ⓓ

Since $5 > 3,$ there are more factors of 3 in the denominator.	
Use Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	
Simplify.	
Simplify.	

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

TRY IT : : 5.25

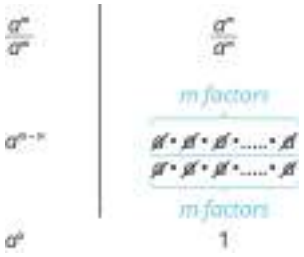
Simplify each expression: ⓐ $\frac{x^{15}}{x^{10}}$ ⓑ $\frac{6^{14}}{6^5}$ ⓒ $\frac{x^{18}}{x^{22}}$ ⓓ $\frac{12^{15}}{12^{30}}$.

TRY IT : : 5.26

Simplify each expression: ⓐ $\frac{y^{43}}{y^{37}}$ ⓑ $\frac{10^{15}}{10^7}$ ⓒ $\frac{m^7}{m^{15}}$ ⓓ $\frac{9^8}{9^{19}}$.

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. We know , $\frac{x}{x} = 1,$ for any $x\,(x \neq 0)$ since any number divided by itself is 1.

The Quotient Property for Exponents shows us how to simplify $\frac{a^m}{a^m}$. when $m > n$ and when $n < m$ by subtracting exponents. What if $m = n$? We will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the **Zero Exponent Property**. In general, for $a \neq 0$:



We see $\frac{a^m}{a^m}$ simplifies to a^0 and to 1. So $a^0 = 1$. Any non-zero base raised to the power of zero equals 1.

Zero Exponent Property

If a is a non-zero number, then $a^0 = 1$.

If a is a non-zero number, then a to the power of zero equals 1.

Any non-zero number raised to the zero power is 1.

In this text, we assume any variable that we raise to the zero power is not zero.

EXAMPLE 5.14

Simplify each expression: ⓐ 9^0 ⓑ n^0.

Solution

The definition says any non-zero number raised to the zero power is 1.

ⓐ

	9^0
Use the definition of he zero exponent.	1

ⓑ

	n^0
Use the definition of he zero exponent.	1

To simplify the expression n raised to the zero power we just use the definition of the zero exponent. The result is 1.

TRY IT : : 5.27 Simplify each expression: ⓐ 11^0 ⓑ q^0.

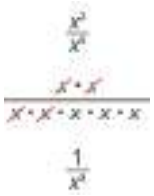
TRY IT : : 5.28 Simplify each expression: ⓐ 23^0 ⓑ r^0.

Use the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents has two forms depending on whether the exponent is larger in the numerator or the denominator. What if we just subtract exponents regardless of which is larger?

Let's consider $\frac{x^2}{x^5}$. We subtract the exponent in the denominator from the exponent in the numerator. We see $\frac{x^2}{x^5}$ is x^{2-5} or x^{-3}.

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:



This implies that $x^{-3} = \frac{1}{x^3}$ and it leads us to the definition of a *negative exponent*. If *n* is an integer and $a \neq 0,$ then $a^{-n} = \frac{1}{a^n}.$

Let's now look at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

	$\frac{1}{a^{-n}}$
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}.$	$\frac{1}{\frac{1}{a^n}}$
Simplify the complex fraction.	$1 \cdot \frac{a^n}{1}$
Multiply.	a^n

This implies $\frac{1}{a^{-n}} = a^n$ and is another form of the definition of **Properties of Negative Exponents**.

Properties of Negative Exponents

If *n* is an integer and $a \neq 0,$ then $a^{-n} = \frac{1}{a^n}$ or $\frac{1}{a^{-n}} = a^n.$

The negative exponent tells us we can rewrite the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression $x^{-3},$ we will take one more step and write $\frac{1}{x^3}.$ The answer is considered to be in simplest form when it has only positive exponents.

EXAMPLE 5.15

Simplify each expression: ⓐ x^{-5} ⓑ 10^{-3} ⓒ $\frac{1}{y^{-4}}$ ⓓ $\frac{1}{3^{-2}}.$

Solution

ⓐ

	x^{-5}
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}.$	$\frac{1}{x^5}$

ⓑ

	10^{-3}
Use the definition of a ne ative exponent, $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{10^3}$
Simplify.	$\frac{1}{1000}$

ⓒ

	$\frac{1}{y^{-4}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}}=a^n$.	y^4

ⓓ

	$\frac{1}{3^{-2}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}}=a^n$.	3^2
Simplify.	9

TRY IT : : 5.29 Simplify each expression: ⓐ z^{-3} ⓑ 10^{-7} ⓒ $\frac{1}{p^{-8}}$ ⓓ $\frac{1}{4^{-3}}$.

TRY IT : : 5.30 Simplify each expression: ⓐ n^{-2} ⓑ 10^{-4} ⓒ $\frac{1}{q^{-7}}$ ⓓ $\frac{1}{2^{-4}}$.

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a ne ative exponent, $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$\frac{16}{9}$

But we know that $\frac{16}{9}$ is $\left(\frac{4}{3}\right)^2$.

This tells us that $\left(\frac{3}{4}\right)^{-2}=\left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the **Quotient to a Negative Power Property**.

Quotient to a Negative Power Property

If *a* and *b* are real numbers, $a\neq 0$, $b\neq 0$ and *n* is an integer, then

$$\text{and } \left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^n$$

EXAMPLE 5.16

Simplify each expression: ⓐ $\left(\frac{5}{7}\right)^{-2}$ ⓑ $\left(-\frac{x}{y}\right)^{-3}$.

Solution

ⓐ

	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}$.	$\left(\frac{7}{5}\right)^{2}$
Take the reciprocal of the fraction and change the sign of the exponent.	
Simplify.	$\frac{49}{25}$

ⓑ

	$\left(-\frac{x}{y}\right)^{-3}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}$.	$\left(-\frac{y}{x}\right)^{3}$
Take the reciprocal of the fraction and change the sign of the exponent.	
Simplify.	$-\frac{y^3}{x^3}$

TRY IT : : 5.31 Simplify each expression: ⓐ $\left(\frac{2}{3}\right)^{-4}$ ⓑ $\left(-\frac{m}{n}\right)^{-2}$.

TRY IT : : 5.32 Simplify each expression: ⓐ $\left(\frac{3}{5}\right)^{-3}$ ⓑ $\left(-\frac{a}{b}\right)^{-4}$.

Now that we have negative exponents, we will use the Product Property with expressions that have negative exponents.

EXAMPLE 5.17

Simplify each expression: ⓐ $z^{-5}\cdot z^{-3}$ ⓑ $\left(m^4n^{-3}\right)\left(m^{-5}n^{-2}\right)$ ⓒ $\left(2x^{-6}y^{8}\right)\left(-5x^{5}y^{-3}\right)$.

Solution

ⓐ

	$z^{-5}\cdot z^{-3}$
Add the exponents, since the bases are the same.	z^{-5-3}
Simplify.	z^{-8}
Use the definition of a ne ative exponent.	$\frac{1}{z^8}$

ⓑ

	$(m^4 n^{-3})(m^{-5} n^{-2})$
Use the Commutative Property to get like bases together.	$m^4 m^{-5} \cdot n^{-2} n^{-3}$
Add the exponents for each base.	$m^{-1} \cdot n^{-5}$
Take reciprocals and change the signs of the exponents.	$\frac{1}{m^1} \cdot \frac{1}{n^5}$
Simplify.	$\frac{1}{mn^5}$

ⓒ

	$(2x^{-6} y^8)(-5x^5 y^{-3})$
Rewrite with the like bases together.	$2(-5) \cdot (x^{-6} x^5) \cdot (y^8 y^{-3})$
Multiply the coefficients and add he exponents of each variable.	$-10 \cdot x^{-1} \cdot y^5$
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$-10 \cdot \frac{1}{x} \cdot y^5$
Simplify.	$\frac{-10y^5}{x}$

TRY IT : : 5.33 Simplify each expression:

ⓐ $z^{-4} \cdot z^{-5}$ ⓑ $(p^6 q^{-2})(p^{-9} q^{-1})$ ⓒ $(3u^{-5} v^7)(-4u^4 v^{-2})$.

TRY IT : : 5.34 Simplify each expression:

ⓐ $c^{-8} \cdot c^{-7}$ ⓑ $(r^5 s^{-3})(r^{-7} s^{-5})$ ⓒ $(-6c^{-6} d^4)(-5c^{-2} d^{-1})$.

Now let's look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

$(x^2)^3$

What does this mean?

$x^2 \cdot x^2 \cdot x^2$

How many factors altogether?



So we have

Notice the 6 is the *product* of the exponents, 2 and 3. We see that $(x^2)^3$ is $x^{2 \cdot 3}$ or x^6.

We multiplied the exponents. This leads to the **Power Property for Exponents.**

Power Property for Exponents

If a is a real number and m and n are integers, then

$$(a^m)^n = a^{m \cdot n}$$

To raise a power to a power, multiply the exponents.

EXAMPLE 5.18

Simplify each expression: ⓐ $(y^5)^9$ ⓑ $(4^4)^7$ ⓒ $(y^3)^6(y^5)^4$.

Solution

ⓐ

	$(y^5)^9$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$y^{5 \cdot 9}$
Simplify.	y^{45}

ⓑ

	$(4^4)^7$
Use the Power Property.	$4^{4 \cdot 7}$
Simplify.	4^{28}

ⓒ

	$(y^3)^6(y^5)^4$
Use the Power Property.	$y^{18} \cdot y^{20}$
Add the exponents.	y^{38}

TRY IT : : 5.35 Simplify each expression: ⓐ $(b^7)^5$ ⓑ $(5^4)^3$ ⓒ $(a^4)^5(a^7)^4$.

TRY IT : : 5.36 Simplify each expression: ⓐ $(z^6)^9$ ⓑ $(3^7)^7$ ⓒ $(q^4)^5(q^3)^3$.

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

	$(2x)^3$
What does this mean?	$2x \cdot 2x \cdot 2x$
We group the like factors together.	$2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x$
How many factors of 2 and of x	$2^3 \cdot x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3 \cdot x^3$.

The exponent applies to each of the factors! This leads to the **Product to a Power Property for Exponents**.

Product to a Power Property for Exponents

If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

To raise a product to a power, raise each factor to that power.

EXAMPLE 5.19

Simplify each expression: ⓐ $(-3mn)^3$ ⓑ $\left(-4a^2b\right)^0$ ⓒ $\left(6k^3\right)^{-2}$ ⓓ $\left(5x^{-3}\right)^2$.

Solution

ⓐ

	$(-3mn)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(-3)^3 m^3 n^3$
Simplify.	$-27m^3n^3$

ⓑ

	$\left(-4a^2 b\right)^0$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(-4)^0\left(a^2\right)^0 (b)^0$
Simplify.	$1 \cdot 1 \cdot 1$
Multiply.	1

ⓒ

	$\left(6k^3\right)^{-2}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$(6)^{-2}\left(k^3\right)^{-2}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$6^{-2}k^{-6}$
Use the Definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{6^2} \cdot \frac{1}{k^6}$
Simplify.	$\frac{1}{36k^6}$

ⓓ

	$\left(5x^{-3}\right)^2$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$5^2\left(x^{-3}\right)^2$
Simplify.	$25 \cdot x^{-6}$
Rewrite x^{-6} using, $a^{-n} = \frac{1}{a^n}$.	$25 \cdot \frac{1}{x^6}$
Simplify.	$\frac{25}{x^6}$

TRY IT : : 5.37 Simplify each expression: ⓐ $(2wx)^5$ ⓑ $\left(-11pq^3\right)^0$ ⓒ $\left(2b^3\right)^{-4}$ ⓓ $\left(8a^{-4}\right)^2$.

TRY IT : : 5.38 Simplify each expression: ⓐ $(-3y)^3$ ⓑ $(-8m^2n^3)^0$ ⓒ $\left(-4x^4\right)^{-2}$ ⓓ $\left(2c^{-4}\right)^3$.

Now we will look at an example that will lead us to the Quotient to a Power Property.

	$\left(\frac{x}{y}\right)^3$
This means	$\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}$
Multiply the fractions.	$\frac{x\cdot x\cdot x}{y\cdot y\cdot y}$
Write with exponents.	$\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We see that $\left(\frac{x}{y}\right)^3$ is $\frac{x^3}{y^3}$.

This leads to the **Quotient to a Power Property for Exponents**.

Quotient to a Power Property for Exponents

If a and b are real numbers, $b\neq 0$, and m is an integer, then

$$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$$

To raise a fraction to a power, raise the numerator and denominator to that power.

EXAMPLE 5.20

Simplify each expression:

ⓐ $\left(\frac{b}{3}\right)^4$ ⓑ $\left(\frac{k}{j}\right)^{-3}$ ⓒ $\left(\frac{2xy^2}{z}\right)^3$ ⓓ $\left(\frac{4p^{-3}}{q^2}\right)^2$.

Solution

ⓐ

	$\left(\frac{b}{3}\right)^4$
Use Quotient to a Power Property, $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$.	$\frac{b^4}{3^4}$
Simplify.	$\frac{b^4}{81}$

ⓑ

	$\left(\frac{k}{j}\right)^{-3}$
Raise the numerator and denominator to the power.	$\frac{k^{-3}}{j^{-3}}$
Use the definition of negative exponent.	$\frac{1}{k^3}\div\frac{j^3}{1}$
Multiply.	$\frac{j^3}{k^3}$

ⓒ

	$\left(\frac{2xy^2}{z}\right)^3$
Use Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{\left(2xy^2\right)^3}{z^3}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$\frac{8x^3y^6}{z^3}$

ⓓ

	$\left(\frac{4p^{-3}}{q^2}\right)^2$
Use Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{\left(4p^{-3}\right)^2}{\left(q^2\right)^2}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$\frac{4^2\left(p^{-3}\right)^2}{\left(q^2\right)^2}$
Simplify using the Power Property, $(a^m)^n = a^{m \cdot n}$.	$\frac{16p^{-6}}{q^4}$
Use the definition of ne ative exponent.	$\frac{16}{q^4} \cdot \frac{1}{p^6}$
Simplify.	$\frac{16}{p^6 q^4}$

TRY IT : : 5.39 Simplify each expression:

ⓐ $\left(\frac{p}{10}\right)^4$ ⓑ $\left(\frac{m}{n}\right)^{-7}$ ⓒ $\left(\frac{3ab^3}{c^2}\right)^4$ ⓓ $\left(\frac{3x^{-2}}{y^3}\right)^3$.

TRY IT : : 5.40 Simplify each expression:

ⓐ $\left(\frac{-2}{q}\right)^3$ ⓑ $\left(\frac{w}{x}\right)^{-4}$ ⓒ $\left(\frac{xy^3}{3z^2}\right)^2$ ⓓ $\left(\frac{2m^{-2}}{n^{-2}}\right)^3$.

We now have several properties for exponents. Let's summarize them and then we'll do some more examples that use more than one of the properties.

Summary of Exponent Properties

If *a* and *b* are real numbers, and *m* and *n* are integers, then

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}$, $a \neq 0$
Zero Exponent Property	$a^0 = 1$, $a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $b \neq 0$
Properties of Negative Exponents	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

EXAMPLE 5.21

Simplify each expression by applying several properties:

ⓐ $(3x^2y)^4(2xy^2)^3$ ⓑ $\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$ ⓒ $\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$.

Solution

ⓐ

$$(3x^2y)^4(2xy^2)^3$$

Use the Product to a Power Property, $(ab)^m = a^m b^m$. $(3^4x^8y^4)(2^3x^3y^6)$

Simplify. $(81x^8y^4)(8x^3y^6)$

Use the Commutative Property. $81 \cdot 8 \cdot x^8 \cdot x^3 \cdot y^4 \cdot y^6$

Multiply the constants and add the exponents. $648x^{11}y^{10}$

ⓑ

$$\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$$

Use the Power Property, $(a^m)^n = a^{m \cdot n}$. $\frac{(x^{12})(x^{-10})}{(x^{30})}$

Add the exponents in the numerator. $\frac{x^2}{x^{30}}$

Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$. $\frac{1}{x^{28}}$

ⓒ

	$\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$
Simplify inside the parentheses fir t.	$\left(\frac{2y^4}{x^2}\right)^2\left(\frac{12y^4}{x^2}\right)^{-1}$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$.	$\frac{\left(2y^4\right)^2\left(12y^4\right)^{-1}}{\left(x^2\right)^2\ \left(x^2\right)^{-1}}$
Use the Product to a Power Property, $(ab)^m=a^m b^m$.	$\frac{4y^8}{x^4}\cdot\frac{12^{-1}y^{-4}}{x^{-2}}$
Simplify.	$\frac{4y^4}{12x^2}$
Simplify.	$\frac{y^4}{3x^2}$

TRY IT : : 5.41 Simplify each expression:

ⓐ $(c^4d^2)^5(3cd^5)^4$ ⓑ $\frac{(a^{-2})^3(a^2)^4}{(a^4)^5}$ ⓒ $\left(\frac{3xy^2}{x^2y^{-3}}\right)^2\left(\frac{9xy^{-3}}{x^3y^2}\right)^{-1}$.

TRY IT : : 5.42 Simplify each expression:

ⓐ $(a^3b^2)^6(4ab^3)^4$ ⓑ $\frac{(p^{-3})^4(p^5)^3}{(p^7)^6}$ ⓒ $\left(\frac{4x^3y^2}{x^2y^{-1}}\right)^2\left(\frac{8xy^{-3}}{x^2y}\right)^{-1}$.

Use Scientific Notation

Working with very large or very small numbers can be awkward. Since our number system is base ten we can use powers of ten to rewrite very large or very small numbers to make them easier to work with. Consider the numbers 4,000 and 0.004.

Using place value, we can rewrite the numbers 4,000 and 0.004. We know that 4,000 means $4\times1{,}000$ and 0.004 means $4\times\frac{1}{1{,}000}$.

If we write the 1,000 as a power of ten in exponential form, we can rewrite these numbers in this way:

4,000	$4\times1{,}000$	4×10^3	
0.004	$4\times\frac{1}{1{,}000}$	$4\times\frac{1}{10^3}$	4×10^{-3}

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than ten, and the second factor is a power of 10 written in exponential form, it is said to be in **scientific notation**.

Scientific Notation

A number is expressed in **scientific notation** when it is of the form

$$a\times10^n \text{ where } 1\le a<10 \text{ and } n \text{ is an integer.}$$

It is customary in scientific notation to use as the $\times$ multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to

scientific notation.

$4000. = 4 \times 10^{3}$	$0.004 = 4 \times 10^{-3}$
Moved the decimal point 3 places to the left.	Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10.

The power of 10 is positive when the number is larger than 1: $4{,}000 = 4 \times 10^{3}$

The power of 10 is negative when the number is between 0 and 1: $0.004 = 4 \times 10^{-3}$

HOW TO : : TO CONVERT A DECIMAL TO SCIENTIFIC NOTATION.

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.

Step 2. Count the number of decimal places, *n*, that the decimal point was moved.

Step 3. Write the number as a product with a power of 10. If the original number is.

- greater than 1, the power of 10 will be 10^{n}.
- between 0 and 1, the power of 10 will be 10^{-n}.

Step 4. Check.

EXAMPLE 5.22

Write in scientific notation: ⓐ 37,000 ⓑ 0.0052.

Solution

ⓐ

The original number, 37,000, is greater than 1 so we will have a positive power of 10.	37,000
Move the decimal point to get 3.7, a number between 1 and 10.	37,000
Count the number of decimal places the point was moved.	4 places
Write as a product with a power of 10.	3.7×10^{4}
Check: 3.7×10^{4} $3.7 \times 10{,}000$ $37{,}000$	
	$37{,}000 = 3.7 \times 10^{4}$

ⓑ

The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10.	0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}

Check:

$$5.2 \times 10^{-3}$$
$$5.2 \times \frac{1}{10^3}$$
$$5.2 \times \frac{1}{1000}$$
$$5.2 \times 0.001$$
$$0.0052$$

$0.0052 = 5.2 \times 10^{-3}$

TRY IT : : 5.43 Write in scientific notation: ⓐ 96,000 ⓑ 0.0078.

TRY IT : : 5.44 Write in scientific notation: ⓐ 48,300 ⓑ 0.0129.

How can we convert from scientific notation to decimal form? Let's look at two numbers written in scientific notation and see.

9.12×10^4	9.12×10^{-4}
$9.12 \times 10{,}000$	9.12×0.0001
$91{,}200$	0.000912

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

$9.12 \times 10^4 = 91{,}200$	$9.12 \times 10^{-4} = 0.000912$
$9.12 \times 10^4 = 91{,}200$	$9.12 \times 10^{-4} = 0.000912$
Move the decimal point 4 places to the right.	Move the decimal point 4 places to the left.

In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

HOW TO : : CONVERT SCIENTIFIC NOTATION TO DECIMAL FORM.

Step 1. Determine the exponent, n, on the factor 10.

Step 2. Move the decimal n places, adding zeros if needed.

- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point $|n|$ places to the left.

Step 3. Check.

EXAMPLE 5.23

Convert to decimal form: ⓐ 6.2×10^3 ⓑ -8.9×10^{-2}.

Solution

ⓐ

	6.2×10^3
Determine the exponent, *n*, on the factor 10.	
	The exponent is 3.
Since the exponent is positive, move the decimal point 3 places to the right.	6.200
Add zeros as needed for placeholders.	6,200
	$6.2 \times 10^3 = 6{,}200$

ⓑ

	-8.9×10^{-2}
Determine the exponent, *n*, on the factor 10.	The exponent is -2.
Since the exponent is negative, move the decimal point 2 places to the left.	−.89
Add zeros as needed for placeholders.	−0.089
	$-8.9 \times 10^{-2} = -0.089$

TRY IT : : 5.45 Convert to decimal form: ⓐ 1.3×10^3 ⓑ -1.2×10^{-4}.

TRY IT : : 5.46 Convert to decimal form: ⓐ -9.5×10^4 ⓑ 7.5×10^{-2}.

When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the Properties of Exponents are used to multiply and divide numbers in scientific notation.

EXAMPLE 5.24

Multiply or divide as indicated. Write answers in decimal form: ⓐ $\left(-4 \times 10^5\right)\left(2 \times 10^{-7}\right)$ ⓑ $\frac{9 \times 10^3}{3 \times 10^{-2}}$.

Solution

ⓐ

	$\left(-4 \times 10^5\right)\left(2 \times 10^{-7}\right)$
Use the Commutative Property to rearrange the factors.	$-4 \cdot 2 \cdot 10^5 \cdot 10^{-7}$
Multiply.	-8×10^{-2}
Change to decimal form by moving the decimal two places left.	-0.08

ⓑ

	$\frac{9\times 10^{3}}{9\times 10^{-2}}$
Separate the factors, rewriting as the product of two fractions.	$\frac{9}{3}\times\frac{10^{3}}{10^{-2}}$
Divide.	3×10^{5}
Change to decimal form by moving the decimal fi e places right.	300,000

TRY IT : : 5.47 Multiply or divide as indicated. Write answers in decimal form:

ⓐ $\left(-3\times 10^{5}\right)\left(2\times 10^{-8}\right)$ ⓑ $\frac{8\times 10^{2}}{4\times 10^{-2}}$.

TRY IT : : 5.48 Multiply or divide as indicated. Write answers in decimal form:

ⓐ; $\left(-3\times 10^{-2}\right)\left(3\times 10^{-1}\right)$ ⓑ $\frac{8\times 10^{4}}{2\times 10^{-1}}$.

MEDIA : :

Access these online resources for additional instruction and practice with using multiplication properties of exponents.

- **Properties of Exponents (https://openstax.org/l/37Propofexpo)**
- **Negative exponents (https://openstax.org/l/37Negativeexpo)**
- **Scientific Notation (https://openstax.org/l/37SciNotation)**

5.2 EXERCISES

Practice Makes Perfect

Simplify Expressions Using the Properties for Exponents

In the following exercises, simplify each expression using the properties for exponents.

81. ⓐ $d^3 \cdot d^6$ ⓑ $4^{5x} \cdot 4^{9x}$ ⓒ $2y \cdot 4y^3$ ⓓ $w \cdot w^2 \cdot w^3$

82. ⓐ $x^4 \cdot x^2$ ⓑ $8^{9x} \cdot 8^3$ ⓒ $3z^{25} \cdot 5z^8$ ⓓ $y \cdot y^3 \cdot y^5$

83. ⓐ $n^{19} \cdot n^{12}$ ⓑ $3^x \cdot 3^6$ ⓒ $7w^5 \cdot 8w$ ⓓ $a^4 \cdot a^3 \cdot a^9$

84. ⓐ $q^{27} \cdot q^{15}$ ⓑ $5^x \cdot 5^{4x}$ ⓒ $9u^{41} \cdot 7u^{53}$ ⓓ $c^5 \cdot c^{11} \cdot c^2$

85. $m^x \cdot m^3$

86. $n^y \cdot n^2$

87. $y^a \cdot y^b$

88. $x^p \cdot x^q$

89. ⓐ $\frac{x^{18}}{x^3}$ ⓑ $\frac{5^{12}}{5^3}$ ⓒ $\frac{q^{18}}{q^{36}}$ ⓓ $\frac{10^2}{10^3}$

90. ⓐ $\frac{y^{20}}{y^{10}}$ ⓑ $\frac{7^{16}}{7^2}$ ⓒ $\frac{t^{10}}{t^{40}}$ ⓓ $\frac{8^3}{8^5}$

91. ⓐ $\frac{p^{21}}{p^7}$ ⓑ $\frac{4^{16}}{4^4}$ ⓒ $\frac{b}{b^9}$ ⓓ $\frac{4}{4^6}$

92. ⓐ $\frac{u^{24}}{u^3}$ ⓑ $\frac{9^{15}}{9^5}$ ⓒ $\frac{x}{x^7}$ ⓓ $\frac{10}{10^3}$

93. ⓐ 20^0 ⓑ b^0

94. ⓐ 13^0 ⓑ k^0

95. ⓐ -27^0 ⓑ $-\left(27^0\right)$

96. ⓐ -15^0 ⓑ $-\left(15^0\right)$

Use the Definition of a Negative Exponent

In the following exercises, simplify each expression.

97. ⓐ a^{-2} ⓑ 10^{-3} ⓒ $\frac{1}{c^{-5}}$ ⓓ $\frac{1}{3^{-2}}$

98. ⓐ b^{-4} ⓑ 10^{-2} ⓒ $\frac{1}{c^{-5}}$ ⓓ $\frac{1}{5^{-2}}$

99. ⓐ r^{-3} ⓑ 10^{-5} ⓒ $\frac{1}{q^{-10}}$ ⓓ $\frac{1}{10^{-3}}$

100. ⓐ s^{-8} ⓑ 10^{-2} ⓒ $\frac{1}{t^{-9}}$ ⓓ $\frac{1}{10^{-4}}$

101. ⓐ $\left(\frac{5}{8}\right)^{-2}$ ⓑ $\left(-\frac{b}{a}\right)^{-2}$

102. ⓐ $\left(\frac{3}{10}\right)^{-2}$ ⓑ $\left(-\frac{2}{z}\right)^{-3}$

103. ⓐ $\left(\frac{4}{9}\right)^{-3}$ ⓑ $\left(-\frac{u}{v}\right)^{-5}$

104. ⓐ $\left(\frac{7}{2}\right)^{-3}$ ⓑ $\left(-\frac{3}{x}\right)^{-3}$

105. ⓐ $(-5)^{-2}$ ⓑ -5^{-2} ⓒ $\left(-\frac{1}{5}\right)^{-2}$ ⓓ $-\left(\frac{1}{5}\right)^{-2}$

106. ⓐ -5^{-3} ⓑ $\left(-\frac{1}{5}\right)^{-3}$ ⓒ $-\left(\frac{1}{5}\right)^{-3}$ ⓓ $(-5)^{-3}$

107. ⓐ $3 \cdot 5^{-1}$ ⓑ $(3 \cdot 5)^{-1}$

108. ⓐ $3 \cdot 4^{-2}$ ⓑ $(3 \cdot 4)^{-2}$

In the following exercises, simplify each expression using the Product Property.

109. ⓐ $b^4 b^{-8}$ ⓑ $\left(w^4 x^{-5}\right)\left(w^{-2} x^{-4}\right)$
ⓒ $\left(-6c^{-3} d^9\right)\left(2c^4 d^{-5}\right)$

110. ⓐ $s^3 \cdot s^{-7}$ ⓑ $\left(m^3 n^{-3}\right)\left(m^{-5} n^{-1}\right)$
ⓒ $\left(-2j^{-5} k^8\right)\left(7j^2 k^{-3}\right)$

111. ⓐ $a^3 \cdot a^{-3}$ ⓑ $\left(uv^{-2}\right)\left(u^{-5} v^{-3}\right)$
ⓒ $\left(-4r^{-2} s^{-8}\right)\left(9r^4 s^3\right)$

112. ⓐ $y^5 \cdot y^{-5}$ ⓑ $\left(pq^{-4}\right)\left(p^{-6} q^{-3}\right)$
ⓒ $\left(-5m^4 n^6\right)\left(8m^{-5} n^{-3}\right)$

113. $p^5 \cdot p^{-2} \cdot p^{-4}$

114. $x^4 \cdot x^{-2} \cdot x^{-3}$

In the following exercises, simplify each expression using the Power Property.

115. ⓐ $(m^4)^2$ ⓑ $(10^3)^6$ ⓒ $\left(x^3\right)^{-4}$

116. ⓐ $(b^2)^7$ ⓑ $(3^8)^2$ ⓒ $\left(k^2\right)^{-5}$

117. ⓐ $\left(y^3\right)^x$ ⓑ $(5^x)^y$ ⓒ $\left(q^6\right)^{-8}$

118. ⓐ $\left(x^2\right)^y$ ⓑ $(7^a)^b$ ⓒ $\left(a^9\right)^{-10}$

In the following exercises, simplify each expression using the Product to a Power Property.

119. ⓐ $(-3xy)^2$ ⓑ $(6a)^0$ ⓒ $\left(5x^2\right)^{-2}$ ⓓ $\left(-4y^{-3}\right)^2$

120. ⓐ $(-4ab)^2$ ⓑ $(5x)^0$ ⓒ $\left(4y^3\right)^{-3}$ ⓓ $\left(-7y^{-3}\right)^2$

121. ⓐ $(-5ab)^3$ ⓑ $(-4pq)^0$ ⓒ $\left(-6x^3\right)^{-2}$ ⓓ $\left(3y^{-4}\right)^2$

122. ⓐ $(-3xyz)^4$ ⓑ $(-7mn)^0$ ⓒ $\left(-3x^3\right)^{-2}$
ⓓ $\left(2y^{-5}\right)^2$

In the following exercises, simplify each expression using the Quotient to a Power Property.

123. ⓐ $\left(\frac{p}{2}\right)^5$ ⓑ $\left(\frac{x}{y}\right)^{-6}$ ⓒ $\left(\frac{2xy^2}{z}\right)^3$ ⓓ $\left(\frac{4p^{-3}}{q^2}\right)^2$

124. ⓐ $\left(\frac{x}{3}\right)^4$ ⓑ $\left(\frac{a}{b}\right)^{-5}$ ⓒ $\left(\frac{2xy^2}{z}\right)^3$ ⓓ $\left(\frac{4p^{-3}}{q^2}\right)^2$

125. ⓐ $\left(\frac{a}{3b}\right)^4$ ⓑ $\left(\frac{5}{4m}\right)^{-2}$ ⓒ $\left(\frac{2xy^2}{z}\right)^3$ ⓓ $\left(\frac{4p^{-3}}{q^2}\right)^2$

126. ⓐ $\left(\frac{x}{2y}\right)^3$ ⓑ $\left(\frac{10}{3q}\right)^{-4}$ ⓒ $\left(\frac{2xy^2}{z}\right)^3$ ⓓ $\left(\frac{4p^{-3}}{q^2}\right)^2$

In the following exercises, simplify each expression by applying several properties.

127. ⓐ $(5t^2)^3 (3t)^2$ ⓑ $\frac{\left(t^2\right)^5 \left(t^{-4}\right)^2}{\left(t^3\right)^7}$
ⓒ $\left(\frac{2xy^2}{x^3 y^{-2}}\right)^2 \left(\frac{12xy^3}{x^3 y^{-1}}\right)^{-1}$

128. ⓐ $(10k^4)^3 (5k^6)^2$ ⓑ $\frac{\left(q^3\right)^6 \left(q^{-2}\right)^3}{\left(q^4\right)^8}$

129. ⓐ $(m^2 n)^2 (2mn^5)^4$ ⓑ $\frac{\left(-2p^{-2}\right)^4 \left(3p^4\right)^2}{\left(-6p^3\right)^2}$

130. ⓐ $(3pq^4)^2 (6p^6 q)^2$ ⓑ $\frac{\left(-2k^{-3}\right)^2 \left(6k^2\right)^4}{\left(9k^4\right)^2}$

Mixed Practice

In the following exercises, simplify each expression.

131. ⓐ $7n^{-1}$ ⓑ $(7n)^{-1}$ ⓒ $(-7n)^{-1}$

132. ⓐ $6r^{-1}$ ⓑ $(6r)^{-1}$ ⓒ $(-6r)^{-1}$

133. ⓐ $(3p)^{-2}$ ⓑ $3p^{-2}$ ⓒ $-3p^{-2}$

134. ⓐ $(2q)^{-4}$ ⓑ $2q^{-4}$ ⓒ $-2q^{-4}$

135. $\left(x^2\right)^4 \cdot \left(x^3\right)^2$

136. $\left(y^4\right)^3 \cdot \left(y^5\right)^2$

137. $\left(a^2\right)^6 \cdot \left(a^3\right)^8$

138. $\left(b^7\right)^5 \cdot \left(b^2\right)^6$

139. $(2m^6)^3$

140. $(3y^2)^4$

141. $(10x^2y)^3$

142. $(2mn^4)^5$

143. $(-2a^3b^2)^4$

144. $(-10u^2v^4)^3$

145. $\left(\frac{2}{3}x^2y\right)^3$

146. $\left(\frac{7}{9}pq^4\right)^2$

147. $(8a^3)^2(2a)^4$

148. $(5r^2)^3(3r)^2$

149. $(10p^4)^3(5p^6)^2$

150. $(4x^3)^3(2x^5)^4$

151. $\left(\frac{1}{2}x^2y^3\right)^4\left(4x^5y^3\right)^2$

152. $\left(\frac{1}{3}m^3n^2\right)^4\left(9m^8n^3\right)^2$

153. $(3m^2n)^2(2mn^5)^4$

154. $(2pq^4)^3(5p^6q)^2$

155. ⓐ $(3x)^2(5x)$ ⓑ $(2y)^3(6y)$

156. ⓐ $(\frac{1}{2}y^2)^3(\frac{2}{3}y)^2$ ⓑ $(\frac{1}{2}j^2)^5(\frac{2}{5}j^3)^2$

157. ⓐ $(2r^{-2})^3(4^{-1}r)^2$ ⓑ $(3x^{-3})^3(3^{-1}x^5)^4$

158. $\left(\frac{k^{-2}k^8}{k^3}\right)^2$

159. $\left(\frac{j^{-2}j^5}{j^4}\right)^3$

160. $\frac{\left(-4m^{-3}\right)^2\left(5m^4\right)^3}{\left(-10m^6\right)^3}$

161. $\frac{\left(-10n^{-2}\right)^3\left(4n^5\right)^2}{\left(2n^8\right)^2}$

Use Scientific Notation

In the following exercises, write each number in scientific notation.

162. ⓐ 57,000 ⓑ 0.026

163. ⓐ 340,000 ⓑ 0.041

164. ⓐ 8,750,000 ⓑ 0.00000871

165. ⓐ 1,290,000 ⓑ 0.00000103

In the following exercises, convert each number to decimal form.

166. ⓐ 5.2×10^{2} ⓑ 2.5×10^{-2}

167. ⓐ -8.3×10^{2} ⓑ 3.8×10^{-2}

168. ⓐ 7.5×10^{6} ⓑ -4.13×10^{-5}

169. ⓐ 1.6×10^{10} ⓑ 8.43×10^{-6}

In the following exercises, multiply or divide as indicated. Write your answer in decimal form.

170. ⓐ $\left(3 \times 10^{-5}\right)\left(3 \times 10^{9}\right)$ ⓑ $\frac{7 \times 10^{-3}}{1 \times 10^{-7}}$

171. ⓐ $\left(2 \times 10^{2}\right)\left(1 \times 10^{-4}\right)$ ⓑ $\frac{5 \times 10^{-2}}{1 \times 10^{-10}}$

172. ⓐ $\left(7.1 \times 10^{-2}\right)\left(2.4 \times 10^{-4}\right)$ ⓑ $\frac{6 \times 10^{4}}{3 \times 10^{-2}}$

173. ⓐ $\left(3.5 \times 10^{-4}\right)\left(1.6 \times 10^{-2}\right)$ ⓑ $\frac{8 \times 10^{6}}{4 \times 10^{-1}}$

Writing Exercises

174. Use the Product Property for Exponents to explain why $x \cdot x = x^{2}$.

175. Jennifer thinks the quotient $\frac{a^{24}}{a^{6}}$ simplifies to a^{4}. What is wrong with her reasoning?

176. Explain why $-5^{3} = (-5)^{3}$ but $-5^{4} \neq (-5)^{4}$.

177. When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions using the properties for exponents.			
use the definition of a Negative Exponents.			
use scientific notation.			

ⓑ *After reviewing this checklist, what will you do to become confident for all goals?*

5.3 Multiply Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Multiply monomials
- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a polynomial by a polynomial
- Multiply special products
- Multiply polynomial functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Distribute: $2(x+3)$.
 If you missed this problem, review **Example 1.50**.
2. Simplify: ⓐ 9^2 ⓑ $(-9)^2$ ⓒ -9^2.
 If you missed this problem, review **Example 1.19**.
3. Evaluate: $2x^2-5x+3$ for $x=-2$.
 If you missed this problem, review **Example 1.21**.

Multiply Monomials

We are ready to perform operations on polynomials. Since monomials are algebraic expressions, we can use the properties of exponents to multiply monomials.

EXAMPLE 5.25

Multiply: ⓐ $\left(3x^2\right)\left(-4x^3\right)$ ⓑ $\left(\frac{5}{6}x^3y\right)\left(12xy^2\right)$.

Solution

ⓐ

	$\left(3x^2\right)\left(-4x^3\right)$
Use the Commutative Property to rearrange the terms.	$3\cdot(-4)\cdot x^2\cdot x^3$
Multiply.	$-12x^5$

ⓑ

	$\left(\frac{5}{6}x^3y\right)\left(12xy^2\right)$
Use the Commutative Property to rearrange the terms.	$\frac{5}{6}\cdot 12\cdot x^3\cdot x\cdot y\cdot y^2$
Multiply.	$10x^4y^3$

TRY IT : : 5.49 Multiply: ⓐ $\left(5y^7\right)\left(-7y^4\right)$ ⓑ $\left(\frac{2}{5}a^4b^3\right)\left(15ab^3\right)$.

TRY IT : : 5.50 Multiply: ⓐ $\left(-6b^4\right)\left(-9b^5\right)$ ⓑ $\left(\frac{2}{3}r^5s\right)\left(12r^6s^7\right)$.

Multiply a Polynomial by a Monomial

Multiplying a polynomial by a monomial is really just applying the Distributive Property.

EXAMPLE 5.26

Multiply: ⓐ $-2y(4y^2+3y-5)$ ⓑ $3x^3y(x^2-8xy+y^2)$.

Solution

ⓐ



Distribute.	$-2y \cdot 4y^2 + (-2y) \cdot 3y - (-2y) \cdot 5$
Multiply.	$-8y^3 - 6y^2 + 10y$

ⓑ

	$3x^3y(x^2-8xy+y^2)$
Distribute.	$3x^3y \cdot x^2 + (3x^3y) \cdot (-8xy) + (3x^3y) \cdot y^2$
Multiply.	$3x^5y - 24x^4y^2 + 3x^3y^3$

TRY IT : : 5.51 Multiply: ⓐ $-3y(5y^2+8y-7)$ ⓑ $4x^2y^2(3x^2-5xy+3y^2)$.

TRY IT : : 5.52 Multiply: ⓐ $4x^2(2x^2-3x+5)$ ⓑ $-6a^3b(3a^2-2ab+6b^2)$.

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

EXAMPLE 5.27

Multiply: ⓐ $(y+5)(y+8)$ ⓑ $(4y+3)(2y-5)$.

Solution

ⓐ



Distribute $(y+8)$.	$y(y+8)+5(y+8)$
Distribute again.	$y^2+8y+5y+40$
Combine like terms.	$y^2+13y+40$

ⓑ

	$(4y+3)(2y-5)$
Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

TRY IT : : 5.53 Multiply: ⓐ $(x+8)(x+9)$ ⓑ $(3c+4)(5c-2)$.

TRY IT : : 5.54 Multiply: ⓐ $(5x+9)(4x+3)$ ⓑ $(5y+2)(6y-3)$.

If you multiply binomials often enough you may notice a pattern. Notice that the first term in the result is the product of the *first* terms in each binomial. The second and third terms are the product of multiplying the two *outer* terms and then the two *inner* terms. And the last term results from multiplying the two *last* terms,

We abbreviate "First, Outer, Inner, Last" as FOIL. The letters stand for 'First, Outer, Inner, Last'. We use this as another method of multiplying binomials. The word FOIL is easy to remember and ensures we find all four products.

Let's multiply $(x+3)(x+7)$ using both methods.

Distributive Property	FOIL
$(x+3)(x+7)$	$(x+3)(x+7)$
$x(x+7)+3(x+7)$	
$x^2+7x+3x+21$	$x^2+7x+3x+21$
F O I L	F O I L
$x^2+10x+21$	$x^2+10x+21$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

HOW TO : : USE THE FOIL METHOD TO MULTIPLY TWO BINOMIALS.

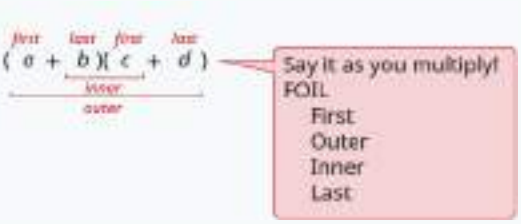
Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

Step 5. Combine like terms, when possible.



When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

EXAMPLE 5.28

Multiply: ⓐ $(y-7)(y+4)$ ⓑ $(4x+3)(2x-5)$.

Solution

ⓐ

$(y-7)(y+4)$

Step 1. Multiply the *First* terms. $y^2 + _ + _ + _$ (F O I L)

Step 2. Multiply the *Outer* terms. $y^2 + 4y + _ + _$

Step 3. Multiply the *Inner* terms. $y^2 + 4y - 7y + _$

Step 4. Multiply the *Last* terms. $y^2 + 4y - 7y - 28$

Step 5. Combine like terms. $y^2 - 3y - 28$

ⓑ

$(4x-3)(2x-5)$

Step 1. Multiply the *First* terms, $4x \cdot 2x$. $8x^2 + _ + _ + _$

Step 2. Multiply the *Outer* terms, $4x \cdot (-5)$. $8x^2 - 20x + _ + _$

Step 3. Multiply the *Inner* terms, $3 \cdot 2x$. $8x^2 - 2x + 6x + _$

Step 4. Multiply the *Last* terms, $3 \cdot (-5)$. $8x^2 - 20x + 6x - 15$

Step 5. Combine like terms. $8x^2 - 14x - 15$

TRY IT : : 5.55 Multiply: ⓐ $(x-7)(x+5)$ ⓑ $(3x+7)(5x-2)$.

TRY IT : : 5.56 Multiply: ⓐ $(b-3)(b+6)$ ⓑ $(4y+5)(4y-10)$.

The final products in the last example were trinomials because we could combine the two middle terms. This is not always the case.

EXAMPLE 5.29

Multiply: ⓐ $\left(n^2+4\right)(n-1)$ ⓑ $(3pq+5)(6pq-11)$.

Solution

ⓐ

	$(n^3+4)(n-1)$
	$(n^3+4)(n-1)$
Step 1. Multiply the *First* terms.	$n^4+__+__+__$
Step 2. Multiply the *Outer* terms.	$n^4-n^3+__+__$
Step 3. Multiply the *Inner* terms.	$n^4-n^3+4n+__$
Step 4. Multiply the *Last* terms.	n^4-n^3+4n-4
Step 5. Combine like terms—there are none.	n^4-n^3+4n-4

ⓑ

	$(3pq+5)(6pq-11)$
	$(3pq+5)(6pq-11)$
Step 1. Multiply the *First* terms.	$18p^2q^2+__+__+__$
Step 2. Multiply the *Outer* terms.	$18p^2q^2-33pq+__+__$
Step 3. Multiply the *Inner* terms.	$18p^2q^2-33pq+30pq+__$
Step 4. Multiply the *Last* terms.	$18p^2q^2-33pq+30pq-55$
Step 5. Combine like terms.	$18p^2q^2-3pq-55$

TRY IT : : 5.57 Multiply: ⓐ $(x^2+6)(x-8)$ ⓑ $(2ab+5)(4ab-4)$.

TRY IT : : 5.58 Multiply: ⓐ $(y^2+7)(y-9)$ ⓑ $(2xy+3)(4xy-5)$.

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

23		
×46		
138	partial product	Start by multiplying 23 by 6 to get 138.
92	partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058	product	Last you add the partial products.

Now we'll apply this same method to multiply two binomials.

EXAMPLE 5.30

Multiply using the Vertical Method: $(3y-1)(2y-6)$.

Solution

It does not matter which binomial goes on the top.

	$3y-1$	
	$\times\ 2y-6$	
Multiply $3y-1$ by -6.	$-18y+6$	partial product
Multiply $3y-1$ by $2y$.	$6y^2-2y$	partial product
Add like terms.	$6y^2-20y+6$	product

Notice the partial products are the same as the terms in the FOIL method.

$$(3y-1)(2y-6)$$
$$6y^2-2y-18y+6$$
$$6y^2-20y+6$$

$$\begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline -18y+6 \\ 6y^2-2y \\ \hline 6y^2-20x+6 \end{array}$$

TRY IT : : 5.59 Multiply using the Vertical Method: $(5m-7)(3m-6)$.

TRY IT : : 5.60 Multiply using the Vertical Method: $(6b-5)(7b-3)$.

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

Multiplying Two Binomials

To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Multiply a Polynomial by a Polynomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a polynomial by a polynomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method.

EXAMPLE 5.31

Multiply $(b+3)(2b^2-5b+8)$ using ⓐ the Distributive Property and ⓑ the Vertical Method.

Solution

ⓐ

	$(b+3)(2b^2-5b+8)$
Distribute.	$b(2b^2-5b+8)+3(2b^2-5b+8)$
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

ⓑ It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2-5b+8)$ by 3. Multiply $(2b^2-5b+8)$ by b.	$\begin{array}{r} 2b^2-5b+8 \\ \times \quad b+3 \\ \hline 6b^2-15b+24 \end{array}$
Add like terms.	$2b^3-5b^2+8b$ $2b^3+b^2-7b+24$

TRY IT : : 5.61 Multiply $(y-3)\left(y^2-5y+2\right)$ using ⓐ the Distributive Property and ⓑ the Vertical Method.

TRY IT : : 5.62

Multiply $(x+4)(2x^2-3x+5)$ using ⓐ the Distributive Property and ⓑ The Vertical Method.

We have now seen two methods you can use to multiply a polynomial by a polynomial. After you practice each method, you'll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

Multiplying a Polynomial by a Polynomial

To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

Multiply Special Products

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and multiplying them, there is less work to do if you learn to use a pattern. Let's start by looking at three examples and look for a pattern.

Look at these results. Do you see any patterns?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

What about the number of terms? In each example we squared a binomial and the result was a trinomial.

$$(a+b)^2 = ___ + ___ + ___$$

Now look at the *first term* in each result. Where did it come from?

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$$(a+b)^2 = a^2 + ___ + ___$$

To get the first term of the product, square the first term.

Where did the *last term* come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$$(a+b)^2 = ___ + ___ + b^2$$

To get the last term of the product, square the last term.

Finally, look at the *middle term*. Notice it came from adding the "outer" and the "inner" terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

$$(a+b)^2 = ___ + 2ab + ___$$
$$(a-b)^2 = ___ - 2ab + ___$$

To get the middle term of the product, multiply the terms and double their product.

Putting it all together:

Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$$

To square a binomial, square the first term, square the last term , double their product.

EXAMPLE 5.32

Multiply: ⓐ $(x+5)^2$ ⓑ $(2x-3y)^2$.

Solution

ⓐ

	$(x+5)^2$
Square the first term.	$x^2 + ___ + ___$
Square the last term.	$x^2 + ___ + 5^2$
Double their product.	$x^2 + 2 \cdot x \cdot 5 + 5^2$
Simplify.	$x^2 + 10x + 25$

ⓑ

	$(2x-3y)^2$
Use the pattern.	$(2x)^2 - 2 \cdot 2x \cdot 3y + (3y)^2$
Simplify.	$4x^2 - 12xy + 9y^2$

TRY IT : : 5.63 Multiply: ⓐ $(x+9)^2$ ⓑ $(2c-d)^2$.

TRY IT : : 5.64 Multiply: ⓐ $(y+11)^2$ ⓑ $(4x-5y)^2$.

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a **conjugate pair** and is of the form $(a-b), (a+b)$.

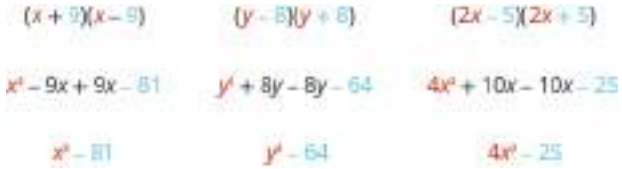
Conjugate Pair

A conjugate pair is two binomials of the form

$$(a-b), (a+b).$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier. Let's look for the pattern by using FOIL to multiply some conjugate pairs.



What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Each *first term* is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b) = a^2 - ___$$

To get the first term, square the first term.

The *last term* came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b) = a^2 - b^2$$

To get the last term, square the last term.

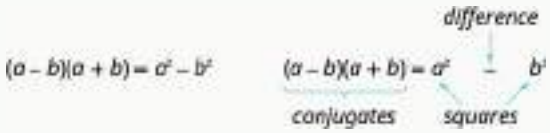
Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form $a^2 - b^2$. This is called a **difference of squares**.

This leads to the pattern:

Product of Conjugates Pattern

If *a* and *b* are real numbers,



The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

EXAMPLE 5.33

Multiply using the product of conjugates pattern: ⓐ $(2x+5)(2x-5)$ ⓑ $(5m-9n)(5m+9n)$.

Solution

ⓐ

Are the binomials conjugates?	$(2x+5)(2x-5)$
It is the product of conjugates.	$(a+b)(a-b)$ $(2x+5)(2x-5)$
Square the first term, $2x$.	a^2-b^2 $(2x)^2-__$
Square the last term, 5.	a^2-b^2 $(2x)^2-5^2$
Simplify. The product is a difference of squares.	a^2-b^2 $4x^2-25$

ⓑ

	$(5m-9n)(5m+9n)$
This fits the pattern.	$(a-b)(a+b)$ $(5m-9n)(5m+9n)$
Use the pattern.	a^2-b^2 $(5m)^2-(9n)^2$
Simplify.	$25m^2-81n^2$

TRY IT : : 5.65 Multiply: ⓐ $(6x+5)(6x-5)$ ⓑ $(4p-7q)(4p+7q)$.

TRY IT : : 5.66 Multiply: ⓐ $(2x+7)(2x-7)$ ⓑ $(3x-y)(3x+y)$.

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

Comparing the Special Product Patterns

Binomial Squares	Product of Conjugates
$(a+b)^2=a^2+2ab+b^2$	$(a-b)(a+b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	
• Squaring a binomial	• Multiplying conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

EXAMPLE 5.34

Choose the appropriate pattern and use it to find the product:

ⓐ $(2x-3)(2x+3)$ ⓑ $(5x-8)^2$ ⓒ $(6m+7)^2$ ⓓ $(5x-6)(6x+5)$.

Solution

ⓐ $(2x-3)(2x+3)$

These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

	$(a-b)(a+b)$ $(2x-3)(2x+3)$
Use the pattern.	a^2-b^2 $(2x)^2-3^2$
Simplify.	$4x^2-9$

ⓑ $(8x-5)^2$

We are asked to square a binomial. It fits the binomial squares pattern.

	$(a-b)^2$ $(8x-5)^2$
Use the pattern.	$a^2-2ab+b^2$ $(8x)^2-2\cdot 8x\cdot 5+5^2$
Simplify.	$64x^2-80x+25$

ⓒ $(6m+7)^2$

Again, we will square a binomial so we use the binomial squares pattern.

	$(a+b)^2$ $(6m+7)^2$
Use the pattern.	$a^2+2ab+b^2$ $(6m)^2+2\cdot 6m\cdot 7+7^2$
Simplify.	$36m^2+84m+49$

ⓓ $(5x-6)(6x+5)$

This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$30x^2+25x-36x-30$
Simplify.	$30x^2-11x-30$

TRY IT : : 5.67 Choose the appropriate pattern and use it to find the product:

ⓐ $(9b-2)(2b+9)$ ⓑ $(9p-4)^2$ ⓒ $(7y+1)^2$ ⓓ $(4r-3)(4r+3)$.

TRY IT : : 5.68 Choose the appropriate pattern and use it to find the product:

ⓐ $(6x+7)^2$ ⓑ $(3x-4)(3x+4)$ ⓒ $(2x-5)(5x-2)$ ⓓ $(6n-1)^2$.

Multiply Polynomial Functions

Just as polynomials can be multiplied, polynomial functions can also be multiplied.

Multiplication of Polynomial Functions

For functions $f(x)$ and $g(x)$,

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

EXAMPLE 5.35

For functions $f(x) = x + 2$ and $g(x) = x^2 - 3x - 4$, find: ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(2)$.

Solution

ⓐ

	$(f \cdot g)(x)$	$=$	$f(x) \cdot g(x)$
Substitute for $f(x)$and $g(x)$.	$(f \cdot g)(x)$	$=$	$(x+2)\left(x^2 - 3x - 4\right)$
Multiply the polynomials.	$(f \cdot g)(x)$	$=$	$x\left(x^2 - 3x - 4\right) + 2\left(x^2 - 3x - 4\right)$
Distribute.	$(f \cdot g)(x)$	$=$	$x^3 - 3x^2 - 4x + 2x^2 - 6x - 8$
Combine like terms.	$(f \cdot g)(x)$	$=$	$x^3 - x^2 - 10x - 8$

ⓑ In part ⓐ we found $(f \cdot g)(x)$ and now are asked to find $(f \cdot g)(2)$.

	$(f \cdot g)(x)$	$=$	$x^3 - x^2 - 10x - 8$
To fin $(f \cdot g)(2)$, substitute $x = 2$.	$(f \cdot g)(2)$	$=$	$2^3 - 2^2 - 10 \cdot 2 - 8$
	$(f \cdot g)(2)$	$=$	$8 - 4 - 20 - 8$
	$(f \cdot g)(2)$	$=$	-24

TRY IT : : 5.69 For functions $f(x) = x - 5$ and $g(x) = x^2 - 2x + 3$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(2)$.

TRY IT : : 5.70 For functions $f(x) = x - 7$ and $g(x) = x^2 + 8x + 4$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(2)$.

MEDIA : :

Access this online resource for additional instruction and practice with multiplying polynomials.

- **Introduction to special products of binomials (https://openstax.org/l/37Introspecprod)**

5.3 EXERCISES

Practice Makes Perfect

Multiply Monomials

In the following exercises, multiply the monomials.

178.
ⓐ $(6y^7)(-3y^4)$
ⓑ $\left(\frac{4}{7}rs^2\right)(14rs^3)$

179.
ⓐ $(-10x^5)(-3x^3)$
ⓑ $\left(\frac{5}{8}x^3y\right)(24x^5y)$

180.
ⓐ $(-8u^6)(-9u)$
ⓑ $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$

181.
ⓐ $(-6c^4)(-12c)$
ⓑ $\left(\frac{3}{5}m^3n^2\right)\left(\frac{5}{9}m^2n^3\right)$

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

182.
ⓐ $-8x(x^2+2x-15)$
ⓑ $5pq^3(p^2-2pq+6q^2)$

183.
ⓐ $-5t(t^2+3t-18)$;
ⓑ $9r^3s(r^2-3rs+5s^2)$

184.
ⓐ $-8y(y^2+2y-15)$
ⓑ $-4y^2z^2(3y^2+12yz-z^2)$

185.
ⓐ $-5m(m^2+3m-18)$
ⓑ $-3x^2y^2(7x^2+10xy-y^2)$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using ⓐ the Distributive Property; ⓑ the FOIL method; ⓒ the Vertical Method.

186. $(w+5)(w+7)$

187. $(y+9)(y+3)$

188. $(4p+11)(5p-4)$

189. $(7q+4)(3q-8)$

In the following exercises, multiply the binomials. Use any method.

190. $(x+8)(x+3)$

191. $(y-6)(y-2)$

192. $(2t-9)(10t+1)$

193. $(6p+5)(p+1)$

194. $(q-5)(q+8)$

195. $(m+11)(m-4)$

196. $(7m+1)(m-3)$

197. $(3r-8)(11r+1)$

198. $(x^2+3)(x+2)$

199. $(y^2-4)(y+3)$

200. $(5ab-1)(2ab+3)$

201. $(2xy+3)(3xy+2)$

202. $(x^2+8)(x^2-5)$

203. $(y^2-7)(y^2-4)$

204. $(6pq-3)(4pq-5)$

205. $(3rs-7)(3rs-4)$

Multiply a Polynomial by a Polynomial

In the following exercises, multiply using ⓐ the Distributive Property; ⓑ the Vertical Method.

206. $(x+5)(x^2+4x+3)$

207. $(u+4)(u^2+3u+2)$

208. $(y+8)(4y^2+y-7)$

209. $(a+10)(3a^2+a-5)$

210. $(y^2-3y+8)(4y^2+y-7)$

211. $(2a^2-5a+10)(3a^2+a-5)$

Multiply Special Products

In the following exercises, multiply. Use either method.

212. $(w-7)(w^2-9w+10)$

213. $(p-4)(p^2-6p+9)$

214. $(3q+1)(q^2-4q-5)$

215. $(6r+1)(r^2-7r-9)$

In the following exercises, square each binomial using the Binomial Squares Pattern.

216. $(w+4)^2$

217. $(q+12)^2$

218. $(3x-y)^2$

219. $(2y-3z)^2$

220. $\left(y+\frac{1}{4}\right)^2$

221. $\left(x+\frac{2}{3}\right)^2$

222. $\left(\frac{1}{5}x-\frac{1}{7}y\right)^2$

223. $\left(\frac{1}{8}x-\frac{1}{9}y\right)^2$

224. $(3x^2+2)^2$

225. $(5u^2+9)^2$

226. $(4y^3-2)^2$

227. $(8p^3-3)^2$

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

228. $(5k+6)(5k-6)$

229. $(8j+4)(8j-4)$

230. $(11k+4)(11k-4)$

231. $(9c+5)(9c-5)$

232. $(9c-2d)(9c+2d)$

233. $(7w+10x)(7w-10x)$

234. $\left(m+\frac{2}{3}n\right)\left(m-\frac{2}{3}n\right)$

235. $\left(p+\frac{4}{5}q\right)\left(p-\frac{4}{5}q\right)$

236. $(ab-4)(ab+4)$

237. $(xy-9)(xy+9)$

238. $(12p^3-11q^2)(12p^3+11q^2)$

239. $(15m^2-8n^4)(15m^2+8n^4)$

In the following exercises, find each product.

240. $(p-3)(p+3)$

241. $(t-9)^2$

242. $(m+n)^2$

243. $(2x+y)(x-2y)$

244. $(2r+12)^2$

245. $(3p+8)(3p-8)$

246. $(7a+b)(a-7b)$

247. $(k-6)^2$

248. $(a^5-7b)^2$

249. $(x^2+8y)(8x-y^2)$

250. $(r^6+s^6)(r^6-s^6)$

251. $(y^4+2z)^2$

252. $(x^5+y^5)(x^5-y^5)$

253. $(m^3-8n)^2$

254. $(9p+8q)^2$

255. $(r^2 - s^3)(r^3 + s^2)$

Mixed Practice

256. $(10y - 6) + (4y - 7)$

257. $(15p - 4) + (3p - 5)$

258. $\left(x^2 - 4x - 34\right) - \left(x^2 + 7x - 6\right)$

259. $\left(j^2 - 8j - 27\right) - \left(j^2 + 2j - 12\right)$

260. $(\frac{1}{5}f^8)(20f^3)$

261. $(\frac{1}{4}d^5)(36d^2)$

262. $(4a^3b)(9a^2b^6)$

263. $(6m^4n^3)(7mn^5)$

264. $-5m(m^2 + 3m - 18)$

265. $5q^3(q^2 - 2q + 6)$

266. $(s - 7)(s + 9)$

267. $\left(y^2 - 2y\right)(y + 1)$

268. $(5x - y)(x - 4)$

269. $(6k - 1)\left(k^2 + 2k - 4\right)$

270. $(3x - 11y)(3x - 11y)$

271. $(11 - b)(11 + b)$

272. $(rs - \frac{2}{7})(rs + \frac{2}{7})$

273. $(2x^2 - 3y^4)(2x^2 + 3y^4)$

274. $(m - 15)^2$

275. $(3d + 1)^2$

276. $(4a + 10)^2$

277. $\left(3z + \frac{1}{5}\right)^2$

Multiply Polynomial Functions

278. For functions $f(x) = x + 2$ and $g(x) = 3x^2 - 2x + 4$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(-1)$

279. For functions $f(x) = x - 1$ and $g(x) = 4x^2 + 3x - 5$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(-2)$

280. For functions $f(x) = 2x - 7$ and $g(x) = 2x + 7$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(-3)$

281. For functions $f(x) = 7x - 8$ and $g(x) = 7x + 8$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(-2)$

282. For functions $f(x) = x^2 - 5x + 2$ and $g(x) = x^2 - 3x - 1$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(-1)$

283. For functions $f(x) = x^2 + 4x - 3$ and $g(x) = x^2 + 2x + 4$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(1)$

Writing Exercises

284. Which method do you prefer to use when multiplying two binomials: the Distributive Property or the FOIL method? Why? Which method do you prefer to use when multiplying a polynomial by a polynomial: the Distributive Property or the Vertical Method? Why?

285. Multiply the following:
$(x + 2)(x - 2)$
$(y + 7)(y - 7)$
$(w + 5)(w - 5)$

Explain the pattern that you see in your answers.

286. Multiply the following:

$(p+3)(p+3)$
$(q+6)(q+6)$
$(r+1)(r+1)$

Explain the pattern that you see in your answers.

287. Why does $(a+b)^2$ result in a trinomial, but $(a-b)(a+b)$ result in a binomial?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
multiply monomials.			
multiply a polynomial by a monomial.			
multiply a binomial by a binomial.			
multiply a polynomial by a polynomial.			
multiply special products.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

5.4 Dividing Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Dividing monomials
- Dividing a polynomial by a monomial
- Dividing polynomials using long division
- Dividing polynomials using synthetic division
- Dividing polynomial functions
- Use the remainder and factor theorems

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{3}{d}+\frac{x}{d}$.
 If you missed this problem, review **Example 1.28**.
2. Simplify: $\frac{30xy^3}{5xy}$.
 If you missed this problem, review **Example 1.25**.
3. Combine like terms: $8a^2+12a+1+3a^2-5a+4$.
 If you missed this problem, review **Example 1.7**.

Dividing Monomials

We are now familiar with all the properties of exponents and used them to multiply polynomials. Next, we'll use these properties to divide monomials and polynomials.

EXAMPLE 5.36

Find the quotient: $54a^2b^3\div\left(-6ab^5\right)$.

Solution

When we divide monomials with more than one variable, we write one fraction for each variable.

$$54a^2b^3\div\left(-6ab^5\right)$$

Rewrite as a fraction.	$\frac{54a^2b^3}{-6ab^5}$
Use fraction multiplication.	$\frac{54}{-6}\cdot\frac{a^2}{a}\cdot\frac{b^3}{b^5}$
Simplify and use the Quotient Property.	$-9\cdot a\cdot\frac{1}{b^2}$
Multiply.	$-\frac{9a}{b^2}$

TRY IT : : 5.71 Find the quotient: $-72a^7b^3\div\left(8a^{12}b^4\right)$.

TRY IT : : 5.72 Find the quotient: $-63c^8d^3\div\left(7c^{12}d^2\right)$.

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

EXAMPLE 5.37

Find the quotient: $\frac{14x^7y^{12}}{21x^{11}y^6}$.

Solution

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

	$\frac{14x^7y^{12}}{21x^{11}y^6}$
Simplify and use the Quotient Property.	$\frac{2y^6}{3x^4}$

TRY IT : : 5.73 Find the quotient: $\frac{28x^5y^{14}}{49x^9y^{12}}$.

TRY IT : : 5.74 Find the quotient: $\frac{30m^5n^{11}}{48m^{10}n^{14}}$.

Divide a Polynomial by a Monomial

Now that we know how to divide a monomial by a monomial, the next procedure is to divide a polynomial of two or more terms by a monomial.

The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition. The sum $\frac{y}{5}+\frac{2}{5}$ simplifies to $\frac{y+2}{5}$.

Now we will do this in reverse to split a single fraction into separate fractions. For example, $\frac{y+2}{5}$ can be written $\frac{y}{5}+\frac{2}{5}$.

This is the "reverse" of fraction addition and it states that if *a*, *b*, and *c* are numbers where $c \neq 0$, then $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$.

We will use this to divide polynomials by monomials.

Division of a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

EXAMPLE 5.38

Find the quotient: $\left(18x^3y-36xy^2\right)\div(-3xy)$.

Solution

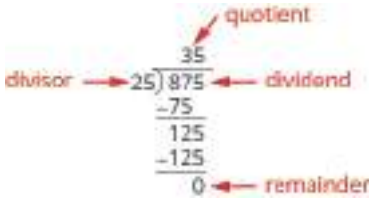
	$\left(18x^3y-36xy^2\right)\div(-3xy)$
Rewrite as a fraction.	$\frac{18x^3y-36xy^2}{-3xy}$
Divide each term by the divisor. Be careful with the signs!	$\frac{18x^3y}{-3xy}-\frac{36xy^2}{-3xy}$
Simplify.	$-6x^2+12y$

TRY IT : : 5.75 Find the quotient: $\left(32a^2b-16ab^2\right)\div(-8ab)$.

TRY IT : : 5.76 Find the quotient: $\left(-48a^8b^4 - 36a^6b^5\right) \div \left(-6a^3b^3\right)$.

Divide Polynomials Using Long Division

Divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. So let's look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.



We check division by multiplying the quotient by the divisor.

If we did the division correctly, the product should equal the dividend.

$$35 \cdot 25$$
$$875 \checkmark$$

Now we will divide a trinomial by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

EXAMPLE 5.39

Find the quotient: $\left(x^2 + 9x + 20\right) \div (x + 5)$.

Solution

	$(x^2 + 9x + 20) \div (x + 5)$
Write it as a long division problem. Be sure the dividend is in standard form.	$x + 5\overline{)x^2 + 9x + 20}$
Divide x^2 by x. It may help to ask yourself, "What do I need to multiply x by to get x^2?"	$\begin{array}{r} x \\ x + 5\overline{)x^2 + 9x + 20} \end{array}$
Put the answer, x, in the quotient over the x term. Multiply x times $x + 5$. Line up the like terms under the dividend.	$\begin{array}{r} x \\ x + 5\overline{)x^2 + 9x + 20} \\ x^2 + 5x \end{array}$
Subtract $x^2 + 5x$ from $x^2 + 9x$. You may find it easier to change the signs and then add. Then bring down the last term, 20.	$\begin{array}{r} x \\ x + 5\overline{)x^2 + 9x + 20} \\ -x^2 + (-5x) \\ \hline 4x + 20 \end{array}$
Divide $4x$ by x. It may help to ask yourself, "What do I need to multiply x by to get $4x$?" Put the answer, 4, in the quotient over the constant term.	$\begin{array}{r} x + 4 \\ x + 5\overline{)x^2 + 9x + 20} \\ -x^2 + (-5x) \\ \hline 4x + 20 \end{array}$
Multiply 4 times $x + 5$.	$\begin{array}{r} x + 4 \\ x + 5\overline{)x^2 + 9x + 20} \\ -x^2 + (-5x) \\ \hline 4x + 20 \\ 4x + 20 \end{array}$
Subtract $4x + 20$ from $4x + 20$.	$\begin{array}{r} x + 4 \\ x + 5\overline{)x^2 + 9x + 20} \\ -x^2 + (-5x) \\ \hline 4x + 20 \\ -4x + (-20) \\ \hline 0 \end{array}$

Check:
Multiply the quotient by the divisor. $(x+4)(x+5)$
You should get the dividend. $x^2+9x+20$ ✓

TRY IT : : 5.77 Find the quotient: $\left(y^2+10y+21\right)\div(y+3)$.

TRY IT : : 5.78 Find the quotient: $\left(m^2+9m+20\right)\div(m+4)$.

When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In the next example, we'll have a division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

Look back at the dividends in previous examples. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in this example will be x^4-x^2+5x-6. It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

EXAMPLE 5.40

Find the quotient: $\left(x^4-x^2+5x-6\right)\div(x+2)$.

Solution

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

$(x^4-x^2+5x-6)\div(x+2)$

Write it as a long division problem. Be sure the dividend is in standard form with placeholders for missing terms.

$x+2\overline{)\,x^4+0x^3-x^2+5x-6}$

Divide x^4 by x.
Put the answer, x^3, in the quotient over the x^3 term.
Multiply x^3 times $x+2$. Line up the like terms.
Subtract and then bring down the next term.

$$\begin{array}{r} x^3 \\ x+2\overline{)\,x^4+0x^3-x^2+5x-6} \\ -(x^4+2x^3) \\ \hline -2x^3-x^2 \end{array}$$

It may be helpful to change the signs and add.

Divide $-2x^3$ by x.
Put the answer, $-2x^2$, in the quotient over the x^2 term.
Multiply $-2x^2$ times $x+1$. Line up the like terms
Subtract and bring down the next term.

$$\begin{array}{r} x^3-2x^2 \\ x+2\overline{)\,x^4+0x^3-x^2+5x-6} \\ -(x^4+2x^3) \\ \hline -2x^3-x^2 \\ -(-2x^3-4x^2) \\ \hline 3x^2+5x \end{array}$$

It may be helpful to change the signs and add.

Divide $3x^2$ by x.
Put the answer, $3x$, in the quotient over the x term.
Multiply $3x$ times $x+1$. Line up the like terms.
Subtract and bring down the next term.

$$\begin{array}{r} x^3-2x^2+3x \\ x+2\overline{)\,x^4+0x^3-x^2+5x-6} \\ -(x^4+2x^3) \\ \hline -2x^3-x^2 \\ -(-2x^3-4x^2) \\ \hline 3x^2+5x \\ -(3x^2+6x) \\ \hline -x-6 \end{array}$$

It may be helpful to change the signs and add.

Divide $-x$ by x.
Put the answer, -1, in the quotient over the constant term.
Multiply -1 times $x+1$. Line up the like terms.
Change the signs, add.

Write the remainder as a fraction with the divisor as the denominator.

$$x+2\overline{)\,x^3+0x^2-x^2+5x-6}\qquad x^3-2x^2+3x-1-\frac{4}{x+2}$$
$$-(x^3+2x^2)$$
$$-2x^2-x^2$$
$$-(-2x^2-4x^2)$$
$$3x^2+5x$$
$$-(3x^2+6x)$$
$$-x-6$$
$$-(-x-2)$$
$$-4$$

It may be helpful to change the signs and add.

To check, multiply
$(x+2)\left(x^3-2x^2+3x-1-\frac{4}{x+2}\right)$.
The result should be x^4-x^2+5x-6.

TRY IT : : 5.79 Find the quotient: $\left(x^4-7x^2+7x+6\right)\div(x+3)$.

TRY IT : : 5.80 Find the quotient: $\left(x^4-11x^2-7x-6\right)\div(x+3)$.

In the next example, we will divide by $2a-3$. As we divide, we will have to consider the constants as well as the variables.

EXAMPLE 5.41

Find the quotient: $\left(8a^3+27\right)\div(2a+3)$.

Solution

This time we will show the division all in one step. We need to add two placeholders in order to divide.

$$(8a^3+27)\div(2a+3)$$
$$2a+3\overline{)\,8a^3+0a^2+0a+27}\qquad 4a^2-6a+9$$
$$-(8a^3+12a^2)\qquad \leftarrow 4a^2(2a+3)$$
$$-12a^2+0a$$
$$-(-12a^2-18a)\qquad \leftarrow -6a(2a+3)$$
$$18a+27$$
$$-(18a+27)\qquad \leftarrow 9(2a+3)$$
$$0$$

To check, multiply $(2a+3)\left(4a^2-6a+9\right)$.

The result should be $8a^3+27$.

TRY IT : : 5.81 Find the quotient: $\left(x^3-64\right)\div(x-4)$.

TRY IT : : 5.82 Find the quotient: $\left(125x^3-8\right)\div(5x-2)$.

Divide Polynomials using Synthetic Division

As we have mentioned before, mathematicians like to find patterns to make their work easier. Since long division can be tedious, let's look back at the long division we did in **Example 5.39** and look for some patterns. We will use this as a basis

for what is called synthetic division. The same problem in the synthetic division format is shown next.



Synthetic division basically just removes unnecessary repeated variables and numbers. Here all the x and x^2 are removed. as well as the $-x^2$ and $-4x$ as they are opposite the term above.

The first row of the synthetic division is the coefficients of the dividend. The -5 is the opposite of the 5 in the divisor.

The second row of the synthetic division are the numbers shown in red in the division problem.

The third row of the synthetic division are the numbers shown in blue in the division problem.

Notice the quotient and remainder are shown in the third row.

Synthetic division only works when the divisor is of the form $x - c$.

The following example will explain the process.

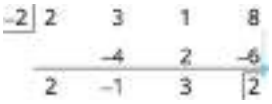
EXAMPLE 5.42

Use synthetic division to find the quotient and remainder when $2x^3 + 3x^2 + x + 8$ is divided by $x + 2$.

Solution

Write the dividend with decreasing powers of x.	$2x^3 + 3x^2 + x + 8$
Write the coefficients of the terms as the first row of the synthetic division.	2 3 1 8
Write the divisor as $x - c$ and place c in the synthetic division in the divisor box.	−2 \| 2 3 1 8
Bring down the first coefficient to the third row.	−2 \| 2 3 1 8 2
Multiply that coefficient by the divisor and place the result in the second row under the second coefficient.	−2 \| 2 3 1 8 −4 2
Add the second column, putting the result in the third row.	−2 \| 2 3 1 8 −4 2 −1
Multiply that result by the divisor and place the result in the second row under the third coefficient.	−2 \| 2 3 1 8 −4 2 2 −1
Add the third column, putting the result in the third row.	−2 \| 2 3 1 8 −4 2 2 −1 3
Multiply that result by the divisor and place the result in the third row under the third coefficient.	−2 \| 2 3 1 8 −4 2 −6 2 −1 −3

Add the final column, putting the result in the third row.



The quotient is $2x^2 - 1x + 3$ and the remainder is 2.

The division is complete. The numbers in the third row give us the result. The $2 \quad -1 \quad 3$ are the coefficients of the quotient. The quotient is $2x^2 - 1x + 3$. The 2 in the box in the third row is the remainder.

Check:

$$\begin{aligned} (\text{quotient})(\text{divisor}) + \text{remainder} &= \text{dividend} \\ \left(2x^2 - 1x + 3\right)(x + 2) + 2 &\stackrel{?}{=} 2x^3 + 3x^2 + x + 8 \\ 2x^3 - x^2 + 3x + 4x^2 - 2x + 6 + 2 &\stackrel{?}{=} 2x^3 + 3x^2 + x + 8 \\ 2x^3 + 3x^2 + x + 8 &= 2x^3 + 3x^2 + x + 8 \checkmark \end{aligned}$$

TRY IT : : 5.83

Use synthetic division to find the quotient and remainder when $3x^3 + 10x^2 + 6x - 2$ is divided by $x + 2$.

TRY IT : : 5.84

Use synthetic division to find the quotient and remainder when $4x^3 + 5x^2 - 5x + 3$ is divided by $x + 2$.

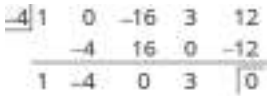
In the next example, we will do all the steps together.

EXAMPLE 5.43

Use synthetic division to find the quotient and remainder when $x^4 - 16x^2 + 3x + 12$ is divided by $x + 4$.

Solution

The polynomial $x^4 - 16x^2 + 3x + 12$ has its term in order with descending degree but we notice there is no x^3 term. We will add a 0 as a placeholder for the x^3 term. In $x - c$ form, the divisor is $x - (-4)$.



We divided a 4th degree polynomial by a 1st degree polynomial so the quotient will be a 3rd degree polynomial.

Reading from the third row, the quotient has the coefficients $1 \quad -4 \quad 0 \quad 3$, which is $x^3 - 4x^2 + 3$. The remainder is 0.

TRY IT : : 5.85

Use synthetic division to find the quotient and remainder when $x^4 - 16x^2 + 5x + 20$ is divided by $x + 4$.

TRY IT : : 5.86

Use synthetic division to find the quotient and remainder when $x^4 - 9x^2 + 2x + 6$ is divided by $x + 3$.

Divide Polynomial Functions

Just as polynomials can be divided, polynomial functions can also be divided.

Division of Polynomial Functions

For functions $f(x)$ and $g(x)$, where $g(x) \neq 0$,

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

EXAMPLE 5.44

For functions $f(x) = x^2 - 5x - 14$ and $g(x) = x + 2$, find: ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-4)$.

Solution

ⓐ

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad x+2\overline{)x^2-5x-14}$$

$$\begin{array}{r} x-7 \\ x+2\overline{)x^2-5x-14} \\ \underline{-(x^2+2x)} \\ -7x-14 \\ \underline{-(-7x-14)} \\ 0 \end{array}$$

Substitute for $f(x)$ and $g(x)$. $\left(\frac{f}{g}\right)(x) = \frac{x^2-5x-14}{x+2}$

Divide the polynomials. $\left(\frac{f}{g}\right)(x) = x - 7$

ⓑ In part ⓐ we found $\left(\frac{f}{g}\right)(x)$ and now are asked to find $\left(\frac{f}{g}\right)(-4)$.

$$\left(\frac{f}{g}\right)(x) = x - 7$$

To fin $\left(\frac{f}{g}\right)(-4)$, substitute $x = -4$. $\left(\frac{f}{g}\right)(-4) = -4 - 7$

$$\left(\frac{f}{g}\right)(-4) = -11$$

TRY IT : : 5.87 For functions $f(x) = x^2 - 5x - 24$ and $g(x) = x + 3$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-3)$.

TRY IT : : 5.88 For functions $f(x) = x^2 - 5x - 36$ and $g(x) = x + 4$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-5)$.

Use the Remainder and Factor Theorem

Let's look at the division problems we have just worked that ended up with a remainder. They are summarized in the chart below. If we take the dividend from each division problem and use it to define a function, we get the functions shown in the chart. When the divisor is written as $x - c$, the value of the function at c, $f(c)$, is the same as the remainder from the division problem.

Dividend	Divisor $x-c$	Remainder	Function	$f(c)$
x^4-x^2+5x-6	$x-(-2)$	-4	$f(x)=x^4-x^2+5x-6$	-4
$3x^3-2x^2-10x+8$	$x-2$	4	$f(x)=3x^3-2x^2-10x+8$	4
$x^4-16x^2+3x+15$	$x-(-4)$	3	$f(x)=x^4-16x^2+3x+15$	3

Table 5.44

To see this more generally, we realize we can check a division problem by multiplying the quotient times the divisor and add the remainder. In function notation we could say, to get the dividend $f(x)$, we multiply the quotient, $q(x)$ times the divisor, $x-c$, and add the remainder, *r*.

$$f(x)=q(x)(x-c)+r$$

If we evaluate this at c, we get:

$$f(c)=q(c)(c-c)+r$$

$$f(c)=q(c)(0)+r$$

$$f(c)=r$$

This leads us to the Remainder Theorem.

Remainder Theorem

If the polynomial function $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

EXAMPLE 5.45

Use the Remainder Theorem to find the remainder when $f(x)=x^3+3x+19$ is divided by $x+2$.

Solution

To use the Remainder Theorem, we must use the divisor in the $x-c$ form. We can write the divisor $x+2$ as $x-(-2)$. So, our c is -2.

To find the remainder, we evaluate $f(c)$ which is $f(-2)$.

	$f(x) = x^3 + 3x + 19$
To evaluate $f(-2)$, substitute $x = -2$.	$f(-2) = (-2)^3 + 3(-2) + 19$
Simplify.	$f(-2) = -8 - 6 + 19$
	$f(-2) = 5$
	The remainder is 5 when $f(x) = x^3 + 3x + 19$ is divided by $x + 2$.
Check: Use synthetic division to check.	
-2 \| 1 0 3 19 −2 4 −14 1 −2 7 \| 5	
The remainder is 5.	

TRY IT : : 5.89

Use the Remainder Theorem to find the remainder when $f(x) = x^3 + 4x + 15$ is divided by $x + 2$.

TRY IT : : 5.90

Use the Remainder Theorem to find the remainder when $f(x) = x^3 - 7x + 12$ is divided by $x + 3$.

When we divided $8a^3 + 27$ by $2a + 3$ in Example 5.41 the result was $4a^2 - 6a + 9$. To check our work, we multiply $4a^2 - 6a + 9$ by $2a + 3$ to get $8a^3 + 27$.

$$(4a^2 - 6a + 9)(2a + 3) = 8a^3 + 27$$

Written this way, we can see that $4a^2 - 6a + 9$ and $2a + 3$ are factors of $8a^3 + 27$. When we did the division, the remainder was zero.

Whenever a divisor, $x - c$, divides a polynomial function, $f(x)$, and resulting in a remainder of zero, we say $x - c$ is a factor of $f(x)$.

The reverse is also true. If $x - c$ is a factor of $f(x)$ then $x - c$ will divide the polynomial function resulting in a remainder of zero.

We will state this in the Factor Theorem.

Factor Theorem

For any polynomial function $f(x)$,

- if $x - c$ is a factor of $f(x)$, then $f(c) = 0$
- if $f(c) = 0$, then $x - c$ is a factor of $f(x)$

EXAMPLE 5.46

Use the Remainder Theorem to determine if $x - 4$ is a factor of $f(x) = x^3 - 64$.

Solution

The Factor Theorem tells us that $x-4$ is a factor of $f(x)=x^3-64$ if $f(4)=0$.

	$f(x)=x^3-64$
To evaluate $f(4)$ substitute $x=4$.	$f(4)=4^3-64$
Simplify.	$f(4)=64-64$
Subtract.	$f(4)=0$

Since $f(4)=0,\quad x-4$ is a factor of $f(x)=x^3-64$.

TRY IT : : 5.91 Use the Factor Theorem to determine if $x-5$ is a factor of $f(x)=x^3-125$.

TRY IT : : 5.92 Use the Factor Theorem to determine if $x-6$ is a factor of $f(x)=x^3-216$.

MEDIA : :

Access these online resources for additional instruction and practice with dividing polynomials.

- **Dividing a Polynomial by a Binomial (https://openstax.org/l/37Polybybinom)**
- **Synthetic Division & Remainder Theorem (https://openstax.org/l/37SynDivision)**

5.4 EXERCISES

Practice Makes Perfect

Divide Monomials

In the following exercises, divide the monomials.

288. $15r^4s^9 \div \left(15r^4s^9\right)$

289. $20m^8n^4 \div \left(30m^5n^9\right)$

290. $\frac{18a^4b^8}{-27a^9b^5}$

291. $\frac{45x^5y^9}{-60x^8y^6}$

292. $\frac{\left(10m^5n^4\right)\left(5m^3n^6\right)}{25m^7n^5}$

293. $\frac{\left(-18p^4q^7\right)\left(-6p^3q^8\right)}{-36p^{12}q^{10}}$

294. $\frac{\left(6a^4b^3\right)\left(4ab^5\right)}{\left(12a^2b\right)\left(a^3b\right)}$

295. $\frac{\left(4u^2v^5\right)\left(15u^3v\right)}{\left(12u^3v\right)\left(u^4v\right)}$

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial.

296. $\left(9n^4+6n^3\right) \div 3n$

297. $\left(8x^3+6x^2\right) \div 2x$

298. $\left(63m^4-42m^3\right) \div \left(-7m^2\right)$

299. $\left(48y^4-24y^3\right) \div \left(-8y^2\right)$

300. $\frac{66x^3y^2-110x^2y^3-44x^4y^3}{11x^2y^2}$

301. $\frac{72r^5s^2+132r^4s^3-96r^3s^5}{12r^2s^2}$

302. $\frac{10x^2+5x-4}{-5x}$

303. $\frac{20y^2+12y-1}{-4y}$

Divide Polynomials using Long Division

In the following exercises, divide each polynomial by the binomial.

304. $\left(y^2+7y+12\right) \div (y+3)$

305. $\left(a^2-2a-35\right) \div (a+5)$

306. $\left(6m^2-19m-20\right) \div (m-4)$

307. $\left(4x^2-17x-15\right) \div (x-5)$

308. $\left(q^2+2q+20\right) \div (q+6)$

309. $\left(p^2+11p+16\right) \div (p+8)$

310. $\left(3b^3+b^2+4\right) \div (b+1)$

311. $\left(2n^3-10n+28\right) \div (n+3)$

312. $\left(z^3+1\right) \div (z+1)$

313. $\left(m^3+1000\right) \div (m+10)$

314. $\left(64x^3-27\right) \div (4x-3)$

315. $\left(125y^3-64\right) \div (5y-4)$

Divide Polynomials using Synthetic Division

In the following exercises, use synthetic Division to find the quotient and remainder.

316. $x^3-6x^2+5x+14$ is divided by $x+1$

317. $x^3-3x^2-4x+12$ is divided by $x+2$

318. $2x^3-11x^2+11x+12$ is divided by $x-3$

319. $2x^3-11x^2+16x-12$ is divided by $x-4$

320. $x^4+13x^2+13x+3$ is divided by $x+3$

321. $x^4+x^2+6x-10$ is divided by $x+2$

322. $2x^4 - 9x^3 + 5x^2 - 3x - 6$ is divided by $x - 4$

323. $3x^4 - 11x^3 + 2x^2 + 10x + 6$ is divided by $x - 3$

Divide Polynomial Functions

In the following exercises, divide.

324. For functions $f(x) = x^2 - 13x + 36$ and $g(x) = x - 4$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-1)$

325. For functions $f(x) = x^2 - 15x + 45$ and $g(x) = x - 9$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-5)$

326. For functions $f(x) = x^3 + x^2 - 7x + 2$ and $g(x) = x - 2$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(2)$

327. For functions $f(x) = x^3 + 2x^2 - 19x + 12$ and $g(x) = x - 3$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(0)$

328. For functions $f(x) = x^2 - 5x + 2$ and $g(x) = x^2 - 3x - 1$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(-1)$

329. For functions $f(x) = x^2 + 4x - 3$ and $g(x) = x^2 + 2x + 4$, find ⓐ $(f \cdot g)(x)$ ⓑ $(f \cdot g)(1)$

Use the Remainder and Factor Theorem

In the following exercises, use the Remainder Theorem to find the remainder.

330. $f(x) = x^3 - 8x + 7$ is divided by $x + 3$

331. $f(x) = x^3 - 4x - 9$ is divided by $x + 2$

332. $f(x) = 2x^3 - 6x - 24$ divided by $x - 3$

333. $f(x) = 7x^2 - 5x - 8$ divided by $x - 1$

In the following exercises, use the Factor Theorem to determine if $x - c$ is a factor of the polynomial function.

334. Determine whether $x + 3$ a factor of $x^3 + 8x^2 + 21x + 18$

335. Determine whether $x + 4$ a factor of $x^3 + x^2 - 14x + 8$

336. Determine whether $x - 2$ a factor of $x^3 - 7x^2 + 7x - 6$

337. Determine whether $x - 3$ a factor of $x^3 - 7x^2 + 11x + 3$

Writing Exercises

338. James divides $48y + 6$ by 6 this way: $\frac{48y + \cancel{6}}{\cancel{6}} = 48y$. What is wrong with his reasoning?

339. Divide $\frac{10x^2 + x - 12}{2x}$ and explain with words how you get each term of the quotient.

340. Explain when you can use synthetic division.

341. In your own words, write the steps for synthetic division for $x^2 + 5x + 6$ divided by $x - 2$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section*

I can...	Confidently	With some help	No-I don't get it!
divide monomials.			
divide a polynomial by a monomial.			
divide polynomials using long division.			
divide polynomials synthetic division.			
divide polynomial functions.			
use the Remainder and Factor Theorem.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

CHAPTER 5 REVIEW

KEY TERMS

binomial A binomial is a polynomial with exactly two terms.

conjugate pair A conjugate pair is two binomials of the form $(a-b)$, $(a+b)$. The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

degree of a constant The degree of any constant is 0.

degree of a polynomial The degree of a polynomial is the highest degree of all its terms.

degree of a term The degree of a term is the sum of the exponents of its variables.

monomial A monomial is an algebraic expression with one term. A monomial in one variable is a term of the form ax^m, where a is a constant and m is a whole number.

polynomial A monomial or two or more monomials combined by addition or subtraction is a polynomial.

polynomial function A polynomial function is a function whose range values are defined by a polynomial.

Power Property According to the Power Property, a to the m to the n equals a to the m times n.

Product Property According to the Product Property, a to the m times a to the n equals a to the m plus n.

Product to a Power According to the Product to a Power Property, a times b in parentheses to the m equals a to the m times b to the m.

Properties of Negative Exponents According to the Properties of Negative Exponents, a to the negative n equals 1 divided by a to the n and 1 divided by a to the negative n equals a to the n.

Quotient Property According to the Quotient Property, a to the m divided by a to the n equals a to the m minus n as long as a is not zero.

Quotient to a Negative Exponent Raising a quotient to a negative exponent occurs when a divided by b in parentheses to the power of negative n equals b divided by a in parentheses to the power of n.

Quotient to a Power Property According to the Quotient to a Power Property, a divided by b in parentheses to the power of m is equal to a to the m divided by b to the m as long as b is not zero.

standard form of a polynomial A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial A trinomial is a polynomial with exactly three terms.

Zero Exponent Property According to the Zero Exponent Property, a to the zero is 1 as long as a is not zero.

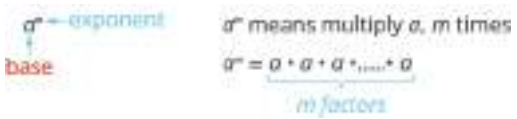
KEY CONCEPTS

5.1 Add and Subtract Polynomials

- **Monomial**
 - A **monomial** is an algebraic expression with one term.
 - A monomial in one variable is a term of the form ax^m, where a is a constant and m is a whole number.
- **Polynomials**
 - **Polynomial**—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
 - **monomial** —A polynomial with exactly one term is called a monomial.
 - **binomial** — A polynomial with exactly two terms is called a binomial.
 - **trinomial** —A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
 - The **degree of a term** is the sum of the exponents of its variables.
 - The **degree of a constant** is 0.
 - The **degree of a polynomial** is the highest degree of all its terms.

5.2 Properties of Exponents and Scientific Notation

- **Exponential Notation**



This is read *a* to the m^{th} power.

In the expression a^m, the *exponent m* tells us how many times we use the *base a* as a factor.

- **Product Property for Exponents**
 If *a* is a real number and *m* and *n* are integers, then

$$a^m \cdot a^n = a^{m+n}$$

 To multiply with like bases, add the exponents.
- **Quotient Property for Exponents**
 If a is a real number, $a \neq 0,$ and *m* and *n* are integers, then

$$\frac{a^m}{a^n} = a^{m-n}, \quad m > n \quad \text{and} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad n > m$$

- **Zero Exponent**
 - If *a* is a non-zero number, then $a^0 = 1.$
 - If *a* is a non-zero number, then *a* to the power of zero equals 1.
 - Any non-zero number raised to the zero power is 1.
- **Negative Exponent**
 - If *n* is an integer and $a \neq 0,$ then $a^{-n} = \frac{1}{a^n}$ or $\frac{1}{a^{-n}} = a^n.$
- **Quotient to a Negative Exponent Property**
 If a, b are real numbers, $a \neq 0, b \neq 0$ and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

- **Power Property for Exponents**
 If a is a real number and m, n are integers, then

$$(a^m)^n = a^{m \cdot n}$$

 To raise a power to a power, multiply the exponents.
- **Product to a Power Property for Exponents**
 If *a* and *b* are real numbers and *m* is a whole number, then

$$(ab)^m = a^m b^m$$

 To raise a product to a power, raise each factor to that power.
- **Quotient to a Power Property for Exponents**
 If a and are real numbers, $b \neq 0,$ and m is an integer, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

 To raise a fraction to a power, raise the numerator and denominator to that power.
- **Summary of Exponent Properties**
 If *a* and *b* are real numbers, and *m* and *n* are integers, then

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n},\ a \neq 0$
Zero Exponent Property	$a^0 = 1,\ a \neq 0$
Quotient to a Power Property:	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$
Properties of Negative Exponents	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

- **Scientific Notation**
 A number is expressed in scientific notation when it is of the form

$$a \times 10^n \text{ where } 1 \leq a < 10 \text{ and } n \text{ is an integer.}$$

- **How to convert a decimal to scientific notation.**

 Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.

 Step 2. Count the number of decimal places, n, that the decimal point was moved.

 Step 3. Write the number as a product with a power of 10. If the original number is.
 - greater than 1, the power of 10 will be 10^n.
 - between 0 and 1, the power of 10 will be 10^{-n}.

 Step 4. Check.

- **How to convert scientific notation to decimal form.**

 Step 1. Determine the exponent, n, on the factor 10.

 Step 2. Move the decimal n places, adding zeros if needed.
 - If the exponent is positive, move the decimal point n places to the right.
 - If the exponent is negative, move the decimal point $|n|$ places to the left.

 Step 3. Check.

5.3 Multiply Polynomials

- **How to use the FOIL method to multiply two binomials.**

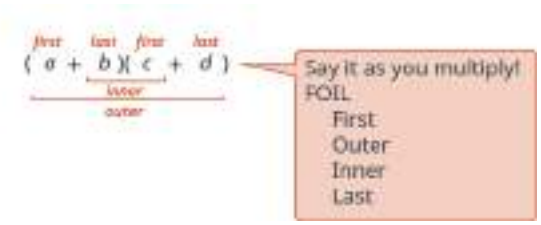
Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

Step 5. Combine like terms, when possible.



- **Multiplying Two Binomials:** To multiply binomials, use the:

- Distributive Property
- FOIL Method

- **Multiplying a Polynomial by a Polynomial:** To multiply a trinomial by a binomial, use the:
 - Distributive Property
 - Vertical Method
- **Binomial Squares Pattern**
 If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

- **Product of Conjugates Pattern**
 If a, b are real numbers

$$(a-b)(a+b) = a^2 - b^2 \qquad \underbrace{(a-b)(a+b)}_{\text{conjugates}} = \underbrace{a^2}_{\text{squares}} - \underbrace{b^2}_{\text{squares}} \quad (\text{difference})$$

The product is called a difference of squares.
To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

- **Comparing the Special Product Patterns**

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)^2 = a^2 - 2ab + b^2$
$(a-b)(a+b) = a^2 - b^2$	
• Squaring a binomial	• Multiplying conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

- **Multiplication of Polynomial Functions:**
 - For functions $f(x)$ and $g(x)$,

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

5.4 Dividing Polynomials

- **Division of a Polynomial by a Monomial**
 - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.
- **Division of Polynomial Functions**
 - For functions $f(x)$ and $g(x)$, where $g(x) \neq 0$,

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

- **Remainder Theorem**
 - If the polynomial function $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.
- **Factor Theorem:** For any polynomial function $f(x)$,
 - if $x-c$ is a factor of $f(x)$, then $f(c)=0$
 - if $f(c)=0$, then $x-c$ is a factor of $f(x)$

REVIEW EXERCISES

5.1 Add and Subtract Polynomials

Determine the Degree of Polynomials

In the following exercises, determine the type of polynomial.

342. $16x^2-40x-25$

343. $5m+9$

344. -15

345. $y^2+6y^3+9y^4$

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

346. $4p+11p$

347. $-8y^3-5y^3$

348. $\left(4a^2+9a-11\right)+\left(6a^2-5a+10\right)$

349. $\left(8m^2+12m-5\right)-\left(2m^2-7m-1\right)$

350. $\left(y^2-3y+12\right)+\left(5y^2-9\right)$

351. $\left(5u^2+8u\right)-(4u-7)$

352. Find the sum of $8q^3-27$ and q^2+6q-2.

353. Find the difference of x^2+6x+8 and $x^2-8x+15$.

In the following exercises, simplify.

354. $17mn^2-(-9mn^2)+3mn^2$

355. $18a-7b-21a$

356. $2pq^2-5p-3q^2$

357. $(6a^2+7)+(2a^2-5a-9)$

358. $(3p^2-4p-9)+(5p^2+14)$

359. $(7m^2-2m-5)-(4m^2+m-8)$

360. $(7b^2-4b+3)-(8b^2-5b-7)$

361. Subtract $\left(8y^2-y+9\right)$ from $\left(11y^2-9y-5\right)$

362. Find the difference of $\left(z^2-4z-12\right)$ and $\left(3z^2+2z-11\right)$

363. $\left(x^3-x^2y\right)-\left(4xy^2-y^3\right)+\left(3x^2y-xy^2\right)$

364. $\left(x^3-2x^2y\right)-\left(xy^2-3y^3\right)-\left(x^2y-4xy^2\right)$

Evaluate a Polynomial Function for a Given Value of the Variable

In the following exercises, find the function values for each polynomial function.

365. For the function $f(x) = 7x^2 - 3x + 5$ find:

ⓐ $f(5)$ ⓑ $f(-2)$ ⓒ $f(0)$

366. For the function $g(x) = 15 - 16x^2$, find:

ⓐ $g(-1)$ ⓑ $g(0)$ ⓒ $g(2)$

367. A pair of glasses is dropped off a bridge 640 feet above a river. The polynomial function $h(t) = -16t^2 + 640$ gives the height of the glasses t seconds after they were dropped. Find the height of the glasses when $t = 6$.

368. A manufacturer of the latest soccer shoes has found that the revenue received from selling the shoes at a cost of p dollars each is given by the polynomial $R(p) = -5p^2 + 360p$. Find the revenue received when $p = 110$ dollars.

Add and Subtract Polynomial Functions

In the following exercises, find ⓐ (f + g)(x) ⓑ (f + g)(3) ⓒ (f − g)(x) ⓓ (f − g)(−2)

369. $f(x) = 2x^2 - 4x - 7$ and $g(x) = 2x^2 - x + 5$

370. $f(x) = 4x^3 - 3x^2 + x - 1$ and $g(x) = 8x^3 - 1$

5.2 Properties of Exponents and Scientific Notation

Simplify Expressions Using the Properties for Exponents

In the following exercises, simplify each expression using the properties for exponents.

371. $p^3 \cdot p^{10}$

372. $2 \cdot 2^6$

373. $a \cdot a^2 \cdot a^3$

374. $x \cdot x^8$

375. $y^a \cdot y^b$

376. $\frac{2^8}{2^2}$

377. $\frac{a^6}{a}$

378. $\frac{n^3}{n^{12}}$

379. $\frac{1}{x^5}$

380. 3^0

381. y^0

382. $(14t)^0$

383. $12a^0 - 15b^0$

Use the Definition of a Negative Exponent

In the following exercises, simplify each expression.

384. 6^{-2}

385. $(-10)^{-3}$

386. $5 \cdot 2^{-4}$

387. $(8n)^{-1}$

388. y^{-5}

389. 10^{-3}

390. $\frac{1}{a^{-4}}$

391. $\frac{1}{6^{-2}}$

392. -5^{-3}

393. $\left(-\frac{1}{5}\right)^{-3}$

394. $-\left(\frac{1}{2}\right)^{-3}$

395. $(-5)^{-3}$

396. $\left(\frac{5}{9}\right)^{-2}$

397. $\left(-\frac{3}{x}\right)^{-3}$

In the following exercises, simplify each expression using the Product Property.

398. $\left(y^4\right)^3$

399. $\left(3^2\right)^5$

400. $\left(a^{10}\right)^y$

401. $x^{-3} \cdot x^9$

402. $r^{-5} \cdot r^{-4}$

403. $\left(uv^{-3}\right)\left(u^{-4}v^{-2}\right)$

404. $\left(m^5\right)^{-1}$

405. $p^5 \cdot p^{-2} \cdot p^{-4}$

In the following exercises, simplify each expression using the Power Property.

406. $(k^{-2})^{-3}$

407. $\frac{q^4}{q^{20}}$

408. $\frac{b^8}{b^{-2}}$

409. $\frac{n^{-3}}{n^{-5}}$

In the following exercises, simplify each expression using the Product to a Power Property.

410. $(-5ab)^3$

411. $(-4pq)^0$

412. $\left(-6x^3\right)^{-2}$

413. $\left(3y^{-4}\right)^2$

In the following exercises, simplify each expression using the Quotient to a Power Property.

414. $\left(\frac{3}{5x}\right)^{-2}$

415. $\left(\frac{3xy^2}{z}\right)^4$

416. $\left(\frac{4p^{-3}}{q^2}\right)^2$

In the following exercises, simplify each expression by applying several properties.

417. $(x^2y)^2(3xy^5)^3$

418. $\frac{\left(-3a^{-2}\right)^4\left(2a^4\right)^2}{\left(-6a^2\right)^3}$

419. $\left(\frac{3xy^3}{4x^4y^{-2}}\right)^2\left(\frac{6xy^4}{8x^3y^{-2}}\right)^{-1}$

In the following exercises, write each number in scientific notation.

420. 2.568

421. 5,300,000

422. 0.00814

In the following exercises, convert each number to decimal form.

423. 2.9×10^4

424. 3.75×10^{-1}

425. 9.413×10^{-5}

In the following exercises, multiply or divide as indicated. Write your answer in decimal form.

426. $\left(3 \times 10^7\right)\left(2 \times 10^{-4}\right)$

427. $\left(1.5 \times 10^{-3}\right)\left(4.8 \times 10^{-1}\right)$

428. $\frac{6 \times 10^9}{2 \times 10^{-1}}$

429. $\frac{9 \times 10^{-3}}{1 \times 10^{-6}}$

5.3 Multiply Polynomials

Multiply Monomials

In the following exercises, multiply the monomials.

430. $(-6p^4)(9p)$

431. $\left(\frac{1}{3}c^2\right)(30c^8)$

432. $\left(8x^2y^5\right)\left(7xy^6\right)$

433. $\left(\frac{2}{3}m^3n^6\right)\left(\frac{1}{6}m^4n^4\right)$

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

434. $7(10-x)$

435. $a^2\left(a^2-9a-36\right)$

436. $-5y\left(125y^3-1\right)$

437. $(4n-5)\left(2n^3\right)$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using:

ⓐ the Distributive Property ⓑ the FOIL method ⓒ the Vertical Method.

438. $(a+5)(a+2)$

439. $(y-4)(y+12)$

440. $(3x+1)(2x-7)$

441. $(6p-11)(3p-10)$

In the following exercises, multiply the binomials. Use any method.

442. $(n+8)(n+1)$

443. $(k+6)(k-9)$

444. $(5u-3)(u+8)$

445. $(2y-9)(5y-7)$

446. $(p+4)(p+7)$

447. $(x-8)(x+9)$

448. $(3c+1)(9c-4)$

449. $(10a-1)(3a-3)$

Multiply a Polynomial by a Polynomial

In the following exercises, multiply using ⓐ the Distributive Property ⓑ the Vertical Method.

450. $(x+1)\left(x^2-3x-21\right)$

451. $(5b-2)\left(3b^2+b-9\right)$

In the following exercises, multiply. Use either method.

452. $(m+6)\left(m^2-7m-30\right)$

453. $(4y-1)\left(6y^2-12y+5\right)$

Multiply Special Products

In the following exercises, square each binomial using the Binomial Squares Pattern.

454. $(2x-y)^2$

455. $\left(x+\frac{3}{4}\right)^2$

456. $(8p^3-3)^2$

457. $(5p+7q)^2$

In the following exercises, multiply each pair of conjugates using the Product of Conjugates.

458. $(3y+5)(3y-5)$

459. $(6x+y)(6x-y)$

460. $\left(a+\frac{2}{3}b\right)\left(a-\frac{2}{3}b\right)$

461. $(12x^3-7y^2)(12x^3+7y^2)$

462. $(13a^2-8b^4)(13a^2+8b^4)$

5.4 Divide Monomials

Divide Monomials

In the following exercises, divide the monomials.

463. $72p^{12} \div 8p^3$

464. $-26a^8 \div \left(2a^2\right)$

465. $\frac{45y^6}{-15y^{10}}$

466. $\frac{-30x^8}{-36x^9}$

467. $\frac{28a^9b}{7a^4b^3}$

468. $\frac{11u^6v^3}{55u^2v^8}$

469. $\frac{\left(5m^9n^3\right)\left(8m^3n^2\right)}{\left(10mn^4\right)\left(m^2n^5\right)}$

470. $\frac{(42r^2s^4)(54rs^2)}{(6rs^3)(9s)}$

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial

471. $\left(54y^4-24y^3\right) \div \left(-6y^2\right)$

472. $\frac{63x^3y^2-99x^2y^3-45x^4y^3}{9x^2y^2}$

473. $\frac{12x^2+4x-3}{-4x}$

Divide Polynomials using Long Division

In the following exercises, divide each polynomial by the binomial.

474. $\left(4x^2-21x-18\right) \div (x-6)$

475. $\left(y^2+2y+18\right) \div (y+5)$

476. $\left(n^3-2n^2-6n+27\right) \div (n+3)$

477. $\left(a^3-1\right) \div (a+1)$

Divide Polynomials using Synthetic Division

In the following exercises, use synthetic Division to find the quotient and remainder.

478. $x^3-3x^2-4x+12$ is divided by $x+2$

479. $2x^3-11x^2+11x+12$ is divided by $x-3$

480. $x^4+x^2+6x-10$ is divided by $x+2$

Divide Polynomial Functions

In the following exercises, divide.

481. For functions $f(x)=x^2-15x+45$ and $g(x)=x-9$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-2)$

482. For functions $f(x)=x^3+x^2-7x+2$ and $g(x)=x-2$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(3)$

Use the Remainder and Factor Theorem

In the following exercises, use the Remainder Theorem to find the remainder.

483. $f(x) = x^3 - 4x - 9$ is divided by $x + 2$

484. $f(x) = 2x^3 - 6x - 24$ divided by $x - 3$

In the following exercises, use the Factor Theorem to determine if $x - c$ is a factor of the polynomial function.

485. Determine whether $x - 2$ is a factor of $x^3 - 7x^2 + 7x - 6$.

486. Determine whether $x - 3$ is a factor of $x^3 - 7x^2 + 11x + 3$.

PRACTICE TEST

487. For the polynomial $8y^4 - 3y^2 + 1$

ⓐ Is it a monomial, binomial, or trinomial? ⓑ What is its degree?

488. $(5a^2 + 2a - 12)(9a^2 + 8a - 4)$

489. $(10x^2 - 3x + 5) - (4x^2 - 6)$

490. $\left(-\frac{3}{4}\right)^3$

491. $x^{-3}x^4$

492. $\frac{5^6}{5^8}$

493. $(47a^{18}b^{23}c^5)^0$

494. 4^{-1}

495. $(2y)^{-3}$

496. $p^{-3} \cdot p^{-8}$

497. $\frac{x^4}{x^{-5}}$

498. $(3x^{-3})^2$

499. $\frac{24r^3s}{6r^2s^7}$

500. $\left(\frac{x^4y^9}{x^{-3}}\right)^2$

501. $(8xy^3)(-6x^4y^6)$

502. $4u(u^2 - 9u + 1)$

503. $(m + 3)(7m - 2)$

504. $(n - 8)(n^2 - 4n + 11)$

505. $(4x - 3)^2$

506. $(5x + 2y)(5x - 2y)$

507. $(15xy^3 - 35x^2y) \div 5xy$

508. $(3x^3 - 10x^2 + 7x + 10) \div (3x + 2)$

509. Use the Factor Theorem to determine if $x + 3$ a factor of $x^3 + 8x^2 + 21x + 18$.

510. ⓐ Convert 112,000 to scientific notation. ⓑ Convert 5.25×10^{-4} to decimal form.

In the following exercises, simplify and write your answer in decimal form.

511. $(2.4 \times 10^8)(2 \times 10^{-5})$

512. $\frac{9 \times 10^4}{3 \times 10^{-1}}$

513. For the function $f(x) = 6x^2 - 3x - 9$ find:

ⓐ $f(3)$ ⓑ $f(-2)$ ⓒ $f(0)$

514. For $f(x) = 2x^2 - 3x - 5$ and $g(x) = 3x^2 - 4x + 1$, find

ⓐ $(f + g)(x)$ ⓑ $(f + g)(1)$

ⓒ $(f - g)(x)$ ⓓ $(f - g)(-2)$

515. For functions $f(x) = 3x^2 - 23x - 36$ and $g(x) = x - 9$, find

ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(3)$

516. A hiker drops a pebble from a bridge 240 feet above a canyon. The function $h(t) = -16t^2 + 240$ gives the height of the pebble t seconds after it was dropped. Find the height when $t = 3$.



6 FACTORING

Figure 6.1 Scientists use factoring to calculate growth rates of infectious diseases such as viruses. (credit: "FotoshopTofs" / Pixabay)

Chapter Outline

6.1 Greatest Common Factor and Factor by Grouping
6.2 Factor Trinomials
6.3 Factor Special Products
6.4 General Strategy for Factoring Polynomials
6.5 Polynomial Equations

Introduction

An epidemic of a disease has broken out. Where did it start? How is it spreading? What can be done to control it? Answers to these and other questions can be found by scientists known as epidemiologists. They collect data and analyze it to study disease and consider possible control measures. Because diseases can spread at alarming rates, these scientists must use their knowledge of mathematics involving factoring. In this chapter, you will learn how to factor and apply factoring to real-life situations.

6.1 Greatest Common Factor and Factor by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

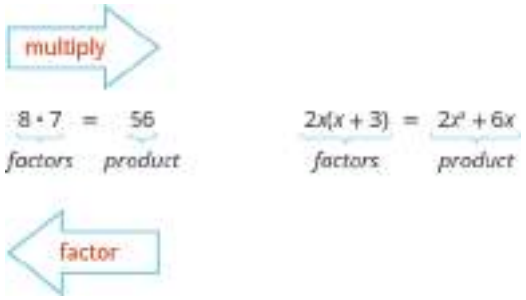
Be Prepared!

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
 If you missed this problem, review Example 1.2.
2. Find the least common multiple (LCM) of 18 and 24.
 If you missed this problem, review Example 1.3.
3. Multiply: $-3a(7a+8b)$.
 If you missed this problem, review Example 5.26.

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

Greatest Common Factor

The **greatest common factor** (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

HOW TO : : FIND THE GREATEST COMMON FACTOR (GCF) OF TWO EXPRESSIONS.

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.
Step 3. Bring down the common factors that all expressions share.
Step 4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

EXAMPLE 6.1

Find the greatest common factor of $21x^3, 9x^2, 15x$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$ $GCF = 3 \cdot x$
Multiply the factors.	$GCF = 3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

TRY IT : : 6.1 Find the greatest common factor: $25m^4, 35m^3, 20m^2$.

TRY IT : : 6.2 Find the greatest common factor: $14x^3, 70x^2, 105x$.

Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with

its factors, $3x(x+5)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

Distributive Property

If *a*, *b*, and *c* are real numbers, then

$$a(b+c)=ab+ac \quad \text{and} \quad ab+ac=a(b+c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

EXAMPLE 6.2 HOW TO USE THE DISTRIBUTIVE PROPERTY TO FACTOR A POLYNOMIAL

Factor: $8m^3-12m^2n+20mn^2$.

Solution

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $8m^3$, $12m^2n$, $20mn^2$	$8m^3=2\cdot2\cdot2\cdot m\cdot m\cdot m$ $12m^2n=2\cdot2\cdot3\cdot m\cdot m\cdot n$ $20mn^2=2\cdot2\cdot5\cdot m\cdot n\cdot n$ $GCF=2\cdot2\cdot m$ $GCF=4m$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $8m^3$, $12m^2n$, $20mn^2$ as products of their GCF, $4m$. $8m^3=4m\cdot2m^2$ $12m^2n=4m\cdot3mn$ $20mn^2=4m\cdot5n^2$	$8m^3-12m^2n+20mn^2$ $4m\cdot2m^2-4m\cdot3mn+4m\cdot5n^2$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4m(2m^2-3mn+5n^2)$
Step 4. Check by multiplying the factors.		$4m(2m^2-3mn+5n^2)$ $4m\cdot2m^2-4m\cdot3mn+4m\cdot5n^2$ $8m^3-12m^2n+20mn^2$ ✓

TRY IT : : 6.3 Factor: $9xy^2+6x^2y^2+21y^3$.

TRY IT : : 6.4 Factor: $3p^3-6p^2q+9pq^3$.

HOW TO : : FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL.

Step 1. Find the GCF of all the terms of the polynomial.
Step 2. Rewrite each term as a product using the GCF.
Step 3. Use the "reverse" Distributive Property to factor the expression.
Step 4. Check by multiplying the factors.

Factor as a Noun and a Verb

We use "factor" as both a noun and a verb:

Noun:	7 is a *factor* of 14
Verb:	*factor* 3 from $3a + 3$

EXAMPLE 6.3

Factor: $5x^3 - 25x^2$.

Solution

Find the GCF of $5x^3$ and $25x^2$.	$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $GCF = 5 \cdot x \cdot x$
	$GCF = 5x^2$
	$5x^2 - 25x^2$
Rewrite each term.	$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$5x^2(x - 5)$

Check:

$$5x^2(x - 5)$$
$$5x^2 \cdot x - 5x^2 \cdot 5$$
$$5x^3 - 25x^2 \checkmark$$

TRY IT : : 6.5 Factor: $2x^3 + 12x^2$.

TRY IT : : 6.6 Factor: $6y^3 - 15y^2$.

EXAMPLE 6.4

Factor: $8x^3y - 10x^2y^2 + 12xy^3$.

Solution

The GCF of $8x^3y$, $-10x^2y^2$, and $12xy^3$ is $2xy$.

$8x^3y = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y$
$10x^2y^2 = 2 \cdot 5 \cdot x \cdot x \cdot y \cdot y$
$12xy^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$
$GCF = 2 \cdot x \cdot y$

$GCF = 2xy$

$8x^3y - 10x^2y^2 + 12xy^3$

Rewrite each term using the GCF, $2xy$.

$2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$

Factor the GCF.

$2xy(4x^2 - 5xy + 6y^2)$

Check:

$$2xy(4x^2 - 5xy + 6y^2)$$
$$2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$$
$$8x^3y - 10x^2y^2 + 12xy^3 \checkmark$$

TRY IT : : 6.7 Factor: $15x^3y - 3x^2y^2 + 6xy^3$.

TRY IT : : 6.8 Factor: $8a^3b + 2a^2b^2 - 6ab^3$.

When the leading coefficient is negative, we factor the negative out as part of the GCF.

EXAMPLE 6.5

Factor: $-4a^3 + 36a^2 - 8a$.

Solution

The leading coefficient is negative, so the GCF will be negative.

$-4a^3 + 36a^2 - 8a$

Rewrite each term using the GCF, $-4a$.

$-4a \cdot a^2 - (-4a) \cdot 9a + (-4a) \cdot 2$

Factor the GCF.

$-4a(a^2 - 9a + 2)$

Check:

$$-4a(a^2 - 9a + 2)$$
$$-4a \cdot a^2 - (-4a) \cdot 9a + (-4a) \cdot 2$$
$$-4a^3 + 36a^2 - 8a \checkmark$$

TRY IT : : 6.9 Factor: $-4b^3 + 16b^2 - 8b$.

TRY IT : : 6.10 Factor: $-7a^3 + 21a^2 - 14a$.

So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

EXAMPLE 6.6

Factor: $3y(y+7)-4(y+7)$.

Solution

The GCF is the binomial $y+7$.

	$3y(y+7)-4(y+7)$
Factor the GCF, $(y+7)$.	$(y+7)(3y-4)$
Check on your own by multiplying.	

TRY IT : : 6.11 Factor: $4m(m+3)-7(m+3)$.

TRY IT : : 6.12 Factor: $8n(n-4)+5(n-4)$.

Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.

EXAMPLE 6.7 HOW TO FACTOR A POLYNOMIAL BY GROUPING

Factor by grouping: $xy+3y+2x+6$.

Solution

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms?	$xy+3y+2x+6$
	No, so let's separate the first two terms from the second two.	$xy+3y+2x+6$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms.	$y(x+3)+2x+6$
	Factor the GCF from the second two terms.	$y(x+3)+2(x+3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x+3)$.	$y(x+3)+2(x+3)$
	Factor out the common factor.	$(x+3)(y+2)$
Step 4. Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $xy+2x+3y+6$ $xy+3y+2x+6$ ✓

TRY IT : : 6.13 Factor by grouping: $xy+8y+3x+24$.

TRY IT : : 6.14 Factor by grouping: $ab+7b+8a+56$.

HOW TO : : FACTOR BY GROUPING.

Step 1. Group terms with common factors.
Step 2. Factor out the common factor in each group.
Step 3. Factor the common factor from the expression.
Step 4. Check by multiplying the factors.

EXAMPLE 6.8

Factor by grouping: ⓐ $x^2+3x-2x-6$ ⓑ $6x^2-3x-4x+2$.

Solution

ⓐ

There is no GCF in all four terms.	$x^2+3x-2x-6$
Separate into two parts.	$x^2+3x \quad -2x-6$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x+3)-2(x+3)$
Factor out the common factor.	$(x+3)(x-2)$
Check on your own by multiplying.	

ⓑ

There is no GCF in all four terms.	$6x^2-3x-4x+2$
Separate into two parts.	$6x^2-3x \quad -4x+2$
Factor the GCF from both parts.	$3x(2x-1)-2(2x-1)$
Factor out the common factor.	$(2x-1)(3x-2)$
Check on your own by multiplying.	

TRY IT : : 6.15 Factor by grouping: ⓐ $x^2+2x-5x-10$ ⓑ $20x^2-16x-15x+12$.

TRY IT : : 6.16 Factor by grouping: ⓐ $y^2+4y-7y-28$ ⓑ $42m^2-18m-35m+15$.

6.1 EXERCISES

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. $10p^3q,\ 12pq^2$

2. $8a^2b^3,\ 10ab^2$

3. $12m^2n^3,\ 30m^5n^3$

4. $28x^2y^4,\ 42x^4y^4$

5. $10a^3,\ 12a^2,\ 14a$

6. $20y^3,\ 28y^2,\ 40y$

7. $35x^3y^2,\ 10x^4y,\ 5x^5y^3$

8. $27p^2q^3,\ 45p^3q^4,\ 9p^4q^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

9. $6m+9$

10. $14p+35$

11. $9n-63$

12. $45b-18$

13. $3x^2+6x-9$

14. $4y^2+8y-4$

15. $8p^2+4p+2$

16. $10q^2+14q+20$

17. $8y^3+16y^2$

18. $12x^3-10x$

19. $5x^3-15x^2+20x$

20. $8m^2-40m+16$

21. $24x^3-12x^2+15x$

22. $24y^3-18y^2-30y$

23. $12xy^2+18x^2y^2-30y^3$

24. $21pq^2+35p^2q^2-28q^3$

25. $20x^3y-4x^2y^2+12xy^3$

26. $24a^3b+6a^2b^2-18ab^3$

27. $-2x-4$

28. $-3b+12$

29. $-2x^3+18x^2-8x$

30. $-5y^3+35y^2-15y$

31. $-4p^3q-12p^2q^2+16pq^2$

32. $-6a^3b-12a^2b^2+18ab^2$

33. $5x(x+1)+3(x+1)$

34. $2x(x-1)+9(x-1)$

35. $3b(b-2)-13(b-2)$

36. $6m(m-5)-7(m-5)$

Factor by Grouping

In the following exercises, factor by grouping.

37. $ab+5a+3b+15$

38. $cd+6c+4d+24$

39. $8y^2+y+40y+5$

40. $6y^2+7y+24y+28$

41. $uv-9u+2v-18$

42. $pq-10p+8q-80$

43. $u^2-u+6u-6$

44. $x^2-x+4x-4$

45. $9p^2-3p-20$

46. $16q^2-8q-35$

47. $mn-6m-4n+24$

48. $r^2-3r-r+3$

49. $2x^2-14x-5x+35$

50. $4x^2-36x-3x+27$

Mixed Practice

In the following exercises, factor.

51. $-18xy^2 - 27x^2y$

52. $-4x^3y^5 - x^2y^3 + 12xy^4$

53. $3x^3 - 7x^2 + 6x - 14$

54. $x^3 + x^2 - x - 1$

55. $x^2 + xy + 5x + 5y$

56. $5x^3 - 3x^2 + 5x - 3$

Writing Exercises

57. What does it mean to say a polynomial is in factored form?

58. How do you check result after factoring a polynomial?

59. The greatest common factor of 36 and 60 is 12. Explain what this means.

60. What is the GCF of y^4, y^5, and y^{10}? Write a general rule that tells you how to find the GCF of y^a, y^b, and y^c.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!*

...with some help. *This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.*

6.2 Factor Trinomials

Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form x^2+bx+c
- Factor trinomials of the form ax^2+bx+c using trial and error
- Factor trinomials of the form ax^2+bx+c using the 'ac' method
- Factor using substitution

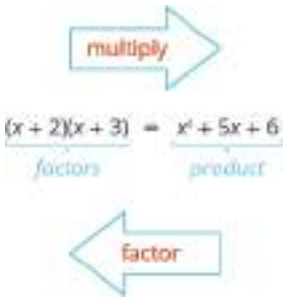
Be Prepared!

Before you get started, take this readiness quiz.

1. Find all the factors of 72.
 If you missed this problem, review **Example 1.2**.
2. Find the product: $(3y+4)(2y+5)$.
 If you missed this problem, review **Example 5.28**.
3. Simplify: $-9(6);\ -9(-6)$.
 If you missed this problem, review **Example 1.18**.

Factor Trinomials of the Form x^2+bx+c

You have already learned how to multiply binomials using FOIL. Now you'll need to "undo" this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a trinomial of the form $x^2+bx+c,$ such as x^2+5x+6 and factor it to $(x+2)(x+3),$ let's start with two general binomials of the form $(x+m)$ and $(x+n)$.

	$(x+m)(x+n)$
Foil to find the product.	$x^2+mx+nx+mn$
Factor the GCF from the middle terms.	$x^2+(m+n)x+mn$
Our trinomial is of the form x^2+bx+c.	$\frac{x^2+\quad bx\quad +\ c}{x^2+(m+n)x+mn}$

This tells us that to factor a trinomial of the form $x^2+bx+c,$ we need two factors $(x+m)$ and $(x+n)$ where the two numbers *m* and *n* multiply to *c* and add to *b*.

EXAMPLE 6.9 HOW TO FACTOR A TRINOMIAL OF THE FORM $x^2 + bx + c$

Factor: $x^2 + 11x + 24$.

Solution

Step 1. Write the factors as two binomials with first terms x.	Write two sets of parentheses and put x as the first term.	$x^2 + 11x + 24$ $(x \quad)(x \quad)$
Step 2. Find two numbers m and n that multiply to c, $m \cdot n = c$ add to b, $m + n = b$	Find two numbers that multiply to 24 and add to 11. Factors of 24: 1, 24 / 2, 12 / 3, 8 / 4, 6 Sum of factors: $1 + 24 = 25$ / $2 + 12 = 14$ / $3 + 8 = 11$* / $4 + 6 = 10$	
Step 3. Use m and n as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$(x + 3)(x + 8)$
Step 4. Check by multiplying the factors.		$(x + 3)(x + 8)$ $x^2 + 8x + 3x + 24$ $x^2 + 11x + 24$ ✓

TRY IT : : 6.17 Factor: $q^2 + 10q + 24$.

TRY IT : : 6.18 Factor: $t^2 + 14t + 24$.

Let's summarize the steps we used to find the factors.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $x^2 + bx + c$.

Step 1. Write the factors as two binomials with first terms x. $x^2 + bx + c$ $(x \quad)(x \quad)$

Step 2. Find two numbers m and n that
- multiply to c, $m \cdot n = c$
- add to b, $m + n = b$

Step 3. Use m and n as the last terms of the factors. $(x + m)(x + n)$

Step 4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

EXAMPLE 6.10

Factor: $y^2 - 11y + 28$.

Solution

Again, with the positive last term, 28, and the negative middle term, $-11y$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

	$y^2 - 11y + 28$
Write the factors as two binomials with fir t terms y.	$(y \quad)(y \quad)$
Find two numbers that: multiply to 28 and add to -11.	

Factors of 28	Sum of factors
$-1, -28$	$-1 + (-28) = -29$
$-2, -14$	$-2 + (-14) = -16$
$-4, -7$	$-4 + (-7) = -11^*$

Use $-4, -7$ as the last terms of the binomials.	$(y - 4)(y - 7)$

Check:

$$(y - 4)(y - 7)$$
$$y^2 - 7y - 4y + 28$$
$$y^2 - 11y + 28 \checkmark$$

TRY IT : : 6.19 Factor: $u^2 - 9u + 18$.

TRY IT : : 6.20 Factor: $y^2 - 16y + 63$.

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

EXAMPLE 6.11

Factor: $2x + x^2 - 48$.

Solution

	$2x + x^2 - 48$
First we put the terms in decreasing degree order.	$x^2 + 2x - 48$
Factors will be two binomials with first terms x.	$(x \quad)(x \quad)$

Factors of -48	Sum of factors
$-1, 48$	$-1 + 48 = 47$
$-2, 24$	$-2 + 24 = 22$
$-3, 16$	$-3 + 16 = 13$
$-4, 12$	$-4 + 12 = 8$
$-6, 8$	$-6 + 8 = 2$*

Use $-6, 8$ as the last terms of the binomials.	$(x - 6)(x + 8)$
Check:	
$(x - 6)(x + 8)$	
$x^2 - 6q + 8q - 48$	
$x^2 + 2x - 48$ ✓	

TRY IT : : 6.21 Factor: $9m + m^2 + 18$.

TRY IT : : 6.22 Factor: $-7n + 12 + n^2$.

Sometimes you'll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain *y*. To get the coefficients *b* and *c*, you use the same process summarized in **How To Factor trinomials**.

EXAMPLE 6.12

Factor: $r^2 - 8rs - 9s^2$.

Solution

We need *r* in the first term of each binomial and *s* in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

	$r^2 - 8rs - 9s^2$
Note that the first terms are r, last terms contain s.	$(r \quad s)(r \quad s)$
Find the numbers that multiply to -9 and add to -8.	

Factors of -9	Sum of factors
$1, -9$	$-1+9=8$
$-1, 9$	$1+(-9)=-8$*
$3, -3$	$3+(-3)=0$

Use $1, -9$ as coefficients of he last terms. $(r+s)(r-9s)$

Check:

$$(r-9s)(r+s)$$
$$r^2+rs-9rs-9s^2$$
$$r^2-8rs-9s^2 \checkmark$$

TRY IT : : 6.23 Factor: $a^2-11ab+10b^2$.

TRY IT : : 6.24 Factor: $m^2-13mn+12n^2$.

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

EXAMPLE 6.13

Factor: $u^2-9uv-12v^2$.

Solution

We need u in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$u^2-9uv-12v^2$$

Note that the fir t terms are u, last terms contain v. $(u \quad v)(u \quad v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	Sum of factors
$1, -12$	$1+(-12)=-11$
$-1, 12$	$-1+12=11$
$2, -6$	$2+(-6)=-4$
$-2, 6$	$-2+6=4$
$3, -4$	$3+(-4)=-1$
$-3, 4$	$-3+4=1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

TRY IT : : 6.25 Factor: $x^2-7xy-10y^2$.

TRY IT : : 6.26 Factor: $p^2+15pq+20q^2$.

Let's summarize the method we just developed to factor trinomials of the form x^2+bx+c.

Strategy for Factoring Trinomials of the Form x^2+bx+c

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$x^2+bx+c$$
$$(x+m)(x+n)$$

When c is positive, m and n have the same sign.

b positive	b negative
$m,\ n$ positive	$m,\ n$ negative
x^2+5x+6	x^2-6x+8
$(x+2)(x+3)$	$(x-4)(x-2)$
same signs	same signs

When c is negative, m and n have opposite signs.

x^2+x-12	$x^2-2x-15$
$(x+4)(x-3)$	$(x-5)(x+3)$
opposite signs	opposite signs

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

Factor Trinomials of the form ax^2+bx+c using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form ax^2+bx+c.

Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.

EXAMPLE 6.14

Factor completely: $4x^3+16x^2-20x$.

Solution

Is there a greatest common factor?
Yes, GCF $=4x$. Factor it. $4x^3+16x^2-20x$
$4x\left(x^2+4x-5\right)$

Binomial, trinomial, or more than three terms?
It is a trinomial. So "undo FOIL." $4x(x\quad)(x\quad)$

Use a table like the one shown to find t o numbers that multiply to -5 and add to 4. $4x(x-1)(x+5)$

Factors of -5	Sum of factors
$-1,\ 5$	$-1+5=4$*
$1,\ -5$	$1+(-5)=-4$

Check:

$$4x(x-1)(x+5)$$
$$4x(x^2+5x-x-5)$$
$$4x(x^2+4x-5)$$
$$4x^3+16x^2-20x\ \checkmark$$

TRY IT : : 6.27 Factor completely: $5x^3 + 15x^2 - 20x$.

TRY IT : : 6.28 Factor completely: $6y^3 + 18y^2 - 60y$.

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2 + 5x + 2$.

From our earlier work, we expect this will factor into two binomials.

$$3x^2 + 5x + 2$$
$$(\quad)(\quad)$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x$, $3x$. We can place them in the binomials.

$$3x^2 + 5x + 2$$
$$(x \quad)(3x \quad)$$

Check: Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1, 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1.

$$3x^2 + 5x + 2 \quad (x+1)(3x+2) \quad \text{or} \quad 3x^2 + 5x + 2 \quad (x+2)(3x+1)$$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$$(x+1)(3x+2) \quad 3x + 2x = 5x \qquad \text{or} \qquad (x+2)(3x+1) \quad 6x + 1x = 7x$$

Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's use FOIL to check.

$$(x+1)(3x+2)$$
$$3x^2 + 2x + 3x + 2$$
$$3x^2 + 5x + 2 \checkmark$$

Our result of the factoring is:

$$3x^2 + 5x + 2$$
$$(x+1)(3x+2)$$

EXAMPLE 6.15 HOW TO FACTOR A TRINOMIAL USING TRIAL AND ERROR

Factor completely using trial and error: $3y^2 + 22y + 7$.

Solution

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$

Step 2. Factor any GCF.	There is no GCF.	
Step 3. Find all the factor pairs of the first term.	The only of $3y^2$ are $1y, 3y$. Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ $1y, 3y$ $3y^2 + 22y + 7$ $1y, 3y$ $(y \quad)(3y \quad)$
Step 4. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ $1y, 3y \qquad 1, 7$ $(y \quad)(3y \quad)$
Step 5. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ $1y, 3y \qquad 1, 7$ $(y + 1)(3y + 7)$ $3y$ $7y$ $10y$ No! We need $22y$ $3y^2 + 22y + 7$ $1y, 3y \qquad 1, 7$ $(y + 7)(3y + 1)$ $21y$ $+y$ $22y$	$3y^2 + 22y + 7$ Possible factors: $(y + 1)(3y + 7)$ — Product: $3y^2 + 10y + 7$ Possible factors: $(y + 7)(3y + 1)$ — Product: $3y^2 + 22y + 7$
Step 6. Check by multiplying.		$(y + 7)(3y + 1)$ $3y^2 + 22y + 7$ ✓

TRY IT : : 6.29 Factor completely using trial and error: $2a^2 + 5a + 3$.

TRY IT : : 6.30 Factor completely using trial and error: $4b^2 + 5b + 1$.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING TRIAL AND ERROR.

Step 1. Write the trinomial in descending order of degrees as needed.
Step 2. Factor any GCF.
Step 3. Find all the factor pairs of the first term.
Step 4. Find all the factor pairs of the third term.
Step 5. Test all the possible combinations of the factors until the correct product is found.
Step 6. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

EXAMPLE 6.16

Factor completely using trial and error: $6b^2 - 13b + 5$.

Solution

The trinomial is already in descending order.	$6b^2-13b+5$
Find the factors of the first term.	$6b^2-13b+5$
Find the factors of the last term. Consider the signs. Since the last term, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2-13b+5$

Consider all the combinations of factors.

$6b^2-13b+5$	
Possible factors	**Product**
$(b-1)(6b-5)$	$6b^2-11b+5$
$(b-5)(6b-1)$	$6b^2-31b+5$
$(2b-1)(3b-5)$	$6b^2-13b+5$*
$(2b-5)(3b-1)$	$6b^2-17b+5$

The correct factors are those whose product is the original trinomial. $(2b-1)(3b-5)$

Check by multiplying:

$$(2b-1)(3b-5)$$
$$6b^2-10b-3b+5$$
$$6b^2-13b+5\ \checkmark$$

TRY IT : : 6.31 Factor completely using trial and error: $8x^2-13x+3$.

TRY IT : : 6.32 Factor completely using trial and error: $10y^2-37y+7$.

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

EXAMPLE 6.17

Factor completely using trial and error: $18x^2-37xy+15y^2$.

Solution

The trinomial is already in descending order.	$18x^2 - 37xy + 15y^2$
Find the factors of the first term.	$18x^2 - 37xy + 15y^2$
Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.	$18x^2 - 37xy + 15y^2$

Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$	
Possible factors	**Product**
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2$*
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. $(2x - 3y)(9x - 5y)$

Check by multiplying:

$$(2x - 3y)(9x - 5y)$$
$$18x^2 - 10xy - 27xy + 15y^2$$
$$18x^2 - 37xy + 15y^2 \checkmark$$

TRY IT : : 6.33 Factor completely using trial and error $18x^2 - 3xy - 10y^2$.

TRY IT : : 6.34 Factor completely using trial and error: $30x^2 - 53xy - 21y^2$.

Don't forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

EXAMPLE 6.18

Factor completely using trial and error: $-10y^4 - 55y^3 - 60y^2$.

Solution

	$-10y^4-55y^3-60y^2$
Notice the greatest common factor, so factor it first.	$-5y^2(2y^2+11y+12)$
Factor the trinomial.	$-5y^2\left(2y^2+11y+12\right)$

Consider all the combinations.

$2y^2+11y+12$	
Possible factors	**Product**
$(y+1)(2y+12)$	Not an option
$(y+12)(2y+1)$	$2y^2+25y+12$
$(y+2)(2y+6)$	Not an option
$(y+6)(2y+2)$	Not an option
$(y+3)(2y+4)$	Not an option
$(y+4)(2y+3)$	$2y^2+11y+12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor $-5y^2$.

$$-5y^2(y+4)(2y+3)$$

Check by multiplying:

$$-5y^2(y+4)(2y+3)$$
$$-5y^2\left(2y^2+8y+3y+12\right)$$
$$-10y^4-55y^3-60y^2 \checkmark$$

TRY IT : : 6.35 Factor completely using trial and error: $15n^3-85n^2+100n$.

TRY IT : : 6.36 Factor completely using trial and error: $56q^3+320q^2-96q$.

Factor Trinomials of the Form ax^2+bx+c using the "ac" Method

Another way to factor trinomials of the form ax^2+bx+c is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

EXAMPLE 6.19 HOW TO FACTOR TRINOMIALS USING THE "AC" METHOD

Factor using the *'ac'* method: $6x^2+7x+2$.

Solution

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2+7x+2$
Step 2. Find the product ac.	$a \cdot c$ $6 \cdot 2$ 12	$6x^2+7x+2$

Step 3. Find two numbers m and n that: Multiply to ac. $m \cdot n = a \cdot c$ Add to b. $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3 + 4 = 7$	
Step 4. Split the middle term using m and n. $ax^2 + bx + c$ bx $ax^2 + mx + nx + c$	Rewrite $7x$ as $3x + 4x$. It would also give the same result if we used $4x + 3x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $6x^2 + 3x + 4x + 2$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$
Step 6. Check by multiplying the factors.		$(2x + 1)(3x + 2)$ $6x^2 + 4x + 3x + 2$ $6x^2 + 7x + 2$ ✓

TRY IT : : 6.37 Factor using the 'ac' method: $6x^2 + 13x + 2$.

TRY IT : : 6.38 Factor using the 'ac' method: $4y^2 + 8y + 3$.

The "ac" method is summarized here.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING THE "AC" METHOD.

Step 1. Factor any GCF.

Step 2. Find the product ac.

Step 3. Find two numbers m and n that:

Multiply to ac $m \cdot n = a \cdot c$

Add to b $m + n = b$

$ax^2 + bx + c$

Step 4. Split the middle term using m and n. $ax^2 + mx + nx + c$

Step 5. Factor by grouping.

Step 6. Check by multiplying the factors.

Don't forget to look for a common factor!

EXAMPLE 6.20

Factor using the *'ac'* method: $10y^2 - 55y + 70$.

Solution

Is there a greatest common factor?		
Yes. The GCF is 5.		$10y^2-55y+70$
Factor it.		$5(2y^2-11y+14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.		$ac+bx+c$ $5(2y^2-11y+14)$
Find the product ac.	$ac=28$	
Find two numbers that multiply to ac	$(-4)(-7)=28$	
and add to b.	$-4+(-7)=-11$	
Split the middle term.		$5(2y^2-11y+14)$
		$5(2y^2-7y-4y+14)$
Factor the trinomial by grouping.		$5(y(2y-7)-2(y-7))$
		$5(y-2)(2y-7)$

Check by multiplying all three factors.

$$5(y-2)(2y-7)$$
$$5\left(2y^2-7y-4y+14\right)$$
$$5\left(2y^2-11y+14\right)$$
$$10y^2-55y+70\ \checkmark$$

TRY IT : : 6.39 Factor using the 'ac' method: $16x^2-32x+12$.

TRY IT : : 6.40 Factor using the 'ac' method: $18w^2-39w+18$.

Factor Using Substitution

Sometimes a trinomial does not appear to be in the ax^2+bx+c form. However, we can often make a thoughtful substitution that will allow us to make it fit the ax^2+bx+c form. This is called factoring by substitution. It is standard to use *u* for the substitution.

In the ax^2+bx+c, the middle term has a variable, *x*, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

EXAMPLE 6.21

Factor by substitution: x^4-4x^2-5.

Solution

The variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2=x^4$). If we let $u=x^2$, we can put our trinomial in the ax^2+bx+c form we need to factor it.

	x^4-4x^2-5
Rewrite the trinomial to prepare for the substitution.	$(x^2)^2-4(x^2)-5$
Let $u=x^2$ and substitute.	u^2-4u-5
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$(x^2+1)(x^2-5)$
Check:	

$$\begin{aligned}&\left(x^2+1\right)\left(x^2-5\right)\\&x^4-5x^2+x^2-5\\&x^4-4x^2-5\ \checkmark\end{aligned}$$

TRY IT : : 6.41 Factor by substitution: h^4+4h^2-12.

TRY IT : : 6.42 Factor by substitution: y^4-y^2-20.

Sometimes the expression to be substituted is not a monomial.

EXAMPLE 6.22

Factor by substitution: $(x-2)^2+7(x-2)+12$

Solution

The binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

	$(x-2)^2+7(x-2)+12$
Rewrite the trinomial to prepare for the substitution.	$(x-2)^2+7(x-2)+12$
Let $u=x-2$ and substitute.	$u^2+7u+12$
Factor the trinomial.	$(u+3)(u+4)$
Replace u with $x-2$.	$((x-2)+3)((x-2)+4)$
Simplify inside the parentheses.	$(x+1)(x+2)$

This could also be factored by first multiplying out the $(x-2)^2$ and the $7(x-2)$ and then combining like terms and then factoring. Most students prefer the substitution method.

TRY IT : : 6.43 Factor by substitution: $(x-5)^2+6(x-5)+8$.

TRY IT : : 6.44 Factor by substitution: $(y-4)^2+8(y-4)+15$.

MEDIA : :

Access this online resource for additional instruction and practice with factoring.

- **Factor a trinomial using the AC method (https://openstax.org/l/37ACmethod)**

6.2 EXERCISES

Practice Makes Perfect

Factor Trinomials of the Form x^2+bx+c

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

61. $p^2+11p+30$

62. $w^2+10x+21$

63. $n^2+19n+48$

64. $b^2+14b+48$

65. $a^2+25a+100$

66. $u^2+101u+100$

67. $x^2-8x+12$

68. $q^2-13q+36$

69. $y^2-18x+45$

70. $m^2-13m+30$

71. x^2-8x+7

72. y^2-5y+6

73. $5p-6+p^2$

74. $6n-7+n^2$

75. $8-6x+x^2$

76. $7x+x^2+6$

77. $x^2-12-11x$

78. $-11-10x+x^2$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

79. $x^2-2xy-80y^2$

80. $p^2-8pq-65q^2$

81. $m^2-64mn-65n^2$

82. $p^2-2pq-35q^2$

83. $a^2+5ab-24b^2$

84. $r^2+3rs-28s^2$

85. $x^2-3xy-14y^2$

86. $u^2-8uv-24v^2$

87. $m^2-5mn+30n^2$

88. $c^2-7cd+18d^2$

Factor Trinomials of the Form ax^2+bx+c Using Trial and Error

In the following exercises, factor completely using trial and error.

89. p^3-8p^2-20p

90. q^3-5q^2-24q

91. $3m^3-21m^2+30m$

92. $11n^3-55n^2+44n$

93. $5x^4+10x^3-75x^2$

94. $6y^4+12y^3-48y^2$

95. $2t^2+7t+5$

96. $5y^2+16y+11$

97. $11x^2+34x+3$

98. $7b^2+50b+7$

99. $4w^2-5w+1$

100. $5x^2-17x+6$

101. $4q^2-7q-2$

102. $10y^2-53y-11$

103. $6p^2-19pq+10q^2$

104. $21m^2-29mn+10n^2$

105. $4a^2+17ab-15b^2$

106. $6u^2+5uv-14v^2$

107. $-16x^2-32x-16$

108. $-81a^2+153a+18$

109. $-30q^3-140q^2-80q$

110. $-5y^3 - 30y^2 + 35y$

Factor Trinomials of the Form $ax^2 + bx + c$ using the 'ac' Method

In the following exercises, factor using the 'ac' method.

111. $5n^2 + 21n + 4$

112. $8w^2 + 25w + 3$

113. $4k^2 - 16k + 15$

114. $5s^2 - 9s + 4$

115. $6y^2 + y - 15$

116. $6p^2 + p - 22$

117. $2n^2 - 27n - 45$

118. $12z^2 - 41z - 11$

119. $60y^2 + 290y - 50$

120. $6u^2 - 46u - 16$

121. $48z^3 - 102z^2 - 45z$

122. $90n^3 + 42n^2 - 216n$

123. $16s^2 + 40s + 24$

124. $24p^2 + 160p + 96$

125. $48y^2 + 12y - 36$

126. $30x^2 + 105x - 60$

Factor Using Substitution

In the following exercises, factor using substitution.

127. $x^4 - x^2 - 12$

128. $x^4 + 2x^2 - 8$

129. $x^4 - 3x^2 - 28$

130. $x^4 - 13x^2 - 30$

131. $(x-3)^2 - 5(x-3) - 36$

132. $(x-2)^2 - 3(x-2) - 54$

133. $(3y-2)^2 - (3y-2) - 2$

134. $(5y-1)^2 - 3(5y-1) - 18$

Mixed Practice

In the following exercises, factor each expression using any method.

135. $u^2 - 12u + 36$

136. $x^2 - 14x - 32$

137. $r^2 - 20rs + 64s^2$

138. $q^2 - 29qr - 96r^2$

139. $12y^2 - 29y + 14$

140. $12x^2 + 36y - 24z$

141. $6n^2 + 5n - 4$

142. $3q^2 + 6q + 2$

143. $13z^2 + 39z - 26$

144. $5r^2 + 25r + 30$

145. $3p^2 + 21p$

146. $7x^2 - 21x$

147. $6r^2 + 30r + 36$

148. $18m^2 + 15m + 3$

149. $24n^2 + 20n + 4$

150. $4a^2 + 5a + 2$

151. $x^4 - 4x^2 - 12$

152. $x^4 - 7x^2 - 8$

153. $(x+3)^2 - 9(x+3) - 36$

154. $(x+2)^2 - 25(x+2) - 54$

Writing Exercises

155. Many trinomials of the form $x^2 + bx + c$ factor into the product of two binomials $(x + m)(x + n)$. Explain how you find the values of *m* and *n*.

156. Tommy factored $x^2 - x - 20$ as $(x + 5)(x - 4)$. Sara factored it as $(x + 4)(x - 5)$. Ernesto factored it as $(x - 5)(x - 4)$. Who is correct? Explain why the other two are wrong.

157. List, in order, all the steps you take when using the "ac" method to factor a trinomial of the form $ax^2 + bx + c$.

158. How is the "ac" method similar to the "undo FOIL" method? How is it different?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form $x^2 + bx + c$			
factor trinomials of the form $ax^2 + bx + c$ using trial and error.			
factor trinomials of the form $ax^2 + bx + c$ with using the "ac" method.			
factor using substitution.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

6.3 Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\left(3x^2\right)^3$.
 If you missed this problem, review **Example 5.18**.
2. Multiply: $(m+4)^2$.
 If you missed this problem, review **Example 5.32**.
3. Multiply: $(x-3)(x+3)$.
 If you missed this problem, review **Example 5.33**.

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$\begin{aligned}&(3x+4)^2\\&a^2+2\cdot a\cdot b+b^2\\&(3x)^2+2(3x\cdot 4)+4^2\\&9x^2+24x+16\end{aligned}$$

The trinomial $9x^2+24x+16$ is called a *perfect square trinomial*. It is the square of the binomial $3x+4$.

In this chapter, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form ax^2+bx+c. But if you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

Perfect Square Trinomials Pattern

If *a* and *b* are real numbers

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it the product, $2ab$? If everything checks, you can easily write the factors.

EXAMPLE 6.23 HOW TO FACTOR PERFECT SQUARE TRINOMIALS

Factor: $9x^2+12x+4$.

Solution

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$? • Is the first term a perfect square? Write it as a square, a^2. • Is the last term a perfect square? Write it as a square, b^2. • Check the middle term. Is it $2ab$?	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$. Is 4 a perfect square? Yes—write it as $(2)^2$. Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$9x^2 + 12x + 4$ $(3x)^2$ $(2)^2$ $(3x)^2$ $(2)^2$ $2(3x)(2)$ $12x$
Step 2. Write the square of the binomial.		$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a+b)^2$ $(3x+2)^2$
Step 3. Check.		$(3x+2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓

TRY IT : : 6.45 Factor: $4x^2 + 12x + 9$.

TRY IT : : 6.46 Factor: $9y^2 + 24y + 16$.

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a-b)^2$.

The steps are summarized here.

HOW TO : : FACTOR PERFECT SQUARE TRINOMIALS.

Step 1. Does the trinomial fit he pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
Is the fir t term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
Is the last term a perfect square? Write it as a square.	$(a)^2$ $(b)^2$	$(a)^2$ $(b)^2$
Check the middle term. Is it $2ab$?	$(a)^2$ $\searrow 2\cdot a\cdot b \swarrow$ $(b)^2$	$(a)^2$ $\searrow 2\cdot a\cdot b \swarrow$ $(b)^2$
Step 2. Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3. Check by multiplying.		

We'll work one now where the middle term is negative.

EXAMPLE 6.24

Factor: $81y^2 - 72y + 16$.

Solution

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a - b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2$ $(4)^2$
Check the middle term.	$(9y)^2$ $(4)^2$ $2(9y)(4)$ $72y$
Does it match $(a - b)^2$? Yes.	$a^2 - 2 \cdot a \cdot b + b^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$
Write as the square of a binomial.	$(9y - 4)^2$
Check by multiplying: $(9y - 4)^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$ $81y^2 - 72y + 16$ ✓	

TRY IT : : 6.47 Factor: $64y^2 - 80y + 25$.

TRY IT : : 6.48 Factor: $16z^2 - 72z + 81$.

The next example will be a perfect square trinomial with two variables.

EXAMPLE 6.25

Factor: $36x^2 + 84xy + 49y^2$.

Solution

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$a^2 + 2 \cdot a \cdot b + b^2$ $(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$
Factor.	$(6x + 7y)^2$
Check by multiplying. $(6x + 7y)^2$ $(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$ $36x^2 + 84xy + 49y^2$ ✓	

TRY IT : : 6.49 Factor: $49x^2 + 84xy + 36y^2$.

TRY IT : : 6.50 Factor: $64m^2 + 112mn + 49n^2$.

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

EXAMPLE 6.26

Factor: $100x^2y - 80xy + 16y$.

Solution

	$100x^2y - 80xy + 16y$
Is there a GCF? Yes, $4y$, so factor it out.	$4y(25x^2 - 20x + 4)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y[(5x)^2 - 2 \cdot 5x \cdot 2 + 2^2]$
Factor.	$4y(5x - 2)^2$

Remember: Keep the factor $4y$ in the final product.

Check:

$$4y(5x-2)^2$$
$$4y\left[(5x)^2 - 2 \cdot 5x \cdot 2 + 2^2\right]$$
$$4y\left(25x^2 - 20x + 4\right)$$
$$100x^2y - 80xy + 16y \checkmark$$

TRY IT : : 6.51 Factor: $8x^2y - 24xy + 18y$.

TRY IT : : 6.52 Factor: $27p^2q + 90pq + 75q$.

Factor Differences of Squares

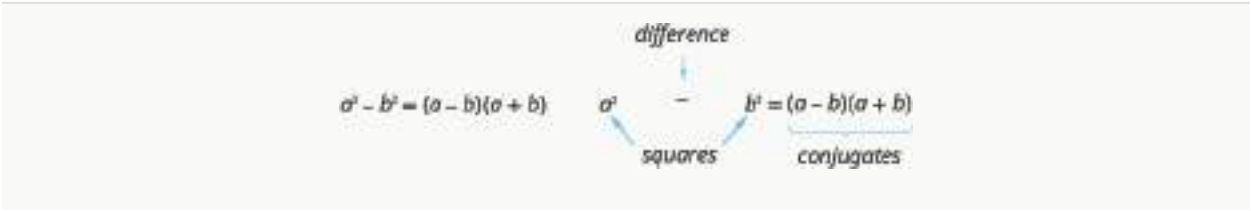
The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:

$$(a-b)(a+b)$$
$$(3x-4)(3x+4)$$
$$(a)^2 - (b)^2$$
$$(3x)^2 - (4)^2$$
$$9x^2 - 16$$

A difference of squares factors to a product of conjugates.

Difference of Squares Pattern

If a and b are real numbers,



Remember, "difference" refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

EXAMPLE 6.27 HOW TO FACTOR A TRINOMIAL USING THE DIFFERENCE OF SQUARES

Factor: $64y^2 - 1$.

Solution

Step 1. Does the binomial fit the pattern?		$64y^2 - 1$
• Is this a difference?	Yes	$64y^2 - 1$
• Are the first and last terms perfect squares?	Yes	
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 - b^2$ $(8y)^2 - 1^2$
Step 3. Write the product of conjugates.		$(a - b)(a + b)$ $(8y - 1)(8y + 1)$
Step 4. Check.		$(8y - 1)(8y + 1)$ $64y^2 - 1$ ✓

TRY IT : : 6.53 Factor: $121m^2 - 1$.

TRY IT : : 6.54 Factor: $81y^2 - 1$.

HOW TO : : FACTOR DIFFERENCES OF SQUARES.

Step 1.	Does the binomial fit he pattern?	$a^2 - b^2$
	Is this a diffe ence?	____ $-$ ____
	Are the fir t and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2 - (b)^2$
Step 3.	Write the product of conjugates.	$(a - b)(a + b)$
Step 4.	Check by multiplying.	

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime!

The next example shows variables in both terms.

EXAMPLE 6.28

Factor: $144x^2 - 49y^2$.

Solution

	$144x^2 - 49y^2$
Is this a diffe ence of squares? Yes.	$(12x)^2 - (7y)^2$
Factor as the product of conjugates.	$(12x - 7y)(12x + 7y)$

Check by multiplying.

$$(12x - 7y)(12x + 7y)$$
$$144x^2 - 49y^2 \checkmark$$

TRY IT : : 6.55 Factor: $196m^2 - 25n^2$.

TRY IT : : 6.56 Factor: $121p^2 - 9q^2$.

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and you won't recognize the perfect squares until you factor the GCF.

Also, to completely factor the binomial in the next example, we'll factor a difference of squares twice!

EXAMPLE 6.29

Factor: $48x^4y^2 - 243y^2$.

Solution

	$48x^4y^2 - 243y^2$
Is there a GCF? Yes, $3y^2$—factor it out!	$3y^2(16x^4 - 81)$
Is the binomial a diffe ence of squares? Yes.	$3y^2\left((4x^2)^2 - (9)^2\right)$
Factor as a product of conjugates.	$3y^2(4x^2 - 9)(4x^2 + 9)$
Notice the fir t binomial is also a diffe ence of squares!	$3y^2((2x)^2 - (3)^2)(4x^2 + 9)$
Factor it as the product of conjugates.	$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

The last factor, the sum of squares, cannot be factored.

Check by multiplying:

$$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$$
$$3y^2(4x^2 - 9)(4x^2 + 9)$$
$$3y^2(16x^4 - 81)$$
$$48x^4y^2 - 243y^2 \checkmark$$

TRY IT : : 6.57 Factor: $2x^4y^2 - 32y^2$.

TRY IT : : 6.58 Factor: $7a^4c^2 - 7b^4c^2$.

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

EXAMPLE 6.30

Factor: $x^2 - 6x + 9 - y^2$.

Solution

Notice that the first three terms form a perfect square trinomial.

	$x^2 - 6x + 9 - y^2$
Factor by grouping the first three terms.	$x^2 - 6x + 9 - y^2$
Use the perfect square trinomial pattern.	$(x-3)^2 - y^2$
Is this a difference of squares? Yes.	
Yes—write them as squares.	$a^2 \quad -b^2$ $(x-3)^2 - y^2$
Factor as the product of conjugates.	$a \quad -b \quad a \quad +b$ $((x-3)-y)((x-3)+y)$
	$(x-3-y)(x-3+y)$

You may want to rewrite the solution as $(x - y - 3)(x + y - 3)$.

TRY IT : : 6.59 Factor: $x^2 - 10x + 25 - y^2$.

TRY IT : : 6.60 Factor: $x^2 + 6x + 9 - 4y^2$.

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

We'll check the first pattern and leave the second to you.

	$(a+b)(a^2 - ab + b^2)$
Distribute.	$a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$
Multiply.	$a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$
Combine like terms.	$a^3 + b^3$

Sum and Difference of Cubes Pattern

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

same sign — opposite signs

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

same sign — opposite signs

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in Table 6.22.

n	1	2	3	4	5	6	7	8	9	10
n^3	1	8	27	64	125	216	343	512	729	1000

Table 6.22

EXAMPLE 6.31 HOW TO FACTOR THE SUM OR DIFFERENCE OF CUBES

Factor: $x^3 + 64$.

Solution

Step 1. Does the binomial fit the sum or difference of cubes pattern? • Is it a sum or difference? • Are the first and last terms perfect cubes?	This is a sum. Yes.	$x^3 + 64$ $x^3 + 64$
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3.	$a^3 + b^3$ $x^3 + 4^3$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$(a + b)(a^2 - ab + b^2)$ $(x + 4)(x^2 - 4x + 4^2)$
Step 4. Simplify inside the parentheses.	It is already simplified.	$(x + 4)(x^2 - 4x + 16)$
Step 5. Check by multiplying the factors.		$x^2 - 4x + 16$ $x + 4$ $4x^2 - 16x + 64$ ✓ $x^3 - 4x^2 + 16x$ $x^3 + 64$

TRY IT : : 6.61 Factor: $x^3 + 27$.

TRY IT : : 6.62 Factor: $y^3 + 8$.

HOW TO : : FACTOR THE SUM OR DIFFERENCE OF CUBES.

Step 1. Does the binomial fit the sum or difference of cubes pattern?
Is it a sum or difference?
Are the first and last terms perfect cubes?

Step 2. Write them as cubes.

Step 3. Use either the sum or difference of cubes pattern.

Step 4. Simplify inside the parentheses.

Step 5. Check by multiplying the factors.

EXAMPLE 6.32

Factor: $27u^3 - 125v^3$.

Solution

	$27u^3 - 125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$(3u)^3 - (5v)^3$
Use the difference of cubes pattern.	$(3u - 5v)((3u)^2 + 3u \cdot 5v + (5v)^2)$
Simplify.	$(3u - 5v)(9u^2 + 15uv + 25v^2)$
Check by multiplying.	We'll leave the check to you.

TRY IT : : 6.63 Factor: $8x^3 - 27y^3$.

TRY IT : : 6.64 Factor: $1000m^3 - 125n^3$.

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

EXAMPLE 6.33

Factor: $6x^3y + 48y^4$.

Solution

	$6x^3y + 48y^4$
Factor the common factor.	$6y(x^3 + 8y^3)$
This binomial is a sum The first and last terms are perfect cubes.	
Write the terms as cubes.	$6y\left(x^3 + (2y)^3\right)$
Use the sum of cubes pattern.	$6y\left(x + 2y\right)\left(x^2 - x \cdot 2y + (2y)^2\right)$
Simplify.	$6y(x + 2y)(x^2 - 2xy + 4y^2)$

Check:

To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by $6y$. We'll leave the multiplication for you.

TRY IT : : 6.65 Factor: $500p^3 + 4q^3$.

TRY IT : : 6.66 Factor: $432c^3 + 686d^3$.

The first term in the next example is a binomial cubed.

EXAMPLE 6.34

Factor: $(x + 5)^3 - 64x^3$.

Solution

	$(x + 5)^3 - 64x^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$(x + 5)^3 - (4x)^3$
Use the difference of cubes pattern.	$\left((x + 5) - 4x\right)\left((x + 5)^2 + (x + 5) \cdot 4x + (4x)^2\right)$
Simplify.	$(x + 5 - 4x)(x^2 + 10x + 25 + 4x^2 + 20x + 16x^2)$
	$(-3x + 5)(21x^2 + 30x + 25)$
Check by multiplying.	We'll leave the check to you.

TRY IT : : 6.67 Factor: $(y + 1)^3 - 27y^3$.

TRY IT : : 6.68 Factor: $(n + 3)^3 - 125n^3$.

MEDIA : :

Access this online resource for additional instruction and practice with factoring special products.

- **Factoring Binomials-Cubes #2 (https://openstax.org/l/37BinomCubes)**

6.3 EXERCISES

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

159. $16y^2 + 24y + 9$

160. $25v^2 + 20v + 4$

161. $36s^2 + 84s + 49$

162. $49s^2 + 154s + 121$

163. $100x^2 - 20x + 1$

164. $64z^2 - 16z + 1$

165. $25n^2 - 120n + 144$

166. $4p^2 - 52p + 169$

167. $49x^2 + 28xy + 4y^2$

168. $25r^2 + 60rs + 36s^2$

169. $100y^2 - 52y + 1$

170. $64m^2 - 34m + 1$

171. $10jk^2 + 80jk + 160j$

172. $64x^2y - 96xy + 36y$

173. $75u^4 - 30u^3v + 3u^2v^2$

174. $90p^4 + 300p^4q + 250p^2q^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

175. $25v^2 - 1$

176. $169q^2 - 1$

177. $4 - 49x^2$

178. $121 - 25s^2$

179. $6p^2q^2 - 54p^2$

180. $98r^3 - 72r$

181. $24p^2 + 54$

182. $20b^2 + 140$

183. $121x^2 - 144y^2$

184. $49x^2 - 81y^2$

185. $169c^2 - 36d^2$

186. $36p^2 - 49q^2$

187. $16z^4 - 1$

188. $m^4 - n^4$

189. $162a^4b^2 - 32b^2$

190. $48m^4n^2 - 243n^2$

191. $x^2 - 16x + 64 - y^2$

192. $p^2 + 14p + 49 - q^2$

193. $a^2 + 6a + 9 - 9b^2$

194. $m^2 - 6m + 9 - 16n^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

195. $x^3 + 125$

196. $n^6 + 512$

197. $z^6 - 27$

198. $v^3 - 216$

199. $8 - 343t^3$

200. $125 - 27w^3$

201. $8y^3 - 125z^3$

202. $27x^3 - 64y^3$

203. $216a^3 + 125b^3$

204. $27y^3 + 8z^3$

205. $7k^3 + 56$

206. $6x^3 - 48y^3$

207. $2x^2 - 16x^2y^3$

208. $-2x^3y^2 - 16y^5$

209. $(x+3)^3 + 8x^3$

210. $(x+4)^3 - 27x^3$

211. $(y-5)^3 - 64y^3$

212. $(y-5)^3 + 125y^3$

Mixed Practice

In the following exercises, factor completely.

213. $64a^2 - 25$

214. $121x^2 - 144$

215. $27q^2 - 3$

216. $4p^2 - 100$

217. $16x^2 - 72x + 81$

218. $36y^2 + 12y + 1$

219. $8p^2 + 2$

220. $81x^2 + 169$

221. $125 - 8y^3$

222. $27u^3 + 1000$

223. $45n^2 + 60n + 20$

224. $48q^3 - 24q^2 + 3q$

225. $x^2 - 10x + 25 - y^2$

226. $x^2 + 12x + 36 - y^2$

227. $(x+1)^3 + 8x^3$

228. $(y-3)^3 - 64y^3$

Writing Exercises

229. Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

230. How do you recognize the binomial squares pattern?

231. Explain why $n^2 + 25 \neq (n+5)^2$. Use algebra, words, or pictures.

232. Maribel factored $y^2 - 30y + 81$ as $(y-9)^2$. Was she right or wrong? How do you know?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

6.4 General Strategy for Factoring Polynomials

Learning Objectives

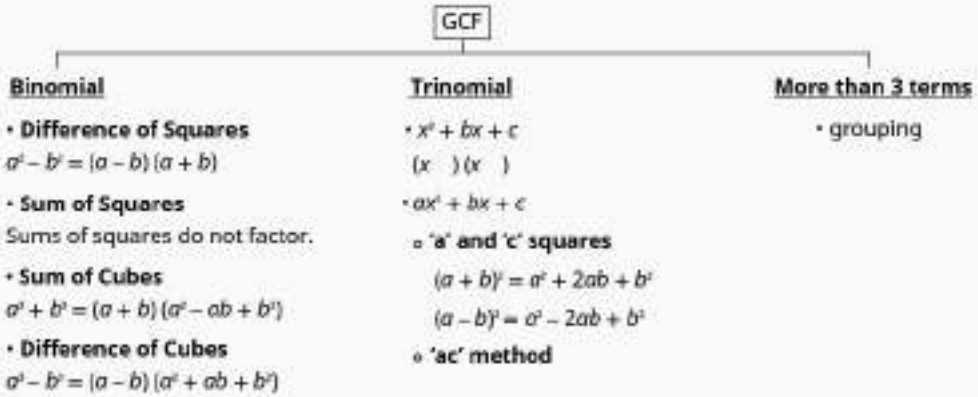
By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered, and outlines a strategy you should use when factoring polynomials.

General Strategy for Factoring Polynomials





HOW TO : : USE A GENERAL STRATEGY FOR FACTORING POLYNOMIALS.

Step 1. Is there a greatest common factor?
Factor it out.

Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
If it is a binomial:

- Is it a sum?
 Of squares? Sums of squares do not factor.
 Of cubes? Use the sum of cubes pattern.
- Is it a difference?
 Of squares? Factor as the product of conjugates.
 Of cubes? Use the difference of cubes pattern.

If it is a trinomial:

- Is it of the form $x^2 + bx + c$? Undo FOIL.
- Is it of the form $ax^2 + bx + c$?
 If *a* and *c* are squares, check if it fits the trinomial square pattern.
 Use the trial and error or "ac" method.

If it has more than three terms:

- Use the grouping method.

Step 3. Check.
Is it factored completely?
Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

EXAMPLE 6.35

Factor completely: $7x^3 - 21x^2 - 70x$.

Solution

	$7x^3 - 21x^2 - 70x$
Is there a GCF? Yes, $7x$.	
Factor out the GCF.	$7x(x^2 - 3x - 10)$
In the parentheses, is it a binomial, trinomial, or are there more terms? Trinomial with leading coefficient	
"Undo" FOIL.	$7x(x \quad)(x \quad)$ $7x(x+2)(x-5)$
Is the expression factored completely? Yes. Neither binomial can be factored.	
Check your answer.	
Multiply.	
	$7x(x+2)(x-5)$
	$7x(x^2 - 5x + 2x - 10)$
	$7x(x^2 - 3x - 10)$
	$7x^3 - 21x^2 - 70x$ ✓

TRY IT : : 6.69 Factor completely: $8y^3 + 16y^2 - 24y$.

TRY IT : : 6.70 Factor completely: $5y^3 - 15y^2 - 270y$.

Be careful when you are asked to factor a binomial as there are several options!

EXAMPLE 6.36

Factor completely: $24y^2 - 150$.

Solution

$$24y^2 - 150$$

Is there a GCF? Yes, 6.

Factor out the GCF. $6\left(4y^2 - 25\right)$

In the parentheses, is it a binomial, trinomial or are there more than three terms? Binomial.
Is it a sum? No.

Is it a diffe ence? Of squares or cubes? Yes, squares. $6\left((2y)^2 - (5)^2\right)$

Write as a product of conjugates. $6(2y - 5)(2y + 5)$

Is the expression factored completely?

Neither binomial can be factored.

Check:

Multiply.

$$6(2y - 5)(2y + 5)$$
$$6\left(4y^2 - 25\right)$$
$$24y^2 - 150 \checkmark$$

TRY IT : : 6.71 Factor completely: $16x^3 - 36x$.

TRY IT : : 6.72 Factor completely: $27y^2 - 48$.

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

EXAMPLE 6.37

Factor completely: $4a^2 - 12ab + 9b^2$.

Solution

$$4a^2 - 12ab + 9b^2$$

Is there a GCF? No.
Is it a binomial, trinomial, or are there more terms?
Trinomial with $a \neq 1$. But the fir t term is a perfect square.

Is the last term a perfect square? Yes. $(2a)^2 - 12ab + (3b)^2$

Does it fit he pattern, $a^2 - 2ab + b^2$? Yes. $(2a)^2 \searrow \underset{-2(2a)(3b)}{-12ab} \swarrow (3b)^2$

Write it as a square. $(2a - 3b)^2$

Is the expression factored completely? Yes.
The binomial cannot be factored.

Check your answer.

Multiply.

$$(2a - 3b)^2$$
$$(2a)^2 - 2 \cdot 2a \cdot 3b + (3b)^2$$
$$4a^2 - 12ab + 9b^2 \checkmark$$

TRY IT : : 6.73 Factor completely: $4x^2 + 20xy + 25y^2$.

TRY IT : : 6.74 Factor completely: $9x^2 - 24xy + 16y^2$.

Remember, sums of squares do not factor, but sums of cubes do!

EXAMPLE 6.38

Factor completely $12x^3y^2 + 75xy^2$.

Solution

$$12x^3y^2 + 75xy^2$$

Is there a GCF? Yes, $3xy^2$.

Factor out the GCF. $3xy^2(4x^2 + 25)$

In the parentheses, is it a binomial, trinomial, or are there more than three terms? Binomial.

Is it a sum? Of squares? Yes. Sums of squares are prime.

Is the expression factored completely? Yes.

Check:

Multiply.

$$3xy^2(4x^2 + 25)$$
$$12x^3y^2 + 75xy^2 \checkmark$$

TRY IT : : 6.75 Factor completely: $50x^3y + 72xy$.

TRY IT : : 6.76 Factor completely: $27xy^3 + 48xy$.

When using the sum or difference of cubes pattern, being careful with the signs.

EXAMPLE 6.39

Factor completely: $24x^3 + 81y^3$.

Solution

Is there a GCF? Yes, 3.	$24x^3 + 81y^3$
Factor it out.	$3(8x^3 + 27y^3)$
In the parentheses, is it a binomial, trinomial, of are there more than three terms? Binomial.	
Is it a sum or difference? Sum.	
Of squares or cubes? Sum of cubes.	$3\left(\underset{(2x)^3}{a^3} + \underset{(3y)^3}{b^3}\right)$
Write it using the sum of cubes pattern.	$3\left(\underset{2x}{a} + \underset{3y}{b}\right)\left(\underset{(2x)^2}{a^2} - \underset{2x \cdot 3y}{ab} + \underset{(3y)^2}{b^2}\right)$
Is the expression factored completely? Yes.	$3(2x + 3y)(4x^2 - 6xy + 9y^2)$
Check by multiplying.	

TRY IT : : 6.77 Factor completely: $250m^3 + 432n^3$.

TRY IT : : 6.78 Factor completely: $2p^3 + 54q^3$.

EXAMPLE 6.40

Factor completely: $3x^5 y - 48xy$.

Solution

	$3x^5y - 48xy$
Is there a GCF? Factor out $3xy$	$3xy(x^4 - 16)$
Is the binomial a sum or diffe ence? Of squares or cubes? Write it as a diffe ence of squares.	$3xy\left((x^2)^2 - (4)^2\right)$
Factor it as a product of conjugates	$3xy(x^2 - 4)(x^2 + 4)$
The fir t binomial is again a diffe ence of squares.	$3xy((x)^2 - (2)^2)(x^2 + 4)$
Factor it as a product of conjugates.	$3xy(x - 2)(x + 2)(x^2 + 4)$

Is the expression factored completely? Yes.
Check your answer.
Multiply.

$$3xy(x - 2)(x + 2)(x^2 + 4)$$
$$3xy(x^2 - 4)(x^2 + 4)$$
$$3xy(x^4 - 16)$$
$$3x^5y - 48xy \checkmark$$

TRY IT : : 6.79 Factor completely: $4a^5b - 64ab$.

TRY IT : : 6.80 Factor completely: $7xy^5 - 7xy$.

EXAMPLE 6.41

Factor completely: $4x^2 + 8bx - 4ax - 8ab$.

Solution

	$4x^2 + 8bx - 4ax - 8ab$
Is there a GCF? Factor out the GCF, 4.	$4(x^2 + 2bx - ax - 2ab)$
There are four terms. Use grouping.	$4[x(x + 2b) - a(x + 2b)]$
	$4(x + 2b)(x - a)$

Is the expression factored completely? Yes.
Check your answer.
Multiply.

$$4(x + 2b)(x - a)$$
$$4(x^2 - ax + 2bx - 2ab)$$
$$4x^2 + 8bx - 4ax - 8ab \checkmark$$

TRY IT : : 6.81 Factor completely: $6x^2 - 12xc + 6bx - 12bc$.

TRY IT : : 6.82 Factor completely: $16x^2 + 24xy - 4x - 6y$.

Taking out the complete GCF in the first step will always make your work easier.

EXAMPLE 6.42

Factor completely: $40x^2y + 44xy - 24y$.

Solution

	$40x^2y + 44xy - 24y$
Is there a GCF? Factor out the GCF, $4y$.	$4y\left(10x^2 + 11x - 6\right)$
Factor the trinomial with $a \neq 1$.	$4y\left(10x^2 + 11x - 6\right)$
	$4y(5x - 2)(2x + 3)$

Is the expression factored completely? Yes.

Check your answer.

Multiply.

$$4y(5x - 2)(2x + 3)$$
$$4y\left(10x^2 + 11x - 6\right)$$
$$40x^2y + 44xy - 24y \checkmark$$

TRY IT : : 6.83 Factor completely: $4p^2q - 16pq + 12q$.

TRY IT : : 6.84 Factor completely: $6pq^2 - 9pq - 6p$.

When we have factored a polynomial with four terms, most often we separated it into two groups of two terms. Remember that we can also separate it into a trinomial and then one term.

EXAMPLE 6.43

Factor completely: $9x^2 - 12xy + 4y^2 - 49$.

Solution

$$9x^2 - 12xy + 4y^2 - 49$$

Is there a GCF? No.
With more than 3 terms, use grouping. Last 2 terms have no GCF. Try grouping fir t 3 terms.

$$9x^2 - 12xy + 4y^2 - 49$$

Factor the trinomial with $a \neq 1$. But the fir t term is a perfect square.

Is the last term of the trinomial a perfect square? Yes.

$$(3x)^2 - 12xy + (2y)^2 - 49$$

Does the trinomial fit he pattern, $a^2 - 2ab + b^2$? Yes.

$$(3x)^2 \underset{-2(3x)(2y)}{-12xy+} (2y)^2 - 49$$

Write the trinomial as a square.

$$(3x - 2y)^2 - 49$$

Is this binomial a sum or diffe ence? Of squares or cubes? Write it as a diffe ence of squares.

$$(3x - 2y)^2 - 7^2$$

Write it as a product of conjugates.

$$((3x - 2y) - 7)((3x - 2y) + 7)$$
$$(3x - 2y - 7)(3x - 2y + 7)$$

Is the expression factored completely? Yes.
Check your answer.
Multiply.

$$(3x - 2y - 7)(3x - 2y + 7)$$
$$9x^2 - 6xy - 21x - 6xy + 4y^2 + 14y + 21x - 14y - 49$$
$$9x^2 - 12xy + 4y^2 - 49 \checkmark$$

TRY IT : : 6.85 Factor completely: $4x^2 - 12xy + 9y^2 - 25$.

TRY IT : : 6.86 Factor completely: $16x^2 - 24xy + 9y^2 - 64$.

6.4 EXERCISES

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

233. $2n^2+13n-7$

234. $8x^2-9x-3$

235. a^5+9a^3

236. $75m^3+12m$

237. $121r^2-s^2$

238. $49b^2-36a^2$

239. $8m^2-32$

240. $36q^2-100$

241. $25w^2-60w+36$

242. $49b^2-112b+64$

243. $m^2+14mn+49n^2$

244. $64x^2+16xy+y^2$

245. $7b^2+7b-42$

246. $30n^2+30n+72$

247. $3x^4y-81xy$

248. $4x^5y-32x^2y$

249. k^4-16

250. m^4-81

251. $5x^5y^2-80xy^2$

252. $48x^5y^2-243xy^2$

253. $15pq-15p+12q-12$

254. $12ab-6a+10b-5$

255. $4x^2+40x+84$

256. $5q^2-15q-90$

257. $4u^5v+4u^2v^3$

258. $5m^4n+320mn^4$

259. $4c^2+20cd+81d^2$

260. $25x^2+35xy+49y^2$

261. $10m^4-6250$

262. $3v^4-768$

263. $36x^2y+15xy-6y$

264. $60x^2y-75xy+30y$

265. $8x^3-27y^3$

266. $64x^3+125y^3$

267. y^6-1

268. y^6+1

269. $9x^2-6xy+y^2-49$

270. $16x^2-24xy+9y^2-64$

271. $(3x+1)^2-6(3x-1)+9$

272. $(4x-5)^2-7(4x-5)+12$

Writing Exercises

273. Explain what it mean to factor a polynomial completely.

274. The difference of squares y^4-625 can be factored as $\left(y^2-25\right)\left(y^2+25\right)$. But it is not completely factored. What more must be done to completely factor.

275. Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, 'ac' method, special products) which is the easiest for you? Which is the hardest? Explain your answers.

276. Create three factoring problems that would be good test questions to measure your knowledge of factoring. Show the solutions.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
recognize and use the appropriate method to factor a polynomial completely.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

6.5 Polynomial Equations

Learning Objectives

By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve equations with polynomial functions
- Solve applications modeled by polynomial equations

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $5y - 3 = 0$.
 If you missed this problem, review **Example 2.2**.
2. Factor completely: $n^3 - 9n^2 - 22n$.
 If you missed this problem, review **Example 3.48**.
3. If $f(x) = 8x - 16$, find $f(3)$ and solve $f(x) = 0$.
 If you missed this problem, review **Example 3.59**.

We have spent considerable time learning how to factor polynomials. We will now look at polynomial equations and solve them using factoring, if possible.

A **polynomial equation** is an equation that contains a polynomial expression. The **degree of the polynomial equation** is the degree of the polynomial.

Polynomial Equation

A **polynomial equation** is an equation that contains a polynomial expression.

The **degree of the polynomial equation** is the degree of the polynomial.

We have already solved polynomial equations of degree one. Polynomial equations of degree one are linear equations are of the form $ax + b = c$.

We are now going to solve polynomial equations of degree two. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2 + 5x + 6 = 0 \qquad 3y^2 + 4y = 10 \qquad 64u^2 - 81 = 0 \qquad n(n+1) = 42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get $n^2 + n$.

The general form of a quadratic equation is $ax^2 + bx + c = 0$, with $a \neq 0$. (If $a = 0$, then $0 \cdot x^2 = 0$ and we are left with no quadratic term.)

Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

$$a,\ b,\ \text{and}\ c\ \text{are real numbers and}\ a \neq 0$$

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Zero Product Property

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a quadratic equation.

EXAMPLE 6.44 HOW TO SOLVE A QUADRATIC EQUATION USING THE ZERO PRODUCT PROPERTY

Solve: $(5n - 2)(6n - 1) = 0$.

Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(5n - 2)(6n - 1) = 0$ $5n - 2 = 0$ or $6n - 1 = 0$
Step 2. Solve the linear equations.	Solve each equation.	$n = \frac{2}{5}$ $n = \frac{1}{6}$
Step 3. Check.	Substitute each solution separately into the original equation.	$n = \frac{2}{5}$ $(5n - 2)(6n - 1) = 0$ $\left(5 \cdot \frac{2}{5} - 2\right)\left(6 \cdot \frac{2}{5} - 1\right) \stackrel{?}{=} 0$ $(2 - 2)\left(\frac{12}{5} - 1\right) \stackrel{?}{=} 0$ $0 \cdot \frac{7}{5} \stackrel{?}{=} 0$ $0 = 0$✓ $n = \frac{1}{6}$ $(5n - 2)(6n - 1) = 0$ $\left(5 \cdot \frac{1}{6} - 2\right)\left(6 \cdot \frac{1}{6} - 1\right) \stackrel{?}{=} 0$ $\left(\frac{5}{6} - \frac{12}{6}\right)(1 - 1) \stackrel{?}{=} 0$ $\left(-\frac{7}{6}\right)(0) \stackrel{?}{=} 0$ $0 = 0$✓

TRY IT : : 6.87 Solve: $(3m - 2)(2m + 1) = 0$.

TRY IT : : 6.88 Solve: $(4p + 3)(4p - 3) = 0$.

HOW TO : : USE THE ZERO PRODUCT PROPERTY.

Step 1. Set each factor equal to zero.
Step 2. Solve the linear equations.
Step 3. Check.

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So we be sure to start with the quadratic equation in standard form, $ax^2 + bx + c = 0$. Then we factor the expression on the left.

EXAMPLE 6.45 HOW TO SOLVE A QUADRATIC EQUATION BY FACTORING

Solve: $2y^2 = 13y + 45$.

Solution

Step 1. Write the quadratic equation in standard form. $ax^2 + bx + c = 0$.	Write the equation in standard form.	$2y^2 = 13y + 45$ $2y^2 - 13y - 45 = 0$
Step 2. Factor the quadratic expression.	Factor $2y^2 - 13y + 45$ $(2y + 5)(y - 9)$	$(2y + 5)(y - 9) = 0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y + 5 = 0$ $y - 9 = 0$
Step 4. Solve the linear equations.		$y = -\frac{5}{2}$ $y = 9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y = -\frac{5}{2}$ $2y^2 = 13y + 45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$ $\frac{25}{2} = \frac{25}{2}$ ✓ $y = 9$ $2y^2 = 13y + 45$ $2(9)^2 \stackrel{?}{=} 13(9) + 45$ $2(81) \stackrel{?}{=} 117 + 45$ $162 = 162$ ✓

TRY IT : : 6.89 Solve: $3c^2 = 10c - 8$.

TRY IT : : 6.90 Solve: $2d^2 - 5d = 3$.

HOW TO : : SOLVE A QUADRATIC EQUATION BY FACTORING.

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
Step 2. Factor the quadratic expression.
Step 3. Use the Zero Product Property.
Step 4. Solve the linear equations.
Step 5. Check. Substitute each solution separately into the original equation.

Before we factor, we must make sure the quadratic equation is in standard form.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

EXAMPLE 6.46

Solve: $169q^2 = 49$.

Solution

	$169x^2 = 49$
Write the quadratic equation in standard form.	$169x^2 - 49 = 0$
Factor. It is a diffe ence of squares.	$(13x - 7)(13x + 7) = 0$
Use the Zero Product Property to set each factor to 0.	$13x - 7 = 0 \qquad 13x + 7 = 0$
Solve each equation.	$13x = 7 \qquad 13x = -7$
	$x = \frac{7}{13} \qquad x = -\frac{7}{13}$

Check:

We leave the check up to you.

TRY IT : : 6.91 Solve: $25p^2 = 49$.

TRY IT : : 6.92 Solve: $36x^2 = 121$.

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

EXAMPLE 6.47

Solve: $(3x - 8)(x - 1) = 3x$.

Solution

	$(3x - 8)(x - 1) = 3x$
Multiply the binomials.	$3x^2 - 11x + 8 = 3x$
Write the quadratic equation in standard form.	$3x^2 - 14x + 8 = 0$
Factor the trinomial.	$(3x - 2)(x - 4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x - 2 = 0 \qquad x - 4 = 0$
Solve each equation.	$3x = 2 \qquad x = 4$
	$x = \frac{2}{3}$
Check your answers.	The check is left to you.

TRY IT : : 6.93 Solve: $(2m + 1)(m + 3) = 12m$.

TRY IT : : 6.94 Solve: $(k + 1)(k - 1) = 8$.

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

EXAMPLE 6.48

Solve: $3x^2 = 12x + 63$.

Solution

$$3x^2 = 12x + 63$$

Write the quadratic equation in standard form.	$3x^2 - 12x - 63 = 0$
Factor the greatest common factor fir t.	$3\left(x^2 - 4x - 21\right) = 0$
Factor the trinomial.	$3(x - 7)(x + 3) = 0$
Use the Zero Product Property to set each factor to 0.	$3 \neq 0 \quad x - 7 = 0 \quad x + 3 = 0$
Solve each equation.	$3 \neq 0 \quad x = 7 \quad x = -3$
Check your answers.	The check is left to you.

TRY IT : : 6.95 Solve: $18a^2 - 30 = -33a$.

TRY IT : : 6.96 Solve: $123b = -6 - 60b^2$.

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

EXAMPLE 6.49

Solve: $9m^3 + 100m = 60m^2$.

Solution

$$9m^3 + 100m = 60m^2$$

Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m = 0$
Factor the greatest common factor fir t.	$m\left(9m^2 - 60m + 100\right) = 0$
Factor the trinomial.	$m(3m - 10)(3m - 10) = 0$
Use the Zero Product Property to set each factor to 0.	$m = 0 \quad 3m - 10 = 0 \quad 3m - 10 = 0$
Solve each equation.	$m = 0 \quad m = \frac{10}{3} \quad m = \frac{10}{3}$
Check your answers.	The check is left to you.

TRY IT : : 6.97 Solve: $8x^3 = 24x^2 - 18x$.

TRY IT : : 6.98 Solve: $16y^2 = 32y^3 + 2y$.

Solve Equations with Polynomial Functions

As our study of polynomial functions continues, it will often be important to know when the function will have a certain value or what points lie on the graph of the function. Our work with the Zero Product Property will be help us find these answers.

EXAMPLE 6.50

For the function $f(x) = x^2 + 2x - 2$,

ⓐ find x when $f(x) = 6$

ⓑ find two points that lie on the graph of the function.

Solution

ⓐ

	$f(x) = x^2 + 2x - 2$
Substitute 6 for $f(x)$.	$6 = x^2 + 2x - 2$
Put the quadratic in standard form.	$x^2 + 2x - 8 = 0$
Factor the trinomial.	$(x + 4)(x - 2) = 0$
Use the zero product property.	$x + 4 = 0$ or $x - 2 = 0$
Solve.	$x = -4$ or $x = 2$

Check:

$$f(x) = x^2 + 2x - 2 \qquad f(x) = x^2 + 2x - 2$$
$$f(-4) = (-4)^2 + 2(-4) - 2 \qquad f(2) = 2^2 + 2 \cdot 2 - 2$$
$$f(-4) = 16 - 8 - 2 \qquad f(2) = 4 + 4 - 2$$
$$f(-4) = 6 \checkmark \qquad f(2) = 6 \checkmark$$

ⓑ Since $f(-4) = 6$ and $f(2) = 6$, the points $(-4, 6)$ and $(2, 6)$ lie on the graph of the function.

TRY IT : : 6.99

For the function $f(x) = x^2 - 2x - 8$,

ⓐ find x when $f(x) = 7$

ⓑ Find two points that lie on the graph of the function.

TRY IT : : 6.100

For the function $f(x) = x^2 - 8x + 3$,

ⓐ find x when $f(x) = -4$

ⓑ Find two points that lie on the graph of the function.

The Zero Product Property also helps us determine where the function is zero. A value of x where the function is 0, is called a **zero of the function**.

Zero of a Function

For any function f, if $f(x) = 0$, then x is a **zero of the function**.

When $f(x) = 0$, the point $(x, 0)$ is a point on the graph. This point is an x-intercept of the graph. It is often important to know where the graph of a function crosses the axes. We will see some examples later.

EXAMPLE 6.51

For the function $f(x) = 3x^2 + 10x - 8$, find

ⓐ the zeros of the function,

ⓑ any x-intercepts of the graph of the function

ⓒ any y-intercepts of the graph of the function

Solution

ⓐ To find the zeros of the function, we need to find when the function value is 0.

$$f(x) = 3x^2 + 10x - 8$$

Substitute 0 for $f(x)$. $0 = 3x^2 + 10x - 8$

Factor the trinomial. $(x+4)(3x-2) = 0$

Use the zero product property. $x+4 = 0$ or $3x-2 = 0$

Solve. $x = -4$ or $x = \frac{2}{3}$

ⓑ An x-intercept occurs when $y = 0$. Since $f(-4) = 0$ and $f\left(\frac{2}{3}\right) = 0$, the points $(-4, 0)$ and $\left(\frac{2}{3}, 0\right)$ lie on the graph. These points are x-intercepts of the function.

ⓒ A y-intercept occurs when $x = 0$. To find the y-intercepts we need to find $f(0)$.

$$f(x) = 3x^2 + 10x - 8$$

Find $f(0)$ by substituting 0 for x. $f(0) = 3 \cdot 0^2 + 10 \cdot 0 - 8$

Simplify. $f(0) = -8$

Since $f(0) = -8$, the point $(0, -8)$ lies on the graph. This point is the y-intercept of the function.

TRY IT : : 6.101

For the function $f(x) = 2x^2 - 7x + 5$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

TRY IT : : 6.102

For the function $f(x) = 6x^2 + 13x - 15$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Solve Applications Modeled by Polynomial Equations

The problem-solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to polynomial equations. We will copy the problem-solving strategy here so we can use it for reference.



HOW TO : : USE A PROBLEM SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what we are looking for.
Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.
Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
Step 5. **Solve** the equation using appropriate algebra techniques.
Step 6. **Check** the answer in the problem and make sure it makes sense.
Step 7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a polynomial equation.

EXAMPLE 6.52

The product of two consecutive odd integers is 323. Find the integers.

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. We are looking for two consecutive integers.

Step 3. Name what we are looking for. Let $n =$ the first integer.

$n + 2 =$ next consecutive odd integer

Step 4. Translate into an equation. Restate the problem in a sentence. The product of the two consecutive odd integers is 323.

$$n(n + 2) = 323$$

Step 5. Solve the equation.

$$n^2 + 2n = 323$$

Bring all the terms to one side. $n^2 + 2n - 323 = 0$

Factor the trinomial. $(n - 17)(n + 19) = 0$

Use the Zero Product Property. $n - 17 = 0 \qquad n + 19 = 0$

Solve the equations. $n = 17 \qquad n = -19$

There are two values for n that are solutions to this problem. So there are two sets of consecutive odd integers that will work.

If the first integer is $n = 17$ then the next odd integer is

$$n + 2$$
$$17 + 2$$
$$19$$

$17,\ 19$

If the first integer is $n = -19$ then the next odd integer is

$$n + 2$$
$$-19 + 2$$
$$-17$$

$-17,\ -19$

Step 6. Check the answer.

The results are consecutive odd integers $17,\ 19$ and $-19,\ -17$.

$17 \cdot 19 = 323$ ✓ $\qquad -19(-17) = 323$ ✓

Both pairs of consecutive integers are solutions.

Step 7. Answer the question. The consecutive integers are $17,\ 19$ and $-19,\ -17$.

TRY IT : : 6.103 The product of two consecutive odd integers is 255. Find the integers.

TRY IT : : 6.104 The product of two consecutive odd integers is 483 Find the integers.

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

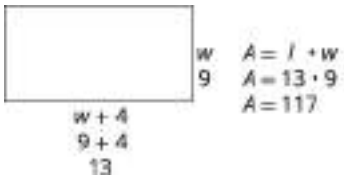
EXAMPLE 6.53

A rectangular bedroom has an area 117 square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

Solution

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.



Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what you are looking for.	Let $w =$ the width of the bedroom.
The length is four feet more than the width.	$w + 4 =$ the length of the garden
Step 4. Translate into an equation.	
Restate the important information in a sentence.	The area of the bedroom is 117 square feet.
Use the formula for the area of a rectangle.	$A = l \cdot w$
Substitute in the variables.	$117 = (w + 4)w$
Step 5. Solve the equation Distribute first.	$117 = w^2 + 4w$
Get zero on one side.	$117 = w^2 + 4w$
Factor the trinomial.	$0 = w^2 + 4w - 117$
Use the Zero Product Property.	$0 = (w^2 + 13)(w - 9)$
Solve each equation.	$0 = w + 13 \quad 0 = w - 9$
Since w is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for w.	$\cancel{-13 = w} \quad 9 = w$
	$w = 9$ Width is 9 feet.
Find the value of the length.	$w + 4$ $9 + 4$ 13 Length is 13 feet.
Step 6. Check the answer. Does the answer make sense?	
 Yes, this makes sense.	
Step 7. Answer the question.	The width of the bedroom is 9 feet and the length is 13 feet.

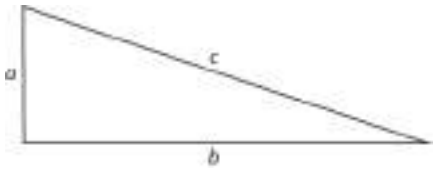
TRY IT : : 6.105

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

TRY IT : : 6.106

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

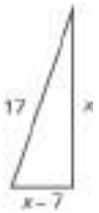
In the next example, we will use the Pythagorean Theorem $\left(a^2+b^2=c^2\right)$. This formula gives the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

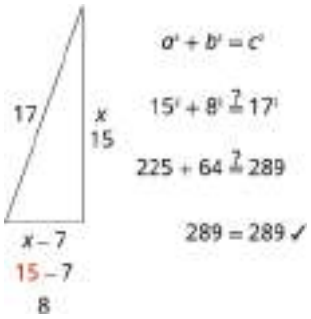
EXAMPLE 6.54

A boat's sail is in the shape of a right triangle as shown. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the sail.



Solution

Step 1. Read the problem	
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the sail.
Step 3. Name what you are looking for. One side is 7 less than the other.	Let $x=$ length of a side of the sail. $x-7=$ length of other side
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2+b^2=c^2$
Substitute in the variables.	$x^2+(x-7)^2=17^2$
Step 5. Solve the equation Simplify.	$x^2+x^2-14x+49=289$
	$2x^2-14x+49=289$
It is a quadratic equation, so get zero on one side.	$2x^2-14x-240=0$
Factor the greatest common factor.	$2(x^2-7x-120)=0$
Factor the trinomial.	$2(x-15)(x+8)=0$
Use the Zero Product Property.	$2\neq 0 \quad x-15=0 \quad x+8=0$
Solve.	$2\neq 0 \quad x=15 \quad x=-8$
Since x is a side of the triangle, $x=-8$ does not make sense.	$2\neq 0 \quad x=15 \quad \cancel{x=-8}$
Find the length of the other side.	

If the length of one side is then the length of the other side is	$x = 15$ $x - 7$ $15 - 7$ 8 is the length of the other side.
Step 6. Check the answer in the problem Do these numbers make sense?  $a^2 + b^2 = c^2$ $15^2 + 8^2 \stackrel{?}{=} 17^2$ $225 + 64 \stackrel{?}{=} 289$ $289 = 289$ ✓	
Step 7. Answer the question	The sides of the sail are 8, 15 and 17 feet.

TRY IT : : 6.107

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the deck.

TRY IT : : 6.108

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

The next example uses the function that gives the height of an object as a function of time when it is thrown from 80 feet above the ground.

EXAMPLE 6.55

Dennis is going to throw his rubber band ball upward from the top of a campus building. When he throws the rubber band ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find:

ⓐ the zeros of this function which tell us when the ball hits the ground

ⓑ when the ball will be 80 feet above the ground

ⓒ the height of the ball at $t = 2$ seconds.

Solution

ⓐ The zeros of this function are found by solving $h(t) = 0$. This will tell us when the ball will hit the ground.

	$h(t) = 0$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 0$
Factor the GCF, -16.	$-16(t^2 - 4t - 5) = 0$
Factor the trinomial.	$-16(t - 5)(t + 1) = 0$
Use the Zero Product Property.	$t - 5 = 0 \qquad t + 1 = 0$
Solve.	$t = 5 \qquad t = -1$

The result $t = 5$ tells us the ball will hit the ground 5 seconds after it is thrown. Since time cannot be negative, the result

$t = -1$ is discarded.

ⓑ The ball will be 80 feet above the ground when $h(t) = 80$.

$$h(t) = 80$$

Substitute in the polynomial for $h(t)$. $-16t^2 + 64t + 80 = 80$

Subtract 80 from both sides. $-16t^2 + 64t = 0$

Factor the GCF, $-16t$. $-16t(t - 4) = 0$

Use the Zero Product Property. $-16t = 0 \quad t - 4 = 0$

Solve. $t = 0 \quad t = 4$

The ball will be at 80 feet the moment Dennis tosses the ball and then 4 seconds later, when the ball is falling.

ⓒ To find the height ball at $t = 2$ seconds we find $h(2)$.

$$h(t) = -16t^2 + 64t + 80$$

To fin $h(2)$ substitute 2 for t. $h(2) = -16(2)^2 + 64 \cdot 2 + 80$

Simplify. $h(2) = 144$

After 2 seconds, the ball will be at 144 feet.

TRY IT : : 6.109

Genevieve is going to throw a rock from the top a trail overlooking the ocean. When she throws the rock upward from 160 feet above the ocean, the function $h(t) = -16t^2 + 48t + 160$ models the height, h, of the rock above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which tell us when the rock will hit the ocean
ⓑ when the rock will be 160 feet above the ocean.
ⓒ the height of the rock at $t = 1.5$ seconds.

TRY IT : : 6.110

Calib is going to throw his lucky penny from his balcony on a cruise ship. When he throws the penny upward from 128 feet above the ground, the function $h(t) = -16t^2 + 32t + 128$ models the height, h, of the penny above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which is when the penny will hit the ocean
ⓑ when the penny will be 128 feet above the ocean.
ⓒ the height the penny will be at $t = 1$ seconds which is when the penny will be at its highest point.

MEDIA : :

Access this online resource for additional instruction and practice with quadratic equations.

- **Beginning Algebra & Solving Quadratics with the Zero Property (https://openstax.org/l/37ZeroProperty)**

6.5 EXERCISES

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

277. $(3a-10)(2a-7)=0$

278. $(5b+1)(6b+1)=0$

279. $6m(12m-5)=0$

280. $2x(6x-3)=0$

281. $(2x-1)^2=0$

282. $(3y+5)^2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

283. $5a^2-26a=24$

284. $4b^2+7b=-3$

285. $4m^2=17m-15$

286. $n^2=5-6n$

287. $7a^2+14a=7a$

288. $12b^2-15b=-9b$

289. $49m^2=144$

290. $625=x^2$

291. $16y^2=81$

292. $64p^2=225$

293. $121n^2=36$

294. $100y^2=9$

295. $(x+6)(x-3)=-8$

296. $(p-5)(p+3)=-7$

297. $(2x+1)(x-3)=-4x$

298. $(y-3)(y+2)=4y$

299. $(3x-2)(x+4)=12x$

300. $(2y-3)(3y-1)=8y$

301. $20x^2-60x=-45$

302. $3y^2-18y=-27$

303. $15x^2-10x=40$

304. $14y^2-77y=-35$

305. $18x^2-9=-21x$

306. $16y^2+12=-32x$

307. $16p^3=24p^2+9p$

308. $m^3-2m^2=-m$

309. $2x^3+72x=24x^2$

310. $3y^3+48y=24y^2$

311. $36x^3+24x^2=-4x$

312. $2y^3+2y^2=12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

313. For the function, $f(x)=x^2-8x+8$, ⓐ find when $f(x)=-4$ ⓑ Use this information to find two points that lie on the graph of the function.

314. For the function, $f(x)=x^2+11x+20$, ⓐ find when $f(x)=-8$ ⓑ Use this information to find two points that lie on the graph of the function.

315. For the function, $f(x)=8x^2-18x+5$, ⓐ find when $f(x)=-4$ ⓑ Use this information to find two points that lie on the graph of the function.

316. For the function, $f(x)=18x^2+15x-10$, ⓐ find when $f(x)=15$ ⓑ Use this information to find two points that lie on the graph of the function.

In the following exercises, for each function, find: ⓐ the zeros of the function ⓑ the x-intercepts of the graph of the function ⓒ the y-intercept of the graph of the function.

317. $f(x)=9x^2-4$

318. $f(x)=25x^2-49$

319. $f(x) = 6x^2 - 7x - 5$

320. $f(x) = 12x^2 - 11x + 2$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

321. The product of two consecutive odd integers is 143. Find the integers.

322. The product of two consecutive odd integers is 195. Find the integers.

323. The product of two consecutive even integers is 168. Find the integers.

324. The product of two consecutive even integers is 288. Find the integers.

325. The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

326. A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

327. The area of a bulletin board is 55 feet. The length is four feet less than three times the width. Find the length and the width of the a bulletin board.

328. A rectangular carport has area 150 square feet. The height of the carport is five feet less than twice its length. Find the height and the length of the carport.

329. A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

330. A stained glass window is shaped like a right triangle. The hypotenuse is 15 feet. One leg is three more than the other. Find the lengths of the legs.

331. A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

332. A goat enclosure is in the shape of a right triangle. One leg of the enclosure is built against the side of the barn. The other leg is 4 feet more than the leg against the barn. The hypotenuse is 8 feet more than the leg along the barn. Find the three sides of the goat enclosure.

333. Juli is going to launch a model rocket in her back yard. When she launches the rocket, the function $h(t) = -16t^2 + 32t$ models the height, *h*, of the rocket above the ground as a function of time, *t*. Find:

ⓐ the zeros of this function which tells us when the penny will hit the ground. ⓑ the time the rocket will be 16 feet above the ground.

334. Gianna is going to throw a ball from the top floor of her middle school. When she throws the ball from 48 feet above the ground, the function $h(t) = -16t^2 + 32t + 48$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find:

ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 48 feet above the ground. ⓒ the height the ball will be at $t = 1$ seconds which is when the ball will be at its highest point.

Writing Exercises

335. Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

336. Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ *Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

CHAPTER 6 REVIEW

KEY TERMS

degree of the polynomial equation The degree of the polynomial equation is the degree of the polynomial.

factoring Splitting a product into factors is called factoring.

greatest common factor The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

polynomial equation A polynomial equation is an equation that contains a polynomial expression.

quadratic equation Polynomial equations of degree two are called quadratic equations.

zero of the function A value of x where the function is 0, is called a zero of the function.

Zero Product Property The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

KEY CONCEPTS

6.1 Greatest Common Factor and Factor by Grouping

- **How to find the greatest common factor (GCF) of two expressions.**

 Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.

 Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.

 Step 3. Bring down the common factors that all expressions share.

 Step 4. Multiply the factors.

- **Distributive Property:** If *a*, *b*, and *c* are real numbers, then

$$a(b+c)=ab+ac \quad \text{and} \quad ab+ac=a(b+c)$$

 The form on the left is used to multiply. The form on the right is used to factor.

- **How to factor the greatest common factor from a polynomial.**

 Step 1. Find the GCF of all the terms of the polynomial.

 Step 2. Rewrite each term as a product using the GCF.

 Step 3. Use the "reverse" Distributive Property to factor the expression.

 Step 4. Check by multiplying the factors.

- **Factor as a Noun and a Verb:** We use "factor" as both a noun and a verb.

Noun: 7 is a *factor* of 14

Verb: *factor* 3 from $3a+3$

- **How to factor by grouping.**

 Step 1. Group terms with common factors.

 Step 2. Factor out the common factor in each group.

 Step 3. Factor the common factor from the expression.

 Step 4. Check by multiplying the factors.

6.2 Factor Trinomials

- **How to factor trinomials of the form x^2+bx+c.**

 Step 1. Write the factors as two binomials with first terms *x*. x^2+bx+c $(x\quad)(x\quad)$

 Step 2. Find two numbers *m* and *n* that
 multiply to c, $m\cdot n=c$
 add to b, $m+n=b$

 Step 3. Use *m* and *n* as the last terms of the factors. $(x+m)(x+n)$

Step 4. Check by multiplying the factors.

- **Strategy for Factoring Trinomials of the Form x^2+bx+c:** When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$x^2+bx+c$$
$$(x+m)(x+n)$$

When c is positive, m and n have the same sign.

b positive	b negative
$m,\ n$ positive	$m,\ n$ negative
x^2+5x+6	x^2-6x+8
$(x+2)(x+3)$	$(x-4)(x-2)$
same signs	same signs

When c is negative, m and n have opposite signs.

x^2+x-12	$x^2-2x-15$
$(x+4)(x-3)$	$(x-5)(x+3)$
opposite signs	opposite signs

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

- **How to factor trinomials of the form ax^2+bx+c using trial and error.**

Step 1. Write the trinomial in descending order of degrees as needed.

Step 2. Factor any GCF.

Step 3. Find all the factor pairs of the first term.

Step 4. Find all the factor pairs of the third term.

Step 5. Test all the possible combinations of the factors until the correct product is found.

Step 6. Check by multiplying.

- **How to factor trinomials of the form ax^2+bx+c using the "ac" method.**

Step 1. Factor any GCF.

Step 2. Find the product ac.

Step 3. Find two numbers m and n that:

Multiply to ac.	$m\cdot n=a\cdot c$
Add to b.	$m+n=b$
	ax^2+bx+c

Step 4. Split the middle term using m and n. $ax^2+mx+nx+c$

Step 5. Factor by grouping.

Step 6. Check by multiplying the factors.

6.3 Factor Special Products

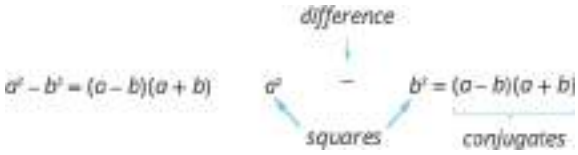
- **Perfect Square Trinomials Pattern:** If a and b are real numbers,

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

- **How to factor perfect square trinomials.**

Step 1. Does the trinomial fit he pattern?	$a^2+2ab+b^2$	$a^2-2ab+b^2$
Is the fir t term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
Is the last term a perfect square? Write it as a square.	$(a)^2$ $(b)^2$	$(a)^2$ $(b)^2$
Check the middle term. Is it $2ab$?	$(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$	$(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$
Step 2. Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3. Check by multiplying.		

- **Difference of Squares Pattern:** If a, b are real numbers,



- **How to factor differences of squares.**

Step 1. Does the binomial fit he pattern?	a^2-b^2
Is this a diffe ence?	____ − ____
Are the fir t and last terms perfect squares?	
Step 2. Write them as squares.	$(a)^2-(b)^2$
Step 3. Write the product of conjugates.	$(a-b)(a+b)$
Step 4. Check by multiplying.	

- **Sum and Difference of Cubes Pattern**

$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$

$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$

- **How to factor the sum or difference of cubes.**

Step 1. Does the binomial fit the sum or difference of cubes pattern?
Is it a sum or difference?
Are the first and last terms perfect cubes?

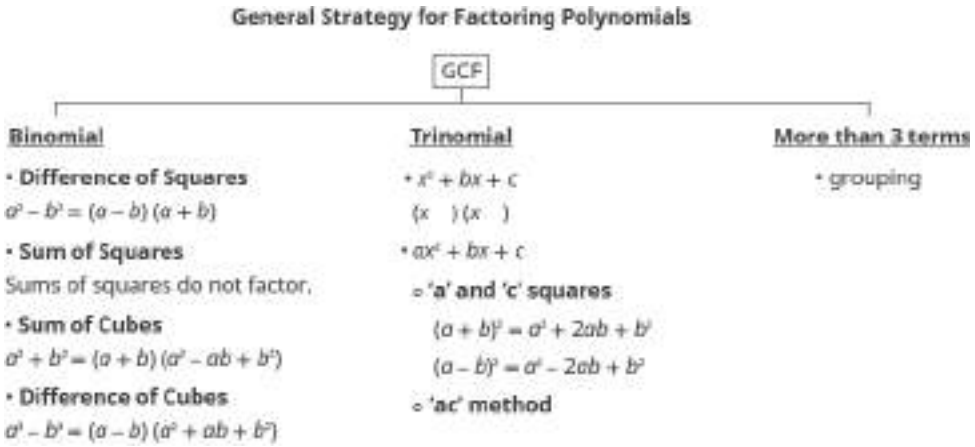
Step 2. Write them as cubes.

Step 3. Use either the sum or difference of cubes pattern.

Step 4. Simplify inside the parentheses

Step 5. Check by multiplying the factors.

6.4 General Strategy for Factoring Polynomials



- **How to use a general strategy for factoring polynomials.**

 Step 1. Is there a greatest common factor?
 Factor it out.

 Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
 If it is a binomial:
 Is it a sum?
 Of squares? Sums of squares do not factor.
 Of cubes? Use the sum of cubes pattern.
 Is it a difference?
 Of squares? Factor as the product of conjugates.
 Of cubes? Use the difference of cubes pattern.
 If it is a trinomial:
 Is it of the form $x^2 + bx + c$? Undo FOIL.
 Is it of the form $ax^2 + bx + c$?
 If *a* and *c* are squares, check if it fits the trinomial square pattern.
 Use the trial and error or "ac" method.
 If it has more than three terms:
 Use the grouping method.

 Step 3. Check.
 Is it factored completely?
 Do the factors multiply back to the original polynomial?

6.5 Polynomial Equations

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

$$a,\ b,\ c \text{ are real numbers and } a \neq 0$$

- **Zero Product Property:** If $a \cdot b = 0,$ then either $a = 0$ or $b = 0$ or both.
- **How to use the Zero Product Property**

 Step 1. Set each factor equal to zero.

 Step 2. Solve the linear equations.

 Step 3. Check.

- **How to solve a quadratic equation by factoring.**

 Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0.$

 Step 2. Factor the quadratic expression.

 Step 3. Use the Zero Product Property.

Step 4. Solve the linear equations.

Step 5. Check. Substitute each solution separately into the original equation.

- **Zero of a Function:** For any function *f*, if $f(x) = 0,$ then *x* is a zero of the function.
- **How to use a problem solving strategy to solve word problems.**

 Step 1. **Read** the problem. Make sure all the words and ideas are understood.

 Step 2. **Identify** what we are looking for.

 Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

 Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.

 Step 5. **Solve** the equation using appropriate algebra techniques.

 Step 6. **Check** the answer in the problem and make sure it makes sense.

 Step 7. **Answer** the question with a complete sentence.

REVIEW EXERCISES

6.1 Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

337. $12a^2b^3,\ 15ab^2$

338. $12m^2n^3,\ 42m^5n^3$

339. $15y^3,\ 21y^2,\ 30y$

340. $45x^3y^2,\ 15x^4y,\ 10x^5y^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

341. $35y + 84$

342. $6y^2 + 12y - 6$

343. $18x^3 - 15x$

344. $15m^4 + 6m^2n$

345. $4x^3 - 12x^2 + 16x$

346. $-3x + 24$

347. $-3x^3 + 27x^2 - 12x$

348. $3x(x - 1) + 5(x - 1)$

Factor by Grouping

In the following exercises, factor by grouping.

349. $ax - ay + bx - by$

350. $x^2y - xy^2 + 2x - 2y$

351. $x^2 + 7x - 3x - 21$

352. $4x^2 - 16x + 3x - 12$

353. $m^3 + m^2 + m + 1$

354. $5x - 5y - y + x$

6.2 Factor Trinomials

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

355. $a^2 + 14a + 33$

356. $k^2 - 16k + 60$

357. $m^2 + 3m - 54$

358. $x^2 - 3x - 10$

In the following examples, factor each trinomial of the form $x^2 + bxy + cy^2$.

359. $x^2 + 12xy + 35y^2$ **360.** $r^2 + 3rs - 28s^2$ **361.** $a^2 + 4ab - 21b^2$

362. $p^2 - 5pq - 36q^2$ **363.** $m^2 - 5mn + 30n^2$

Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

364. $x^3 + 5x^2 - 24x$ **365.** $3y^3 - 21y^2 + 30y$ **366.** $5x^4 + 10x^3 - 75x^2$

367. $5y^2 + 14y + 9$ **368.** $8x^2 + 25x + 3$ **369.** $10y^2 - 53y - 11$

370. $6p^2 - 19pq + 10q^2$ **371.** $-81a^2 + 153a + 18$

Factor Trinomials of the Form $ax^2 + bx + c$ using the 'ac' Method

In the following exercises, factor.

372. $2x^2 + 9x + 4$ **373.** $18a^2 - 9a + 1$ **374.** $15p^2 + 2p - 8$

375. $15x^2 + 6x - 2$ **376.** $8a^2 + 32a + 24$ **377.** $3x^2 + 3x - 36$

378. $48y^2 + 12y - 36$ **379.** $18a^2 - 57a - 21$ **380.** $3n^4 - 12n^3 - 96n^2$

Factor using substitution

In the following exercises, factor using substitution.

381. $x^4 - 13x^2 - 30$ **382.** $(x - 3)^2 - 5(x - 3) - 36$

6.3 Factor Special Products

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

383. $25x^2 + 30x + 9$ **384.** $36a^2 - 84ab + 49b^2$ **385.** $40x^2 + 360x + 810$

386. $5k^3 - 70k^2 + 245k$ **387.** $75u^4 - 30u^3v + 3u^2v^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

388. $81r^2 - 25$ **389.** $169m^2 - n^2$ **390.** $25p^2 - 1$

391. $9 - 121y^2$ **392.** $20x^2 - 125$ **393.** $169n^3 - n$

394. $6p^2q^2 - 54p^2$ **395.** $24p^2 + 54$ **396.** $49x^2 - 81y^2$

397. $16z^4 - 1$ **398.** $48m^4n^2 - 243n^2$ **399.** $a^2 + 6a + 9 - 9b^2$

400. $x^2 - 16x + 64 - y^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

401. $a^3 - 125$

402. $b^3 - 216$

403. $2m^3 + 54$

404. $81m^3 + 3$

6.4 General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

405. $24x^3 + 44x^2$

406. $24a^4 - 9a^3$

407. $16n^2 - 56mn + 49m^2$

408. $6a^2 - 25a - 9$

409. $5u^4 - 45u^2$

410. $n^4 - 81$

411. $64j^2 + 225$

412. $5x^2 + 5x - 60$

413. $b^3 - 64$

414. $m^3 + 125$

415. $2b^2 - 2bc + 5cb - 5c^2$

416. $48x^5y^2 - 243xy^2$

417. $5q^2 - 15q - 90$

418. $4u^5v + 4u^2v^3$

419. $10m^4 - 6250$

420. $60x^2y - 75xy + 30y$

421. $16x^2 - 24xy + 9y^2 - 64$

6.5 Polynomial Equations

Use the Zero Product Property

In the following exercises, solve.

422. $(a - 3)(a + 7) = 0$

423. $(5b + 1)(6b + 1) = 0$

424. $6m(12m - 5) = 0$

425. $(2x - 1)^2 = 0$

426. $3m(2m - 5)(m + 6) = 0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

427. $x^2 + 9x + 20 = 0$

428. $y^2 - y - 72 = 0$

429. $2p^2 - 11p = 40$

430. $q^3 + 3q^2 + 2q = 0$

431. $144m^2 - 25 = 0$

432. $4n^2 = 36$

433. $(x + 6)(x - 3) = -8$

434. $(3x - 2)(x + 4) = 12x$

435. $16p^3 = 24p^2 + 9p$

436. $2y^3 + 2y^2 = 12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

437. For the function, $f(x) = x^2 + 11x + 20$, ⓐ find when $f(x) = -8$ ⓑ Use this information to find two points that lie on the graph of the function.

438. For the function, $f(x) = 9x^2 - 18x + 5$, ⓐ find when $f(x) = -3$ ⓑ Use this information to find two points that lie on the graph of the function.

In each function, find: ⓐ the zeros of the function ⓑ the x-intercepts of the graph of the function ⓒ the y-intercept of the graph

of the function.

439. $f(x) = 64x^2 - 49$

440. $f(x) = 6x^2 - 13x - 5$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

441. The product of two consecutive numbers is 399. Find the numbers.

442. The area of a rectangular shaped patio 432 square feet. The length of the patio is 6 feet more than its width. Find the length and width.

443. A ladder leans against the wall of a building. The length of the ladder is 9 feet longer than the distance of the bottom of the ladder from the building. The distance of the top of the ladder reaches up the side of the building is 7 feet longer than the distance of the bottom of the ladder from the building. Find the lengths of all three sides of the triangle formed by the ladder leaning against the building.

444. Shruti is going to throw a ball from the top of a cliff. When she throws the ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 80 feet above the ground. ⓒ the height the ball will be at $t = 2$ seconds which is when the ball will be at its highest point.

PRACTICE TEST

In the following exercises, factor completely.

445. $80a^2 + 120a^3$

446. $5m(m-1) + 3(m-1)$

447. $x^2 + 13x + 36$

448. $p^2 + pq - 12q^2$

449. $xy - 8y + 7x - 56$

450. $40r^2 + 810$

451. $9s^2 - 12s + 4$

452. $6x^2 - 11x - 10$

453. $3x^2 - 75y^2$

454. $6u^2 + 3u - 18$

455. $x^3 + 125$

456. $32x^5y^2 - 162xy^2$

457. $6x^4 - 19x^2 + 15$

458. $3x^3 - 36x^2 + 108x$

In the following exercises, solve

459. $5a^2 + 26a = 24$

460. The product of two consecutive integers is 156. Find the integers.

461. The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

462. Jing is going to throw a ball from the balcony of her condo. When she throws the ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 128 feet above the ground. ⓒ the height the ball will be at $t = 4$ seconds.

463. For the function, $f(x) = x^2 - 7x + 5$, ⓐ find when $f(x) = -7$ ⓑ Use this information to find two points that lie on the graph of the function.

464. For the function $f(x) = 25x^2 - 81$, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.



7 RATIONAL EXPRESSIONS AND FUNCTIONS

Figure 7.1 American football is the most watched spectator sport in the United States. People around the country are constantly tracking statistics for football and other sports. (credit: "keijj44" / Pixabay)

Chapter Outline

7.1 Multiply and Divide Rational Expressions
7.2 Add and Subtract Rational Expressions
7.3 Simplify Complex Rational Expressions
7.4 Solve Rational Equations
7.5 Solve Applications with Rational Equations
7.6 Solve Rational Inequalities

Introduction

Twelve goals last season. Fifteen home runs. Nine touchdowns. Whatever the statistics, sports analysts know it. Their jobs depend on it. Compiling and analyzing sports data not only help fans appreciate their teams but also help owners and coaches decide which players to recruit, how to best use them in games, how much they should be paid, and which players to trade. Understanding this kind of data requires a knowledge of specific types of expressions and functions. In this chapter, you will work with rational expressions and perform operations on them. And you will use rational expressions and inequalities to solve real-world problems.

7.1 Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{90y}{15y^2}$.

 If you missed this problem, review Example 5.13.

2. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
 If you missed this problem, review **Example 1.25**.
3. Divide: $\frac{12}{10} \div \frac{8}{25}$.
 If you missed this problem, review **Example 1.26**.

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

Rational Expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{24}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{24}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.



HOW TO : : DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

Step 1. Set the denominator equal to zero.

Step 2. Solve the equation.

EXAMPLE 7.1

Determine the value for which each rational expression is undefined:

ⓐ $\frac{8a^2b}{3c}$ ⓑ $\frac{4b-3}{2b+5}$ ⓒ $\frac{x+4}{x^2+5x+6}$.

Solution

The expression will be undefined when the denominator is zero.

ⓐ

$$\frac{8a^2b}{3c}$$

Set the denominator equal to zero and solve for the variable.

$$3c = 0$$

$$c = 0$$

$\frac{8a^2b}{3c}$ is undefined or $c = 0$.

ⓑ

$$\frac{4b-3}{2b+5}$$

Set the denominator equal to zero and solve for the variable.

$$\begin{aligned} 2b+5 &= 0 \\ 2b &= -5 \\ b &= -\frac{5}{2} \end{aligned}$$

$\frac{4b-3}{2b+5}$ is undefined or $b = -\frac{5}{2}$.

ⓒ

$$\frac{x+4}{x^2+5x+6}$$

Set the denominator equal to zero and solve for the variable.

$$x^2+5x+6 = 0$$

$$(x+2)(x+3) = 0$$

$$x+2 = 0 \text{ or } x+3 = 0$$

$$x = -2 \text{ or } x = -3$$

$\frac{x+4}{x^2+5x+6}$ is undefined or $x = -2$ or $x = -3$.

TRY IT : : 7.1 Determine the value for which each rational expression is undefined.

ⓐ $\frac{3y^2}{8x}$ ⓑ $\frac{8n-5}{3n+1}$ ⓒ $\frac{a+10}{a^2+4a+3}$

TRY IT : : 7.2 Determine the value for which each rational expression is undefined.

ⓐ $\frac{4p}{5q}$ ⓑ $\frac{y-1}{3y+2}$ ⓒ $\frac{m-5}{m^2+m-6}$

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

Simplified Rational Expression

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$\frac{x+2}{x+3}$ is simplified because here are no common factors of $x+2$ and $x+3$.

$\frac{2x}{3x}$ is not simplified becaus x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Equivalent Fractions Property

If *a*, *b*, and *c* are numbers where $b \neq 0$, $c \neq 0$,

$$\text{then } \frac{a}{b}=\frac{a\cdot c}{b\cdot c} \quad \text{and} \quad \frac{a\cdot c}{b\cdot c}=\frac{a}{b}.$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b\neq 0,\ c\neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2\cdot 3\cdot 7}{3\cdot 5\cdot 7}$ $\frac{3x(x-9)}{5(x-9)}$ where $x\neq 9$ $\frac{x+5}{x}$

$\frac{2}{5}$ $\frac{3x}{5}$ NO COMMON FACTORS

We removed the common factors 3 and 7. They are factors of the product. We removed the common factor $(x-9)$. It is a factor of the product. While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

EXAMPLE 7.2 HOW TO SIMPLIFY A RATIONAL EXPRESSION

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor x^2+5x+6 and $x^2+8x+12$.	$\frac{x^2+5x+6}{x^2+8x+12}$ $\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Step 2. Simplify by dividing out common factors.	Remove the common factor $x+2$ from the numerator and the denominator.	$\frac{(x+2)(x+3)}{(x+2)(x+6)}$ $\frac{(x+3)}{(x+6)}$ $x\neq -2\ x\neq -6$

TRY IT : : 7.3 Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

TRY IT : : 7.4 Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

We now summarize the steps you should follow to simplify rational expressions.

HOW TO : : SIMPLIFY A RATIONAL EXPRESSION.

Step 1. Factor the numerator and denominator completely.

Step 2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all*

the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

EXAMPLE 7.3

Simplify: $\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$.

Solution

$$\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$$

Factor the numerator and denominator, fir t factoring out the GCF.

$$\frac{3\left(a^2-4ab+4b^2\right)}{6\left(a^2-4b^2\right)}$$

$$\frac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$$

Remove the common factors of $a-2b$ and 3.

$$\frac{\cancel{3}(a-2b)\cancel{(a-2b)}}{\cancel{3}\cdot 2(a+2b)\cancel{(a-2b)}}$$

$$\frac{a-2b}{2(a+2b)}$$

TRY IT : : 7.5 Simplify: $\frac{2x^2-12xy+18y^2}{3x^2-27y^2}$.

TRY IT : : 7.6 Simplify: $\frac{5x^2-30xy+25y^2}{2x^2-50y^2}$.

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a=-1\cdot a$.

The numerical fraction, say $\frac{7}{-7}$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $\frac{a}{-a}$, whose numerator and denominator are opposites also simplifies to -1.

Let's look at the expression $b-a$.	$b-a$
Rewrite.	$-a+b$
Factor out –1.	$-1(a-b)$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

Opposites in a Rational Expression

The opposite of $a-b$ is $b-a$.

$$\frac{a-b}{b-a}=-1 \quad a\neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a \neq -b$, then $\frac{a+b}{b+a}=1$.

EXAMPLE 7.4

Simplify: $\frac{x^2-4x-32}{64-x^2}$.

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Simplify.	$\frac{x+4}{x+8}$

TRY IT : : 7.7 Simplify: $\frac{x^2-4x-5}{25-x^2}$.

TRY IT : : 7.8 Simplify: $\frac{x^2+x-2}{1-x^2}$.

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Multiplication of Rational Expressions

If *p*, *q*, *r*, and *s* are polynomials where $q \neq 0,\ s \neq 0,$ then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0,\ x \neq 3,$ and $x \neq 4$.

EXAMPLE 7.5 HOW TO MULTIPLY RATIONAL EXPRESSIONS

Simplify: $\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$.

Solution

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$ $\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$

Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^1(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\frac{2 \cdot x(x-3)(x+3)}{2 \cdot 3 \cdot x \cdot x(x-3)(x-4)}$ $\frac{(x+3)}{3x(x-4)}$

TRY IT : : 7.9 Simplify: $\frac{5x}{x^2+5x+6} \cdot \frac{x^2-4}{10x}$.

TRY IT : : 7.10 Simplify: $\frac{9x^2}{x^2+11x+30} \cdot \frac{x^2-36}{3x^2}$.

HOW TO : : MULTIPLY RATIONAL EXPRESSIONS.

Step 1. Factor each numerator and denominator completely.
Step 2. Multiply the numerators and denominators.
Step 3. Simplify by dividing out common factors.

EXAMPLE 7.6

Multiply: $\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$.

Solution

$$\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$$

Factor the numerators and denominators and then multiply.
$$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

Simplify by dividing out common factors.
$$\frac{\cancel{(3a+1)}(a-3)\cancel{(a+5)}(a+5)}{(a-5)\cancel{(a+5)}\cancel{(3a+1)}(a-5)}$$

Simplify.
$$\frac{(a-3)(a+5)}{(a-5)(a-5)}$$

Rewrite $(a-5)(a-5)$ using an exponent.
$$\frac{(a-3)(a+5)}{(a-5)^2}$$

TRY IT : : 7.11 Simplify: $\frac{2x^2+5x-12}{x^2-16} \cdot \frac{x^2-8x+16}{2x^2-13x+15}$.

TRY IT : : 7.12 Simplify: $\frac{4b^2+7b-2}{1-b^2} \cdot \frac{b^2-2b+1}{4b^2+15b-4}$.

Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

Division of Rational Expressions

If *p*, *q*, *r*, and *s* are polynomials where $q \neq 0,\ r \neq 0,\ s \neq 0,$ then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

EXAMPLE 7.7 HOW TO DIVIDE RATIONAL EXPRESSIONS

Divide: $\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \div \dfrac{p^2-q^2}{6}$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \div \dfrac{p^2-q^2}{6}$ $\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \dfrac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\dfrac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \dfrac{2 \cdot 3}{(p-q)(p+q)}$
Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\dfrac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\dfrac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$ $\dfrac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

TRY IT : : 7.13 Simplify: $\dfrac{x^3-8}{3x^2-6x+12} \div \dfrac{x^2-4}{6}$.

TRY IT : : 7.14 Simplify: $\dfrac{2z^2}{z^2-1} \div \dfrac{z^3-z^2+z}{z^3+1}$.

HOW TO : : DIVIDE RATIONAL EXPRESSIONS.

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.

Step 2. Factor the numerators and denominators completely.

Step 3. Multiply the numerators and denominators together.

Step 4. Simplify by dividing out common factors.

Recall from **Use the Language of Algebra** that a complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Also, remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

EXAMPLE 7.8

Divide: $\dfrac{\dfrac{6x^2-7x+2}{4x-8}}{\dfrac{2x^2-7x+3}{x^2-5x+6}}$.

Solution

$$\frac{\dfrac{6x^2-7x+2}{4x-8}}{\dfrac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign. $\dfrac{6x^2-7x+2}{4x-8} \div \dfrac{2x^2-7x+3}{x^2-5x+6}$

Rewrite as product of fir t times reciprocal of second. $\dfrac{6x^2-7x+2}{4x-8} \cdot \dfrac{x^2-5x+6}{2x^2-7x+3}$

Factor the numerators and the denominators, and then multiply. $\dfrac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$

Simplify by dividing out common factors. $\dfrac{\cancel{(2x-1)}(3x-2)\cancel{(x-2)}\cancel{(x-3)}}{4\cancel{(x-2)}\cancel{(2x-1)}\cancel{(x-3)}}$

Simplify. $\dfrac{3x-2}{4}$

TRY IT : : 7.15

Simplify: $\dfrac{\dfrac{3x^2+7x+2}{4x+24}}{\dfrac{3x^2-14x-5}{x^2+x-30}}$.

TRY IT : : 7.16

Simplify: $\dfrac{\dfrac{y^2-36}{2y^2+11y-6}}{\dfrac{2y^2-2y-60}{8y-4}}$.

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

EXAMPLE 7.9

Perform the indicated operations: $\dfrac{3x-6}{4x-4} \cdot \dfrac{x^2+2x-3}{x^2-3x-10} \div \dfrac{2x+12}{8x+16}$.

Solution

	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \cdot \frac{8x+16}{2x+12}$
Factor the numerators and the denominators.	$\frac{3(x-2)}{4(x-1)} \cdot \frac{(x+3)(x-1)}{(x+2)(x-5)} \cdot \frac{8(x+2)}{2(x+6)}$
Multiply the fractions. Bringing the constants to the front will help when removing common factors.	
Simplify by dividing out common factors.	$\frac{3 \cdot 8(x-2)(x+3)(x-1)(x+2)}{4 \cdot 2(x-1)(x+2)(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

TRY IT : : 7.17 Perform the indicated operations: $\frac{4m+4}{3m-15} \cdot \frac{m^2-3m-10}{m^2-4m-32} \div \frac{12m-36}{6m-48}$.

TRY IT : : 7.18 Perform the indicated operations: $\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

Multiply and Divide Rational Functions

We started this section stating that a rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$. Similarly, we define a **rational function** as a function of the form $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

Rational Function

A rational function is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

The domain of a rational function is all real numbers except for those values that would cause division by zero. We must eliminate any values that make $q(x) = 0$.

HOW TO : : DETERMINE THE DOMAIN OF A RATIONAL FUNCTION.

Step 1. Set the denominator equal to zero.
Step 2. Solve the equation.
Step 3. The domain is all real numbers excluding the values found in Step 2.

EXAMPLE 7.10

Find the domain of $R(x) = \dfrac{2x^2 - 14x}{4x^2 - 16x - 48}$.

Solution

The domain will be all real numbers except those values that make the denominator zero. We will set the denominator equal to zero , solve that equation, and then exclude those values from the domain.

Set the denominator to zero.	$4x^2 - 16x - 48 = 0$
Factor, fir t factor out the GCF.	$4\left(x^2 - 4x - 12\right) = 0$
	$4(x - 6)(x + 2) = 0$
Use the Zero Product Property.	$4 \neq 0 \quad x - 6 = 0 \quad x + 2 = 0$
Solve.	$x = 6 \quad x = -2$
	The domain of $R(x)$ is all real numbers where $x \neq 6$ and $x \neq -2$.

TRY IT : : 7.19 Find the domain of $R(x) = \dfrac{2x^2 - 10x}{4x^2 - 16x - 20}$.

TRY IT : : 7.20 Find the domain of $R(x) = \dfrac{4x^2 - 16x}{8x^2 - 16x - 64}$.

To multiply rational functions, we multiply the resulting rational expressions on the right side of the equation using the same techniques we used to multiply rational expressions.

EXAMPLE 7.11

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{2x - 6}{x^2 - 8x + 15}$ and $g(x) = \dfrac{x^2 - 25}{2x + 10}$.

Solution

$$R(x) = f(x) \cdot g(x)$$

$$R(x) = \frac{2x - 6}{x^2 - 8x + 15} \cdot \frac{x^2 - 25}{2x + 10}$$

Factor each numerator and denominator. $$R(x) = \frac{2(x - 3)}{(x - 3)(x - 5)} \cdot \frac{(x - 5)(x + 5)}{2(x + 5)}$$

Multiply the numerators and denominators. $$R(x) = \frac{2(x - 3)(x - 5)(x + 5)}{2(x - 3)(x - 5)(x + 5)}$$

Remove common factors. $$R(x) = \frac{\cancel{2}\cancel{(x - 3)}\cancel{(x - 5)}\cancel{(x + 5)}}{\cancel{2}\cancel{(x - 3)}\cancel{(x - 5)}\cancel{(x + 5)}}$$

Simplify. $$R(x) = 1$$

TRY IT : : 7.21 Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{3x - 21}{x^2 - 9x + 14}$ and $g(x) = \dfrac{2x^2 - 8}{3x + 6}$.

TRY IT : : 7.22 Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{x^2 - x}{3x^2 + 27x - 30}$ and $g(x) = \dfrac{x^2 - 100}{x^2 - 10x}$.

To divide rational functions, we divide the resulting rational expressions on the right side of the equation using the same techniques we used to divide rational expressions.

EXAMPLE 7.12

Find $R(x)=\frac{f(x)}{g(x)}$ where $f(x)=\frac{3x^2}{x^2-4x}$ and $g(x)=\frac{9x^2-45x}{x^2-7x+10}$.

Solution

$$R(x)=\frac{f(x)}{g(x)}$$

Substitute in the functions $f(x)$, $g(x)$.

$$R(x)=\frac{\frac{3x^2}{x^2-4x}}{\frac{9x^2-45x}{x^2-7x+10}}$$

Rewrite the division as the product of $f(x)$ and the reciprocal of $g(x)$.

$$R(x)=\frac{3x^2}{x^2-4x}\cdot\frac{x^2-7x+10}{9x^2-45x}$$

Factor the numerators and denominators and then multiply.

$$R(x)=\frac{3\cdot x\cdot x\cdot(x-5)(x-2)}{x(x-4)\cdot 3\cdot 3\cdot x\cdot(x-5)}$$

Simplify by dividing out common factors.

$$R(x)=\frac{\cancel{3}\cdot\cancel{x}\cdot\cancel{x}\cancel{(x-5)}(x-2)}{\cancel{x}(x-4)\cdot\cancel{3}\cdot 3\cdot\cancel{x}\cancel{(x-5)}}$$

$$R(x)=\frac{x-2}{3(x-4)}$$

TRY IT : : 7.23 Find $R(x)=\frac{f(x)}{g(x)}$ where $f(x)=\frac{2x^2}{x^2-8x}$ and $g(x)=\frac{8x^2+24x}{x^2+x-6}$.

TRY IT : : 7.24 Find $R(x)=\frac{f(x)}{g(x)}$ where $f(x)=\frac{15x^2}{3x^2+33x}$ and $g(x)=\frac{5x-5}{x^2+9x-22}$.

7.1 EXERCISES

Practice Makes Perfect

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

1.

ⓐ $\frac{2x^2}{z}$

ⓑ $\frac{4p-1}{6p-5}$

ⓒ $\frac{n-3}{n^2+2n-8}$

2.

ⓐ $\frac{10m}{11n}$

ⓑ $\frac{6y+13}{4y-9}$

ⓒ $\frac{b-8}{b^2-36}$

3.

ⓐ $\frac{4x^2y}{3y}$

ⓑ $\frac{3x-2}{2x+1}$

ⓒ $\frac{u-1}{u^2-3u-28}$

4.

ⓐ $\frac{5pq^2}{9q}$

ⓑ $\frac{7a-4}{3a+5}$

ⓒ $\frac{1}{x^2-4}$

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

5. $-\frac{44}{55}$

6. $\frac{56}{63}$

7. $\frac{8m^3n}{12mn^2}$

8. $\frac{36v^3w^2}{27vw^3}$

9. $\frac{8n-96}{3n-36}$

10. $\frac{12p-240}{5p-100}$

11. $\frac{x^2+4x-5}{x^2-2x+1}$

12. $\frac{y^2+3y-4}{y^2-6y+5}$

13. $\frac{a^2-4}{a^2+6a-16}$

14. $\frac{y^2-2y-3}{y^2-9}$

15. $\frac{p^3+3p^2+4p+12}{p^2+p-6}$

16. $\frac{x^3-2x^2-25x+50}{x^2-25}$

17. $\frac{8b^2-32b}{2b^2-6b-80}$

18. $\frac{-5c^2-10c}{-10c^2+30c+100}$

19. $\frac{3m^2+30mn+75n^2}{4m^2-100n^2}$

20. $\frac{5r^2+30rs-35s^2}{r^2-49s^2}$

21. $\frac{a-5}{5-a}$

22. $\frac{5-d}{d-5}$

23. $\frac{20-5y}{y^2-16}$

24. $\frac{4v-32}{64-v^2}$

25. $\frac{w^3+216}{w^2-36}$

26. $\frac{v^3+125}{v^2-25}$

27. $\frac{z^2-9z+20}{16-z^2}$

28. $\frac{a^2-5z-36}{81-a^2}$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

29. $\frac{12}{16} \cdot \frac{4}{10}$

30. $\frac{32}{5} \cdot \frac{16}{24}$

31. $\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

32. $\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

33. $\frac{5p^2}{p^2-5p-36} \cdot \frac{p^2-16}{10p}$

34. $\frac{3q^2}{q^2+q-6} \cdot \frac{q^2-9}{9q}$

35. $\frac{2y^2-10y}{y^2+10y+25} \cdot \frac{y+5}{6y}$

36. $\frac{z^2+3z}{z^2-3z-4} \cdot \frac{z-4}{z^2}$

37. $\frac{28-4b}{3b-3} \cdot \frac{b^2+8b-9}{b^2-49}$

38. $\frac{72m-12m^2}{8m+32} \cdot \frac{m^2+10m+24}{m^2-36}$

39. $\frac{5c^2+9c+2}{c^2-25} \cdot \frac{c^2+10c+25}{3c^2-14c-5}$

40. $\frac{2d^2+d-3}{d^2-16} \cdot \frac{d^2-8d+16}{2d^2-9d-18}$

41. $\frac{6m^2-2m-10}{9-m^2} \cdot \frac{m^2-6m+9}{6m^2+29m-20}$

42. $\frac{2n^2-3n-14}{25-n^2} \cdot \frac{n^2-10n+25}{2n^2-13n+21}$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

43. $\frac{v-5}{11-v} \div \frac{v^2-25}{v-11}$

44. $\frac{10+w}{w-8} \div \frac{100-w^2}{8-w}$

45. $\frac{3s^2}{s^2-16} \div \frac{s^3-4s^2+16s}{s^3-64}$

46. $\frac{r^2-9}{15} \div \frac{r^3-27}{5r^2+15r+45}$

47. $\frac{p^3+q^3}{3p^2+3pq+3q^2} \div \frac{p^2-q^2}{12}$

48. $\frac{v^3-8w^3}{2v^2+4vw+8w^2} \div \frac{v^2-4w^2}{4}$

49. $\frac{x^2+3x-10}{4x} \div (2x^2+20x+50)$

50. $\frac{2y^2-10yz-48z^2}{2y-1} \div (4y^2-32yz)$

51. $\frac{\frac{2a^2-a-21}{5a+20}}{\frac{a^2+7a+12}{a^2+8a+16}}$

52. $\frac{\frac{3b^2+2b-8}{12b+18}}{\frac{3b^2+2b-8}{2b^2-7b-15}}$

53. $\frac{\frac{12c^2-12}{2c^2-3c+1}}{\frac{4c+4}{6c^2-13c+5}}$

54. $\frac{\frac{4d^2+7d-2}{35d+10}}{\frac{d^2-4}{7d^2-12d-4}}$

For the following exercises, perform the indicated operations.

55. $\frac{10m^2+80m}{3m-9}\cdot\frac{m^2+4m-21}{m^2-9m+20}\div\frac{5m^2+10m}{2m-10}$

56. $\frac{4n^2+32n}{3n+2}\cdot\frac{3n^2-n-2}{n^2+n-30}\div\frac{108n^2-24n}{n+6}$

57. $\frac{12p^2+3p}{p+3}\div\frac{p^2+2p-63}{p^2-p-12}\cdot\frac{p-7}{9p^3-9p^2}$

58. $\frac{6q+3}{9q^2-9q}\div\frac{q^2+14q+33}{q^2+4q-5}\cdot\frac{4q^2+12q}{12q+6}$

Multiply and Divide Rational Functions

In the following exercises, find the domain of each function.

59. $R(x)=\frac{x^3-2x^2-25x+50}{x^2-25}$

60. $R(x)=\frac{x^3+3x^2-4x-12}{x^2-4}$

61. $R(x)=\frac{3x^2+15x}{6x^2+6x-36}$

62. $R(x)=\frac{8x^2-32x}{2x^2-6x-80}$

For the following exercises, find $R(x)=f(x)\cdot g(x)$ where $f(x)$ and $g(x)$ are given.

63. $f(x)=\frac{6x^2-12x}{x^2+7x-18}$
$g(x)=\frac{x^2-81}{3x^2-27x}$

64. $f(x)=\frac{x^2-2x}{x^2+6x-16}$
$g(x)=\frac{x^2-64}{x^2-8x}$

65. $f(x)=\frac{4x}{x^2-3x-10}$
$g(x)=\frac{x^2-25}{8x^2}$

66. $f(x)=\frac{2x^2+8x}{x^2-9x+20}$
$g(x)=\frac{x-5}{x^2}$

For the following exercises, find $R(x)=\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are given.

67. $f(x)=\frac{27x^2}{3x-21}$
$g(x)=\frac{3x^2+18x}{x^2+13x+42}$

68. $f(x)=\frac{24x^2}{2x-8}$
$g(x)=\frac{4x^3+28x^2}{x^2+11x+28}$

69. $f(x)=\frac{16x^2}{4x+36}$
$g(x)=\frac{4x^2-24x}{x^2+4x-45}$

70. $f(x)=\frac{24x^2}{2x-4}$
$g(x)=\frac{12x^2+36x}{x^2-11x+18}$

Writing Exercises

71. Explain how you find the values of x for which the rational expression $\frac{x^2-x-20}{x^2-4}$ is undefined.

72. Explain all the steps you take to simplify the rational expression $\frac{p^2+4p-21}{9-p^2}$.

73. ⓐ Multiply $\frac{7}{4} \cdot \frac{9}{10}$ and explain all your steps.

ⓑ Multiply $\frac{n}{n-3} \cdot \frac{9}{n+3}$ and explain all your steps.

ⓒ Evaluate your answer to part ⓑ when $n = 7$. Did you get the same answer you got in part ⓐ? Why or why not?

74. ⓐ Divide $\frac{24}{5} \div 6$ and explain all your steps.

ⓑ Divide $\frac{x^2 - 1}{x} \div (x + 1)$ and explain all your steps.

ⓒ Evaluate your answer to part ⓑ when $x = 5$. Did you get the same answer you got in part ⓐ? Why or why not?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
simplify rational expressions.			
multiply rational expressions.			
divide rational expressions.			
multiply and divide rational functions.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!*

...with some help. *This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.*

7.2 Add and Subtract Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
 If you missed this problem, review **Example 1.29**.
2. Subtract: $\frac{3x}{4}-\frac{8}{9}$.
 If you missed this problem, review **Example 1.28**.
3. Subtract: $6(2x+1)-4(x-5)$.
 If you missed this problem, review **Example 1.56**.

Add and Subtract Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Rational Expression Addition and Subtraction

If *p*, *q*, and *r* are polynomials where $r \neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of *x* should be excluded in the next example?

EXAMPLE 7.13

Add: $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$.

Solution

Since the denominator is $x+4$, we must exclude the value $x=-4$.

$$\frac{11x+28}{x+4}+\frac{x^2}{x+4}, \quad x\neq -4$$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator.

$$\frac{11x+28+x^2}{x+4}$$

Write the degrees in descending order.

$$\frac{x^2+11x+28}{x+4}$$

Factor the numerator.

$$\frac{(x+4)(x+7)}{x+4}$$

Simplify by removing common factors.

$$\frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$$

Simplify.

$$x+7$$

The expression simplifies to $x+7$ but the original expression had a denominator of $x+4$ so $x\neq -4$.

TRY IT : : 7.25 Simplify: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

TRY IT : : 7.26 Simplify: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

EXAMPLE 7.14

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$.

Solution

$$\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$$

Subtract the numerators and place the difference over the common denominator.

$$\frac{5x^2-7x+3-\left(4x^2+x-9\right)}{x^2-3x+18}$$

Distribute the sign in the numerator.

$$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$$

Combine like terms.

$$\frac{x^2-8x+12}{x^2-3x-18}$$

Factor the numerator and the denominator.

$$\frac{(x-2)(x-6)}{(x+3)(x-6)}$$

Simplify by removing common factors.

$$\frac{(x-2)\cancel{(x-6)}}{(x+3)\cancel{(x-6)}}$$

$$\frac{(x-2)}{(x+3)}$$

TRY IT : : 7.27 Subtract: $\frac{4x^2-11x+8}{x^2-3x+2}-\frac{3x^2+x-3}{x^2-3x+2}$.

TRY IT : : 7.28 Subtract: $\frac{6x^2-x+20}{x^2-81}-\frac{5x^2+11x-7}{x^2-81}$.

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

$$\frac{7}{a}+\frac{5}{-a}$$

Multiply the second fraction by $\frac{-1}{-1}$.

$$\frac{7}{a}+\frac{(-1)5}{(-1)(-a)}$$

The denominators are the same.

$$\frac{7}{a}+\frac{-5}{a}$$

Simplify.

$$\frac{2}{a}$$

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

EXAMPLE 7.15

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify.	$\frac{m-2}{m+1}$

TRY IT : : 7.29 Subtract: $\frac{y^2-5y}{y^2-4}-\frac{6y-6}{4-y^2}$.

TRY IT : : 7.30 Subtract: $\frac{2n^2+8n-1}{n^2-1}-\frac{n^2-7n-1}{1-n^2}$.

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12}+\frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\frac{7}{12}+\frac{5}{18}$$
$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$
$$\frac{21}{36}+\frac{10}{36}$$

$$12=2\cdot 2\cdot 3$$
$$18=2\cdot \quad 3\cdot 3$$
$$\text{LCD}=2\cdot 2\cdot 3\cdot 3$$
$$\text{LCD}=36$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

HOW TO : : FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

Step 1. Factor each denominator completely.
Step 2. List the factors of each denominator. Match factors vertically when possible.
Step 3. Bring down the columns by including all factors, but do not include common factors twice.
Step 4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

EXAMPLE 7.16

ⓐ Find the LCD for the expressions $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$ and ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

Solution

ⓐ

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.

Factor each denominator completely, lining up common factors.	$x^2-2x-3=(x+1)(x-3)$ $x^2+4x+3=(x+1)(x+3)$
Bring down the columns.	$\text{LCD}=(x+1)(x-3)(x+3)$
Write the LCD as the product of the factors.	The LCD is $(x+1)(x-3)(x+3)$.

ⓑ

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

TRY IT : : 7.31

ⓐ Find the LCD for the expressions $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$ ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

TRY IT : : 7.32

ⓐ Find the LCD for the expressions $\frac{3x}{x^2-3x+10}, \frac{5}{x^2+3x+2}$ ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

EXAMPLE 7.17 HOW TO ADD RATIONAL EXPRESSIONS WITH UNLIKE DENOMINATORS

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. • Find the LCD. • Rewrite each rational expression as an equivalent rational expression with the LCD.	No. Find the LCD of $(x-3)$ and $(x-2)$. Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$. Keep the denominators factored!	$x-3:(x-3)$ $x-2$ ◯ : $(x-2)$ LCD : $(x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

TRY IT : : 7.33 Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

TRY IT : : 7.34 Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

The steps used to add rational expressions are summarized here.

HOW TO : : ADD OR SUBTRACT RATIONAL EXPRESSIONS.

Step 1. Determine if the expressions have a common denominator.
- **Yes** – go to step 2.
- **No** – Rewrite each rational expression with the LCD.
 - Find the LCD.
 - Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add or subtract the rational expressions.

Step 3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after you have combined the numerators.

EXAMPLE 7.18

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $x^2-2x-3=(x+1)(x-3)$; $x^2+4x+3=(x+1)\quad(x+3)$; $\text{LCD}=(x+1)(x-3)(x+3)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

TRY IT : : 7.35 Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

TRY IT : : 7.36 Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

EXAMPLE 7.19

Subtract: $\frac{8y}{y^2-16}-\frac{4}{y-4}$.

Solution

	$\frac{8y}{y^2-16}-\frac{4}{y-4}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD.	$y^2-16=(y-4)(y+4)$ $y-4=y-4$ $\text{LCD}=(y-4)(y+4)$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)}-\frac{4y+16}{(y-4)(y+4)}$
Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerator.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors	$\frac{4(y-4)}{(y-4)(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

TRY IT : : 7.37 Subtract: $\frac{2x}{x^2-4}-\frac{1}{x+2}$.

TRY IT : : 7.38 Subtract: $\frac{3}{z+3}-\frac{6z}{z^2-9}$.

There are lots of negative signs in the next example. Be extra careful.

EXAMPLE 7.20

Subtract: $\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(-1)(2-n)}$
Write $(-1)(2-n)$ as $n-2$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(n-2)}$
Simplify. Remember, $a-(-b)=a+b$.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)}{(n-2)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $n^2+n-6=(n-2)(n+3)$; $n-2=(n-2)$; LCD $=(n-2)(n+3)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{n^2+6n+9}{(n-2)(n+3)}$
Add the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n(n+3)}{(n-2)(n+3)}$
Simplify.	$\frac{n}{(n-2)}$

TRY IT : : 7.39 Subtract : $\frac{3x-1}{x^2-5x-6}-\frac{2}{6-x}$.

TRY IT : : 7.40 Subtract: $\frac{-2y-2}{y^2+2y-8}-\frac{y-1}{2-y}$.

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

EXAMPLE 7.21

Subtract: $\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$.

Solution

	$\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$
Factor the denominators.	$\frac{4}{(a+1)(a+5)}-\frac{3}{(a+2)(a+5)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $a^2+6a+5=(a+1)(a+5)$ $a^2+7a+10=(a+5)(a+2)$ $\text{LCD}=(a+1)(a+5)(a+2)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{4(a+2)}{(a+1)(a+5)(a+2)}-\frac{3(a+1)}{(a+2)(a+5)(a+1)}$
Simplify the numerators.	$\frac{4a+8}{(a+1)(a+5)(a+2)}-\frac{3a+3}{(a+2)(a+5)(a+1)}$
Subtract the rational expressions.	$\frac{4a+8-(3a+3)}{(a+1)(a+5)(a+2)}$
Simplify the numerator.	$\frac{4a+8-3a+3}{(a+1)(a+5)(a+2)}$
	$\frac{a+5}{(a+1)(a+5)(a+2)}$
Look for common factors.	$\frac{(a+5)}{(a+1)(a+5)(a+2)}$
Simplify.	$\frac{1}{(a+1)(a+2)}$

TRY IT : : 7.41 Subtract: $\frac{3}{b^2-4b-5}-\frac{2}{b^2-6b+5}$.

TRY IT : : 7.42 Subtract: $\frac{4}{x^2-4}-\frac{3}{x^2-x-2}$.

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example, we will start by factoring all three denominators to find their LCD.

EXAMPLE 7.22

Simplify: $\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$.

Solution

	$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	

Find the LCD.

$$\begin{aligned} u-1 &= (u-1)\\ u &= u\\ u^2-u &= u(u-1)\\ \text{LCD} &= u(u-1) \end{aligned}$$

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u\cdot u}{(u-1)u}+\frac{1\cdot(u-1)}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u}+\frac{u-1}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{u(2u-1)}{u(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

TRY IT : : 7.43 Simplify: $\frac{v}{v+1}+\frac{3}{v-1}-\frac{6}{v^2-1}$.

TRY IT : : 7.44 Simplify: $\frac{3w}{w+2}+\frac{2}{w+7}-\frac{17w+4}{w^2+9w+14}$.

Add and subtract rational functions

To add or subtract rational functions, we use the same techniques we used to add or subtract polynomial functions.

EXAMPLE 7.23

Find $R(x)=f(x)-g(x)$ where $f(x)=\frac{x+5}{x-2}$ and $g(x)=\frac{5x+18}{x^2-4}$.

Solution

	$R(x)=f(x)-g(x)$
Substitute in the functions $f(x)$, $g(x)$.	$R(x)=\frac{x+5}{x-2}-\frac{5x+18}{x^2-4}$
Factor the denominators.	$R(x)=\frac{x+5}{x-2}-\frac{5x+18}{(x-2)(x+2)}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	

Find the LCD.

$$\begin{aligned} x-2 &= (x-2)\\ x^2-4 &= (x-2)(x+2)\\ \text{LCD} &= (x-2)(x+2) \end{aligned}$$

Rewrite each rational expression as an equivalent rational expression with the LCD.	$R(x)=\frac{(x+5)(x+2)}{(x-2)(x+2)}-\frac{5x+18}{(x-2)(x+2)}$

Write as one rational expression.	$R(x)=\frac{(x+5)(x+2)-(5x+18)}{(x-2)(x+2)}$
Simplify.	$R(x)=\frac{x^2+7x+10-5x-18}{(x-2)(x+2)}$
	$R(x)=\frac{x^2+2x-8}{(x-2)(x+2)}$
Factor the numerator, and remove common factors.	$R(x)=\frac{(x+4)(x-2)}{(x-2)(x+2)}$
Simplify.	$R(x)=\frac{(x+4)}{(x+2)}$

TRY IT : : 7.45 Find $R(x)=f(x)-g(x)$ where $f(x)=\frac{x+1}{x+3}$ and $g(x)=\frac{x+17}{x^2-x-12}$.

TRY IT : : 7.46 Find $R(x)=f(x)+g(x)$ where $f(x)=\frac{x-4}{x+3}$ and $g(x)=\frac{4x+6}{x^2-9}$.

MEDIA : :

Access this online resource for additional instruction and practice with adding and subtracting rational expressions.

- **Add and Subtract Rational Expressions- Unlike Denominators (https://openstax.org/l/37AddSubRatExp)**

7.2 EXERCISES

Practice Makes Perfect

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, add.

75. $\frac{2}{15}+\frac{7}{15}$

76. $\frac{7}{24}+\frac{11}{24}$

77. $\frac{3c}{4c-5}+\frac{5}{4c-5}$

78. $\frac{7m}{2m+n}+\frac{4}{2m+n}$

79. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

80. $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

81. $\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$

82. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

In the following exercises, subtract.

83. $\frac{9a^2}{3a-7}-\frac{49}{3a-7}$

84. $\frac{25b^2}{5b-6}-\frac{36}{5b-6}$

85. $\frac{3m^2}{6m-30}-\frac{21m-30}{6m-30}$

86. $\frac{2n^2}{4n-32}-\frac{18n-16}{4n-32}$

87. $\frac{6p^2+3p+4}{p^2+4p-5}-\frac{5p^2+p+7}{p^2+4p-5}$

88. $\frac{5q^2+3q-9}{q^2+6q+8}-\frac{4q^2+9q+7}{q^2+6q+8}$

89. $\frac{5r^2+7r-33}{r^2-49}-\frac{4r^2+5r+30}{r^2-49}$

90. $\frac{7t^2-t-4}{t^2-25}-\frac{6t^2+12t-44}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add or subtract.

91. $\frac{10v}{2v-1}+\frac{2v+4}{1-2v}$

92. $\frac{20w}{5w-2}+\frac{5w+6}{2-5w}$

93. $\frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{3-8x}$

94. $\frac{6y^2+2y-11}{3y-7}+\frac{3y^2-3y+17}{7-3y}$

95. $\frac{z^2+6z}{z^2-25}-\frac{3z+20}{25-z^2}$

96. $\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2}$

97. $\frac{2b^2+30b-13}{b^2-49}-\frac{2b^2-5b-8}{49-b^2}$

98. $\frac{c^2+5c-10}{c^2-16}-\frac{c^2-8c-10}{16-c^2}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, ⓐ find the LCD for the given rational expressions ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

99. $\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$

100. $\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$

101. $\frac{9}{z^2+2z-8}, \frac{4z}{z^2-4}$

102. $\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$

103. $\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$

104. $\frac{5}{c^2-4c+4}, \frac{3c}{c^2-7c+10}$

105. $\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$

106. $\frac{3}{5m^2-3m-2}, \frac{6m}{5m^2+17m+6}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

107. $\frac{7}{10x^2y}+\frac{4}{15xy^2}$

108. $\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

109. $\frac{3}{r+4}+\frac{2}{r-5}$

110. $\frac{4}{s-7}+\frac{5}{s+3}$

111. $\frac{5}{3w-2}+\frac{2}{w+1}$

112. $\frac{4}{2x+5}+\frac{2}{x-1}$

113. $\frac{2y}{y+3}+\frac{3}{y-1}$

114. $\frac{3z}{z-2}+\frac{1}{z+5}$

115. $\frac{5b}{a^2b-2a^2}+\frac{2b}{b^2-4}$

116. $\frac{4}{cd+3c}+\frac{1}{d^2-9}$

117. $\frac{-3m}{3m-3}+\frac{5m}{m^2+3m-4}$

118. $\frac{8}{4n+4}+\frac{6}{n^2-n-2}$

119. $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

120. $\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

121. $\frac{t}{t-6}-\frac{t-2}{t+6}$

122. $\frac{x-3}{x+6}-\frac{x}{x+3}$

123. $\frac{5a}{a+3}-\frac{a+2}{a+6}$

124. $\frac{3b}{b-2}-\frac{b-6}{b-8}$

125. $\frac{6}{m+6}-\frac{12m}{m^2-36}$

126. $\frac{4}{n+4}-\frac{8n}{n^2-16}$

127. $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

128. $\frac{7q+8}{q^2-2q-24}-\frac{q+2}{4-q}$

129. $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$

130. $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

131. $\frac{2x+7}{10x-1}+3$

132. $\frac{8y-4}{5y+2}-6$

133. $\frac{3}{x^2-3x-4}-\frac{2}{x^2-5x+4}$

134. $\frac{4}{x^2-6x+5}-\frac{3}{x^2-7x+10}$

135. $\frac{5}{x^2+8x-9}-\frac{4}{x^2+10x+9}$

136. $\frac{3}{2x^2+5x+2}-\frac{1}{2x^2+3x+1}$

137. $\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$

138. $\frac{2b}{b-5}+\frac{3}{2b}-\frac{2b-15}{2b^2-10b}$

139. $\frac{c}{c+2}+\frac{5}{c-2}-\frac{11c}{c^2-4}$

140. $\frac{6d}{d-5}+\frac{1}{d+4}-\frac{7d-5}{d^2-d-20}$

141. $\frac{3d}{d+2}+\frac{4}{d}-\frac{d+8}{d^2+2d}$

142. $\frac{2q}{q+5}+\frac{3}{q-3}-\frac{13q+15}{q^2+2q-15}$

Add and Subtract Rational Functions

In the following exercises, find ⓐ $R(x) = f(x) + g(x)$ ⓑ $R(x) = f(x) - g(x)$.

143. $f(x) = \frac{-5x-5}{x^2+x-6}$ and $g(x) = \frac{x+1}{2-x}$

144. $f(x) = \frac{-4x-24}{x^2+x-30}$ and $g(x) = \frac{x+7}{5-x}$

145. $f(x) = \frac{6x}{x^2-64}$ and $g(x) = \frac{3}{x-8}$

146. $f(x) = \frac{5}{x+7}$ and $g(x) = \frac{10x}{x^2-49}$

Writing Exercises

147. Donald thinks that $\frac{3}{x} + \frac{4}{x}$ is $\frac{7}{2x}$. Is Donald correct? Explain.

148. Explain how you find the Least Common Denominator of $x^2 + 5x + 4$ and $x^2 - 16$.

149. Felipe thinks $\frac{1}{x} + \frac{1}{y}$ is $\frac{2}{x+y}$.

ⓐ Choose numerical values for *x* and *y* and evaluate $\frac{1}{x} + \frac{1}{y}$.

ⓑ Evaluate $\frac{2}{x+y}$ for the same values of *x* and *y* you used in part ⓐ.

ⓒ Explain why Felipe is wrong.

ⓓ Find the correct expression for $\frac{1}{x} + \frac{1}{y}$.

150. Simplify the expression $\frac{4}{n^2+6n+9} - \frac{1}{n^2-9}$ and explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
add and subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			
find the least common denominator of rational expressions.			
add and subtract rational expressions with unlike denominators.			
add or subtract rational functions.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

7.3 Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{\frac{3}{5}}{\frac{9}{10}}$.
 If you missed this problem, review Example 1.27.
2. Simplify: $\frac{1-\frac{1}{3}}{4^2+4\cdot 5}$.
 If you missed this problem, review Example 1.31.
3. Solve: $\frac{1}{2x}+\frac{1}{4}=\frac{1}{8}$.
 If you missed this problem, review Example 2.9.

Simplify a Complex Rational Expression by Writing it as Division

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem:

$$\left(\frac{6x^2-7x+2}{4x-8}\right)\div\left(\frac{2x^2-8x+3}{x^2-5x+6}\right).$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form

where one fraction is over one fraction. We then write it as if we were dividing two fractions.

EXAMPLE 7.24

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$.

Solution

$$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$$

Rewrite the complex fraction as division.	$\frac{6}{x-4} \div \frac{3}{x^2-16}$
Rewrite as the product of fir t times the reciprocal of the second.	$\frac{6}{x-4} \cdot \frac{x^2-16}{3}$
Factor.	$\frac{3 \cdot 2}{x-4} \cdot \frac{(x-4)(x+4)}{3}$
Multiply.	$\frac{3 \cdot 2(x-4)(x+4)}{3(x-4)}$
Remove common factors.	$\frac{\cancel{3} \cdot 2\cancel{(x-4)}(x+4)}{\cancel{3}\cancel{(x-4)}}$
Simplify.	$2(x+4)$

Are there any value(s) of x that should not be allowed? The original complex rational expression had denominators of $x-4$ and x^2-16. This expression would be undefined if $x=4$ or $x=-4$.

TRY IT : : 7.47

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$.

TRY IT : : 7.48

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$.

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

EXAMPLE 7.25

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Simplify the numerator and denominator. Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.	$\frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$
Simplify the numerator and denominator.	$\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$
Rewrite the complex rational expression as a division problem.	$\frac{3}{6}\div\frac{1}{6}$
Multiply the first by the reciprocal of the second.	$\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	3

TRY IT : : 7.49 Simplify the complex rational expression by writing it as division: $\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$.

TRY IT : : 7.50 Simplify the complex rational expression by writing it as division: $\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$.

We follow the same procedure when the complex rational expression contains variables.

EXAMPLE 7.26 HOW TO SIMPLIFY A COMPLEX RATIONAL EXPRESSION USING DIVISION

Simplify the complex rational expression by writing it as division: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution

Step 1. Simplify the numerator and denominator.	We will simplify the sum in the numerator and difference in the denominator.	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find a common denominator and add the fractions in the numerator.	$\frac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
		$\frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	Find a common denominator and subtract the fractions in the denominator.	$\frac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
	We now have just one rational expression in the numerator and one in the denominator.	
Step 2. Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Step 3. Divide the expressions.	Multiply the first by the reciprocal of the second.	$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Simplify.	$\frac{1}{x-y}$

TRY IT : : 7.51 Simplify the complex rational expression by writing it as division: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.

TRY IT : : 7.52 Simplify the complex rational expression by writing it as division: $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$.

We summarize the steps here.

HOW TO : : SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY WRITING IT AS DIVISION.

Step 1. Simplify the numerator and denominator.
Step 2. Rewrite the complex rational expression as a division problem.
Step 3. Divide the expressions.

EXAMPLE 7.27

Simplify the complex rational expression by writing it as division: $\dfrac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$.

Solution

	$\dfrac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$
Simplify the numerator and denominator. Find common denominators for the numerator and denominator.	$\dfrac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$\dfrac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$
Subtract the rational expressions in the numerator and add in the denominator.	$\dfrac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$
Simplify. (We now have one rational expression over one rational expression.)	$\dfrac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$
Rewrite as fraction division.	$\dfrac{n^2+n}{n+5}\div\dfrac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$\dfrac{n^2+n}{n+5}\cdot\dfrac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$\dfrac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$\dfrac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Simplify.	$\dfrac{(n+1)(n-5)}{2}$

TRY IT : : 7.53 Simplify the complex rational expression by writing it as division: $\dfrac{b-\frac{3b}{b+5}}{\frac{2}{b+5}+\frac{1}{b-5}}$.

TRY IT : : 7.54 Simplify the complex rational expression by writing it as division: $\dfrac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}$.

Simplify a Complex Rational Expression by Using the LCD

We "cleared" the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let's look at the complex rational expression we simplified one way in Example 7.25. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{\text{LCD}}{\text{LCD}}$ we are multiplying by 1, so the value stays the same.

EXAMPLE 7.28

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6.	
Clear the fractions by multiplying the numerator and denominator by that LCD.	$\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$\frac{2+1}{3-2}$
	$\frac{3}{1}$
	3

TRY IT : : 7.55 Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$.

TRY IT : : 7.56 Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$.

We will use the same example as in **Example 7.26**. Decide which method works better for you.

EXAMPLE 7.29 HOW TO SIMPLIFY A COMPLEX RATIONAL EXPRESSING USING THE LCD

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution

Step 1. Find the LCD of all fractions in the complex rational expression.	The LCD of all the fractions is xy.	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Step 2. Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by xy.	$\frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$

Step 3. Simplify the expression.	Distribute.	$\frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$
		$\frac{y+x}{x^2-y^2}$
	Simplify.	$\frac{(y+x)}{(x-y)(x+y)}$
	Remove common factors.	$\frac{1}{x-y}$

TRY IT : : 7.57

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$.

TRY IT : : 7.58

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$.

HOW TO : : SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY USING THE LCD.

Step 1. Find the LCD of all fractions in the complex rational expression.
Step 2. Multiply the numerator and denominator by the LCD.
Step 3. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

EXAMPLE 7.30

Simplify the complex rational expression by using the LCD: $\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$.

Solution

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Find the LCD of all fractions in the complex rational expression. The LCD is $x^2-36=(x+6)(x-6)$.	
Multiply the numerator and denominator by the LCD.	$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$
Simplify the expression.	

Distribute in the denominator.	$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$
Simplify.	$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$
Simplify.	$\frac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$\frac{2(x-6)}{4x+20}$
Factor the denominator.	$\frac{2(x-6)}{4(x+5)}$
Remove common factors.	$\frac{2(x-6)}{2\cdot 2(x+5)}$
Simplify.	$\frac{x-6}{2(x+5)}$
Notice that there are no more factors common to the numerator and denominator.	

TRY IT : : 7.59

Simplify the complex rational expression by using the LCD: $\frac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$.

TRY IT : : 7.60

Simplify the complex rational expression by using the LCD: $\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$.

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

EXAMPLE 7.31

Simplify the complex rational expression by using the LCD: $\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$.

Solution

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$$

Find the LCD of all fractions in the complex rational expression.	
The LCD is $(m-3)(m-4)$.	
Multiply the numerator and denominator by the LCD.	$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$
Simplify.	$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}\right)-(m-3)(m-4)\left(\frac{2}{m-4}\right)}$
Simplify.	$\frac{4}{3(m-4)-2(m-3)}$
Distribute.	$\frac{4}{3m-12-2m+6}$
Combine like terms.	$\frac{4}{m-6}$

TRY IT : : 7.61

Simplify the complex rational expression by using the LCD: $\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$.

TRY IT : : 7.62

Simplify the complex rational expression by using the LCD: $\frac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$.

EXAMPLE 7.32

Simplify the complex rational expression by using the LCD: $\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$.

Solution

$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$

Find the LCD of all fractions in the complex rational expression.	
The LCD is $(y+1)(y-1)$.	

Multiply the numerator and denominator by the LCD.	$\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$
Distribute in the denominator and simplify.	$\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)(1)+(y+1)(y-1)\left(\frac{1}{y-1}\right)}$
Simplify.	$\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$
Simplify the denominator and leave the numerator factored.	$\frac{y(y-1)}{y^2-1+y+1}$
	$\frac{y(y-1)}{y^2+y}$
Factor the denominator and remove factors common with the numerator.	$\frac{y(y-1)}{y(y+1)}$
Simplify.	$\frac{y-1}{y+1}$

TRY IT : : 7.63

Simplify the complex rational expression by using the LCD: $\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$.

TRY IT : : 7.64

Simplify the complex rational expression by using the LCD: $\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$.

MEDIA : :

Access this online resource for additional instruction and practice with complex fractions.

- **Complex Fractions (https://openstax.org/l/37CompFrac)**

7.3 EXERCISES

Practice Makes Perfect

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify each complex rational expression by writing it as division.

151. $\dfrac{\dfrac{2a}{a+4}}{\dfrac{4a^2}{a^2-16}}$

152. $\dfrac{\dfrac{3b}{b-5}}{\dfrac{b^2}{b^2-25}}$

153. $\dfrac{\dfrac{5}{c^2+5c-14}}{\dfrac{10}{c+7}}$

154. $\dfrac{\dfrac{8}{d^2+9d+18}}{\dfrac{12}{d+6}}$

155. $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

156. $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

157. $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

158. $\dfrac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

159. $\dfrac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

160. $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

161. $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

162. $\dfrac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

163. $\dfrac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

164. $\dfrac{y-\frac{2y}{y-4}}{\frac{2}{y-4}+\frac{2}{y+4}}$

165. $\dfrac{2-\frac{2}{a+3}}{\frac{1}{a+3}+\frac{a}{2}}$

166. $\dfrac{4+\frac{4}{b-5}}{\frac{1}{b-5}+\frac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify each complex rational expression by using the LCD.

167. $\dfrac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

168. $\dfrac{\frac{1}{4}+\frac{1}{9}}{\frac{1}{6}+\frac{1}{12}}$

169. $\dfrac{\frac{5}{6}+\frac{2}{9}}{\frac{7}{18}-\frac{1}{3}}$

170. $\dfrac{\frac{1}{6}+\frac{4}{15}}{\frac{3}{5}-\frac{1}{2}}$

171. $\dfrac{\frac{c}{d}+\frac{1}{d}}{\frac{1}{d}-\frac{d}{c}}$

172. $\dfrac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

173. $\dfrac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$

174. $\dfrac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

175. $\dfrac{\dfrac{2}{x+5}}{\dfrac{3}{x-5}+\dfrac{1}{x^2-25}}$

176. $\dfrac{\dfrac{5}{y-4}}{\dfrac{3}{y+4}+\dfrac{2}{y^2-16}}$

177. $\dfrac{\dfrac{5}{z^2-64}+\dfrac{3}{z+8}}{\dfrac{1}{z+8}+\dfrac{2}{z-8}}$

178. $\dfrac{\dfrac{3}{s+6}+\dfrac{5}{s-6}}{\dfrac{1}{s^2-36}+\dfrac{4}{s+6}}$

179. $\dfrac{\dfrac{4}{a^2-2a-15}}{\dfrac{1}{a-5}+\dfrac{2}{a+3}}$

180. $\dfrac{\dfrac{5}{b^2-6b-27}}{\dfrac{3}{b-9}+\dfrac{1}{b+3}}$

181. $\dfrac{\dfrac{5}{c+2}-\dfrac{3}{c+7}}{\dfrac{5c}{c^2+9c+14}}$

182. $\dfrac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$

183. $\dfrac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$

184. $\dfrac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$

185. $\dfrac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$

186. $\dfrac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$

In the following exercises, simplify each complex rational expression using either method.

187. $\dfrac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$

188. $\dfrac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$

189. $\dfrac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$

190. $\dfrac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$

191. $\dfrac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$

192. $\dfrac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$

193. $\dfrac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$

194. $\dfrac{\frac{y}{y+3}}{2+\frac{1}{y-3}}$

Writing Exercises

195. In this section, you learned to simplify the complex fraction $\dfrac{\frac{3}{x+2}}{\frac{x}{x^2-4}}$ two ways: rewriting it as a division problem or multiplying the numerator and denominator by the LCD. Which method do you prefer? Why?

196. Efraim wants to start simplifying the complex fraction $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator, $\dfrac{\frac{1}{\cancel{a}}+\frac{1}{\cancel{b}}}{\frac{1}{\cancel{a}}-\frac{1}{\cancel{b}}}$. Explain what is wrong with Efraim's plan.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify a complex rational expression by writing it as division.			
simplify a complex rational expression by using the LCD.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

7.4 Solve Rational Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$.
 If you missed this problem, review **Example 2.9**.
2. Solve: $n^2-5n-36=0$.
 If you missed this problem, review **Example 6.45**.
3. Solve the formula $5x+2y=10$ for y.
 If you missed this problem, review **Example 2.31**.

After defining the terms 'expression' and 'equation' earlier, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will *solve* a **rational equation**.

Rational Equation

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to "clear" the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, *c*, by writing $x\neq c$ next to the equation.

EXAMPLE 7.33 HOW TO SOLVE A RATIONAL EQUATION

Solve: $\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$.

Solution

Step 1. Note any value of the variable that would make any denominator zero.	If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x\neq 0$ next to the equation.	$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x\neq 0$
Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$ and $\frac{5}{6}$.	The LCD is $6x$.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$. Use the Distributive Property. Simplify – and notice, no more fractions!	$6x\cdot\left(\frac{1}{x}+\frac{1}{3}\right)=6x\cdot\left(\frac{5}{6}\right)$ $6x\cdot\frac{1}{x}+6x\cdot\frac{1}{3}=6x\cdot\left(\frac{5}{6}\right)$ $6+2x=5x$
Step 4. Solve the resulting equation.	Simplify.	$6=3x$ $2=x$
Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution. We substitute $x=2$ into the original equation.	$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$ $\frac{1}{2}+\frac{1}{3}\stackrel{?}{=}\frac{5}{6}$ $\frac{3}{6}+\frac{2}{6}\stackrel{?}{=}\frac{5}{6}$ $\frac{5}{6}=\frac{5}{6}\checkmark$ The solution is $x=2$.

TRY IT : : 7.65 Solve: $\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$.

TRY IT : : 7.66 Solve: $\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$.

The steps of this method are shown.

HOW TO : : SOLVE EQUATIONS WITH RATIONAL EXPRESSIONS.

Step 1. Note any value of the variable that would make any denominator zero.
Step 2. Find the least common denominator of *all* denominators in the equation.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
Step 4. Solve the resulting equation.
Step 5. Check:
- If any values found in Step 1 are algebraic solutions, discard them.
- Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

EXAMPLE 7.34 HOW TO SOLVE A RATIONAL EQUATION USING THE ZERO PRODUCT PROPERTY

Solve: $1-\frac{5}{y}=-\frac{6}{y^2}$.

Solution

	$1-\frac{5}{y}=-\frac{6}{y^2}$
Note any value of the variable that would make any denominator zero.	$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$
Find the least common denominator of all denominators in the equation. The LCD is y^2.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Distribute.	$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Multiply.	$y^2-5y=-6$
Solve the resulting equation. First write the quadratic equation in standard form.	$y^2-5y+6=0$
Factor.	$(y-2)(y-3)=0$
Use the Zero Product Property.	$y-2=0$ or $y-3=0$
Solve.	$y=2$ or $y=3$

Check.
We did not get 0 as an algebraic solution.

Check $y = 2$ and $y = 3$ in the original equation.

$1 - \frac{5}{y} = \frac{6}{y^2}$ $\quad$ $1 - \frac{5}{y} = \frac{6}{y^2}$

$1 - \frac{5}{2} \stackrel{?}{=} \frac{6}{2^2}$ $\quad$ $1 - \frac{5}{3} \stackrel{?}{=} \frac{6}{3^2}$

$1 - \frac{5}{2} \stackrel{?}{=} \frac{6}{4}$ $\quad$ $1 - \frac{5}{3} \stackrel{?}{=} \frac{6}{9}$

$\frac{2}{2} - \frac{5}{2} \stackrel{?}{=} \frac{6}{4}$ $\quad$ $\frac{3}{3} - \frac{5}{3} \stackrel{?}{=} \frac{6}{9}$

$-\frac{3}{2} \stackrel{?}{=} \frac{6}{4}$ $\quad$ $-\frac{2}{3} \stackrel{?}{=} \frac{6}{9}$

$-\frac{3}{2} = -\frac{3}{2}$ ✓ $\quad$ $-\frac{2}{3} = -\frac{2}{3}$ ✓

The solution is $y = 2, \quad y = 3.$

TRY IT : : 7.67 Solve: $1 - \frac{2}{x} = \frac{15}{x^2}.$

TRY IT : : 7.68 Solve: $1 - \frac{4}{y} = \frac{12}{y^2}.$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

EXAMPLE 7.35

Solve: $\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{x^2-4}.$

Solution

	$\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{x^2-4}$
Note any value of the variable that would make any denominator zero.	$\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{(x+2)(x-2)}, x \neq -2, x \neq 2$
Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(x+2)(x-2)\left(\frac{2}{x+2} + \frac{4}{x-2}\right) = (x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$
Distribute.	$(x+2)(x-2)\frac{2}{x+2} + (x+2)(x-2)\frac{4}{x-2} = (x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$

Remove common factors.	$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$
Simplify.	$2(x-2)+4(x+2)=x-1$
Distribute.	$2x-4+4x+8=x-1$
Solve.	$6x+4=x-1$
	$5x=-5$
	$x=-1$

Check:
We did not get 2 or −2 as algebraic solutions.

Check $x=-1$ in the original equation.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$

$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$

$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$

$$\frac{6}{3}-\frac{4}{3}\stackrel{?}{=}\frac{2}{3}$$

$$\frac{2}{3}=\frac{2}{3}\checkmark$$

The solution is $x=-1$.

TRY IT : : 7.69 Solve: $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.

TRY IT : : 7.70 Solve: $\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$.

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

EXAMPLE 7.36

Solve: $\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$.

Solution

	$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$
Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.	$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1},\ m\neq 4, m\neq 1$
Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$
Distribute.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$
Remove common factors.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$
Simplify.	$m+11=5(m-1)-3(m-4)$
Solve the resulting equation.	$m+11=5m-5-3m+12$ $4=m$
Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution.	
	There is no solution to this equation.

TRY IT : : 7.71 Solve: $\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$.

TRY IT : : 7.72 Solve: $\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

EXAMPLE 7.37

Solve: $\frac{y}{y+6}=\frac{72}{y^2-36}+4$.

Solution

$$\frac{y}{y+6}=\frac{72}{y^2-36}+4$$

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4,\ y\neq 6, y\neq -6$
Find the least common denominator. The LCD is $(y-6)(y+6)$.	
Clear the fractions.	$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$
Simplify.	$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$
Simplify.	$y(y-6)=72+4(y^2-36)$
Solve the resulting equation.	$y^2-6y=72+4y^2-144$ $0=3y^2+6y-72$ $0=3(y^2+2y-24)$ $0=3(y+6)(y-4)$ $y=-6, y=4$
Check. $y=-6$ is an extraneous solution. Check $y=4$ in the original equation. $\frac{y}{y+6}=\frac{72}{y^2-36}+4$ $\frac{4}{4+6}\stackrel{?}{=}\frac{72}{4^2-36}+4$ $\frac{4}{10}\stackrel{?}{=}\frac{72}{-20}+4$ $\frac{4}{10}\stackrel{?}{=}-\frac{36}{10}+\frac{40}{10}$ $\frac{4}{10}=\frac{4}{10}$ ✓	
	The solution is $y=4$.

TRY IT : : 7.73 Solve: $\frac{x}{x+4}=\frac{32}{x^2-16}+5.$

TRY IT : : 7.74 Solve: $\frac{y}{y+8}=\frac{128}{y^2-64}+9.$

In some cases, all the algebraic solutions are extraneous.

EXAMPLE 7.38

Solve: $\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$.

Solution

	$\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$
We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.	$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}$
Note any value of the variable that would make any denominator zero.	$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}, x\neq 1, x\neq -1$
Find the least common denominator. The LCD is $12(x-1)(x+1)$.	
Clear the fractions.	$12(x-1)(x+1)\left(\frac{x}{2(x-1)}-\frac{2}{3(x+1)}\right)=12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$
Simplify.	$6(x+1)\cdot x-4(x-1)\cdot 2=5x^2-2x+9$
Simplify.	$6x(x+1)-4\cdot 2(x-1)=5x^2-2x+9$
Solve the resulting equation.	$6x^2+6x-8x+8=5x^2-2x+9$ $x^2-1=0$ $(x-1)(x+1)=0$ $x=1$ or $x=-1$
Check. $x=1$ and $x=-1$ are extraneous solutions.	
	The equation has no solution.

TRY IT : : 7.75 Solve: $\frac{y}{5y-10}-\frac{5}{3y+6}=\frac{2y^2-19y+54}{15y^2-60}$.

TRY IT : : 7.76 Solve: $\frac{z}{2z+8}-\frac{3}{4z-8}=\frac{3z^2-16z-16}{8z^2+2z-64}$.

EXAMPLE 7.39

Solve: $\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$.

Solution

	$\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}=\frac{2}{(x-3)(x+1)}$ $x\neq -1,\ x\neq \frac{1}{3},\ x\neq 3$
Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.	
Clear the fractions.	
	$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}\right)=(3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$
Simplify.	$4(x+1)+3(x-3)=2(3x-1)$
Distribute.	$4x+4+3x-9=6x-2$
Simplify.	$7x-5=6x-2$
	$x=3$
The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.	
	There is no solution to this equation.

TRY IT : : 7.77 Solve: $\frac{15}{x^2+x-6}-\frac{3}{x-2}=\frac{2}{x+3}$.

TRY IT : : 7.78 Solve: $\frac{5}{x^2+2x-3}-\frac{3}{x^2+x-2}=\frac{1}{x^2+5x+6}$.

Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

EXAMPLE 7.40

For rational function, $f(x)=\frac{2x-6}{x^2-8x+15}$, ⓐ find the domain of the function, ⓑ solve $f(x)=1$, and ⓒ find the points on the graph at this function value.

Solution

ⓐ The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

	$x^2 - 8x + 15 = 0$
Factor the trinomial.	$(x-3)(x-5) = 0$
Use the Zero Product Property.	$x-3=0 \quad x-5=0$.
Solve.	$x = 3 \quad x = 5$

The domain is all real numbers except $x \neq 3$, $x \neq 5$.

ⓑ

	$f(x) = 1$
Substitute in the rational expression.	$\frac{2x-6}{x^2-8x+15} = 1$
Factor the denominator.	$\frac{2x-6}{(x-3)(x-5)} = 1$
Multiply both sides by the LCD, $(x-3)(x-5)$.	$(x-3)(x-5)\left(\frac{2x-6}{(x-3)(x-5)}\right) = (x-3)(x-5)(1)$
Simplify.	$2x-6 = x^2-8x+15$
Solve.	$0 = x^2-10x+21$
Factor.	$0 = (x-7)(x-3)$
Use the Zero Product Property.	$x-7=0 \quad x-3=0$
Solve.	$x=7 \quad x=3$

ⓒ The value of the function is 1 when $x = 7$, $x = 3$. So the points on the graph of this function when $f(x) = 1$, will be $(7, 1)$, $(3, 1)$.

TRY IT : : 7.79

For rational function, $f(x) = \frac{8-x}{x^2-7x+12}$, ⓐ find the domain of the function ⓑ solve $f(x) = 3$ ⓒ find the points on the graph at this function value.

TRY IT : : 7.80

For rational function, $f(x) = \frac{x-1}{x^2-6x+5}$, ⓐ find the domain of the function ⓑ solve $f(x) = 4$ ⓒ find the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

	$m = \frac{y-y_1}{x-x_1}$
Multiply both sides of the equation by $x-x_1$.	$m(x-x_1) = \left(\frac{y-y_1}{x-x_1}\right)(x-x_1)$
Simplify.	$m(x-x_1) = y-y_1$
Rewrite the equation with the y terms on the left.	$y-y_1 = m(x-x_1)$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2, 3)$. We will add one more step to solve for y.

EXAMPLE 7.41

Solve: $m = \frac{y-2}{x-3}$ for y.

Solution

	$m = \frac{y-2}{x-3}$
Note any value of the variable that would make any denominator zero.	$m = \frac{y-2}{x-3}, x \neq 3$
Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.	$(x-3)m = (x-3)\left(\frac{y-2}{x-3}\right)$
Simplify.	$xm - 3m = y - 2$
Isolate the term with *y*.	$xm - 3m + 2 = y$

TRY IT : : 7.81 Solve: $m = \frac{y-5}{x-4}$ for y.

TRY IT : : 7.82 Solve: $m = \frac{y-1}{x+5}$ for y.

Remember to multiply both sides by the LCD in the next example.

EXAMPLE 7.42

Solve: $\frac{1}{c} + \frac{1}{m} = 1$ for *c*.

Solution

	$\frac{1}{c} + \frac{1}{m} = 1$ for c
Note any value of the variable that would make any denominator zero.	$\frac{1}{c} + \frac{1}{m} = 1, c \neq 0, m \neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, *cm*.	$cm\left(\frac{1}{c} + \frac{1}{m}\right) = cm(1)$
Distribute.	$cm\left(\frac{1}{c}\right) + cm\frac{1}{m} = cm(1)$
Simplify.	$m + c = cm$
Collect the terms with *c* to the right.	$m = cm - c$
Factor the expression on the right.	$m = c(m-1)$
To isolate *c*, divide both sides by $m-1$.	$\frac{m}{m-1} = \frac{c(m-1)}{m-1}$
Simplify by removing common factors.	$\frac{m}{m-1} = c$
Notice that even though we excluded $c = 0, m = 0$ from the original equation, we must also now state that $m \neq 1$.	

TRY IT : : 7.83 Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

TRY IT : : 7.84 Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y.

MEDIA : :

Access this online resource for additional instruction and practice with equations with rational expressions.

- **Equations with Rational Expressions (https://openstax.org/l/37EqRatExp)**

7.4 EXERCISES

Practice Makes Perfect

Solve Rational Equations

In the following exercises, solve each rational equation.

197. $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

198. $\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

199. $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

200. $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

201. $1-\frac{2}{m}=\frac{8}{m^2}$

202. $1+\frac{4}{n}=\frac{21}{n^2}$

203. $1+\frac{9}{p}=\frac{-20}{p^2}$

204. $1-\frac{7}{q}=\frac{-6}{q^2}$

205. $\frac{5}{3v-2}=\frac{7}{4v}$

206. $\frac{8}{2w+1}=\frac{3}{w}$

207. $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

208. $\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

209. $\frac{8}{z-10}-\frac{7}{z+10}=\frac{5}{z^2-100}$

210. $\frac{9}{a+11}-\frac{6}{a-11}=\frac{6}{a^2-121}$

211. $\frac{-10}{q-2}-\frac{7}{q+4}=1$

212. $\frac{2}{s+7}-\frac{3}{s-3}=1$

213. $\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$

214. $\frac{w+8}{w^2-11w+28}=\frac{5}{w-7}+\frac{2}{w-4}$

215. $\frac{x-10}{x^2+8x+12}=\frac{3}{x+2}+\frac{4}{x+6}$

216. $\frac{y-5}{y^2-4y-5}=\frac{1}{y+1}+\frac{1}{y-5}$

217. $\frac{b+3}{3b}+\frac{b}{24}=\frac{1}{b}$

218. $\frac{c+3}{12c}+\frac{c}{36}=\frac{1}{4c}$

219. $\frac{d}{d+3}=\frac{18}{d^2-9}+4$

220. $\frac{m}{m+5}=\frac{50}{m^2-25}+6$

221. $\frac{n}{n+2}-3=\frac{8}{n^2-4}$

222. $\frac{p}{p+7}-8=\frac{98}{p^2-49}$

223. $\frac{q}{3q-9}-\frac{3}{4q+12}=\frac{7q^2+6q+63}{24q^2-216}$

224. $\frac{r}{3r-15}-\frac{1}{4r+20}=\frac{3r^2+17r+40}{12r^2-300}$

225. $\frac{s}{2s+6}-\frac{2}{5s+5}=\frac{5s^2-3s-7}{10s^2+40s+30}$

226. $\frac{t}{6t-12}-\frac{5}{2t+10}=\frac{t^2-23t+70}{12t^2+36t-120}$

227. $\frac{2}{x^2+2x-8}-\frac{1}{x^2+9x+20}=\frac{4}{x^2+3x-10}$

228. $\frac{5}{x^2+4x+3}+\frac{2}{x^2+x-6}=\frac{3}{x^2-x-2}$

229. $\frac{3}{x^2-5x-6}+\frac{3}{x^2-7x+6}=\frac{6}{x^2-1}$

230. $\frac{2}{x^2+2x-3}+\frac{3}{x^2+4x+3}=\frac{6}{x^2-1}$

Solve Rational Equations that Involve Functions

231. For rational function, $f(x)=\frac{x-2}{x^2+6x+8}$,

ⓐ find the domain of the function

ⓑ solve $f(x)=5$

ⓒ find the points on the graph at this function value.

232. For rational function, $f(x)=\frac{x+1}{x^2-2x-3}$,

ⓐ find the domain of the function

ⓑ solve $f(x)=1$

ⓒ find the points on the graph at this function value.

233. For rational function, $f(x)=\frac{2-x}{x^2-7x+10}$,

ⓐ find the domain of the function

ⓑ solve $f(x)=2$

ⓒ find the points on the graph at this function value.

234. For rational function, $f(x)=\frac{5-x}{x^2+5x+6}$,

ⓐ find the domain of the function

ⓑ solve $f(x)=3$

ⓒ the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

In the following exercises, solve.

235. $\frac{C}{r}=2\pi$ for r.

236. $\frac{I}{r}=P$ for r.

237. $\frac{v+3}{w-1}=\frac{1}{2}$ for w.

238. $\frac{x+5}{2-y}=\frac{4}{3}$ for y.

239. $a=\frac{b+3}{c-2}$ for c.

240. $m=\frac{n}{2-n}$ for n.

241. $\frac{1}{p}+\frac{2}{q}=4$ for p.

242. $\frac{3}{s}+\frac{1}{t}=2$ for s.

243. $\frac{2}{v}+\frac{1}{5}=\frac{3}{w}$ for w.

244. $\frac{6}{x}+\frac{2}{3}=\frac{1}{y}$ for y.

245. $\frac{m+3}{n-2}=\frac{4}{5}$ for n.

246. $r=\frac{s}{3-t}$ for t.

247. $\frac{E}{c}=m^2$ for c.

248. $\frac{R}{T}=W$ for T.

249. $\frac{3}{x}-\frac{5}{y}=\frac{1}{4}$ for y.

250. $c=\frac{2}{a}+\frac{b}{5}$ for a.

Writing Exercises

251. Your class mate is having trouble in this section. Write down the steps you would use to explain how to solve a rational equation.

252. Alek thinks the equation $\frac{y}{y+6}=\frac{72}{y^2-36}+4$ has two solutions, $y=-6$ and $y=4$. Explain why Alek is wrong.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve rational equations.			
solve rational equations involving functions.			
solve rational equations for a specific variable.			

ⓑ *On a scale of* $1-10$, *how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

7.5 Solve Applications with Rational Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve proportions
- Solve similar figure applications
- Solve uniform motion applications
- Solve work applications
- Solve direct variation problems
- Solve inverse variation problems

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $2(n-1)-4=-10$.
 If you missed this problem, review Example 2.13.
2. An express train and a charter bus leave Chicago to travel to Champaign. The express train can make the trip in two hours and the bus takes five hours for the trip. The speed of the express train is 42 miles per hour faster than the speed of the bus. Find the speed of the bus.
 If you missed this problem, review Example 2.43.
3. Solve $\frac{1}{3}x+\frac{1}{4}x=\frac{5}{6}$.
 If you missed this problem, review Example 2.9.

Solve Proportions

When two rational expressions are equal, the equation relating them is called a **proportion**.

Proportion

A **proportion** is an equation of the form $\frac{a}{b}=\frac{c}{d}$, where $b\neq 0, d\neq 0$.

The proportion is read "*a* is to *b* as *c* is to *d*."

The equation $\frac{1}{2}=\frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2}=\frac{4}{8}$ is read "1 is to 2 as 4 is to 8."

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved rational equations. We'll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

EXAMPLE 7.43

Solve: $\frac{n}{n+14}=\frac{5}{7}$.

Solution

	$\frac{n}{n+14}=\frac{5}{7}$ $\quad n\neq -14$
Multiply both sides by LCD.	$7(n+14)\left(\frac{n}{n+14}\right)=7(n+14)\left(\frac{5}{7}\right)$
Remove common factors on each side.	$7n=5(n+14)$
Simplify.	$7n=5n+70$
Solve for *n*.	$2n=70$

$n = 35$

Check.	
	$\frac{n}{n+14} = \frac{5}{7}$
Substitute $n = 35$.	$\frac{35}{35+14} \stackrel{?}{=} \frac{5}{7}$
Simplify.	$\frac{35}{49} \stackrel{?}{=} \frac{5}{7}$
Show common factors.	$\frac{5 \cdot 7}{7 \cdot 7} \stackrel{?}{=} \frac{5}{7}$
Simplify.	$\frac{5}{7} = \frac{5}{7}$ ✓

TRY IT : : 7.85 Solve the proportion: $\frac{y}{y+55} = \frac{3}{8}$.

TRY IT : : 7.86 Solve the proportion: $\frac{z}{z-84} = -\frac{1}{5}$.

Notice in the last example that when we cleared the fractions by multiplying by the LCD, the result is the same as if we had cross-multiplied.

$$\frac{n}{n+14} = \frac{5}{7} \qquad \frac{n}{n+14} = \frac{5}{7}$$

$$7(n+14)\left(\frac{n}{n+14}\right) = 7(n+14)\left(\frac{5}{7}\right) \qquad \frac{n}{n+14} = \frac{5}{7}$$

$$7n = 5(n+14) \qquad 7n = 5(n+14)$$

For any proportion, $\frac{a}{b} = \frac{c}{d}$, we get the same result when we clear the fractions by multiplying by the LCD as when we cross-multiply.

$$\frac{a}{b} = \frac{c}{d} \qquad \frac{a}{b} = \frac{c}{d}$$

$$bd\left(\frac{a}{b} = \frac{c}{d}\right)bd \qquad \frac{a}{b} = \frac{c}{d}$$

$$ad = bc \qquad ad = bc$$

To solve applications with proportions, we will follow our usual strategy for solving applications But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match each other and the units in the denominators must also match each other.

EXAMPLE 7.44

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution

Identify what we are asked to find, and choose a variable to represent it.	How many ml of acetaminophen will the doctor prescribe?
	Let a = ml of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?
Translate into a proportion—be careful of the units.	
$\frac{\text{ml}}{\text{pounds}} = \frac{\text{ml}}{\text{pounds}}$	$\frac{5}{25} = \frac{a}{80}$
Multiply both sides by the LCD, 400.	$400\left(\frac{5}{25}\right) = 400\left(\frac{a}{80}\right)$
Remove common factors on each side.	$25 \cdot 16\left(\frac{5}{25}\right) = 80 \cdot 5\left(\frac{a}{80}\right)$
Simplify, but don't multiply on the left. Notice what the next step will be.	$16 \cdot 5 = 5a$
Solve for a.	$\frac{16 \cdot 5}{5} = \frac{5a}{5}$
	$16 = a$
Check. Is the answer reasonable? Yes, since 80 is about 3 times 25, the medicine should be about 3 times 5. So 16 ml makes sense. Substitute $a = 16$ in the original proportion. $\frac{5}{25} = \frac{a}{80}$ $\frac{5}{25} \stackrel{?}{=} \frac{16}{80}$ $\frac{1}{5} = \frac{1}{5}$ ✓	
Write a complete sentence.	The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

TRY IT : : 7.87

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

TRY IT : : 7.88

For every 1 kilogram (kg) of a child's weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

Solve similar figure applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with **similar figures**. If two figures have exactly the same shape, but different sizes, they are said to be similar. One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides have the same ratio.

Similar Figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

For example, the two triangles in Figure 7.2 are similar. Each side of ΔABC is four times the length of the corresponding side of ΔXYZ.

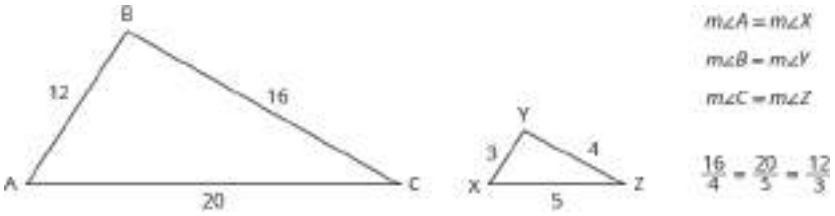
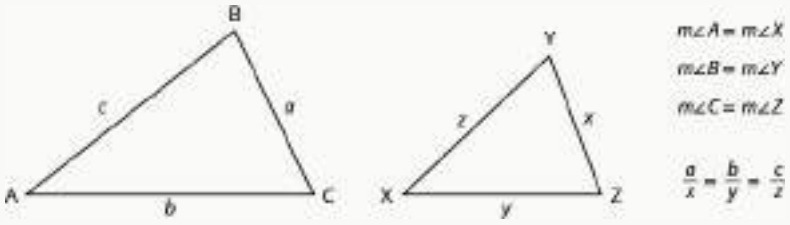


Figure 7.2

This is summed up in the Property of Similar Triangles.

Property of Similar Triangles

If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides have the same ratio.



To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

EXAMPLE 7.45

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. If the actual distance from Los Angeles to Las Vegas is 270 miles, find the distance from Los Angeles to San Francisco.

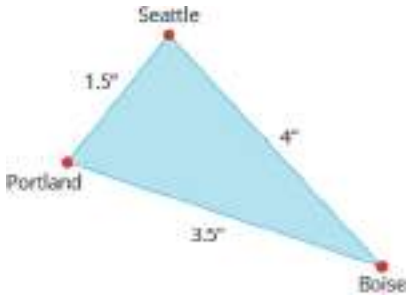


Solution

Since the triangles are similar, the corresponding sides are proportional.

Read the problem. Draw the figures and label it with the given information.	The figures are shown above.
Identify what we are looking for.	the actual distance from Los Angeles to San Francisco
Name the variables.	Let x = distance from Los Angeles to San Francisco.
Translate into an equation. Since the triangles are similar, the corresponding sides are proportional. We'll make the numerators "miles" and the denominators "inches".	$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$
Solve the equation.	$1.3\left(\frac{x}{1.3}\right) = 1.3\left(\frac{270}{1}\right)$
	$x = 351$
Check. On the map, the distance from Los Angeles to San Francisco is more than the distance from Los Angeles to Las Vegas. Since 351 is more than 270 the answer makes sense. Check $x = 351$ in the original proportion. Use a calculator. $\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{351 \text{ miles}}{1.3 \text{ inches}} \stackrel{?}{=} \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{270 \text{ miles}}{1 \text{ inch}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ ✓	
Answer the question.	The distance from Los Angeles to San Francisco is 351 miles.

On the map, Seattle, Portland, and Boise form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. The actual distance from Seattle to Boise is 400 miles.



TRY IT : : 7.89 Find the actual distance from Seattle to Portland.

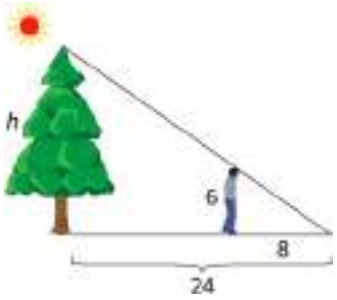
TRY IT : : 7.90 Find the actual distance from Portland to Boise.

We can use similar figures to find heights that we cannot directly measure.

EXAMPLE 7.46

Tyler is 6 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a tree was 24 feet long. Find the height of the tree.

Solution

Read the problem and draw a figure.	
We are looking for h, the height of the tree.	
We will use similar triangles to write an equation.	
The small triangle is similar to the large triangle.	$\frac{h}{24} = \frac{6}{8}$
Solve the proportion.	$24\left(\frac{6}{8}\right) = 24\left(\frac{h}{24}\right)$
	$18 = h$
Simplify.	
Check.	
Tyler's height is less than his shadow's length so it makes sense that the tree's height is less than the length of its shadow. Check $h = 18$ in the original proportion.	$\frac{6}{8} = \frac{h}{24}$ $\frac{6}{8} \stackrel{?}{=} \frac{18}{24}$ $\frac{3}{4} = \frac{3}{4}$ ✓

TRY IT : : 7.91

A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a shadow that is 10 feet long. How tall is the telephone pole?

TRY IT : : 7.92

A pine tree casts a shadow of 80 feet next to a 30 foot tall building which casts a 40 feet shadow. How tall is the pine tree?

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate	•	Time	=	Distance

The formula $D = rt$ assumes we know *r* and *t* and use them to find *D*. If we know *D* and *r* and need to find *t*, we would solve the equation for *t* and get the formula $t = \frac{D}{r}$.

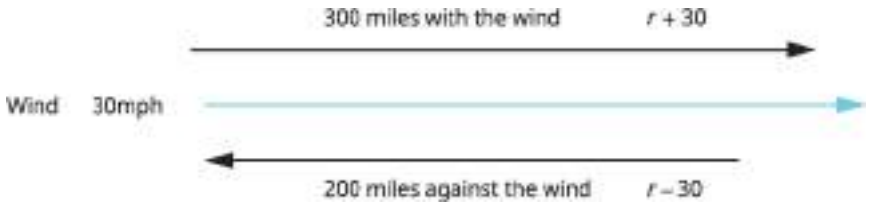
We have also explained how flying with or against the wind affects the speed of a plane. We will revisit that idea in the next example.

EXAMPLE 7.47

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the airplane.

Let *r* = the speed of the airplane.

When the plane flies with the wind, the wind increases its speed and so the rate is *r* + 30.

When the plane flies against the wind, the wind decreases its speed and the rate is *r* − 30.

Write in the rates.
Write in the distances.

Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$.

We divide the distance by the rate in each row, and place the expression in the time column.

	Rate	•	Time	=	Distance
Headwind	$r - 30$		$\frac{200}{r-30}$		200
Tailwind	$r + 30$		$\frac{300}{r+30}$		300

We know the times are equal and so we write our equation.

$$\frac{200}{r-30} = \frac{300}{r+30}$$

We multiply both sides by the LCD.

$$(r+30)(r-30)\left(\frac{200}{r-30}\right) = (r+30)(r-30)\left(\frac{300}{r+30}\right)$$

Simplify.

$$(r+30)(200) = (r-30)300$$

$$200r + 6000 = 300r - 9000$$

Solve.

$$15000 = 100r$$

Check.
Is 150 mph a reasonable speed for an airplane? Yes.
If the plane is traveling 150 mph and the wind is 30 mph,

Tailwind $150 + 30 = 180$ mph $\frac{300}{180} = \frac{5}{3}$ hours

Headwind $150 - 30 = 120$ mph $\frac{200}{120} = \frac{5}{3}$ hours

The times are equal, so it checks.	The plane was traveling 150 mph.

TRY IT : : 7.93

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

TRY IT : : 7.94

Danica can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Danica's boat without a wind?

In the next example, we will know the total time resulting from travelling different distances at different speeds.

EXAMPLE 7.48

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.	
We are looking for Jazmine's running speed.	Let r = Jazmine's running speed.
Her biking speed is 4 miles faster than her running speed.	$r + 4$ = her biking speed
The distances are given, enter them into the chart. Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	
Write a word sentence.	Her time plus the time biking is 3 hours.
Translate the sentence to get the equation.	$\frac{8}{r} + \frac{24}{r+4} = 3$

Solve.

$$\begin{aligned} r\left(r+4\right)\left(\frac{8}{r}+\frac{24}{r+4}\right) &= 3\cdot r(r+4) \\ 8(r+4)+24r &= 3r(r+4) \\ 8r+32+24r &= 3r^2+12r \\ 32+32r &= 3r^2+12r \\ 0 &= 3r^2-20r-32 \\ 0 &= (3r+4)(r-8) \end{aligned}$$

$$(3r+4)=0 \quad (r-8)=0$$

$$r=\cancel{-\frac{4}{3}} \quad r=8$$

Check.
A negative speed does not make sense in this problem, so $r=8$ is the solution.

Is 8 mph a reasonable running speed? Yes.
If Jazmine's running rate is 4, then her biking rate, $r+4$, which is $8+4=12$.

$$\text{Run 8 mph} \quad \frac{8 \text{ miles}}{8 \text{ mph}} = 1 \text{ hour}$$

$$\text{Bike 12 mph} \quad \frac{24 \text{ miles}}{12 \text{ mph}} = 2 \text{ hours}$$

Total 3 hours.

Jazmine's running speed is 8 mph.

TRY IT : : 7.95

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

TRY IT : : 7.96

Joon drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

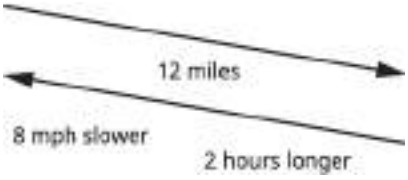
Once again, we will use the uniform motion formula solved for the variable t.

EXAMPLE 7.49

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Hamilton's downhill speed.

Let h = Hamilton's downhill speed.

His uphill speed is 8 miles per hour slower.
Enter the rates into the chart.

$h - 8$ = Hamilton's uphill speed

The distance is the same in both directions.
12 miles.

Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$.

We divide the distance by the rate in each row, and place the expression in the time column.

	Rate	Time	Distance
Downhill	h	$\frac{12}{h}$	12
Uphill	$h-8$	$\frac{12}{h-8}$	12

Write a word sentence about the line.

He took 2 hours longer uphill than downhill.
The uphill time is 2 more than the downhill time.

Translate the sentence to get the equation.

$$\frac{12}{h-8} = \frac{12}{h} + 2$$

Solve.

$$\begin{aligned} h(h-8)\left(\frac{12}{h-8}\right) &= h(h-8)\left(\frac{12}{h}+2\right) \\ 12h &= 12(h-8) + 2h(h-8) \\ 12h &= 12h - 96 + 2h^2 - 16h \\ 0 &= 2h^2 - 16h - 96 \\ 0 &= 2(h^2 - 8h - 48) \\ 0 &= 2(h-12)(h+4) \\ h - 12 = 0 \quad & h + 4 = 0 \\ h = 12 \quad & \cancel{h = -4} \end{aligned}$$

Check.
Is 12 mph a reasonable speed for biking downhill? Yes.

Downhill 12 mph $\quad \frac{12 \text{ miles}}{12 \text{ mph}} = 1$ hour

Uphill $12 - 8 = 4$ mph $\quad \frac{12 \text{ miles}}{4 \text{ mph}} = 3$ hours.

The uphill time is 2 hours more that the downhill time.

Hamilton's downhill speed is 12 mph.

TRY IT : : 7.97

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla's biking speed. Find Kayla's biking speed.

TRY IT : : 7.98

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 20 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

Solve Work Applications

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

This is a typical 'work' application. There are three quantities involved here—the time it would take each of the two presses to do the job alone and the time it would take for them to do the job together.

If Press #1 can complete the job in 6 hours, in one hour it would complete $\frac{1}{6}$ of the job.

If Press #2 can complete the job in 12 hours, in one hour it would complete $\frac{1}{12}$ of the job.

We will let t be the number of hours it would take the presses to print the magazines with both presses running together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take the presses to complete the job if they worked together, we solve for t.

A chart will help us organize the information. We are looking for how many hours it would take to complete the job with both presses running together.

Let t = the number of hours needed to complete the job together.

Enter the hours per job for Press #1, Press #2, and when they work together.

If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed.

Similarly find the part of the job completed/ hours for Press #2 and when thet both together.

	Number of hours to complete the job.	Part of job completed/hour
Press #1	6	$\frac{1}{6}$
Press #2	12	$\frac{1}{12}$
Together	t	$\frac{1}{t}$

Write a word sentence.

The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.

Translate into an equation.

$$\underbrace{\text{Work completed by Press \#1} + \text{Press \#2}}_{} = \text{Together}$$

$$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$$

Solve.

$$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$$

Mutiply by the LCD, $12t$

$$12t\left(\frac{1}{6} + \frac{1}{12}\right) = 12t\left(\frac{1}{t}\right)$$

Simplify.

$$\begin{aligned} 2t + t &= 12 \\ 3t &= 12 \\ t &= 4 \end{aligned}$$

When both presses are running it takes 4 hours to do the job.

Keep in mind, it should take less time for two presses to complete a job working together than for either press to do it

alone.

EXAMPLE 7.50

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint $\frac{1}{8}$ of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

Solution

This is a 'work' application. A chart will help us organize the information. We are looking for the numbers of hours it will take them to paint the room together.

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

Let t be the number of hours needed to paint the room together.

Enter the hours per job for Pete, Alicia, and when they work together.

In 1 hour working together, they have completed $\frac{1}{t}$ of the job.

Similarly, find the part of the job completed/hour by Pete and then by Alicia.

	Number of complete hours to the job.	Part of job completed/hour
Pete	10	$\frac{1}{10}$
Alicia	8	$\frac{1}{8}$
Together	t	$\frac{1}{t}$

Write a word sentence.

The work completed by Pete plus the work completed by Alicia equals the total work completed.

Work completed by:

$$\underbrace{\text{Pete} + \text{Alicia}}_{\frac{1}{10} + \frac{1}{8}} = \text{Together}$$

$$\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$$

Multiply by the LCD, 40t.

$$40t\left(\frac{1}{10} + \frac{1}{8}\right) = 40t\left(\frac{1}{t}\right)$$

Distribute.

$$40t \cdot \frac{1}{10} + 40t \cdot \frac{1}{8} = 40t\left(\frac{1}{t}\right)$$

Simplify and solve.

$$4t + 5t = 40$$
$$9t = 40$$
$$t = \frac{40}{9}$$

We'll write as a mixed number so that we can convert it to hours and minutes.

$$t = 4\frac{4}{9} \text{ hours}$$

Remember, 1 hour = 60 minutes.

$$t = 4 \text{ hours} + \frac{4}{9}(60 \text{ minutes})$$

Multiply, and then round to the nearest minute.

$$t = 4 \text{ hours} + 27 \text{ minutes}$$

It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

TRY IT : : 7.99

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

TRY IT : : 7.100

Daria can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

EXAMPLE 7.51

Ra'shon can clean the house in 7 hours. When his sister helps him it takes 3 hours. How long does it take his sister when she cleans the house alone?

Solution

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take Ra'shon's sister to complete the job by herself.

Let s be the number of hours Ra'shon's sister takes to clean the house alone.

Enter the hours per job for Ra'shon, his sister, and when they work together.
If Ra'shon takes 7 hours, then in 1 hour $\frac{1}{7}$ of the job is completed.
If Ra'shon's sister takes s hours, then in 1 hour $\frac{1}{s}$ of the job is completed.

	Number of hours to clean the house	Part of job completed/hour
Ra'shon	7	$\frac{1}{7}$
His sister	s	$\frac{1}{s}$
Together	3	$\frac{1}{3}$

Write a word sentence.

The part completed by Ra'shon plus the part by his sister equals the amount completed together.

Translate to an equation.

$$\text{Work completed by Ra'shon} + \text{His sister} = \text{Together}$$
$$\frac{1}{7} + \frac{1}{s} = \frac{1}{3}$$

Solve.

$$\frac{1}{7} + \frac{1}{s} = \frac{1}{3}$$

Multiply by the LCD, $21s$.

$$21s\left(\frac{1}{7} + \frac{1}{s}\right) = \left(\frac{1}{3}\right)21s$$
$$3s + 21 = 7s$$

Simplify.

$$-4s = -21$$
$$s = \frac{-21}{-4} = \frac{21}{4}$$

Write as a mixed number to convert it to hours and minutes.

$$s = 5\frac{1}{4} \text{ hours}$$

There are 60 minutes in 1 hour.

$$s = 5 \text{ hours} + \frac{1}{4}(60 \text{ minutes})$$
$$s = 5 \text{ hours} + 15 \text{ minutes}$$

It would take Ra'shon's sister 5 hours and 15 minutes to clean the house alone.

TRY IT : : 7.101

Alice can paint a room in 6 hours. If Kristina helps her it takes them 4 hours to paint the room. How long would it take Kristina to paint the room by herself?

TRY IT : : 7.102

Tracy can lay a slab of concrete in 3 hours, with Jordan's help they can do it in 2 hours. If Jordan works alone, how long will it take?

Solve Direct Variation Problems

When two quantities are related by a proportion, we say they are *proportional* to each other. Another way to express this relation is to talk about the *variation* of the two quantities. We will discuss direct variation and inverse variation in this section.

Lindsay gets paid $15 per hour at her job. If we let s be her salary and h be the number of hours she has worked, we could model this situation with the equation

$$s = 15h$$

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary *varies directly* with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

Direct Variation

For any two variables x and y, y varies directly with x if

$$y = kx, \text{ where } k \neq 0$$

The constant k is called the constant of variation.

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates x and y. Then we can use that equation to find values of y for other values of x.

We'll list the steps here.

HOW TO : : SOLVE DIRECT VARIATION PROBLEMS.

Step 1. Write the formula for direct variation.
Step 2. Substitute the given values for the variables.
Step 3. Solve for the constant of variation.
Step 4. Write the equation that relates x and y using the constant of variation.

Now we'll solve an application of direct variation.

EXAMPLE 7.52

When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes.

ⓐ Write the equation that relates c and m. ⓑ How many calories would he burn if he ran on the treadmill for 25 minutes?

Solution

ⓐ

	The number of calories, c, varies directly with the number of minutes, m, on the treadmill, and $c = 315$ when $m = 18$.
Write the formula for direct variation.	$y = kx$

We will use c in place of y and m in place of x.	$c = km$
Substitute the given values for the variables.	$315 = k \cdot 18$
Solve for the constant of variation.	$\frac{315}{18} = \frac{k \cdot 18}{18}$ $17.5 = k$
Write the equation that relates c and m.	$c = km$
Substitute in the constant of variation.	$c = 17.5m$

ⓑ

	Find c when $m = 25$.
Write the equation that relates c and m.	$c = 17.5m$
Substitute the given value for m.	$c = 17.5(25)$
Simplify.	$c = 437.5$
	Raoul would burn 437.5 calories if he used the treadmill for 25 minutes.

TRY IT : : 7.103

The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising.

ⓐ Write the equation that relates c and t. ⓑ How many calories would he burn if he exercises for 90 minutes?

TRY IT : : 7.104

The distance a moving body travels, d, varies directly with time, t, it moves. A train travels 100 miles in 2 hours

ⓐ Write the equation that relates d and t. ⓑ How many miles would it travel in 5 hours?

Solve Inverse Variation Problems

Many applications involve two variable that *vary inversely*. As one variable increases, the other decreases. The equation that relates them is $y = \frac{k}{x}$.

Inverse Variation

For any two variables x and y, y varies inversely with x if

$$y = \frac{k}{x}, \text{ where } k \neq 0$$

The constant k is called the constant of variation.

The word ‘inverse’ in inverse variation refers to the multiplicative inverse. The multiplicative inverse of x is $\frac{1}{x}$.

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change ‘direct’ to ‘inverse’.

HOW TO : : SOLVE INVERSE VARIATION PROBLEMS.

Step 1. Write the formula for inverse variation.
Step 2. Substitute the given values for the variables.
Step 3. Solve for the constant of variation.
Step 4. Write the equation that relates x and y using the constant of variation.

EXAMPLE 7.53

The frequency of a guitar string varies inversely with its length. A 26 in.-long string has a frequency of 440 vibrations per second.

ⓐ Write the equation of variation. ⓑ How many vibrations per second will there be if the string's length is reduced to 20 inches by putting a finger on a fret?

Solution

ⓐ

	The frequency varies inversely with the length.
Name the variables.	Let f = frequency. L = length
Write the formula for inverse variation.	$y=\frac{k}{x}$
We will use f in place of y and L in place of x.	$f=\frac{k}{L}$
Substitute the given values for the variables.	$f=440$ when $L=26$ $440=\frac{k}{26}$
Solve for the constant of variation	$26(440)=26\left(\frac{k}{26}\right)$ $11{,}440=k$
Write the equation that relates f and L.	$f=\frac{k}{L}$
Substitute the constant of variation	$f=\frac{11{,}440}{L}$

ⓑ

	Find f when $L=20$.
Write the equation that relates f and L.	$f=\frac{11{,}440}{L}$
Substitute the given value for L.	$f=\frac{11{,}440}{20}$
Simplify.	$f=572$
	A 20″-guitar string has frequency 572 vibrations per second.

TRY IT : : 7.105

The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees Celsius.

ⓐ Write the equation of variation. ⓑ How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

TRY IT : : 7.106

Xander's new business found that the daily demand for its product was inversely proportional to the price, p. When the price is $5, the demand is 700 units.

ⓐ Write the equation of variation. ⓑ What is the demand if the price is raised to $7?

MEDIA : :

Access this online resource for additional instruction and practice with applications of rational expressions

- **Applications of Rational Expressions (https://openstax.org/l/37AppRatExp)**

7.5 EXERCISES

Practice Makes Perfect

Solve Proportions

In the following exercises, solve each proportion.

253. $\frac{x}{56} = \frac{7}{8}$

254. $\frac{56}{72} = \frac{y}{9}$

255. $\frac{98}{154} = \frac{-7}{p}$

256. $\frac{72}{156} = \frac{-6}{q}$

257. $\frac{a}{a+12} = \frac{4}{7}$

258. $\frac{b}{b-16} = \frac{11}{9}$

259. $\frac{m+90}{25} = \frac{m+30}{15}$

260. $\frac{n+10}{4} = \frac{40-n}{6}$

261. $\frac{2p+4}{8} = \frac{p+18}{6}$

262. $\frac{q-2}{2} = \frac{2q-7}{18}$

In the following exercises, solve.

263. Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.

ⓐ How many beats per minute is this?

ⓑ Has Kevin met his target heart rate?

264. Jesse's car gets 30 miles per gallon of gas.

ⓐ If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home?

ⓑ If gas is $3.09 per gallon, what is the total cost of the gas for the trip?

265. Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

266. A veterinarian prescribed Sunny, a 65-pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?

267. A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

268. One 12-ounce can of soda has 150 calories. If Josiah drinks the big 32-ounce size from the local mini-mart, how many calories does he get?

269. Kyra is traveling to Canada and will change $250 US dollars into Canadian dollars. At the current exchange rate, $1 US is equal to $1.3 Canadian. How many Canadian dollars will she get for her trip?

270. Maurice is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?

271. Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

272. Sonya drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?

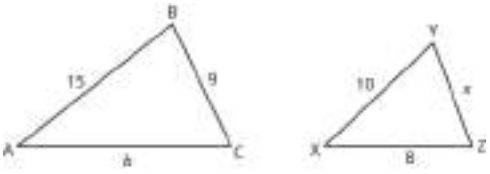
273. Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

274. An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

Solve Similar Figure Applications

In the following exercises, the triangles are similar. Find the length of the indicated side.

275.



ⓐ side x

ⓑ side b

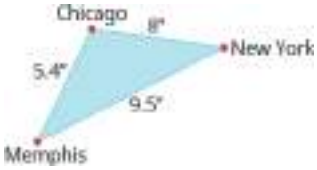
276.



ⓐ side d

ⓑ side q

In the following exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle. The actual distance from New York to Chicago is 800 miles.



277. Find the actual distance from New York to Memphis.

278. Find the actual distance from Chicago to Memphis.

In the following exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle. The actual distance from Atlanta to New Orleans is 420 miles.



279. Find the actual distance from New Orleans to Miami.

280. Find the actual distance from Atlanta to Miami.

In the following exercises, answer each question.

281. A 2-foot-tall dog casts a 3-foot shadow at the same time a cat casts a one foot shadow. How tall is the cat ?

282. Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry's shadow was 8 feet and Tom's was 7.75 feet long. What is Tom's height?

283. The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven-foot shadow. How long is the windmill's shadow?

284. The height of the Statue of Liberty is 305 feet. Nikia, who is standing next to the statue, casts a 6-foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Solve Uniform Motion Applications

In the following exercises, solve the application problem provided.

285. Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35-mph headwind in the same amount of time it can travel 702 miles with a 35-mph tailwind. Find the speed of the helicopter.

286. A private jet can fly 1,210 miles against a 25-mph headwind in the same amount of time it can fly 1694 miles with a 25-mph tailwind. Find the speed of the jet.

287. A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

288. Darrin can skateboard 2 miles against a 4-mph wind in the same amount of time he skateboards 6 miles with a 4-mph wind. Find the speed Darrin skateboards with no wind.

289. Jane spent 2 hours exploring a mountain with a dirt bike. First, she rode 40 miles uphill. After she reached the peak she rode for 12 miles along the summit. While going uphill, she went 5 mph slower than when she was on the summit. What was her rate along the summit?

290. Laney wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

291. Byron wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

292. Nancy took a 3-hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?

293. Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, what was his uphill rate?

294. Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?

295. Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

296. Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel.

297. John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

298. Jim's speedboat can travel 20 miles upstream against a 3-mph current in the same amount of time it travels 22 miles downstream with a 3-mph current speed . Find the speed of the Jim's boat.

299. Hazel needs to get to her granddaughter's house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

300. Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?

301. When driving the 9-hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

302. Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?

303. Dana enjoys taking her dog for a walk, but sometimes her dog gets away, and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

304. Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Solve Work Applications

305. Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

306. It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?

307. Mia can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

308. Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?

309. Josephine can correct her students test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

310. Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take Levi's dad to wash the car by himself?

311. At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

312. Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

Solve Direct Variation Problems

In the following exercises, solve.

313. If y varies directly as x and $y = 14$, when $x = 3$. find the equation that relates x and y.

314. If a varies directly as b and $a = 16$, when $b = 4$. find the equation that relates a and b.

315. If p varies directly as q and $p = 9.6$, when $q = 3$. find the equation that relates p and q.

316. If v varies directly as w and $v = 8$, when $w = \frac{1}{2}$. find the equation that relates v and w.

317. The price, P, that Eric pays for gas varies directly with the number of gallons, g, he buys. It costs him \$50 to buy 20 gallons of gas.

ⓐ Write the equation that relates P and g.

ⓑ How much would 33 gallons cost Eric?

318. Joseph is traveling on a road trip. The distance, d, he travels before stopping for lunch varies directly with the speed, v, he travels. He can travel 120 miles at a speed of 60 mph.

ⓐ Write the equation that relates d and v.

ⓑ How far would he travel before stopping for lunch at a rate of 65 mph?

319. The mass of a liquid varies directly with its volume. A liquid with mass 16 kilograms has a volume of 2 liters.

ⓐ Write the equation that relates the mass to the volume.

ⓑ What is the volume of this liquid if its mass is 128 kilograms?

320. The length that a spring stretches varies directly with a weight placed at the end of the spring. When Sarah placed a 10-pound watermelon on a hanging scale, the spring stretched 5 inches.

ⓐ Write the equation that relates the length of the spring to the weight.

ⓑ What weight of watermelon would stretch the spring 6 inches?

321. The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 6 inch will support a maximum load of 108 pounds.

ⓐ Write the equation that relates the load to the diagonal of the cross-section.

ⓑ What load will a beam with a 10-inch diagonal support?

322. The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

ⓐ Write the equation that relates the area to the radius.

ⓑ What is the area of a personal pizza with a radius 4 inches?

Solve Inverse Variation Problems

In the following exercises, solve.

323. If y varies inversely with x and $y = 5$ when $x = 4$, find the equation that relates x and y.

324. If p varies inversely with q and $p = 2$ when $q = 1$, find the equation that relates p and q.

325. If v varies inversely with w and $v = 6$ when $w = \frac{1}{2}$, find the equation that relates v and w.

326. If a varies inversely with b and $a = 12$ when $b = \frac{1}{3}$, find the equation that relates a and b.

In the following exercises, write an inverse variation equation to solve the following problems.

327. The fuel consumption (mpg) of a car varies inversely with its weight. A Toyota Corolla weighs 2800 pounds getting 33 mpg on the highway.

ⓐ Write the equation that relates the mpg to the car's weight.

ⓑ What would the fuel consumption be for a Toyota Sequoia that weighs 5500 pounds?

328. A car's value varies inversely with its age. Jackie bought a 10-year-old car for $2,400.

ⓐ Write the equation that relates the car's value to its age.

ⓑ What will be the value of Jackie's car when it is 15 years old?

329. The time required to empty a tank varies inversely as the rate of pumping. It took Ada 5 hours to pump her flooded basement using a pump that was rated at 200 gpm (gallons per minute).

ⓐ Write the equation that relates the number of hours to the pump rate.

ⓑ How long would it take Ada to pump her basement if she used a pump rated at 400 gpm?

330. On a string instrument, the length of a string varies inversely as the frequency of its vibrations. An 11-inch string on a violin has a frequency of 400 cycles per second.

ⓐ Write the equation that relates the string length to its frequency.

ⓑ What is the frequency of a 10 inch string?

331. Paul, a dentist, determined that the number of cavities that develops in his patient's mouth each year varies inversely to the number of minutes spent brushing each night. His patient, Lori, had four cavities when brushing her teeth 30 seconds (0.5 minutes) each night.

ⓐ Write the equation that relates the number of cavities to the time spent brushing.

ⓑ How many cavities would Paul expect Lori to have if she had brushed her teeth for 2 minutes each night?

332. Boyle's law states that if the temperature of a gas stays constant, then the pressure varies inversely to the volume of the gas. Braydon, a scuba diver, has a tank that holds 6 liters of air under a pressure of 220 psi.

ⓐ Write the equation that relates pressure to volume.

ⓑ If the pressure increases to 330 psi, how much air can Braydon's tank hold?

333. The cost of a ride service varies directly with the distance traveled. It costs $35 for a ride from the city center to the airport, 14 miles away.

ⓐ Write the equation that relates the cost, c, with the number of miles, m.

ⓑ What would it cost to travel 22 miles with this service?

334. The number of hours it takes Jack to drive from Boston to Bangor is inversely proportional to his average driving speed. When he drives at an average speed of 40 miles per hour, it takes him 6 hours for the trip.

ⓐ Write the equation that relates the number of hours, h, with the speed, s.

ⓑ How long would the trip take if his average speed was 75 miles per hour?

Writing Exercises

335. Marisol solves the proportion $\frac{144}{a}=\frac{9}{4}$ by 'cross multiplying,' so her first step looks like $4\cdot 144=9\cdot a$. Explain how this differs from the method of solution shown in **Example 7.44**.

336. Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3}+\frac{1}{4}=\frac{1}{t}$ can be used to find t, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.

337. In your own words, explain the difference between direct variation and inverse variation.

338. Make up an example from your life experience of inverse variation.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve proportions.			
solve similar figure applications.			
solve uniform motion applications.			
solve work applications.			
solve direct variation problems.			
solve inverse variation problems.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

7.6 Solve Rational Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve rational inequalities
- Solve an inequality with rational functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Find the value of $x-5$ when ⓐ $x=6$ ⓑ $x=-3$ ⓒ $x=5$.
 If you missed this problem, review **Example 1.6**.
2. Solve: $8-2x<12$.
 If you missed this problem, review **Example 2.52**.
3. Write in interval notation: $-3\le x<5$.
 If you missed this problem, review **Example 2.49**.

Solve Rational Inequalities

We learned to solve linear inequalities after learning to solve linear equations. The techniques were very much the same with one major exception. When we multiplied or divided by a negative number, the inequality sign reversed.

Having just learned to solve rational equations we are now ready to solve rational inequalities. A **rational inequality** is an inequality that contains a rational expression.

Rational Inequality

A **rational inequality** is an inequality that contains a rational expression.

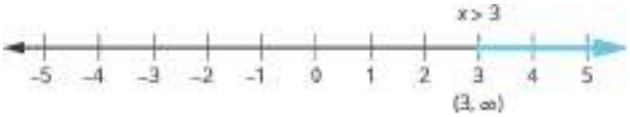
Inequalities such as $\frac{3}{2x}>1$, $\frac{2x}{x-3}<4$, $\frac{2x-3}{x-6}\ge x$, and $\frac{1}{4}-\frac{2}{x^2}\le\frac{3}{x}$ are rational inequalities as they each contain a rational expression.

When we solve a rational inequality, we will use many of the techniques we used solving linear inequalities. We especially must remember that when we multiply or divide by a negative number, the inequality sign must reverse.

Another difference is that we must carefully consider what value might make the rational expression undefined and so must be excluded.

When we solve an equation and the result is $x=3$, we know there is one solution, which is 3.

When we solve an inequality and the result is $x>3$, we know there are many solutions. We graph the result to better help show all the solutions, and we start with 3. Three becomes a **critical point** and then we decide whether to shade to the left or right of it. The numbers to the right of 3 are larger than 3, so we shade to the right.



To solve a rational inequality, we first must write the inequality with only one quotient on the left and 0 on the right.

Next we determine the critical points to use to divide the number line into intervals. A **critical point** is a number which make the rational expression zero or undefined.

We then will evaluate the factors of the numerator and denominator, and find the quotient in each interval. This will identify the interval, or intervals, that contains all the solutions of the rational inequality.

We write the solution in interval notation being careful to determine whether the endpoints are included.

EXAMPLE 7.54

Solve and write the solution in interval notation: $\frac{x-1}{x+3}\ge 0$.

Solution

Step 1. Write the inequality as one quotient on the left and zero on the right.

Our inequality is in this form. $\frac{x-1}{x+3} \geq 0$

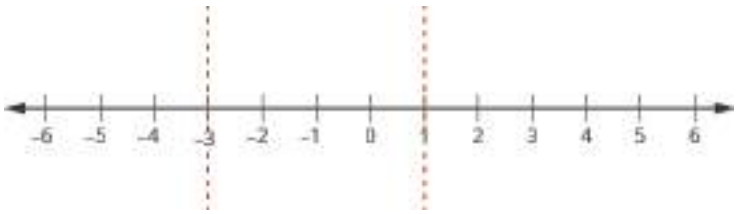
Step 2. Determine the critical points—the points where the rational expression will be zero or undefined.

The rational expression will be zero when the numerator is zero. Since $x-1=0$ when $x=1$, then 1 is a critical point.

The rational expression will be undefined when the denominator is zero. Since $x+3=0$ when $x=-3$, then -3 is a critical point.

The critical points are 1 and -3.

Step 3. Use the critical points to divide the number line into intervals.



The number line is divided into three intervals:

$(-\infty, -3)$ $(-3, 1)$ $(1, \infty)$

Step 4. Test a value in each interval. Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.

To find the sign of each factor in an interval, we choose any point in that interval and use it as a test point. Any point in the interval will give the expression the same sign, so we can choose any point in the interval.

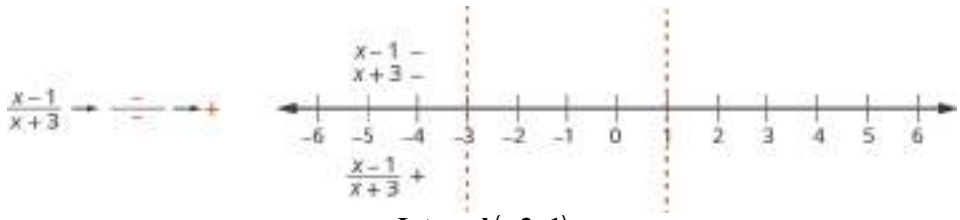
Interval $(-\infty, -3)$

The number -4 is in the interval $(-\infty, -3)$. Test $x=-4$ in the expression in the numerator and the denominator.

the numerator	$x-1$	the denominator	$x+3$
	$-4-1$		$-4+3$
	-5		-1
	Negative		Negative

Above the number line, mark the factor $x-1$ negative and mark the factor $x+3$ negative.

Since a negative divided by a negative is positive, mark the quotient positive in the interval $(-\infty, -3)$.



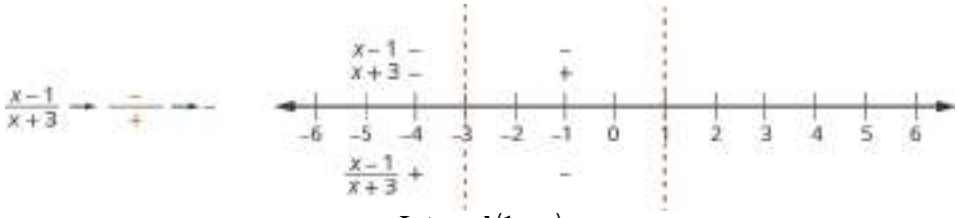
Interval $(-3, 1)$

The number 0 is in the interval $(-3, 1)$. Test $x=0$.

the numerator	$x-1$	the denominator	$x+3$
	$0-1$		$0+3$
	-1		3
	Negative		Positive

Above the number line, mark the factor $x-1$ negative and mark $x+3$ positive.

Since a negative divided by a positive is negative, the quotient is marked negative in the interval $(-3, 1)$.



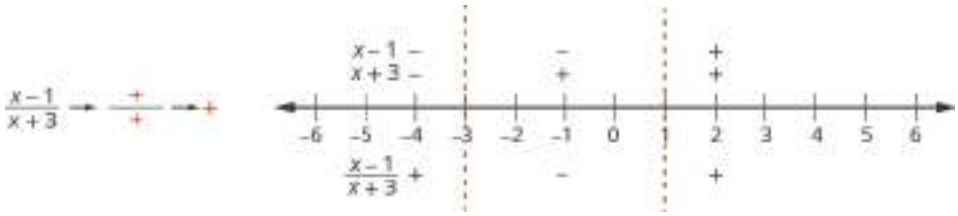
Interval $(1, \infty)$

The number 2 is in the interval $(1, \infty)$. Test $x = 2$.

the numerator	$x - 1$	the denominator	$x + 3$
	$2 - 1$		$2 + 3$
	1		5
	Positive		Positive

Above the number line, mark the factor $x - 1$ positive and mark $x + 3$ positive.

Since a positive divided by a positive is positive, mark the quotient positive in the interval $(1, \infty)$.



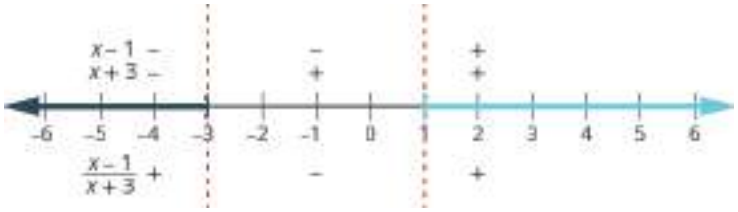
Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

We want the quotient to be greater than or equal to zero, so the numbers in the intervals $(-\infty, -3)$ and $(1, \infty)$ are solutions.

But what about the critical points?

The critical point $x = -3$ makes the denominator 0, so it must be excluded from the solution and we mark it with a parenthesis.

The critical point $x = 1$ makes the whole rational expression 0. The inequality requires that the rational expression be greater than or equal to. So, 1 is part of the solution and we will mark it with a bracket.



Recall that when we have a solution made up of more than one interval we use the union symbol, $\cup$, to connect the two intervals. The solution in interval notation is $(-\infty, -3) \cup [1, \infty)$.

TRY IT : : 7.107 Solve and write the solution in interval notation: $\frac{x-2}{x+4} \geq 0$.

TRY IT : : 7.108 Solve and write the solution in interval notation: $\frac{x+2}{x-4} \geq 0$.

We summarize the steps for easy reference.

HOW TO : : SOLVE A RATIONAL INEQUALITY.

Step 1. Write the inequality as one quotient on the left and zero on the right.

Step 2. Determine the critical points–the points where the rational expression will be zero or undefined.

Step 3. Use the critical points to divide the number line into intervals.

Step 4. Test a value in each interval. Above the number line show the sign of each factor of the numerator and denominator in each interval. Below the number line show the sign of the quotient.

Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

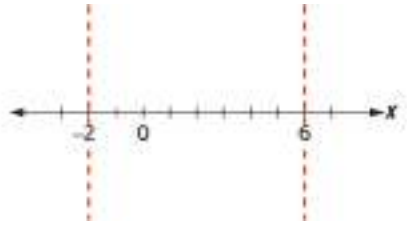
The next example requires that we first get the rational inequality into the correct form.

EXAMPLE 7.55

Solve and write the solution in interval notation: $\frac{4x}{x-6} < 1$.

Solution

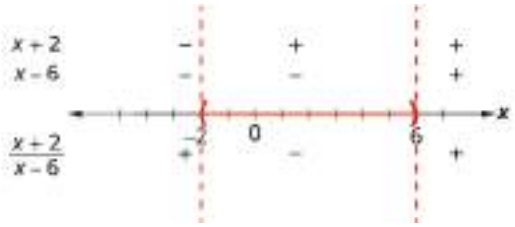
	$\frac{4x}{x-6} < 1$
Subtract 1 to get zero on the right.	$\frac{4x}{x-6} - 1 < 0$
Rewrite 1 as a fraction using the LCD.	$\frac{4x}{x-6} - \frac{x-6}{x-6} < 0$
Subtract the numerators and place the difference over the common denominator.	$\frac{4x-(x-6)}{x-6} < 0$
Simplify.	$\frac{3x+6}{x-6} < 0$
Factor the numerator to show all factors.	$\frac{3(x+2)}{x-6} < 0$
Find the critical points.	
The quotient will be zero when the numerator is zero. The quotient is undefined when the denominator is zero.	$x+2 = 0 \quad x-6 = 0$ $x = -2 \quad x = 6$
Use the critical points to divide the number line into intervals.	



Test a value in each interval.

	$(-\infty, -2)$	$(-2, 6)$	$(6, \infty)$
$x+2$	$x+2$ $-3+2$ -1 $-$	$x+2$ $0+2$ 2 $+$	$x+2$ $7+2$ 9 $+$
$x-6$	$x-6$ $-3-6$ -9 $-$	$x-6$ $0-6$ -6 $-$	$x-6$ $7-6$ 1 $+$

Above the number line show the sign of each factor of the rational expression in each interval.
Below the number line show the sign of the quotient.



Determine the intervals where the inequality is correct. We want the quotient to be negative, so the solution includes the points between −2 and 6. Since the inequality is strictly less than, the endpoints are not included.

We write the solution in interval notation as (−2, 6).

TRY IT : : 7.109 Solve and write the solution in interval notation: $\frac{3x}{x-3} < 1$.

TRY IT : : 7.110 Solve and write the solution in interval notation: $\frac{3x}{x-4} < 2$.

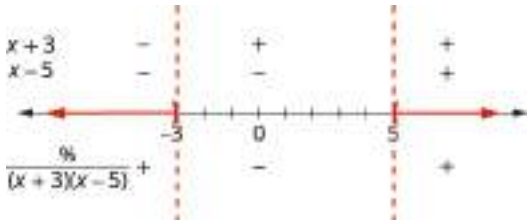
In the next example, the numerator is always positive, so the sign of the rational expression depends on the sign of the denominator.

EXAMPLE 7.56

Solve and write the solution in interval notation: $\frac{5}{x^2-2x-15} > 0$.

Solution

The inequality is in the correct form.	$\frac{5}{x^2-2x-15} > 0$
Factor the denominator.	$\frac{5}{(x+3)(x-5)} > 0$
Find the critical points. The quotient is 0 when the numerator is 0. Since the numerator is always 5, the quotient cannot be 0.	

The quotient will be undefined when the denominator is zero.	$(x+3)(x-5)=0$ $x=-3,\ x=5$
Use the critical points to divide the number line into intervals.	
	
Test values in each interval. Above the number line show the sign of each factor of the denominator in each interval. Below the number line, show the sign of the quotient.	
Write the solution in interval notation.	$(-\infty,\ -3)\cup(5,\ \infty)$

TRY IT : : 7.111 Solve and write the solution in interval notation: $\frac{1}{x^2+2x-8}>0.$

TRY IT : : 7.112 Solve and write the solution in interval notation: $\frac{3}{x^2+x-12}>0.$

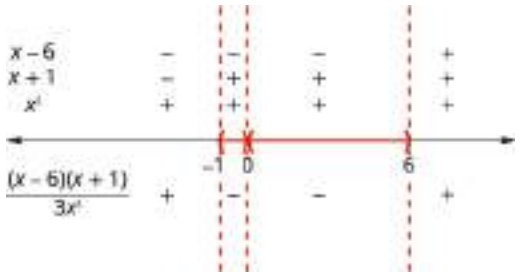
The next example requires some work to get it into the needed form.

EXAMPLE 7.57

Solve and write the solution in interval notation: $\frac{1}{3}-\frac{2}{x^2}<\frac{5}{3x}.$

Solution

	$\frac{1}{3}-\frac{2}{x^2}<\frac{5}{3x}$
Subtract $\frac{5}{3x}$ to get zero on the right.	$\frac{1}{3}-\frac{2}{x^2}-\frac{5}{3x}<0$
Rewrite to get each fraction with the LCD $3x^2$.	$\frac{1\cdot x^2}{3\cdot x^2}-\frac{2\cdot 3}{x^2\cdot 3}-\frac{5\cdot x}{3x\cdot x}<0$
Simplify.	$\frac{x^2}{3x^2}-\frac{6}{3x^2}-\frac{5x}{3x^2}<0$
Subtract the numerators and place the difference over the common denominator.	$\frac{x^2-5x-6}{3x^2}<0$
Factor the numerator.	$\frac{(x-6)(x+1)}{3x^2}<0$

Find the critical points.	$3x^2 = 0 \quad x-6 = 0 \quad x+1 = 0$ $x = 0 \quad x = 6 \quad x = -1$
Use the critical points to divide the number line into intervals.	
	
Above the number line show the sign of each factor in each interval. Below the number line, show the sign of the quotient.	
Since, 0 is excluded, the solution is the two intervals, $(-1, 0)$ and $(0, 6)$.	$(-1, 0) \cup (0, 6)$

TRY IT : : 7.113 Solve and write the solution in interval notation: $\frac{1}{2} + \frac{4}{x^2} < \frac{3}{x}$.

TRY IT : : 7.114 Solve and write the solution in interval notation: $\frac{1}{3} + \frac{6}{x^2} < \frac{3}{x}$.

Solve an Inequality with Rational Functions

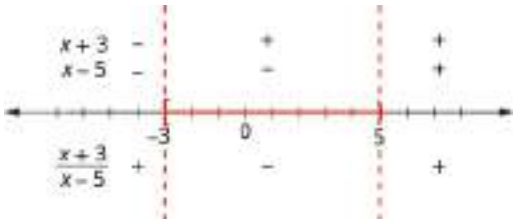
When working with rational functions, it is sometimes useful to know when the function is greater than or less than a particular value. This leads to a rational inequality.

EXAMPLE 7.58

Given the function $R(x) = \frac{x+3}{x-5}$, find the values of x that make the function less than or equal to 0.

Solution

We want the function to be less than or equal to 0.

	$R(x) \le 0$
Substitute the rational expression for $R(x)$.	$\frac{x+3}{x-5} \le 0 \qquad x \ne 5$
Find the critical points.	$x+3 = 0 \qquad x-5 = 0$ $x = -3 \qquad x = 5$
Use the critical points to divide the number line into intervals.	
	
Test values in each interval. Above the number line, show the sign of each factor in each interval. Below the number line, show the sign of the quotient	
Write the solution in interval notation. Since 5 is excluded we, do not include it in the interval.	$[-3, 5)$

TRY IT : : 7.115

Given the function $R(x) = \frac{x-2}{x+4}$, find the values of x that make the function less than or equal to 0.

TRY IT : : 7.116

Given the function $R(x) = \frac{x+1}{x-4}$, find the values of x that make the function less than or equal to 0.

In economics, the function $C(x)$ is used to represent the cost of producing x units of a commodity. The average cost per unit can be found by dividing $C(x)$ by the number of items x. Then, the average cost per unit is $c(x) = \frac{C(x)}{x}$.

EXAMPLE 7.59

The function $C(x) = 10x + 3000$ represents the cost to produce x, number of items. Find ⓐ the average cost function, $c(x)$ ⓑ how many items should be produced so that the average cost is less than $40.

Solution

ⓐ

The average cost function is $c(x) = \frac{C(x)}{x}$.

$C(x) = 10x + 3000$

To find he average cost function, divide the cost function by x.

$c(x) = \frac{C(x)}{x}$

$c(x) = \frac{10x + 3000}{x}$

The average cost function is $c(x) = \frac{10x + 3000}{x}$.

ⓑ

We want the function $c(x)$ to be less than 40.

$$c(x) < 40$$

Substitute the rational expression for $c(x)$.

$$\frac{10x+3000}{x} < 40 \quad x \neq 0$$

Subtract 40 to get 0 on the right.

$$\frac{10x+3000}{x} - 40 < 0$$

Rewrite the left side as one quotient by findin the LCD and performing the subtraction.

$$\frac{10x+3000}{x} - 40\left(\frac{x}{x}\right) < 0$$

$$\frac{10x+3000}{x} - \frac{40x}{x} < 0$$

$$\frac{10x+3000-40x}{x} < 0$$

$$\frac{-30x+3000}{x} < 0$$

Factor the numerator to show all factors.

$$\frac{-30(x-100)}{x} < 0$$

Find the critical points.

$$-30(x-100) = 0 \qquad x = 0$$

$$-30 \neq 0 \quad x-100 = 0$$

$$x = 100$$

More than 100 items must be produced to keep the average cost below $40 per item.

TRY IT : : 7.117

The function $C(x) = 20x + 6000$ represents the cost to produce x, number of items. Find ⓐ the average cost function, $c(x)$ ⓑ how many items should be produced so that the average cost is less than $60?

TRY IT : : 7.118

The function $C(x) = 5x + 900$ represents the cost to produce x, number of items. Find ⓐ the average cost function, $c(x)$ ⓑ how many items should be produced so that the average cost is less than $20?

7.6 EXERCISES

Practice Makes Perfect

Solve Rational Inequalities

In the following exercises, solve each rational inequality and write the solution in interval notation.

339. $\frac{x-3}{x+4} \geq 0$

340. $\frac{x+6}{x-5} \geq 0$

341. $\frac{x+1}{x-3} \leq 0$

342. $\frac{x-4}{x+2} \leq 0$

343. $\frac{x-7}{x-1} > 0$

344. $\frac{x+8}{x+3} > 0$

345. $\frac{x-6}{x+5} < 0$

346. $\frac{x+5}{x-2} < 0$

347. $\frac{3x}{x-5} < 1$

348. $\frac{5x}{x-2} < 1$

349. $\frac{6x}{x-6} > 2$

350. $\frac{3x}{x-4} > 2$

351. $\frac{2x+3}{x-6} \leq 1$

352. $\frac{4x-1}{x-4} \leq 1$

353. $\frac{3x-2}{x-4} \geq 2$

354. $\frac{4x-3}{x-3} \geq 2$

355. $\frac{1}{x^2+7x+12} > 0$

356. $\frac{1}{x^2-4x-12} > 0$

357. $\frac{3}{x^2-5x+4} < 0$

358. $\frac{4}{x^2+7x+12} < 0$

359. $\frac{2}{2x^2+x-15} \geq 0$

360. $\frac{6}{3x^2-2x-5} \geq 0$

361. $\frac{-2}{6x^2-13x+6} \leq 0$

362. $\frac{-1}{10x^2+11x-6} \leq 0$

363. $\frac{1}{2}+\frac{12}{x^2} > \frac{5}{x}$

364. $\frac{1}{3}+\frac{1}{x^2} > \frac{4}{3x}$

365. $\frac{1}{2}-\frac{4}{x^2} \leq \frac{1}{x}$

366. $\frac{1}{2}-\frac{3}{2x^2} \geq \frac{1}{x}$

367. $\frac{1}{x^2-16} < 0$

368. $\frac{4}{x^2-25} > 0$

369. $\frac{4}{x-2} \geq \frac{3}{x+1}$

370. $\frac{5}{x-1} \leq \frac{4}{x+2}$

Solve an Inequality with Rational Functions

In the following exercises, solve each rational function inequality and write the solution in interval notation.

371. Given the function $R(x)=\frac{x-5}{x-2}$, find the values of x that make the function less than or equal to 0.

372. Given the function $R(x)=\frac{x+1}{x+3}$, find the values of x that make the function less than or equal to 0.

373. Given the function $R(x)=\frac{x-6}{x+2}$, find the values of x that make the function less than or equal to 0.

374. Given the function $R(x)=\frac{x+1}{x-4}$, find the values of x that make the function less than or equal to 0.

Writing Exercises

375. Write the steps you would use to explain solving rational inequalities to your little brother.

376. Create a rational inequality whose solution is $(-\infty, -2] \cup [4, \infty)$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve rational inequalities.			
solve an inequality with rational functions.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

CHAPTER 7 REVIEW

KEY TERMS

complex rational expression A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

critical point of a rational inequality The critical point of a rational inequality is a number which makes the rational expression zero or undefined.

extraneous solution to a rational equation An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

proportion When two rational expressions are equal, the equation relating them is called a proportion.

rational equation A rational equation is an equation that contains a rational expression.

rational expression A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

rational function A rational function is a function of the form $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

rational inequality A rational inequality is an inequality that contains a rational expression.

similar figures Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

simplified rational expression A simplified rational expression has no common factors, other than 1, in its numerator and denominator.

KEY CONCEPTS

7.1 Multiply and Divide Rational Expressions

- **Determine the values for which a rational expression is undefined.**

 Step 1. Set the denominator equal to zero.

 Step 2. Solve the equation.

- **Equivalent Fractions Property**
 If a, b, and c are numbers where $b \neq 0, c \neq 0,$ then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

- **How to simplify a rational expression.**

 Step 1. Factor the numerator and denominator completely.

 Step 2. Simplify by dividing out common factors.

- **Opposites in a Rational Expression**
 The opposite of $a - b$ is $b - a$.
 $\frac{a-b}{b-a} = -1 \qquad a \neq b$
 An expression and its opposite divide to -1.

- **Multiplication of Rational Expressions**
 If p, q, r, and s are polynomials where $q \neq 0, s \neq 0,$ then
 $$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

- **How to multiply rational expressions.**

 Step 1. Factor each numerator and denominator completely.

 Step 2. Multiply the numerators and denominators.

 Step 3. Simplify by dividing out common factors.

- **Division of Rational Expressions**
 If p, q, r, and s are polynomials where $q \neq 0, r \neq 0, s \neq 0,$ then
 $$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

- **How to divide rational expressions.**

 Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.

 Step 2. Factor the numerators and denominators completely.

 Step 3. Multiply the numerators and denominators together.

 Step 4. Simplify by dividing out common factors.
- **How to determine the domain of a rational function.**

 Step 1. Set the denominator equal to zero.

 Step 2. Solve the equation.

 Step 3. The domain is all real numbers excluding the values found in Step 2.

7.2 Add and Subtract Rational Expressions

- **Rational Expression Addition and Subtraction**
 If *p*, *q*, and *r* are polynomials where $r \neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \text{ and } \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

- **How to find the least common denominator of rational expressions.**

 Step 1. Factor each expression completely.

 Step 2. List the factors of each expression. Match factors vertically when possible.

 Step 3. Bring down the columns.

 Step 4. Write the LCD as the product of the factors.
- **How to add or subtract rational expressions.**

 Step 1. Determine if the expressions have a common denominator.
 - Yes – go to step 2.
 - No – Rewrite each rational expression with the LCD.
 - Find the LCD.
 - Rewrite each rational expression as an equivalent rational expression with the LCD.

 Step 2. Add or subtract the rational expressions.

 Step 3. Simplify, if possible.

7.3 Simplify Complex Rational Expressions

- **How to simplify a complex rational expression by writing it as division.**

 Step 1. Simplify the numerator and denominator.

 Step 2. Rewrite the complex rational expression as a division problem.

 Step 3. Divide the expressions.
- **How to simplify a complex rational expression by using the LCD.**

 Step 1. Find the LCD of all fractions in the complex rational expression.

 Step 2. Multiply the numerator and denominator by the LCD.

 Step 3. Simplify the expression.

7.4 Solve Rational Equations

- **How to solve equations with rational expressions.**

 Step 1. Note any value of the variable that would make any denominator zero.

 Step 2. Find the least common denominator of all denominators in the equation.

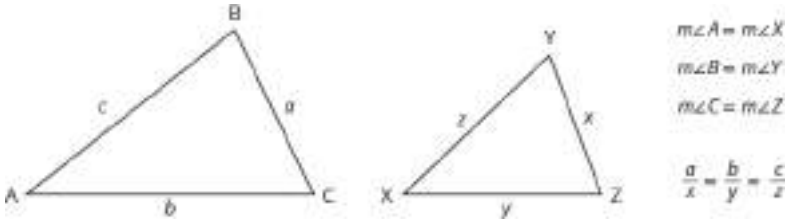
 Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

 Step 4. Solve the resulting equation.

 Step 5. Check:
 - If any values found in Step 1 are algebraic solutions, discard them.
 - Check any remaining solutions in the original equation.

7.5 Solve Applications with Rational Equations

- A proportion is an equation of the form $\frac{a}{b}=\frac{c}{d}$, where $b\neq 0, d\neq 0$. The proportion is read "*a* is to *b* as *c* is to *d*."
- **Property of Similar Triangles**
 If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides have the same ratio.



- **Direct Variation**
 - For any two variables *x* and *y*, *y* varies directly with *x* if $y=kx$, where $k\neq 0$. The constant *k* is called the constant of variation.
 - How to solve direct variation problems.
 Step 1. Write the formula for direct variation.
 Step 2. Substitute the given values for the variables.
 Step 3. Solve for the constant of variation.
 Step 4. Write the equation that relates x and y.
- **Inverse Variation**
 - For any two variables *x* and *y*, *y* varies inversely with *x* if $y=\frac{k}{x}$, where $k\neq 0$. The constant *k* is called the constant of variation.
 - How to solve inverse variation problems.
 Step 1. Write the formula for inverse variation.
 Step 2. Substitute the given values for the variables.
 Step 3. Solve for the constant of variation.
 Step 4. Write the equation that relates *x* and *y*.

7.6 Solve Rational Inequalities

- **Solve a rational inequality.**
 Step 1. Write the inequality as one quotient on the left and zero on the right.
 Step 2. Determine the critical points–the points where the rational expression will be zero or undefined.
 Step 3. Use the critical points to divide the number line into intervals.
 Step 4. Test a value in each interval. Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.
 Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

REVIEW EXERCISES

7.1 Simplify, Multiply, and Divide Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

377. $\frac{5a+3}{3a-2}$

378. $\frac{b-7}{b^2-25}$

379. $\frac{5x^2y^2}{8y}$

380. $\frac{x-3}{x^2-x-30}$

Simplify Rational Expressions

In the following exercises, simplify.

381. $\frac{18}{24}$

382. $\frac{9m^4}{18mn^3}$

383. $\frac{x^2+7x+12}{x^2+8x+16}$

384. $\frac{7v-35}{25-v^2}$

Multiply Rational Expressions

In the following exercises, multiply.

385. $\frac{5}{8}\cdot\frac{4}{15}$

386. $\frac{3xy^2}{8y^3}\cdot\frac{16y^2}{24x}$

387. $\frac{72x-12x^2}{8x+32}\cdot\frac{x^2+10x+24}{x^2-36}$

388. $\frac{6y^2-2y-10}{9-y^2}\cdot\frac{y^2-6y+9}{6y^2+29y-20}$

Divide Rational Expressions

In the following exercises, divide.

389. $\frac{x^2-4x+12}{x^2+8x+12}\div\frac{x^2-36}{3x}$

390. $\frac{y^2-16}{4}\div\frac{y^3-64}{2y^2+8y+32}$

391. $\frac{11+w}{w-9}\div\frac{121-w^2}{9-w}$

392. $\frac{3y^2-12y-63}{4y+3}\div(6y^2-42y)$

393. $\frac{\frac{c^2-64}{3c^2+26c+16}}{\frac{c^2-4c-32}{15c+10}}$

394. $\frac{8a^2+16a}{a-4}\cdot\frac{a^2+2a-24}{a^2+7a+10}\div\frac{2a^2-6a}{a+5}$

Multiply and Divide Rational Functions

395. Find $R(x)=f(x)\cdot g(x)$ where $f(x)=\frac{9x^2+9x}{x^2-3x-4}$ and $g(x)=\frac{x^2-16}{3x^2+12x}$.

396. Find $R(x)=\frac{f(x)}{g(x)}$ where $f(x)=\frac{27x^2}{3x-21}$ and $g(x)=\frac{9x^2+54x}{x^2-x-42}$.

7.2 Add and Subtract Rational Expressions

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, perform the indicated operations.

397. $\frac{7}{15}+\frac{8}{15}$

398. $\frac{4a^2}{2a-1}-\frac{1}{2a-1}$

399. $\frac{y^2+10y}{y+5}+\frac{25}{y+5}$

400. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

401. $\frac{x^2}{x-7}-\frac{3x+28}{x-7}$

402. $\frac{y^2}{y+11}-\frac{121}{y+11}$

403. $\frac{4q^2-q+3}{q^2+6q+5}-\frac{3q^2-q-6}{q^2+6q+5}$

404. $\frac{5t+4t+3}{t^2-25}-\frac{4t^2-8t-32}{t^2-25}$

Add and Subtract Rational Expressions Whose Denominators Are Opposites

In the following exercises, add and subtract.

405. $\frac{18w}{6w-1}+\frac{3w-2}{1-6w}$

406. $\frac{a^2+3a}{a^2-4}-\frac{3a-8}{4-a^2}$

407. $\frac{2b^2+3b-15}{b^2-49}-\frac{b^2+16b-1}{49-b^2}$

408. $\frac{8y^2-10y+7}{2y-5}+\frac{2y^2+7y+2}{5-2y}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

409. $\frac{7}{a^2-3a-10}, \frac{3a}{a^2-a-20}$

410. $\frac{6}{n^2-4}, \frac{2n}{n^2-4n+4}$

411. $\frac{5}{3p^2+17p-6}, \frac{2m}{3p^2-23p-8}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

412. $\frac{7}{5a}+\frac{3}{2b}$

413. $\frac{2}{c-2}+\frac{9}{c+3}$

414. $\frac{3x}{x^2-9}+\frac{5}{x^2+6x+9}$

415. $\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$

416. $\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$

417. $\frac{3y}{y+2}-\frac{y+2}{y+8}$

418. $\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$

419. $\frac{7m+3}{m+2}-5$

420. $\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$

421. $\frac{8a}{a^2-64}-\frac{4}{a+8}$

422. $\frac{5}{12x^2y}+\frac{7}{20xy^3}$

Add and Subtract Rational Functions

In the following exercises, find $R(x)=f(x)+g(x)$ where $f(x)$ and $g(x)$ are given.

423. $f(x)=\frac{2x^2+12x-11}{x^2+3x-10}, \quad g(x)=\frac{x+1}{2-x}$

424. $f(x)=\frac{-4x+31}{x^2+x-30}, \quad g(x)=\frac{5}{x+6}$

In the following exercises, find $R(x)=f(x)-g(x)$ where $f(x)$ and $g(x)$ are given.

425. $f(x)=\frac{4x}{x^2-121}, \quad g(x)=\frac{2}{x-11}$

426. $f(x)=\frac{7}{x+6}, \quad g(x)=\frac{14x}{x^2-36}$

7.3 Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing It as Division

In the following exercises, simplify.

427. $\dfrac{\frac{7x}{x+2}}{\frac{14x^2}{x^2-4}}$

428. $\dfrac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$

429. $\dfrac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$

430. $\dfrac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

431. $\dfrac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

432. $\dfrac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$

433. $\dfrac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$

434. $\dfrac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$

7.4 7.4 Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.

435. $\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$

436. $1-\frac{2}{m}=\frac{8}{m^2}$

437. $\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$

438. $\frac{3}{q+8}-\frac{2}{q-2}=1$

439. $\frac{v-15}{v^2-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$

440. $\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$

Solve Rational Equations that Involve Functions

441. For rational function, $f(x)=\dfrac{x+2}{x^2-6x+8}$, ⓐ find the domain of the function ⓑ solve $f(x)=1$ ⓒ find the points on the graph at this function value.

442. For rational function, $f(x)=\dfrac{2-x}{x^2+7x+10}$, ⓐ find the domain of the function ⓑ solve $f(x)=2$ ⓒ find the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.

443. $\frac{V}{l}=hw$ for l.

444. $\frac{1}{x}-\frac{2}{y}=5$ for y.

445. $x=\frac{y+5}{z-7}$ for z.

446. $P=\frac{k}{V}$ for V.

7.5 Solve Applications with Rational Equations

Solve Proportions

In the following exercises, solve.

447. $\frac{x}{4}=\frac{3}{5}$

448. $\frac{3}{y}=\frac{9}{5}$

449. $\frac{s}{s+20}=\frac{3}{7}$

450. $\frac{t-3}{5}=\frac{t+2}{9}$

Solve Using Proportions

In the following exercises, solve.

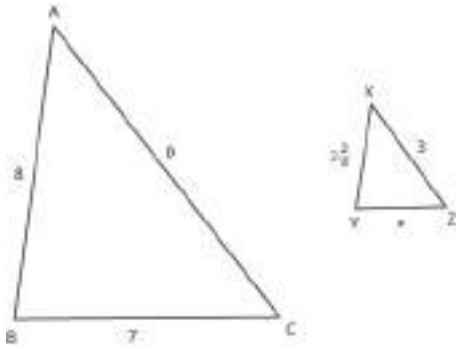
451. Rachael had a 21-ounce strawberry shake that has 739 calories. How many calories are there in a 32-ounce shake?

452. Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

Solve Similar Figure Applications

In the following exercises, solve.

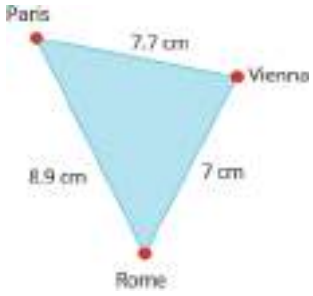
453. ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.



454. On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from

ⓐ Paris to Rome

ⓑ Paris to Vienna



455. Francesca is 5.75 feet tall. Late one afternoon, her shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

456. The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5-foot-tall Natasha cast a 1.1-foot shadow. How long would the shadow of the lighthouse be?

Solve Uniform Motion Applications

In the following exercises, solve.

457. When making the 5-hour drive home from visiting her parents, Lolo ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles when it turned bad. How fast did she drive when the weather was bad?

458. Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

459. Josue can ride his bicycle 8 mph faster than Arjun can ride his bike. It takes Luke 3 hours longer than Josue to ride 48 miles. How fast can John ride his bike?

460. Curtis was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

461. Brandy can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

462. Prem takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

463. Jeffrey can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

464. Marta and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

Solve Direct Variation Problems

In the following exercises, solve.

465. If y varies directly as x when $y = 9$ and $x = 3$, find x when $y = 21$.

466. If y varies inversely as x when $y = 20$ and $x = 2$, find y when $x = 4$.

467. Vanessa is traveling to see her fiancé. The distance, d, varies directly with the speed, v, she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

468. If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs $12, how much would a 6” diameter pizza cost?

469. The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Solve Inverse Variation Problems

In the following exercises, solve.

470. If m varies inversely with the square of n, when $m = 4$ and $n = 6$ find m when $n = 2$.

471. The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for $6, how many tickets can Madelyn afford to buy if the price increased to $8?

472. On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12-inch string have?

7.6 Solve Rational Inequalities

Solve Rational Inequalities

In the following exercises, solve each rational inequality and write the solution in interval notation.

473. $\frac{x-3}{x+4} \leq 0$

474. $\frac{5x}{x-2} > 1$

475. $\frac{3x-2}{x-4} \leq 2$

476. $\frac{1}{x^2-4x-12} < 0$

477. $\frac{1}{2} - \frac{4}{x^2} \geq \frac{1}{x}$

478. $\frac{4}{x-2} < \frac{3}{x+1}$

Solve an Inequality with Rational Functions

In the following exercises, solve each rational function inequality and write the solution in interval notation

479. Given the function, $R(x) = \frac{x-5}{x-2}$, find the values of x that make the function greater than or equal to 0.

480. Given the function, $R(x) = \frac{x+1}{x+3}$, find the values of x that make the function less than or equal to 0.

481. The function $C(x) = 150x + 100{,}000$ represents the cost to produce x, number of items. Find ⓐ the average cost function, $c(x)$ ⓑ how many items should be produced so that the average cost is less than $160.

482. Tillman is starting his own business by selling tacos at the beach. Accounting for the cost of his food truck and ingredients for the tacos, the function $C(x) = 2x + 6{,}000$ represents the cost for Tillman to produce x, tacos. Find ⓐ the average cost function, $c(x)$ for Tillman's Tacos ⓑ how many tacos should Tillman produce so that the average cost is less than $4.

PRACTICE TEST

In the following exercises, simplify.

483. $\frac{4a^2b}{12ab^2}$

484. $\frac{6x-18}{x^2-9}$

In the following exercises, perform the indicated operation and simplify.

485. $\frac{4x}{x+2} \cdot \frac{x^2+5x+6}{12x^2}$

486. $\frac{2y^2}{y^2-1} \div \frac{y^3-y^2+y}{y^3-1}$

487. $\frac{6x^2-x+20}{x^2-81} - \frac{5x^2+11x-7}{x^2-81}$

488. $\frac{-3a}{3a-3} + \frac{5a}{a^2+3a-4}$

489. $\frac{2n^2+8n-1}{n^2-1} - \frac{n^2-7n-1}{1-n^2}$

490. $\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

491. $\frac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$

In the following exercises, solve each equation.

492. $\frac{1}{x}+\frac{3}{4}=\frac{5}{8}$

493. $\frac{1}{z-5}+\frac{1}{z+5}=\frac{1}{z^2-25}$

494. $\frac{z}{2z+8}-\frac{3}{4z-8}=\frac{3z^2-16z-16}{8z^2+2z-64}$

In the following exercises, solve each rational inequality and write the solution in interval notation.

495. $\frac{6x}{x-6} \le 2$

496. $\frac{2x+3}{x-6} > 1$

497. $\frac{1}{2}+\frac{12}{x^2} \ge \frac{5}{x}$

In the following exercises, find $R(x)$ given $f(x)=\frac{x-4}{x^2-3x-10}$ and $g(x)=\frac{x-5}{x^2-2x-8}$.

498. $R(x)=f(x)-g(x)$

499. $R(x)=f(x)\cdot g(x)$

500. $R(x)=f(x)\div g(x)$

501. Given the function, $R(x)=\frac{2}{2x^2+x-15}$, find the values of x that make the function less than or equal to 0.

In the following exercises, solve.

502. If y varies directly with x, and $x=5$ when $y=30$, find x when $y=42$.

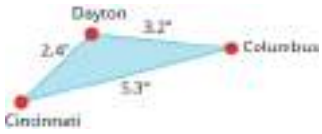
503. If y varies inversely with the square of x and $x=3$ when $y=9$, find y when $x=4$.

504. Matheus can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Matheus' speed on his bike?

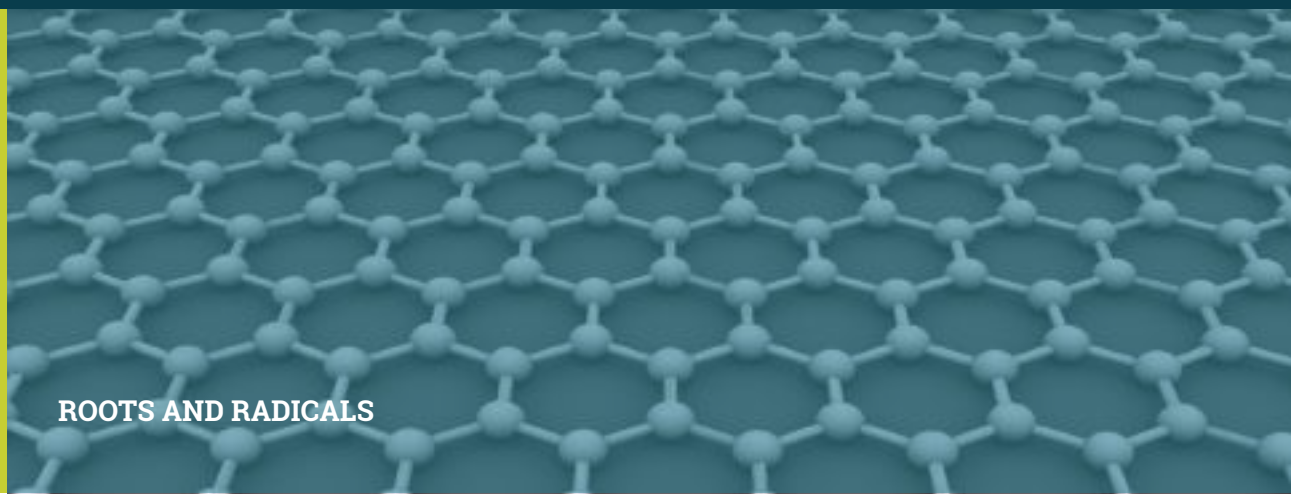
505. Oliver can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Oliver's dad working alone to split the logs?

506. The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

507. The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio. The diagram gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?



8 ROOTS AND RADICALS

Figure 8.1 Graphene is an incredibly strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: "AlexanderAIUS" / Wikimedia Commons)

Chapter Outline

8.1 Simplify Expressions with Roots
8.2 Simplify Radical Expressions
8.3 Simplify Rational Exponents
8.4 Add, Subtract, and Multiply Radical Expressions
8.5 Divide Radical Expressions
8.6 Solve Radical Equations
8.7 Use Radicals in Functions
8.8 Use the Complex Number System

Introduction

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

8.1 Simplify Expressions with Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: ⓐ $(-9)^2$ ⓑ -9^2 ⓒ $(-9)^3$.
 If you missed this problem, review Example 2.21.

2. Round 3.846 to the nearest hundredth.
 If you missed this problem, review **Example 1.34**.
3. Simplify: ⓐ $x^3 \cdot x^3$ ⓑ $y^2 \cdot y^2 \cdot y^2$ ⓒ $z^3 \cdot z^3 \cdot z^3 \cdot z^3$.
 If you missed this problem, review **Example 5.12**.

Simplify Expressions with Roots

In **Foundations**, we briefly looked at square roots. Remember that when a real number *n* is multiplied by itself, we write n^2 and read it '*n* squared'. This number is called the **square** of *n*, and *n* is called the **square root**. For example,

$$13^2 \text{ is read "13 squared"}$$
$$169 \text{ is called the } \textit{square} \text{ of } 13, \text{ since } 13^2 = 169$$
$$13 \text{ is a } \textit{square root} \text{ of } 169$$

Square and Square Root of a number

Square

$$\text{If } n^2 = m, \text{ then } m \text{ is the } \textbf{square} \text{ of } n.$$

Square Root

$$\text{If } n^2 = m, \text{ then } n \text{ is a } \textbf{square root} \text{ of } m.$$

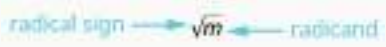
Notice $(-13)^2 = 169$ also, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of *m*. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Square Root Notation

$$\sqrt{m} \text{ is read "the square root of } m\text{"}.$$
$$\text{If } n^2 = m, \text{ then } n = \sqrt{m}, \text{ for } n \geq 0.$$



We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169} = 13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169} = -13$.

EXAMPLE 8.1

Simplify: ⓐ $\sqrt{144}$ ⓑ $-\sqrt{289}$.

Solution

ⓐ

Since $12^2 = 144$. $\sqrt{144}$ → 12

ⓑ

Since $17^2 = 289$ and the negative is in front of the radical sign. $-\sqrt{289}$ → -17

TRY IT : : 8.1 Simplify: ⓐ $-\sqrt{64}$ ⓑ $\sqrt{225}$.

TRY IT : : 8.2 Simplify: ⓐ $\sqrt{100}$ ⓑ $-\sqrt{121}$.

Can we simplify $\sqrt{-49}$? Is there a number whose square is -49?

$$(\quad)^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

EXAMPLE 8.2

Simplify: ⓐ $\sqrt{-196}$ ⓑ $-\sqrt{64}$.

Solution

ⓐ

There is no real number whose square is -196.

$\sqrt{-196}$

$\sqrt{-196}$ is not a real number.

ⓑ

The negative is in front of the radical.

$-\sqrt{64}$

-8

TRY IT : : 8.3 Simplify: ⓐ $\sqrt{-169}$ ⓑ $-\sqrt{81}$.

TRY IT : : 8.4 Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{-121}$.

So far we have only talked about squares and square roots. Let's now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth power
n^5	n to the fi th power

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from −5 to 5. See **Figure 8.2**.

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
−1	1	−1	1	−1
−2	4	−8	16	−32
−3	9	−27	81	−243
−4	16	−64	256	−1024
−5	25	−125	625	−3125

Figure 8.2

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of −2 to help you see

this.

n	n^2	n^3	n^4	n^5
−2	4	−8	16	−32

Even power
Positive result

Odd power
Negative result

We will now extend the square root definition to higher roots.

n^{th} Root of a Number

If $b^n = a$, then b is an n^{th} root of a.
The principal n^{th} root of a is written $\sqrt[n]{a}$.
n is called the **index** of the radical.

Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We can refer to Figure 8.2 to help find higher roots.

$$4^3 = 64 \qquad \sqrt[3]{64} = 4$$
$$3^4 = 81 \qquad \sqrt[4]{81} = 3$$
$$(-2)^5 = -32 \qquad \sqrt[5]{-32} = -2$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Properties of $\sqrt[n]{a}$

When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

We will apply these properties in the next two examples.

EXAMPLE 8.3

Simplify: ⓐ $\sqrt[3]{64}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{32}$.

Solution

ⓐ

Since $4^3 = 64$. $\sqrt[3]{64}$
4

ⓑ

Since $(3)^4 = 81$. $\sqrt[4]{81}$
3

ⓒ

Since $(2)^5 = 32$. $\sqrt[5]{32}$
2

TRY IT : : 8.5 Simplify: ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{256}$ ⓒ $\sqrt[5]{243}$.

TRY IT : : 8.6 Simplify: ⓐ $\sqrt[3]{1000}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$.

In this example be alert for the negative signs as well as even and odd powers.

EXAMPLE 8.4

Simplify: ⓐ $\sqrt[3]{-125}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{-243}$.

Solution

ⓐ

Since $(-5)^3 = -125$. $\sqrt[3]{-125}$ -5

ⓑ

Think, $(?)^4 = -16$. No real number raised to the fourth power is negative. $\sqrt[4]{-16}$ Not a real number.

ⓒ

Since $(-3)^5 = -243$. $\sqrt[5]{-243}$ -3

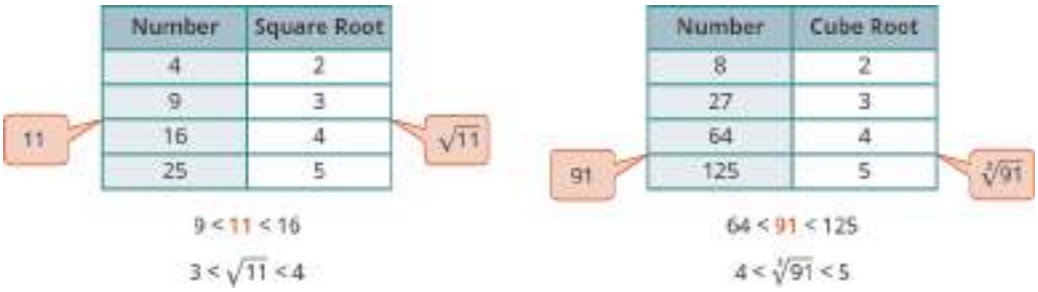
TRY IT : : 8.7 Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-256}$ ⓒ $\sqrt[5]{-32}$.

TRY IT : : 8.8 Simplify: ⓐ $\sqrt[3]{-216}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1024}$.

Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4} = 2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for perfect square numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.



Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

EXAMPLE 8.5

Estimate each root between two consecutive whole numbers: ⓐ $\sqrt{105}$ ⓑ $\sqrt[3]{43}$.

Solution

ⓐ Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

$\sqrt{105}$

Number	Square Root
81	9
100	10
121	11
144	12

105 — $\sqrt{105}$

Locate 105 between two consecutive perfect squares.	$100 < 105 < 121$
$\sqrt{105}$ is between their square roots.	$10 < \sqrt{105} < 11$

ⓑ Similarly we locate 43 between two perfect cube numbers.

$\sqrt[3]{43}$

Number	Cube Root
8	2
27	3
64	4
125	5

43 — $\sqrt[3]{43}$

Locate 43 between two consecutive perfect cubes.	$27 < 43 < 64$
$\sqrt[3]{43}$ is between their cube roots.	$3 < \sqrt[3]{43} < 4$

TRY IT : : 8.9 Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{38}$ ⓑ $\sqrt[3]{93}$

TRY IT : : 8.10 Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{84}$ ⓑ $\sqrt[3]{152}$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{ rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{ rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$(2.236067978)^2 = 5.000000002 \qquad (3.105422799)^4 = 92.999999991$$
$$(2.24)^2 = 5.0176 \qquad (3.11)^4 = 93.54951841$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

EXAMPLE 8.6

Round to two decimal places: ⓐ $\sqrt{17}$ ⓑ $\sqrt[3]{49}$ ⓒ $\sqrt[4]{51}$.

Solution

ⓐ

	$\sqrt{17}$
Use the calculator square root key.	4.123105626…
Round to two decimal places.	4.12

$\sqrt{17} \approx 4.12$

ⓑ

	$\sqrt[3]{49}$
Use the calculator $\sqrt[y]{x}$ key.	3.659305710…
Round to two decimal places.	3.66

$\sqrt[3]{49} \approx 3.66$

ⓒ

	$\sqrt[4]{51}$
Use the calculator $\sqrt[y]{x}$ key.	2.6723451177…
Round to two decimal places.	2.67

$\sqrt[4]{51} \approx 2.67$

TRY IT : : 8.11 Round to two decimal places:

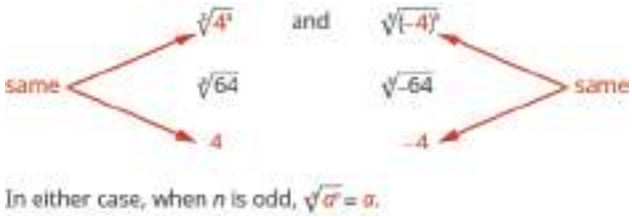
ⓐ $\sqrt{11}$ ⓑ $\sqrt[3]{71}$ ⓒ $\sqrt[4]{127}$.

TRY IT : : 8.12 Round to two decimal places:

ⓐ $\sqrt{13}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{98}$.

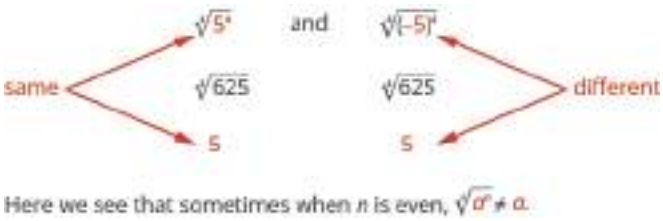
Simplify Variable Expressions with Roots

The odd root of a number can be either positive or negative. For example,



But what about an even root? We want the principal root, so $\sqrt[4]{625} = 5$.

But notice,



How can we make sure the fourth root of −5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even $\sqrt[n]{a^n} = |a|$. This guarantees the principal root is positive.

Simplifying Odd and Even Roots

For any integer $n \geq 2$,

when the index n is odd	$\sqrt[n]{a^n} = a$
when the index n is even	$\sqrt[n]{a^n} = \|a\|$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

EXAMPLE 8.7

Simplify: ⓐ $\sqrt{x^2}$ ⓑ $\sqrt[3]{n^3}$ ⓒ $\sqrt[4]{p^4}$ ⓓ $\sqrt[5]{y^5}$.

Solution

ⓐ We use the absolute value to be sure to get the positive root.

Since the index n is even, $\sqrt[n]{a^n} = |a|$. $\sqrt{x^2}$ $|x|$

ⓑ This is an odd indexed root so there is no need for an absolute value sign.

Since the index n is odd, $\sqrt[n]{a^n} = a$. $\sqrt[3]{m^3}$ m

ⓒ

Since the index n is even $\sqrt[n]{a^n} = |a|$. $\sqrt[4]{p^4}$ $|p|$

ⓓ

Since the index n is odd, $\sqrt[n]{a^n} = a$. $\sqrt[5]{y^5}$ y

TRY IT : : 8.13 Simplify: ⓐ $\sqrt{b^2}$ ⓑ $\sqrt[3]{w^3}$ ⓒ $\sqrt[4]{m^4}$ ⓓ $\sqrt[5]{q^5}$.

TRY IT : : 8.14 Simplify: ⓐ $\sqrt{y^2}$ ⓑ $\sqrt[3]{p^3}$ ⓒ $\sqrt[4]{z^4}$ ⓓ $\sqrt[5]{q^5}$.

What about square roots of higher powers of variables? The Power Property of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root,

	$\sqrt{a^{2m}}$
Since $(a^m)^2 = a^{2m}$.	$\sqrt{(a^m)^2}$
Since n is even $\sqrt[n]{a^n} = \lvert a\rvert$.	$\lvert a^m\rvert$

So $\sqrt{a^{2m}} = \lvert a^m\rvert$.

We apply this concept in the next example.

EXAMPLE 8.8

Simplify: ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$.

Solution

ⓐ

	$\sqrt{x^6}$
Since $\left(x^3\right)^2 = x^6$.	$\sqrt{\left(x^3\right)^2}$
Since the index n is even $\sqrt{a^n} = \lvert a\rvert$.	$\lvert x^3\rvert$

ⓑ

	$\sqrt{y^{16}}$
Since $\left(y^8\right)^2 = y^{16}$.	$\sqrt{\left(y^8\right)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a\rvert$.	y^8
In this case the absolute value sign is not needed as y^8 is positive.	

TRY IT : : 8.15 Simplify: ⓐ $\sqrt{y^{18}}$ ⓑ $\sqrt{z^{12}}$.

TRY IT : : 8.16 Simplify: ⓐ $\sqrt{m^4}$ ⓑ $\sqrt{b^{10}}$.

The next example uses the same idea for higher roots.

EXAMPLE 8.9

Simplify: ⓐ $\sqrt[3]{y^{18}}$ ⓑ $\sqrt[4]{z^8}$.

Solution

ⓐ

	$\sqrt[3]{y^{18}}$
Since $\left(y^6\right)^3 = y^{18}$.	$\sqrt[3]{\left(y^6\right)^3}$
Since n is odd, $\sqrt[n]{a^n} = a$.	y^6

ⓑ

	$\sqrt[4]{z^8}$
Since $\left(z^2\right)^4 = z^8$.	$\sqrt[4]{\left(z^2\right)^4}$
Since z^2 is positive, we do not need an absolute value sign.	z^2

TRY IT : : 8.17 Simplify: ⓐ $\sqrt[4]{u^{12}}$ ⓑ $\sqrt[3]{v^{15}}$.

TRY IT : : 8.18 Simplify: ⓐ $\sqrt[5]{c^{20}}$ ⓑ $\sqrt[6]{d^{24}}$

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = |a^m|$ works in much the same way.

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as u^4 is always positive.

EXAMPLE 8.10

Simplify: ⓐ $\sqrt{16n^2}$ ⓑ $-\sqrt{81c^2}$.

Solution

ⓐ

	$\sqrt{16n^2}$
Since $(4n)^2 = 16n^2$.	$\sqrt{(4n)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a\rvert$.	$4\lvert n\rvert$

ⓑ

	$-\sqrt{81c^2}$
Since $(9c)^2 = 81c^2$.	$-\sqrt{(9c)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a\rvert$.	$-9\lvert c\rvert$

TRY IT : : 8.19 Simplify: ⓐ $\sqrt{64x^2}$ ⓑ $-\sqrt{100p^2}$.

TRY IT : : 8.20 Simplify: ⓐ $\sqrt{169y^2}$ ⓑ $-\sqrt{121y^2}$.

This example just takes the idea farther as it has roots of higher index.

EXAMPLE 8.11

Simplify: ⓐ $\sqrt[3]{64p^6}$ ⓑ $\sqrt[4]{16q^{12}}$.

Solution

ⓐ

	$\sqrt[3]{64p^6}$
Rewrite $64p^6$ as $\left(4p^2\right)^3$.	$\sqrt[3]{\left(4p^2\right)^3}$
Take the cube root.	$4p^2$

ⓑ

	$\sqrt[4]{16q^{12}}$
Rewrite the radicand as a fourth power.	$\sqrt[4]{\left(2q^3\right)^4}$
Take the fourth root.	$2\left\|q^3\right\|$

TRY IT : : 8.21 Simplify: ⓐ $\sqrt[3]{27x^{27}}$ ⓑ $\sqrt[4]{81q^{28}}$.

TRY IT : : 8.22 Simplify: ⓐ $\sqrt[3]{125q^9}$ ⓑ $\sqrt[5]{243q^{25}}$.

The next examples have two variables.

EXAMPLE 8.12

Simplify: ⓐ $\sqrt{36x^2y^2}$ ⓑ $\sqrt{121a^6b^8}$ ⓒ $\sqrt[3]{64p^{63}q^9}$.

Solution

ⓐ

	$\sqrt{36x^2y^2}$
Since $(6xy)^2 = 36x^2y^2$	$\sqrt{(6xy)^2}$
Take the square root.	$6\|xy\|$

ⓑ

	$\sqrt{121a^6b^8}$
Since $\left(11a^3b^4\right)^2 = 121a^6b^8$	$\sqrt{\left(11a^3b^4\right)^2}$
Take the square root.	$11\left\|a^3\right\|b^4$

ⓒ

	$\sqrt[3]{64p^{63}q^9}$
Since $\left(4p^{21}q^3\right)^3 = 64p^{63}q^9$	$\sqrt[3]{\left(4p^{21}q^3\right)^3}$
Take the cube root.	$4p^{21}q^3$

TRY IT : : 8.23 Simplify: ⓐ $\sqrt{100a^2b^2}$ ⓑ $\sqrt{144p^{12}q^{20}}$ ⓒ $\sqrt[3]{8x^{30}y^{12}}$

TRY IT : : 8.24 Simplify: ⓐ $\sqrt{225m^2n^2}$ ⓑ $\sqrt{169x^{10}y^{14}}$ ⓒ $\sqrt[3]{27w^{36}z^{15}}$

MEDIA : :

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- **Simplifying Variables Exponents with Roots using Absolute Values (https://openstax.org/l/37SimVarAbVal)**

8.1 EXERCISES

Practice Makes Perfect

Simplify Expressions with Roots

In the following exercises, simplify.

1. ⓐ $\sqrt{64}$ ⓑ $-\sqrt{81}$

2. ⓐ $\sqrt{169}$ ⓑ $-\sqrt{100}$

3. ⓐ $\sqrt{196}$ ⓑ $-\sqrt{1}$

4. ⓐ $\sqrt{144}$ ⓑ $-\sqrt{121}$

5. ⓐ $\sqrt{\frac{4}{9}}$ ⓑ $-\sqrt{0.01}$

6. ⓐ $\sqrt{\frac{64}{121}}$ ⓑ $-\sqrt{0.16}$

7. ⓐ $\sqrt{-121}$ ⓑ $-\sqrt{289}$

8. ⓐ $-\sqrt{400}$ ⓑ $\sqrt{-36}$

9. ⓐ $-\sqrt{225}$ ⓑ $\sqrt{-9}$

10. ⓐ $\sqrt{-49}$ ⓑ $-\sqrt{256}$

11. ⓐ $\sqrt[3]{216}$ ⓑ $\sqrt[4]{256}$

12. ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$

13. ⓐ $\sqrt[3]{512}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{1}$

14. ⓐ $\sqrt[3]{125}$ ⓑ $\sqrt[4]{1296}$ ⓒ $\sqrt[5]{1024}$

15. ⓐ $\sqrt[3]{-8}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-32}$

16.
ⓐ $\sqrt[3]{-64}$
ⓑ $\sqrt[4]{-16}$
ⓒ $\sqrt[5]{-243}$

17.
ⓐ $\sqrt[3]{-125}$
ⓑ $\sqrt[4]{-1296}$
ⓒ $\sqrt[5]{-1024}$

18.
ⓐ $\sqrt[3]{-512}$
ⓑ $\sqrt[4]{-81}$
ⓒ $\sqrt[5]{-1}$

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

19. ⓐ $\sqrt{70}$ ⓑ $\sqrt[3]{71}$

20. ⓐ $\sqrt{55}$ ⓑ $\sqrt[3]{119}$

21. ⓐ $\sqrt{200}$ ⓑ $\sqrt[3]{137}$

22. ⓐ $\sqrt{172}$ ⓑ $\sqrt[3]{200}$

In the following exercises, approximate each root and round to two decimal places.

23. ⓐ $\sqrt{19}$ ⓑ $\sqrt[3]{89}$ ⓒ $\sqrt[4]{97}$

24. ⓐ $\sqrt{21}$ ⓑ $\sqrt[3]{93}$ ⓒ $\sqrt[4]{101}$

25. ⓐ $\sqrt{53}$ ⓑ $\sqrt[3]{147}$ ⓒ $\sqrt[4]{452}$

26. ⓐ $\sqrt{47}$ ⓑ $\sqrt[3]{163}$ ⓒ $\sqrt[4]{527}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

27. ⓐ $\sqrt[5]{u^5}$ ⓑ $\sqrt[8]{v^8}$

28. ⓐ $\sqrt[3]{a^3}$ ⓑ $\sqrt[9]{b^9}$

29. ⓐ $\sqrt[4]{y^4}$ ⓑ $\sqrt[7]{m^7}$

30. ⓐ $\sqrt[8]{k^8}$ ⓑ $\sqrt[6]{p^6}$

31. ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$

32. ⓐ $\sqrt{a^{14}}$ ⓑ $\sqrt{w^{24}}$

33. ⓐ $\sqrt{x^{24}}$ ⓑ $\sqrt{y^{22}}$

34. ⓐ $\sqrt{a^{12}}$ ⓑ $\sqrt{b^{26}}$

35. ⓐ $\sqrt[3]{x^9}$ ⓑ $\sqrt[4]{y^{12}}$

36. ⓐ $\sqrt[5]{a^{10}}$ ⓑ $\sqrt[3]{b^{27}}$

37. ⓐ $\sqrt[4]{m^8}$ ⓑ $\sqrt[5]{n^{20}}$

38. ⓐ $\sqrt[6]{r^{12}}$ ⓑ $\sqrt[3]{s^{30}}$

39. ⓐ $\sqrt{49x^2}$ ⓑ $-\sqrt{81x^{18}}$

40. ⓐ $\sqrt{100y^2}$ ⓑ $-\sqrt{100m^{32}}$

41. ⓐ $\sqrt{121m^{20}}$ ⓑ $-\sqrt{64a^2}$

42.
ⓐ $\sqrt{81x^{36}}$
ⓑ $-\sqrt{25x^2}$

43.
ⓐ $\sqrt[4]{16x^8}$
ⓑ $\sqrt[6]{64y^{12}}$

44.
ⓐ $\sqrt[3]{-8c^9}$
ⓑ $\sqrt[3]{125d^{15}}$

45.
ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

46.
ⓐ $\sqrt[7]{128r^{14}}$
ⓑ $\sqrt[4]{81s^{24}}$

47.
ⓐ $\sqrt{144x^2y^2}$
ⓑ $\sqrt{169w^8y^{10}}$
ⓒ $\sqrt[3]{8a^{51}b^6}$

48.
ⓐ $\sqrt{196a^2b^2}$
ⓑ $\sqrt{81p^{24}q^6}$
ⓒ $\sqrt[3]{27p^{45}q^9}$

49.
ⓐ $\sqrt{121a^2b^2}$
ⓑ $\sqrt{9c^8d^{12}}$
ⓒ $\sqrt[3]{64x^{15}y^{66}}$

50.
ⓐ $\sqrt{225x^2y^2z^2}$
ⓑ $\sqrt{36r^6s^{20}}$
ⓒ $\sqrt[3]{125y^{18}z^{27}}$

Writing Exercises

51. Why is there no real number equal to $\sqrt{-64}$?

52. What is the difference between 9^2 and $\sqrt{9}$?

53. Explain what is meant by the n^{th} root of a number.

54. Explain the difference of finding the n^{th} root of a number when the index is even compared to when the index is odd.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with roots.			
estimate and approximate roots.			
simplify variable expressions with roots.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

8.2 Simplify Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{x^9}{x^4}$.
 If you missed this problem, review **Example 5.13**.
2. Simplify: $\frac{y^3}{y^{11}}$.
 If you missed this problem, review **Example 5.13**.
3. Simplify: $\left(n^2\right)^6$.
 If you missed this problem, review **Example 5.17**.

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A radical expression, $\sqrt[n]{a}$, is considered simplified if it has no factors of m^n. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

Simplified Radical Expression

For real numbers a and m, and $n \geq 2$,

$$\sqrt[n]{a} \text{ is considered simplified i } a \text{ has no factors of } m^n$$

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

Similarly, $\sqrt[3]{4}$ is simplified because there are no perfect cube factors in 4. But $\sqrt[3]{24}$ is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that $(ab)^n = a^n b^n$. The corresponding of **Product Property of Roots** says that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Product Property of n^{th} Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $n \geq 2$ is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

We use the Product Property of Roots to remove all perfect square factors from a square root.

EXAMPLE 8.13 SIMPLIFY SQUARE ROOTS USING THE PRODUCT PROPERTY OF ROOTS

Simplify: $\sqrt{98}$.

Solution

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.	We see that 49 is the largest factor of 98 that has a power of 2. In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{98}$ $\sqrt{49 \cdot 2}$
Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Step 3. Simplify the root of the perfect power.		$7\sqrt{2}$

TRY IT : : 8.25 Simplify: $\sqrt{48}$.

TRY IT : : 8.26 Simplify: $\sqrt{45}$.

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

HOW TO : : SIMPLIFY A RADICAL EXPRESSION USING THE PRODUCT PROPERTY.

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.

Step 2. Use the product rule to rewrite the radical as the product of two radicals.

Step 3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

EXAMPLE 8.14

Simplify: ⓐ $\sqrt{500}$ ⓑ $\sqrt[3]{16}$ ⓒ $\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt{500}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{100 \cdot 5}$
Rewrite the radical as the product of two radicals	$\sqrt{100} \cdot \sqrt{5}$
Simplify.	$10\sqrt{5}$

ⓑ

	$\sqrt[3]{16}$
Rewrite the radicand as a product using the greatest perfect cube factor. $2^3 = 8$	$\sqrt[3]{8 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8} \cdot \sqrt[3]{2}$
Simplify.	$2\sqrt[3]{2}$

ⓒ

	$\sqrt[4]{243}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{81 \cdot 3}$
$3^4 = 81$	
Rewrite the radical as the product of two radicals	$\sqrt[4]{81} \cdot \sqrt[4]{3}$
Simplify.	$3\sqrt[4]{3}$

TRY IT : : 8.27 Simplify: ⓐ $\sqrt{288}$ ⓑ $\sqrt[3]{81}$ ⓒ $\sqrt[4]{64}$.

TRY IT : : 8.28 Simplify: ⓐ $\sqrt{432}$ ⓑ $\sqrt[3]{625}$ ⓒ $\sqrt[4]{729}$.

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

EXAMPLE 8.15

Simplify: ⓐ $\sqrt{x^3}$ ⓑ $\sqrt[3]{x^4}$ ⓒ $\sqrt[4]{x^7}$.

Solution

ⓐ

	$\sqrt{x^3}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{x^2 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt{x^2} \cdot \sqrt{x}$
Simplify.	$\lvert x \rvert \sqrt{x}$

ⓑ

	$\sqrt[3]{x^4}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{x^3 \cdot x}.$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$
Simplify.	$x\sqrt[3]{x}$

ⓒ

	$\sqrt[4]{x^7}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{x^4 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$
Simplify.	$\lvert x \rvert \sqrt[4]{x^3}$

TRY IT : : 8.29 Simplify: ⓐ $\sqrt{b^5}$ ⓑ $\sqrt[4]{y^6}$ ⓒ $\sqrt[3]{z^5}$

TRY IT : : 8.30 Simplify: ⓐ $\sqrt{p^9}$ ⓑ $\sqrt[5]{y^8}$ ⓒ $\sqrt[6]{q^{13}}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

EXAMPLE 8.16

Simplify: ⓐ $\sqrt{72n^7}$ ⓑ $\sqrt[3]{24x^7}$ ⓒ $\sqrt[4]{80y^{14}}$.

Solution

ⓐ

	$\sqrt{72n^7}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{36n^6 \cdot 2n}$
Rewrite the radical as the product of two radicals.	$\sqrt{36n^6} \cdot \sqrt{2n}$
Simplify.	$6\lvert n^3 \rvert \sqrt{2n}$

ⓑ

	$\sqrt[3]{24x^7}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{8x^6 \cdot 3x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$
Rewrite the fir t radicand as $(2x^2)^3$.	$\sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}$
Simplify.	$2x^2 \sqrt[3]{3x}$

ⓒ

	$\sqrt[4]{80y^{14}}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{16y^{12}\cdot 5\,y^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{16y^{12}}\cdot\sqrt[4]{5y^2}$
Rewrite the fir t radicand as $\left(2y^3\right)^4$.	$\sqrt[4]{\left(2\,y^3\right)^4}\cdot\sqrt[4]{5y^2}$
Simplify.	$2\left\|y^3\right\|\sqrt[4]{5\,y^2}$

TRY IT : : 8.31 Simplify: ⓐ $\sqrt{32y^5}$ ⓑ $\sqrt[3]{54p^{10}}$ ⓒ $\sqrt[4]{64q^{10}}$.

TRY IT : : 8.32 Simplify: ⓐ $\sqrt{75a^9}$ ⓑ $\sqrt[3]{128m^{11}}$ ⓒ $\sqrt[4]{162n^7}$.

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

EXAMPLE 8.17

Simplify: ⓐ $\sqrt{63u^3v^5}$ ⓑ $\sqrt[3]{40x^4y^5}$ ⓒ $\sqrt[4]{48x^4y^7}$.

Solution

ⓐ

	$\sqrt{63u^3\,v^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9u^2v^4\cdot 7uv}$
Rewrite the radical as the product of two radicals.	$\sqrt{9u^2v^4}\cdot\sqrt{7uv}$
Rewrite the fir t radicand as $\left(3uv^2\right)^2$.	$\sqrt{\left(3uv^2\right)^2}\cdot\sqrt{7uv}$
Simplify.	$3\|u\|v^2\ \sqrt{7uv}$

ⓑ

	$\sqrt[3]{40x^4\,y^5}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{8x^3\,y^3\cdot 5xy^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8x^3\,y^3}\cdot\sqrt[3]{5xy^2}$
Rewrite the fir t radicand as $(2xy)^3$.	$\sqrt[3]{(2xy)^3}\cdot\sqrt[3]{5xy^2}$
Simplify.	$2xy\sqrt[3]{5xy^2}$

ⓒ

	$\sqrt[4]{48x^4y^7}$
Rewrite the radicand as a product using the largest perfect fourth power factor.	$\sqrt[4]{16x^4y^4 \cdot 3y^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{16x^4y^4} \cdot \sqrt[4]{3y^3}$
Rewrite the fir t radicand as $(2xy)^4$.	$\sqrt[4]{(2xy)^4} \cdot \sqrt[4]{3y^3}$
Simplify.	$2\lvert xy\rvert \sqrt[4]{3y^3}$

TRY IT : : 8.33 Simplify: ⓐ $\sqrt{98a^7b^5}$ ⓑ $\sqrt[3]{56x^5y^4}$ ⓒ $\sqrt[4]{32x^5y^8}$.

TRY IT : : 8.34 Simplify: ⓐ $\sqrt{180m^9n^{11}}$ ⓑ $\sqrt[3]{72x^6y^5}$ ⓒ $\sqrt[4]{80x^7y^4}$.

EXAMPLE 8.18

Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-16}$.

Solution

ⓐ

	$\sqrt[3]{-27}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3}$
Take the cube root.	-3

ⓑ

	$\sqrt[4]{-16}$
There is no real number n where $n^4 = -16$.	Not a real number.

TRY IT : : 8.35 Simplify: ⓐ $\sqrt[3]{-64}$ ⓑ $\sqrt[4]{-81}$.

TRY IT : : 8.36 Simplify: ⓐ $\sqrt[3]{-625}$ ⓑ $\sqrt[4]{-324}$.

We have seen how to use the order of operations to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

EXAMPLE 8.19

Simplify: ⓐ $3 + \sqrt{32}$ ⓑ $\dfrac{4 - \sqrt{48}}{2}$.

Solution

ⓐ

	$3+\sqrt{32}$
Rewrite the radicand as a product using the largest perfect square factor.	$3+\sqrt{16\cdot 2}$
Rewrite the radical as the product of two radicals.	$3+\sqrt{16}\cdot\sqrt{2}$
Simplify.	$3+4\sqrt{2}$

The terms cannot be added as one has a radical and the other does not. Trying to add an integer and a radical is like trying to add an integer and a variable. They are not like terms!

ⓑ

	$\frac{4-\sqrt{48}}{2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\frac{4-\sqrt{16\cdot 3}}{2}$
Rewrite the radical as the product of two radicals.	$\frac{4-\sqrt{16}\cdot\sqrt{3}}{2}$
Simplify.	$\frac{4-4\sqrt{3}}{2}$
Factor the common factor from the numerator.	$\frac{4(1-\sqrt{3})}{2}$
Remove the common factor, 2, from the numerator and denominator.	$\frac{\cancel{2}\cdot 2(1-\sqrt{3})}{\cancel{2}}$
Simplify.	$2(1-\sqrt{3})$

TRY IT : : 8.37 Simplify: ⓐ $5+\sqrt{75}$ ⓑ $\frac{10-\sqrt{75}}{5}$

TRY IT : : 8.38 Simplify: ⓐ $2+\sqrt{98}$ ⓑ $\frac{6-\sqrt{45}}{3}$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

EXAMPLE 8.20

Simplify: ⓐ $\sqrt{\frac{45}{80}}$ ⓑ $\sqrt[3]{\frac{16}{54}}$ ⓒ $\sqrt[4]{\frac{5}{80}}$.

Solution

ⓐ

	$\sqrt{\frac{45}{80}}$
Simplify inside the radical fir t.	
Rewrite showing the common factors of the numerator and denominator.	$\sqrt{\frac{5\cdot 9}{5\cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt{\frac{9}{16}}$
Simplify. Note $\left(\frac{3}{4}\right)^2=\frac{9}{16}$.	$\frac{3}{4}$

ⓑ

	$\sqrt[3]{\frac{16}{54}}$
Simplify inside the radical fir t.	
Rewrite showing the common factors of the numerator and denominator.	$\sqrt[3]{\frac{2\cdot 8}{2\cdot 27}}$
Simplify the fraction by removing common factors.	$\sqrt[3]{\frac{8}{27}}$
Simplify. Note $\left(\frac{2}{3}\right)^3=\frac{8}{27}$.	$\frac{2}{3}$

ⓒ

	$\sqrt[4]{\frac{5}{80}}$
Simplify inside the radical fir t.	
Rewrite showing the common factors of the numerator and denominator.	$\sqrt[4]{\frac{5\cdot 1}{5\cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt[4]{\frac{1}{16}}$
Simplify. Note $\left(\frac{1}{2}\right)^4=\frac{1}{16}$.	$\frac{1}{2}$

TRY IT : : 8.39 Simplify: ⓐ $\sqrt{\frac{75}{48}}$ ⓑ $\sqrt[3]{\frac{54}{250}}$ ⓒ $\sqrt[4]{\frac{32}{162}}$.

TRY IT : : 8.40 Simplify: ⓐ $\sqrt{\frac{98}{162}}$ ⓑ $\sqrt[3]{\frac{24}{375}}$ ⓒ $\sqrt[4]{\frac{4}{324}}$.

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n}=a^{m-n},\quad a\neq 0$$

EXAMPLE 8.21

Simplify: ⓐ $\sqrt{\frac{m^6}{m^4}}$ ⓑ $\sqrt[3]{\frac{a^8}{a^5}}$ ⓒ $\sqrt[4]{\frac{a^{10}}{a^2}}$.

Solution

ⓐ

	$\sqrt{\frac{m^6}{m^4}}$
Simplify the fraction inside the radical fir t.	
Divide the like bases by subtracting the exponents.	$\sqrt{m^2}$
Simplify.	$\lvert m\rvert$

ⓑ

$$\sqrt[3]{\frac{a^8}{a^5}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical fir t. $\sqrt[3]{a^3}$

Simplify. a

ⓒ

$$\sqrt[4]{\frac{a^{10}}{a^2}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical fir t. $\sqrt[4]{a^8}$

Rewrite the radicand using perfect fourth power factors. $\sqrt[4]{(a^2)^4}$

Simplify. a^2

TRY IT : : 8.41 Simplify: ⓐ $\sqrt{\frac{a^8}{a^6}}$ ⓑ $\sqrt[4]{\frac{x^7}{x^3}}$ ⓒ $\sqrt[4]{\frac{y^{17}}{y^5}}$.

TRY IT : : 8.42 Simplify: ⓐ $\sqrt{\frac{x^{14}}{x^{10}}}$ ⓑ $\sqrt[3]{\frac{m^{13}}{m^7}}$ ⓒ $\sqrt[5]{\frac{n^{12}}{n^2}}$.

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$$

We can use a similar property to simplify a root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect power of the index, we simplify the numerator and denominator separately.

Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

EXAMPLE 8.22 HOW TO SIMPLIFY THE QUOTIENT OF RADICAL EXPRESSIONS

Simplify: $\sqrt{\frac{27m^3}{196}}$.

Solution

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$

Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

TRY IT : : 8.43 Simplify: $\sqrt[4]{\frac{24p^3}{49}}$.

TRY IT : : 8.44 Simplify: $\sqrt[4]{\frac{48x^5}{100}}$.

HOW TO : : SIMPLIFY A SQUARE ROOT USING THE QUOTIENT PROPERTY.

Step 1. Simplify the fraction in the radicand, if possible.
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
Step 3. Simplify the radicals in the numerator and the denominator.

EXAMPLE 8.23

Simplify: ⓐ $\sqrt[4]{\frac{45x^5}{y^4}}$ ⓑ $\sqrt[3]{\frac{24x^7}{y^3}}$ ⓒ $\sqrt[4]{\frac{48x^{10}}{y^8}}$.

Solution

ⓐ

$$\sqrt[4]{\frac{45x^5}{y^4}}$$

We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.

$$\frac{\sqrt[4]{45x^5}}{\sqrt[4]{y^4}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt[4]{9x^4}\cdot\sqrt[4]{5x}}{y^2}$$

Simplify.

$$\frac{3x^2\sqrt[4]{5x}}{y^2}$$

ⓑ

$$\sqrt[3]{\frac{24x^7}{y^3}}$$

The fraction in the radicand cannot be simplified. Use he Quotient Property to write as two radicals.

$$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$$

Rewrite each radicand as a product using perfect cube factors.

$$\frac{\sqrt[3]{8x^6 \cdot 3x}}{\sqrt[3]{y^3}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}}{\sqrt[3]{y^3}}$$

Simplify.

$$\frac{2x^2\sqrt[3]{3x}}{y}$$

ⓒ

$$\sqrt[4]{\frac{48x^{10}}{y^8}}$$

The fraction in the radicand cannot be simplified

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{(2x^2)^4} \cdot \sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$$

Simplify.

$$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$$

TRY IT : : 8.45 Simplify: ⓐ $\sqrt{\frac{80m^3}{n^6}}$ ⓑ $\sqrt[3]{\frac{108c^{10}}{d^6}}$ ⓒ $\sqrt[4]{\frac{80x^{10}}{y^4}}$.

TRY IT : : 8.46 Simplify: ⓐ $\sqrt{\frac{54u^7}{v^8}}$ ⓑ $\sqrt[3]{\frac{40r^3}{s^6}}$ ⓒ $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$.

Be sure to simplify the fraction in the radicand first, if possible.

EXAMPLE 8.24

Simplify: ⓐ $\sqrt{\frac{18p^5q^7}{32pq^2}}$ ⓑ $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$.

Solution

ⓐ

$$\sqrt{\frac{18p^5q^7}{32pq^2}}$$

Simplify the fraction in the radicand, if possible. $\sqrt{\frac{9p^4q^5}{16}}$

Rewrite using the Quotient Property. $\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$

Simplify the radicals in the numerator and the denominator. $\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$

Simplify. $\frac{3p^2q^2\sqrt{q}}{4}$

ⓑ

$$\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$$

Simplify the fraction in the radicand, if possible. $\sqrt[3]{\frac{8x^3y^5}{27}}$

Rewrite using the Quotient Property. $\frac{\sqrt[3]{8x^3y^5}}{\sqrt[3]{27}}$

Simplify the radicals in the numerator and the denominator. $\frac{\sqrt[3]{8x^3y^3}\cdot\sqrt[3]{y^2}}{\sqrt[3]{27}}$

Simplify. $\frac{2xy\sqrt[3]{y^2}}{3}$

ⓒ

$$\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt[4]{\frac{a^5b^4}{16}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt[4]{a^4b^4}\cdot\sqrt[4]{a}}{\sqrt[4]{16}}$$

Simplify.

$$\frac{|ab|\sqrt[4]{a}}{2}$$

TRY IT : : 8.47

Simplify: ⓐ $\sqrt{\frac{50x^5y^3}{72x^4y}}$ ⓑ $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$.

TRY IT : : 8.48

Simplify: ⓐ $\sqrt{\frac{48m^7n^2}{100m^5n^8}}$ ⓑ $\sqrt[3]{\frac{54x^7y^5}{250x^2y^2}}$ ⓒ $\sqrt[4]{\frac{32a^9b^7}{162a^3b^3}}$.

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the Quotient Property again, to combine them into one radical. We will then look to see if we can simplify the expression.

EXAMPLE 8.25

Simplify: ⓐ $\frac{\sqrt{48a^7}}{\sqrt{3a}}$ ⓑ $\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$ ⓒ $\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$.

Solution

ⓐ

$$\frac{\sqrt{48a^7}}{\sqrt{3a}}$$

The denominator cannot be simplified, s use the Quotient Property to write as one radical.

$$\sqrt{\frac{48a^7}{3a}}$$

Simplify the fraction under the radical.

$$\sqrt{16a^6}$$

Simplify.

$$4|a^3|$$

ⓑ

$$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$$

The denominator cannot be simplified, s use the Quotient Property to write as one radical.

$$\sqrt[3]{\frac{-108}{2}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{-54}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{(-3)^3 \cdot 2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{(-3)^3} \cdot \sqrt[3]{2}$$

Simplify.

$$-3\sqrt[3]{2}$$

ⓒ

$$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$$

The denominator cannot be simplified, s use the Quotient Property to write as one radical.

$$\sqrt[4]{\frac{96x^7}{3x^2}}$$

Simplify the fraction under the radical.

$$\sqrt[4]{32x^5}$$

Rewrite the radicand as a product using perfect fourth power factors.

$$\sqrt[4]{16x^4} \cdot \sqrt[4]{2x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$$

Simplify.

$$2|x|\sqrt[4]{2x}$$

TRY IT : : 8.49

Simplify: ⓐ $\frac{\sqrt{98z^5}}{\sqrt{2z}}$ ⓑ $\frac{\sqrt[3]{-500}}{\sqrt[3]{2}}$ ⓒ $\frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$.

TRY IT : : 8.50

Simplify: ⓐ $\frac{\sqrt{128m^9}}{\sqrt{2m}}$ ⓑ $\frac{\sqrt[3]{-192}}{\sqrt[3]{3}}$ ⓒ $\frac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$.

MEDIA : :

Access these online resources for additional instruction and practice with simplifying radical expressions.

- **Simplifying Square Root and Cube Root with Variables (https://openstax.org/l/37SimRtwithVar1)**
- **Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents (https://openstax.org/l/37SimRtwithVar2)**
- **Simplifying Cube Roots (https://openstax.org/l/37SimRtwithVar3)**

8.2 EXERCISES

Practice Makes Perfect

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

55. $\sqrt{27}$

56. $\sqrt{80}$

57. $\sqrt{125}$

58. $\sqrt{96}$

59. $\sqrt{147}$

60. $\sqrt{450}$

61. $\sqrt{800}$

62. $\sqrt{675}$

63. ⓐ $\sqrt[4]{32}$ ⓑ $\sqrt[5]{64}$

64. ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

65. ⓐ $\sqrt[5]{64}$ ⓑ $\sqrt[3]{256}$

66. ⓐ $\sqrt[4]{3125}$ ⓑ $\sqrt[3]{81}$

In the following exercises, simplify using absolute value signs as needed.

67.
ⓐ $\sqrt{y^{11}}$
ⓑ $\sqrt[3]{r^5}$
ⓒ $\sqrt[4]{s^{10}}$

68.
ⓐ $\sqrt{m^{13}}$
ⓑ $\sqrt[5]{u^7}$
ⓒ $\sqrt[6]{v^{11}}$

69.
ⓐ $\sqrt{n^{21}}$
ⓑ $\sqrt[3]{q^8}$
ⓒ $\sqrt[8]{n^{10}}$

70.
ⓐ $\sqrt{r^{25}}$
ⓑ $\sqrt[5]{p^8}$
ⓒ $\sqrt[4]{m^5}$

71.
ⓐ $\sqrt{125r^{13}}$
ⓑ $\sqrt[3]{108x^5}$
ⓒ $\sqrt[4]{48y^6}$

72.
ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
ⓒ $\sqrt[6]{128b^7}$

73.
ⓐ $\sqrt{242m^{23}}$
ⓑ $\sqrt[4]{405m10}$
ⓒ $\sqrt[5]{160n^8}$

74.
ⓐ $\sqrt{175n^{13}}$
ⓑ $\sqrt[5]{512p^5}$
ⓒ $\sqrt[4]{324q^7}$

75.
ⓐ $\sqrt{147m^7n^{11}}$
ⓑ $\sqrt[3]{48x^6y^7}$
ⓒ $\sqrt[4]{32x^5y^4}$

76.
ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
ⓒ $\sqrt[4]{80x^8y^9}$

77.
ⓐ $\sqrt{192q^3r^7}$
ⓑ $\sqrt[3]{54m^9n^{10}}$
ⓒ $\sqrt[4]{81a^9b^8}$

78.
ⓐ $\sqrt{150m^9n^3}$
ⓑ $\sqrt[3]{81p^7q^8}$
ⓒ $\sqrt[4]{162c^{11}d^{12}}$

79.
ⓐ $\sqrt[3]{-864}$
ⓑ $\sqrt[4]{-256}$

80.
ⓐ $\sqrt[5]{-486}$
ⓑ $\sqrt[6]{-64}$

81.
ⓐ $\sqrt[5]{-32}$
ⓑ $\sqrt[8]{-1}$

82.
ⓐ $\sqrt[3]{-8}$
ⓑ $\sqrt[4]{-16}$

83.
ⓐ $5+\sqrt{12}$
ⓑ $\frac{10-\sqrt{24}}{2}$

84.
ⓐ $8+\sqrt{96}$
ⓑ $\frac{8-\sqrt{80}}{4}$

85.
ⓐ $1+\sqrt{45}$
ⓑ $\frac{3+\sqrt{90}}{3}$

86.
ⓐ $3+\sqrt{125}$
ⓑ $\frac{15+\sqrt{75}}{5}$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

87. ⓐ $\sqrt{\frac{45}{80}}$ ⓑ $\sqrt[3]{\frac{8}{27}}$ ⓒ $\sqrt[4]{\frac{1}{81}}$

88. ⓐ $\sqrt{\frac{72}{98}}$ ⓑ $\sqrt[3]{\frac{24}{81}}$ ⓒ $\sqrt[4]{\frac{6}{96}}$

89. ⓐ $\sqrt{\frac{100}{36}}$ ⓑ $\sqrt[3]{\frac{81}{375}}$ ⓒ $\sqrt[4]{\frac{1}{256}}$

90. ⓐ $\sqrt{\frac{121}{16}}$ ⓑ $\sqrt[3]{\frac{16}{250}}$ ⓒ $\sqrt[4]{\frac{32}{162}}$

91. ⓐ $\sqrt{\frac{x^{10}}{x^6}}$ ⓑ $\sqrt[3]{\frac{p^{11}}{p^2}}$ ⓒ $\sqrt[4]{\frac{q^{17}}{q^{13}}}$

92. ⓐ $\sqrt{\frac{p^{20}}{p^{10}}}$ ⓑ $\sqrt[5]{\frac{d^{12}}{d^7}}$ ⓒ $\sqrt[8]{\frac{m^{12}}{m^4}}$

93. ⓐ $\sqrt{\frac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

94. ⓐ $\sqrt{\frac{q^8}{q^{14}}}$ ⓑ $\sqrt[3]{\frac{r^{14}}{r^5}}$ ⓒ $\sqrt[4]{\frac{c^{21}}{c^9}}$

95. $\sqrt{\frac{96x^7}{121}}$

96. $\sqrt{\frac{108y^4}{49}}$

97. $\sqrt{\frac{300m^5}{64}}$

98. $\sqrt{\frac{125n^7}{169}}$

99. $\sqrt{\frac{98r^5}{100}}$

100. $\sqrt{\frac{180s^{10}}{144}}$

101. $\sqrt{\frac{28q^6}{225}}$

102. $\sqrt{\frac{150r^3}{256}}$

103.
ⓐ $\sqrt{\frac{75r^9}{s^8}}$
ⓑ $\sqrt[3]{\frac{54a^8}{b^3}}$
ⓒ $\sqrt[4]{\frac{64c^5}{d^4}}$

104.
ⓐ $\sqrt{\frac{72x^5}{y^6}}$
ⓑ $\sqrt[5]{\frac{96r^{11}}{s^5}}$
ⓒ $\sqrt[6]{\frac{128u^7}{v^{12}}}$

105.
ⓐ $\sqrt{\frac{28p^7}{q^2}}$
ⓑ $\sqrt[3]{\frac{81s^8}{t^3}}$
ⓒ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

106.
ⓐ $\sqrt{\frac{45r^3}{s^{10}}}$
ⓑ $\sqrt[3]{\frac{625u^{10}}{v^3}}$
ⓒ $\sqrt[4]{\frac{729c^{21}}{d^8}}$

107.
ⓐ $\sqrt{\frac{32x^5y^3}{18x^3y}}$
ⓑ $\sqrt[3]{\frac{5x^6y^9}{40x^5y^3}}$
ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

108.
ⓐ $\sqrt{\frac{75r^6s^8}{48rs^4}}$
ⓑ $\sqrt[3]{\frac{24x^8y^4}{81x^2y}}$
ⓒ $\sqrt[4]{\frac{32m^9n^2}{162mn^2}}$

109.
ⓐ $\sqrt{\frac{27p^2q}{108p^4q^3}}$
ⓑ $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$
ⓒ $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

110.
ⓐ $\sqrt{\frac{50r^5s^2}{128r^2s^6}}$
ⓑ $\sqrt[3]{\frac{24m^9n^7}{375m^4n}}$
ⓒ $\sqrt[4]{\frac{81m^2n^8}{256m^1n^2}}$

111.

ⓐ $\frac{\sqrt{45p^9}}{\sqrt{5q^2}}$

ⓑ $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$

ⓒ $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

112.

ⓐ $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

ⓑ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

ⓒ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

113.

ⓐ $\frac{\sqrt{50m^7}}{\sqrt{2m}}$

ⓑ $\sqrt[3]{\frac{1250}{2}}$

ⓒ $\sqrt[4]{\frac{486y^9}{2y^3}}$

114.

ⓐ $\frac{\sqrt{72n^{11}}}{\sqrt{2n}}$

ⓑ $\sqrt[3]{\frac{162}{6}}$

ⓒ $\sqrt[4]{\frac{160r^{10}}{5r^3}}$

Writing Exercises

115. Explain why $\sqrt{x^4} = x^2$. Then explain why $\sqrt{x^{16}} = x^8$.

116. Explain why $7 + \sqrt{9}$ is not equal to $\sqrt{7+9}$.

117. Explain how you know that $\sqrt[5]{x^{10}} = x^2$.

118. Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify radical expressions.			
use the Quotient Property to simplify radical expressions.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

8.3 Simplify Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15}+\frac{5}{12}$.
 If you missed this problem, review **Example 1.28**.
2. Simplify: $\left(4x^2y^5\right)^3$.
 If you missed this problem, review **Example 5.18**.
3. Simplify: 5^{-3}.
 If you missed this problem, review **Example 5.14**.

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n=a^{m\cdot n}$ when *m* and *n* are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $(8^p)^3=8$. We will use the Power Property of Exponents to find the value of *p*.

	$(8^p)^3 = 8$
Multiply the exponents on the left.	$8^{3p} = 8$
Write the exponent 1 on the right.	$8^{3p} = 8^1$
Since the bases are the same, the exponents must be equal.	$3p = 1$
Solve for p.	$p = \frac{1}{3}$

So $\left(8^{\frac{1}{3}}\right)^3=8$. But we know also $\left(\sqrt[3]{8}\right)^3=8$. Then it must be that $8^{\frac{1}{3}}=\sqrt[3]{8}$.

This same logic can be used for any positive integer exponent *n* to show that $a^{\frac{1}{n}}=\sqrt[n]{a}$.

Rational Exponent $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n\geq 2$, then

$$a^{\frac{1}{n}}=\sqrt[n]{a}$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

EXAMPLE 8.26

Write as a radical expression: ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$.

Solution

We want to write each expression in the form $\sqrt[n]{a}$.

ⓐ

	$x^{\frac{1}{2}}$
The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.	$\sqrt{x}$

ⓑ

	$y^{\frac{1}{3}}$
The denominator of the exponent is 3, so the index is 3.	$\sqrt[3]{y}$

ⓒ

	$z^{\frac{1}{4}}$
The denominator of the exponent is 4, so the index is 4.	$\sqrt[4]{z}$

TRY IT : : 8.51

Write as a radical expression: ⓐ $t^{\frac{1}{2}}$ ⓑ $m^{\frac{1}{3}}$ ⓒ $r^{\frac{1}{4}}$.

TRY IT : : 8.52

Write as a radial expression: ⓐ $b^{\frac{1}{6}}$ ⓑ $z^{\frac{1}{5}}$ ⓒ $p^{\frac{1}{4}}$.

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

EXAMPLE 8.27

Write with a rational exponent: ⓐ $\sqrt{5y}$ ⓑ $\sqrt[3]{4x}$ ⓒ $3\sqrt[4]{5z}$.

Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

	$\sqrt{5y}$
No index is shown, so it is 2. The denominator of the exponent will be 2. Put parentheses around the entire expression $5y$.	$(5y)^{\frac{1}{2}}$

ⓑ

	$\sqrt[3]{4x}$
The index is 3, so the denominator of the exponent is 3. Include parentheses $(4x)$.	$(4x)^{\frac{1}{3}}$

ⓒ

	$3\sqrt[4]{5z}$
The index is 4, so the denominator of the exponent is 4. Put parentheses only around the $5z$ since 3 is not under the radical sign.	$3(5z)^{\frac{1}{4}}$

TRY IT : : 8.53 Write with a rational exponent: ⓐ $\sqrt{10m}$ ⓑ $\sqrt[5]{3n}$ ⓒ $3\sqrt[4]{6y}$.

TRY IT : : 8.54 Write with a rational exponent: ⓐ $\sqrt[7]{3k}$ ⓑ $\sqrt[4]{5j}$ ⓒ $8\sqrt[3]{2a}$.

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

EXAMPLE 8.28

Simplify: ⓐ $25^{\frac{1}{2}}$ ⓑ $64^{\frac{1}{3}}$ ⓒ $256^{\frac{1}{4}}$.

Solution

ⓐ

	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5

ⓑ

	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

ⓒ

	$256^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{256}$
Recognize 256 is a perfect fourth power.	$\sqrt[4]{4^4}$
Simplify.	4

TRY IT : : 8.55 Simplify: ⓐ $36^{\frac{1}{2}}$ ⓑ $8^{\frac{1}{3}}$ ⓒ $16^{\frac{1}{4}}$.

TRY IT : : 8.56 Simplify: ⓐ $100^{\frac{1}{2}}$ ⓑ $27^{\frac{1}{3}}$ ⓒ $81^{\frac{1}{4}}$.

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n}=\frac{1}{a^n}$ in one case.

EXAMPLE 8.29

Simplify: ⓐ $(-16)^{\frac{1}{4}}$ ⓑ $-16^{\frac{1}{4}}$ ⓒ $(16)^{-\frac{1}{4}}$.

Solution

ⓐ

	$(-16)^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{-16}$
	$\sqrt[4]{(-2)^4}$
Simplify.	No real solution.

ⓑ

	$-16^{\frac{1}{4}}$
The exponent only applies to the 16. Rewrite as a fouth root.	$-\sqrt[4]{16}$
Rewrite 16 as 2^4.	$-\sqrt[4]{2^4}$
Simplify.	-2

ⓒ

	$(16)^{-\frac{1}{4}}$
Rewrite using the property $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{(16)^{\frac{1}{4}}}$
Rewrite as a fourth root.	$\frac{1}{\sqrt[4]{16}}$
Rewrite 16 as 2^4.	$\frac{1}{\sqrt[4]{2^4}}$
Simplify.	$\frac{1}{2}$

TRY IT : : 8.57 Simplify: ⓐ $(-64)^{-\frac{1}{2}}$ ⓑ $-64^{\frac{1}{2}}$ ⓒ $(64)^{-\frac{1}{2}}$.

TRY IT : : 8.58 Simplify: ⓐ $(-256)^{\frac{1}{4}}$ ⓑ $-256^{\frac{1}{4}}$ ⓒ $(256)^{-\frac{1}{4}}$.

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and

$(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the following definition.

Rational Exponent $a^{\frac{m}{n}}$

For any positive integers *m* and *n*,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

EXAMPLE 8.30

Write with a rational exponent: ⓐ $\sqrt{y^3}$ ⓑ $\left(\sqrt[3]{2x}\right)^4$ ⓒ $\sqrt{\left(\frac{3a}{4b}\right)^3}$.

Solution

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$.

ⓐ

$\sqrt{y^3}$

The numerator of the exponent is the exponent, 3.

The denominator of the exponent is the index of the radical, 2. $y^{\frac{3}{2}}$

ⓑ

$\left(\sqrt[3]{2x}\right)^4$

The numerator of the exponent is the exponent, 4.

The denominator of the exponent is the index of the radical, 3. $(2x)^{\frac{4}{3}}$

ⓒ

$\sqrt{\left(\frac{3a}{4b}\right)^3}$

The numerator of the exponent is the exponent, 3.

The denominator of the exponent is the index of the radical, 2. $\left(\frac{3a}{4b}\right)^{\frac{3}{2}}$

TRY IT : : 8.59 Write with a rational exponent: ⓐ $\sqrt{x^5}$ ⓑ $\left(\sqrt[4]{3y}\right)^3$ ⓒ $\sqrt{\left(\frac{2m}{3n}\right)^5}$.

TRY IT : : 8.60 Write with a rational exponent: ⓐ $\sqrt[5]{a^2}$ ⓑ $\left(\sqrt[3]{5ab}\right)^5$ ⓒ $\sqrt{\left(\frac{7xy}{z}\right)^3}$.

Remember that $a^{-n}=\frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

EXAMPLE 8.31

Simplify: ⓐ $125^{\frac{2}{3}}$ ⓑ $16^{-\frac{3}{2}}$ ⓒ $32^{-\frac{2}{5}}$.

Solution

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}}=(\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

	$125^{\frac{2}{3}}$
The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.	$\left(\sqrt[3]{125}\right)^2$
Simplify.	$(5)^2$
	25

ⓑ We will rewrite each expression first using $a^{-n}=\frac{1}{a^n}$ and then change to radical form.

	$16^{-\frac{3}{2}}$
Rewrite using $a^{-n}=\frac{1}{a^n}$	$\frac{1}{16^{\frac{3}{2}}}$
Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.	$\frac{1}{\left(\sqrt{16}\right)^3}$
Simplify.	$\frac{1}{4^3}$
	$\frac{1}{64}$

ⓒ

$$32^{-\frac{2}{5}}$$

Rewrite using $a^{-n}=\frac{1}{a^n}$. $$\frac{1}{32^{\frac{2}{5}}}$$

Change to radical form. $$\frac{1}{\left(\sqrt[5]{32}\right)^2}$$

Rewrite the radicand as a power. $$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$$

Simplify. $$\frac{1}{2^2}$$

$$\frac{1}{4}$$

TRY IT : : 8.61 Simplify: ⓐ $27^{\frac{2}{3}}$ ⓑ $81^{-\frac{3}{2}}$ ⓒ $16^{-\frac{3}{4}}$.

TRY IT : : 8.62 Simplify: ⓐ $4^{\frac{3}{2}}$ ⓑ $27^{-\frac{2}{3}}$ ⓒ $625^{-\frac{3}{4}}$.

EXAMPLE 8.32

Simplify: ⓐ $-25^{\frac{3}{2}}$ ⓑ $-25^{-\frac{3}{2}}$ ⓒ $(-25)^{\frac{3}{2}}$.

Solution

ⓐ

$$-25^{\frac{3}{2}}$$

Rewrite in radical form. $$-\left(\sqrt{25}\right)^3$$

Simplify the radical. $$-(5)^3$$

Simplify. $$-125$$

ⓑ

	$-25^{-\frac{3}{2}}$
Rewrite using $a^{-n}=\frac{1}{a^n}$.	$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$
Rewrite in radical form.	$-\left(\frac{1}{\left(\sqrt{25}\right)^3}\right)$
Simplify the radical.	$-\left(\frac{1}{(5)^3}\right)$
Simplify.	$-\frac{1}{125}$

ⓒ

	$(-25)^{\frac{3}{2}}$
Rewrite in radical form.	$\left(\sqrt{-25}\right)^3$
There is no real number whose square root is -25.	Not a real number.

TRY IT : : 8.63 Simplify: ⓐ $-16^{\frac{3}{2}}$ ⓑ $-16^{-\frac{3}{2}}$ ⓒ $(-16)^{-\frac{3}{2}}$.

TRY IT : : 8.64 Simplify: ⓐ $-81^{\frac{3}{2}}$ ⓑ $-81^{-\frac{3}{2}}$ ⓒ $(-81)^{-\frac{3}{2}}$.

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponenets here to have them for reference as we simplify expressions.

Properties of Exponents

If a and b are real numbers and m and n are rational numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n},\ a \neq 0$
Zero Exponent Definitio	$a^0 = 1,\ a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$
Negative Exponent Property	$a^{-n} = \frac{1}{a^n},\ a \neq 0$

We will apply these properties in the next example.

EXAMPLE 8.33

Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$ ⓑ $\left(z^9\right)^{\frac{2}{3}}$ ⓒ $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$.

Solution

ⓐ The Product Property tells us that when we multiply the same base, we add the exponents.

$$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$$

The bases are the same, so we add the exponents.

$$x^{\frac{1}{2}+\frac{5}{6}}$$

Add the fractions.

$$x^{\frac{8}{6}}$$

Simplify the exponent.

$$x^{\frac{4}{3}}$$

ⓑ The Power Property tells us that when we raise a power to a power, we multiply the exponents.

$$\left(z^{9}\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiply the exponents.

$$z^{9 \cdot \frac{2}{3}}$$

Simplify.

$$z^{6}$$

ⓒ The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

$$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$$

$$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$$

To divide with the same base, we subtract the exponents.

$$\frac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$$

Simplify.

$$\frac{1}{x^{\frac{4}{3}}}$$

TRY IT : : 8.65

Simplify: ⓐ $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$ ⓑ $\left(x^{6}\right)^{\frac{4}{3}}$ ⓒ $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$.

TRY IT : : 8.66

Simplify: ⓐ $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$ ⓑ $\left(m^{9}\right)^{\frac{2}{9}}$ ⓒ $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$.

Sometimes we need to use more than one property. In the next example, we will use both the Product to a Power Property and then the Power Property.

EXAMPLE 8.34

Simplify: ⓐ $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ ⓑ $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$.

Solution

ⓐ

	$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
Rewrite 27 as a power of 3.	$\left(3^3\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$\left(3^2\right)\left(u^{\frac{1}{3}}\right)$
Simplify.	$9u^{\frac{1}{3}}$

ⓑ

	$\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$
First we use the Product to a Power Property.	$\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$
To raise a power to a power, we multiply the exponents.	$mn^{\frac{3}{4}}$

TRY IT : : 8.67

Simplify: ⓐ $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ ⓑ $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$.

TRY IT : : 8.68

Simplify: ⓐ $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$ ⓑ $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$.

We will use both the Product Property and the Quotient Property in the next example.

EXAMPLE 8.35

Simplify: ⓐ $\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ ⓑ $\left(\frac{16\,x^{\frac{4}{3}} y^{-\frac{5}{6}}}{x^{-\frac{2}{3}} y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$.

Solution

ⓐ

$$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$$

Use the Quotient Property, subtract the exponents.

$$x^{\frac{8}{4}}$$

Simplify.

$$x^2$$

ⓑ Follow the order of operations to simplify inside the parenthese first.

$$\left(\frac{16\,x^{\frac{4}{3}} y^{-\frac{5}{6}}}{x^{-\frac{2}{3}} y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$$

Use the Quotient Property, subtract the exponents.

$$\left(\frac{16x^{\frac{6}{3}}}{y^{\frac{6}{6}}}\right)^{\frac{1}{2}}$$

Simplify.

$$\left(\frac{16x^2}{y}\right)^{\frac{1}{2}}$$

Use the Product to a Power Property, multiply the exponents.

$$\frac{4x}{y^{\frac{1}{2}}}$$

TRY IT : : 8.69

Simplify: ⓐ $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$ ⓑ $\left(\frac{25m^{\frac{1}{6}} n^{\frac{11}{6}}}{m^{\frac{2}{3}} n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$.

TRY IT : : 8.70

Simplify: ⓐ $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$ ⓑ $\left(\frac{27x^{\frac{4}{5}} y^{\frac{1}{6}}}{x^{\frac{1}{5}} y^{-\frac{5}{6}}}\right)^{\frac{1}{3}}$.

MEDIA : :

Access these online resources for additional instruction and practice with simplifying rational exponents.

- **Review-Rational Exponents (https://openstax.org/l/37RatExpont1)**
- **Using Laws of Exponents on Radicals: Properties of Rational Exponents (https://openstax.org/l/37RatExpont2)**

8.3 EXERCISES

Practice Makes Perfect

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

119. ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$

120. ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

121. ⓐ $u^{\frac{1}{5}}$ ⓑ $v^{\frac{1}{9}}$ ⓒ $w^{\frac{1}{20}}$

122. ⓐ $g^{\frac{1}{7}}$ ⓑ $h^{\frac{1}{5}}$ ⓒ $j^{\frac{1}{25}}$

In the following exercises, write with a rational exponent.

123. ⓐ $\sqrt[7]{x}$ ⓑ $\sqrt[9]{y}$ ⓒ $\sqrt[5]{f}$

124. ⓐ $\sqrt[8]{r}$ ⓑ $\sqrt[10]{s}$ ⓒ $\sqrt[4]{t}$

125. ⓐ $\sqrt[3]{7c}$ ⓑ $\sqrt[7]{12d}$ ⓒ $2\sqrt[4]{6b}$

126. ⓐ $\sqrt[4]{5x}$ ⓑ $\sqrt[8]{9y}$ ⓒ $7\sqrt[5]{3z}$

127. ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $4\sqrt[6]{36r}$

128. ⓐ $\sqrt[3]{25a}$ ⓑ $\sqrt{3b}$ ⓒ $\sqrt[8]{40c}$

In the following exercises, simplify.

129.
ⓐ $81^{\frac{1}{2}}$
ⓑ $125^{\frac{1}{3}}$
ⓒ $64^{\frac{1}{2}}$

130.
ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
ⓒ $32^{\frac{1}{5}}$

131.
ⓐ $16^{\frac{1}{4}}$
ⓑ $16^{\frac{1}{2}}$
ⓒ $625^{\frac{1}{4}}$

132.
ⓐ $64^{\frac{1}{3}}$
ⓑ $32^{\frac{1}{5}}$
ⓒ $81^{\frac{1}{4}}$

133.
ⓐ $(-216)^{\frac{1}{3}}$
ⓑ $-216^{\frac{1}{3}}$
ⓒ $(216)^{-\frac{1}{3}}$

134.
ⓐ $(-1000)^{\frac{1}{3}}$
ⓑ $-1000^{\frac{1}{3}}$
ⓒ $(1000)^{-\frac{1}{3}}$

135.
ⓐ $(-81)^{\frac{1}{4}}$
ⓑ $-81^{\frac{1}{4}}$
ⓒ $(81)^{-\frac{1}{4}}$

136.
ⓐ $(-49)^{\frac{1}{2}}$
ⓑ $-49^{\frac{1}{2}}$
ⓒ $(49)^{-\frac{1}{2}}$

137.
ⓐ $(-36)^{\frac{1}{2}}$
ⓑ $-36^{\frac{1}{2}}$
ⓒ $(36)^{-\frac{1}{2}}$

138.
ⓐ $(-16)^{\frac{1}{4}}$
ⓑ $-16^{\frac{1}{4}}$
ⓒ $16^{-\frac{1}{4}}$

139.
ⓐ $(-100)^{\frac{1}{2}}$
ⓑ $-100^{\frac{1}{2}}$
ⓒ $(100)^{-\frac{1}{2}}$

140.
ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

141.

ⓐ $\sqrt{m^5}$

ⓑ $\left(\sqrt[3]{3y}\right)^7$

ⓒ $\sqrt[5]{\left(\frac{4x}{5y}\right)^3}$

142.

ⓐ $\sqrt[4]{r^7}$

ⓑ $\left(\sqrt[5]{2pq}\right)^3$

ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

143.

ⓐ $\sqrt[5]{u^2}$

ⓑ $\left(\sqrt[3]{6x}\right)^5$

ⓒ $\sqrt[4]{\left(\frac{18a}{5b}\right)^7}$

144.

ⓐ $\sqrt[3]{a}$

ⓑ $\left(\sqrt[4]{21v}\right)^3$

ⓒ $\sqrt[4]{\left(\frac{2xy}{5z}\right)^2}$

In the following exercises, simplify.

145.

ⓐ $64^{\frac{5}{2}}$

ⓑ $81^{\frac{-3}{2}}$

ⓒ $(-27)^{\frac{2}{3}}$

146.

ⓐ $25^{\frac{3}{2}}$

ⓑ $9^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{2}{3}}$

147.

ⓐ $32^{\frac{2}{5}}$

ⓑ $27^{-\frac{2}{3}}$

ⓒ $(-25)^{\frac{1}{2}}$

148.

ⓐ $100^{\frac{3}{2}}$

ⓑ $49^{-\frac{5}{2}}$

ⓒ $(-100)^{\frac{3}{2}}$

149.

ⓐ $-9^{\frac{3}{2}}$

ⓑ $-9^{-\frac{3}{2}}$

ⓒ $(-9)^{\frac{3}{2}}$

150.

ⓐ $-64^{\frac{3}{2}}$

ⓑ $-64^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{3}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

151.

ⓐ $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$

ⓑ $\left(p^{12}\right)^{\frac{3}{4}}$

ⓒ $\frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

152.

ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$

ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$

ⓒ $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

153.

ⓐ $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$

ⓑ $\left(x^{12}\right)^{\frac{2}{3}}$

ⓒ $\frac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$

154.

ⓐ $q^{\frac{2}{3}} \cdot q^{\frac{5}{6}}$

ⓑ $\left(h^{6}\right)^{\frac{4}{3}}$

ⓒ $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

155.

ⓐ $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$

ⓑ $\left(a^{\frac{1}{3}}b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

156.

ⓐ $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$

ⓑ $\left(m^{\frac{4}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{4}}$

157.

ⓐ $\left(16\,u^{\frac{1}{3}}\right)^{\frac{3}{4}}$

ⓑ $\left(4\,p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$

158.

ⓐ $\left(625\,n^{\frac{8}{3}}\right)^{\frac{3}{4}}$

ⓑ $\left(9\,x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$

159.

ⓐ $\dfrac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$

ⓑ $\left(\dfrac{36\,s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$

160.

ⓐ $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

ⓑ $\left(\dfrac{27\,b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

161.

ⓐ $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$

ⓑ $\left(\dfrac{8\,x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27\,x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$

162.

ⓐ $\dfrac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

ⓑ $\left(\dfrac{16\,m^{\frac{1}{5}}n^{\frac{3}{2}}}{81\,m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

Writing Exercises

163. Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

164. Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with $a^{\frac{1}{n}}$.			
simplify expressions with $a^{\frac{m}{n}}$.			
use the Laws of Exponents to simply expressions with rational exponents.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

8.4 Add, Subtract, and Multiply Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $3x^2+9x-5-\left(x^2-2x+3\right)$.
 If you missed this problem, review Example 5.5.
2. Simplify: $(2+a)(4-a)$.
 If you missed this problem, review Example 5.28.
3. Simplify: $(9-5y)^2$.
 If you missed this problem, review Example 5.31.

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Like Radicals

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x+8x$ is $11x$. Similarly we add $3\sqrt{x}+8\sqrt{x}$ and the result is $11\sqrt{x}$.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

EXAMPLE 8.36

Simplify: ⓐ $2\sqrt{2}-7\sqrt{2}$ ⓑ $5\sqrt[3]{y}+4\sqrt[3]{y}$ ⓒ $7\sqrt[4]{x}-2\sqrt[4]{y}$.

Solution

ⓐ

Since the radicals are like, we subtract the coefficient

$$2\sqrt{2}-7\sqrt{2}$$
$$-5\sqrt{2}$$

ⓑ

Since the radicals are like, we add the coefficient

$$5\sqrt[3]{y}+4\sqrt[3]{y}$$
$$9\sqrt[3]{y}$$

ⓒ

$$7\sqrt[4]{x}-2\sqrt[4]{y}$$

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

TRY IT : : 8.71 Simplify: ⓐ $8\sqrt{2}-9\sqrt{2}$ ⓑ $4\sqrt[3]{x}+7\sqrt[3]{x}$ ⓒ $3\sqrt[4]{x}-5\sqrt[4]{y}$.

TRY IT : : 8.72 Simplify: ⓐ $5\sqrt{3}-9\sqrt{3}$ ⓑ $5\sqrt[3]{y}+3\sqrt[3]{y}$ ⓒ $5\sqrt[4]{m}-2\sqrt[3]{m}$.

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

EXAMPLE 8.37

Simplify: ⓐ $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$ ⓑ $\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$.

Solution

ⓐ

	$2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$
Since the radicals are like, we combine them.	$0\sqrt{5n}$
Simplify.	0

ⓑ

	$\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$
Since the radicals are like, we combine them.	$2\sqrt[4]{3xy}$

TRY IT : : 8.73 Simplify: ⓐ $\sqrt{7x}-7\sqrt{7x}+4\sqrt{7x}$ ⓑ $4\sqrt[4]{5xy}+2\sqrt[4]{5xy}-7\sqrt[4]{5xy}$.

TRY IT : : 8.74 Simplify: ⓐ $4\sqrt{3y}-7\sqrt{3y}+2\sqrt{3y}$ ⓑ $6\sqrt[3]{7mn}+\sqrt[3]{7mn}-4\sqrt[3]{7mn}$.

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

EXAMPLE 8.38

Simplify: ⓐ $\sqrt{20}+3\sqrt{5}$ ⓑ $\sqrt[3]{24}-\sqrt[3]{375}$ ⓒ $\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt{20}+3\sqrt{5}$
Simplify the radicals, when possible.	$\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$
	$2\sqrt{5}+3\sqrt{5}$
Combine the like radicals.	$5\sqrt{5}$

ⓑ

	$\sqrt[3]{24}-\sqrt[3]{375}$
Simplify the radicals.	$\sqrt[3]{8}\cdot\sqrt[3]{3}-\sqrt[3]{125}\cdot\sqrt[3]{3}$
	$2\sqrt[3]{3}-5\sqrt[3]{3}$
Combine the like radicals.	$-3\sqrt[3]{3}$

ⓒ

Simplify the radicals.

$$\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$$
$$\frac{1}{2}\sqrt[4]{16}\cdot\sqrt[4]{3}-\frac{2}{3}\sqrt[4]{81}\cdot\sqrt[4]{3}$$
$$\frac{1}{2}\cdot 2\cdot\sqrt[4]{3}-\frac{2}{3}\cdot 3\cdot\sqrt[4]{3}$$
$$\sqrt[4]{3}-2\sqrt[4]{3}$$

Combine the like radicals.

$$-\sqrt[4]{3}$$

TRY IT : : 8.75 Simplify: ⓐ $\sqrt{18}+6\sqrt{2}$ ⓑ $6\sqrt[3]{16}-2\sqrt[3]{250}$ ⓒ $\frac{2}{3}\sqrt[3]{81}-\frac{1}{2}\sqrt[3]{24}$.

TRY IT : : 8.76 Simplify: ⓐ $\sqrt{27}+4\sqrt{3}$ ⓑ $4\sqrt[3]{5}-7\sqrt[3]{40}$ ⓒ $\frac{1}{2}\sqrt[3]{128}-\frac{5}{3}\sqrt[3]{54}$.

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

EXAMPLE 8.39

Simplify: ⓐ $9\sqrt{50m^2}-6\sqrt{48m^2}$ ⓑ $\sqrt[3]{54n^5}-\sqrt[3]{16n^5}$.

Solution

ⓐ

Simplify the radicals.

$$9\sqrt{50m^2}-6\sqrt{48m^2}$$
$$9\sqrt{25m^2}\cdot\sqrt{2}-6\sqrt{16m^2}\cdot\sqrt{3}$$
$$9\cdot 5m\cdot\sqrt{2}-6\cdot 4m\cdot\sqrt{3}$$
$$45m\sqrt{2}-24m\sqrt{3}$$

The radicals are not like and so cannot be combined.

ⓑ

Simplify the radicals.

$$\sqrt[3]{54n^5}-\sqrt[3]{16n^5}$$
$$\sqrt[3]{27n^3}\cdot\sqrt[3]{2n^2}-\sqrt[3]{8n^3}\cdot\sqrt[3]{2n^2}$$
$$3n\sqrt[3]{2n^2}-2n\sqrt[3]{2n^2}$$

Combine the like radicals.

$$n\sqrt[3]{2n^2}$$

TRY IT : : 8.77 Simplify: ⓐ $\sqrt{32m^7}-\sqrt{50m^7}$ ⓑ $\sqrt[3]{135x^7}-\sqrt[3]{40x^7}$.

TRY IT : : 8.78 Simplify: ⓐ $\sqrt{27p^3}-\sqrt{48p^3}$ ⓑ $\sqrt[3]{256y^5}-\sqrt[3]{32n^5}$.

Multiply Radical Expressions

We have used the Product Property of Roots to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots 'in reverse' to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Product Property of Roots

For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is 12*xy*. Keep this in mind as you do these examples.

EXAMPLE 8.40

Simplify: ⓐ $(6\sqrt{2})(3\sqrt{10})$ ⓑ $\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$.

Solution

ⓐ

	$(6\sqrt{2})(3\sqrt{10})$
Multiply using the Product Property.	$18\sqrt{20}$
Simplify the radical.	$18\sqrt{4}\cdot\sqrt{5}$
Simplify.	$18\cdot 2\cdot\sqrt{5}$
	$36\sqrt{5}$

ⓑ

	$\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$
Multiply using the Product Property.	$20\sqrt[3]{24}$
Simplify the radical.	$20\sqrt[3]{8}\cdot\sqrt[3]{3}$
Simplify.	$20\cdot 2\cdot\sqrt[3]{3}$
	$40\sqrt[3]{3}$

TRY IT : : 8.79 Simplify: ⓐ $(3\sqrt{2})(2\sqrt{30})$ ⓑ $\left(2\sqrt[3]{18}\right)\left(-3\sqrt[3]{6}\right)$.

TRY IT : : 8.80 Simplify: ⓐ $(3\sqrt{3})(3\sqrt{6})$ ⓑ $\left(-4\sqrt[3]{9}\right)\left(3\sqrt[3]{6}\right)$.

We follow the same procedures when there are variables in the radicands.

EXAMPLE 8.41

Simplify: ⓐ $\left(10\sqrt{6p^3}\right)(4\sqrt{3p})$ ⓑ $\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$.

Solution

ⓐ

$$\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$$

Multiply.	$40\sqrt{18p^4}$
Simplify the radical.	$40\sqrt{9p^4}\cdot\sqrt{2}$
Simplify.	$40\cdot 3p^2\cdot\sqrt{3}$
	$120p^2\sqrt{3}$

ⓑ When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

$$\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$$

Multiply.	$6\sqrt[4]{4\cdot 5\cdot 4\cdot 7y^5}$
Simplify the radical.	$6\sqrt[4]{16y^4}\cdot\sqrt[4]{35y}$
Simplify.	$6\cdot 2y\sqrt[4]{35y}$
Multiply.	$12y\sqrt[4]{35y}$

TRY IT : : 8.81 Simplify: ⓐ $\left(6\sqrt{6x^2}\right)\left(8\sqrt{30x^4}\right)$ ⓑ $\left(-4\sqrt[4]{12y^3}\right)\left(-\sqrt[4]{8y^3}\right)$.

TRY IT : : 8.82 Simplify: ⓐ $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$ ⓑ $\left(-4\sqrt[4]{9a^3}\right)\left(3\sqrt[4]{27a^2}\right)$.

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the Distributive Property to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

EXAMPLE 8.42

Simplify: ⓐ $\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$ ⓑ $\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$.

Solution

ⓐ

$$\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$$

Multiply.	$\sqrt{12}+\sqrt{108}$
Simplify.	$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$
Simplify.	$2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$8\sqrt{3}$

ⓑ

$$\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$$

Distribute.	$5\sqrt[3]{9}-\sqrt[3]{162}$
Simplify.	$5\sqrt[3]{9}-\sqrt[3]{27}\cdot\sqrt[3]{6}$
Simplify.	$5\sqrt[3]{9}-3\sqrt[3]{6}$

TRY IT : : 8.83 Simplify: ⓐ $\sqrt{6}(1+3\sqrt{6})$ ⓑ $\sqrt[3]{4}\left(-2-\sqrt[3]{6}\right)$.

TRY IT : : 8.84 Simplify: ⓐ $\sqrt{8}(2-5\sqrt{8})$ ⓑ $\sqrt[3]{3}\left(-\sqrt[3]{9}-\sqrt[3]{6}\right)$.

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

EXAMPLE 8.43

Simplify: ⓐ $(3-2\sqrt{7})(4-2\sqrt{7})$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$.

Solution

ⓐ

	$(3-2\sqrt{7})(4-2\sqrt{7})$
Multiply	$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$
Simplify.	$12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$40-14\sqrt{7}$

ⓑ

	$\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$
Multiply.	$\sqrt[3]{x^2}+4\sqrt[3]{x}-2\sqrt[3]{x}-8$
Combine like terms.	$\sqrt[3]{x^2}+2\sqrt[3]{x}-8$

TRY IT : : 8.85 Simplify: ⓐ $(6-3\sqrt{7})(3+4\sqrt{7})$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}-3\right)$.

TRY IT : : 8.86 Simplify: ⓐ $(2-3\sqrt{11})(4-\sqrt{11})$ ⓑ $\left(\sqrt[3]{x}+1\right)\left(\sqrt[3]{x}+3\right)$.

EXAMPLE 8.44

Simplify: $\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$.

Solution

	$\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$
Multiply.	$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$
Simplify.	$6+12\sqrt{10}-\sqrt{10}-20$
Combine like terms.	$-14+11\sqrt{10}$

TRY IT : : 8.87 Simplify: $\left(5\sqrt{3}-\sqrt{7}\right)\left(\sqrt{3}+2\sqrt{7}\right)$

TRY IT : : 8.88 Simplify: $\left(\sqrt{6}-3\sqrt{8}\right)\left(2\sqrt{6}+\sqrt{8}\right)$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Special Products

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a+b)(a-b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

We will use the special product formulas in the next few examples. We will start with the Product of Binomial Squares Pattern.

EXAMPLE 8.45

Simplify: ⓐ $(2+\sqrt{3})^2$ ⓑ $(4-2\sqrt{5})^2$.

Solution

Be sure to include the $2ab$ term when squaring a binomial.

ⓐ

	$(a+b)^2$ $(2+\sqrt{3})^2$
Multiply, using the Product of Binomial Squares Pattern.	$a^2+2\ a\ b+b^2$ $2^2+2\cdot2\cdot\sqrt{3}+(\sqrt{3})^2$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

ⓑ

	$(a-b)^2$ $(4-2\sqrt{5})^2$
Multiply, using the Product of Binomial Squares Pattern.	$a^2-2\ a\ b+b^2$ $4^2-2\cdot4\cdot2\sqrt{5}+(2\sqrt{5})^2$
Simplify.	$16-16\sqrt{5}+4\cdot5$
	$16-16\sqrt{5}+20$
Combine like terms.	$36-16\sqrt{5}$

TRY IT : : 8.89 Simplify: ⓐ $(10+\sqrt{2})^2$ ⓑ $(1+3\sqrt{6})^2$.

TRY IT : : 8.90 Simplify: ⓐ $(6-\sqrt{5})^2$ ⓑ $(9-2\sqrt{10})^2$.

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

EXAMPLE 8.46

Simplify: $(5-2\sqrt{3})(5+2\sqrt{3})$.

Solution

	$(a-b)(a+b)$
	$(5-2\sqrt{3})(5+2\sqrt{3})$
Multiply, using the Product of Conjugates Pattern.	a^2-b^2
	$5^2-(2\sqrt{3})^2$
Simplify.	$25-4\cdot 3$
	13

TRY IT : : 8.91 Simplify: $(3-2\sqrt{5})(3+2\sqrt{5})$

TRY IT : : 8.92 Simplify: $(4+5\sqrt{7})(4-5\sqrt{7})$.

MEDIA : :

Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- **Multiplying Adding Subtracting Radicals (https://openstax.org/l/37Radicals1)**
- **Multiplying Special Products: Square Binomials Containing Square Roots (https://openstax.org/l/37Radicals2)**
- **Multiplying Conjugates (https://openstax.org/l/37Radicals3)**

8.4 EXERCISES

Practice Makes Perfect

Add and Subtract Radical Expressions

In the following exercises, simplify.

165.
ⓐ $8\sqrt{2}-5\sqrt{2}$
ⓑ $5\sqrt[3]{m}+2\sqrt[3]{m}$
ⓒ $8\sqrt[4]{m}-2\sqrt[4]{n}$

166.
ⓐ $7\sqrt{2}-3\sqrt{2}$
ⓑ $7\sqrt[3]{p}+2\sqrt[3]{p}$
ⓒ $5\sqrt[3]{x}-3\sqrt[3]{x}$

167.
ⓐ $3\sqrt{5}+6\sqrt{5}$
ⓑ $9\sqrt[3]{a}+3\sqrt[3]{a}$
ⓒ $5\sqrt[4]{2z}+\sqrt[4]{2z}$

168.
ⓐ $4\sqrt{5}+8\sqrt{5}$
ⓑ $\sqrt[3]{m}-4\sqrt[3]{m}$
ⓒ $\sqrt{n}+3\sqrt{n}$

169.
ⓐ $3\sqrt{2a}-4\sqrt{2a}+5\sqrt{2a}$
ⓑ $5\sqrt[4]{3ab}-3\sqrt[4]{3ab}-2\sqrt[4]{3ab}$

170.
ⓐ $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$
ⓑ $8\sqrt[4]{11cd}+5\sqrt[4]{11cd}-9\sqrt[4]{11cd}$

171.
ⓐ $8\sqrt{3c}+2\sqrt{3c}-9\sqrt{3c}$
ⓑ $2\sqrt[3]{4pq}-5\sqrt[3]{4pq}+4\sqrt[3]{4pq}$

172.
ⓐ $3\sqrt{5d}+8\sqrt{5d}-11\sqrt{5d}$
ⓑ $11\sqrt[3]{2rs}-9\sqrt[3]{2rs}+3\sqrt[3]{2rs}$

173.
ⓐ $\sqrt{27}-\sqrt{75}$
ⓑ $\sqrt[3]{40}-\sqrt[3]{320}$
ⓒ $\frac{1}{2}\sqrt[4]{32}+\frac{2}{3}\sqrt[4]{162}$

174.
ⓐ $\sqrt{72}-\sqrt{98}$
ⓑ $\sqrt[3]{24}+\sqrt[3]{81}$
ⓒ $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$

175.
ⓐ $\sqrt{48}+\sqrt{27}$
ⓑ $\sqrt[3]{54}+\sqrt[3]{128}$
ⓒ $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$

176.
ⓐ $\sqrt{45}+\sqrt{80}$
ⓑ $\sqrt[3]{81}-\sqrt[3]{192}$
ⓒ $\frac{5}{2}\sqrt[4]{80}+\frac{7}{3}\sqrt[4]{405}$

177.
ⓐ $\sqrt{72a^5}-\sqrt{50a^5}$
ⓑ $9\sqrt[4]{80p^4}-6\sqrt[4]{405p^4}$

178.
ⓐ $\sqrt{48b^5}-\sqrt{75b^5}$
ⓑ $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$

179.
ⓐ $\sqrt{80c^7}-\sqrt{20c^7}$
ⓑ $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$

180.
ⓐ $\sqrt{96d^9}-\sqrt{24d^9}$
ⓑ $5\sqrt[4]{243s^6}+2\sqrt[4]{3s^6}$

181.
$3\sqrt{128y^2}+4y\sqrt{162}-8\sqrt{98y^2}$

182. $3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Multiply Radical Expressions

In the following exercises, simplify.

183.
ⓐ $(-2\sqrt{3})(3\sqrt{18})$
ⓑ $\left(8\sqrt[3]{4}\right)\left(-4\sqrt[3]{18}\right)$

184.
ⓐ $(-4\sqrt{5})(5\sqrt{10})$
ⓑ $\left(-2\sqrt[3]{9}\right)\left(7\sqrt[3]{9}\right)$

185.
ⓐ $(5\sqrt{6})(-\sqrt{12})$
ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

186.
ⓐ $(-2\sqrt{7})(-2\sqrt{14})$
ⓑ $\left(-3\sqrt[4]{8}\right)\left(-5\sqrt[4]{6}\right)$

187.
ⓐ $\left(4\sqrt{12z^3}\right)\left(3\sqrt{9z}\right)$
ⓑ $\left(5\sqrt[3]{3x^3}\right)\left(3\sqrt[3]{18x^3}\right)$

188.
ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$
ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

189.
ⓐ $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$
ⓑ $\left(2\sqrt[4]{8y^2}\right)\left(-2\sqrt[4]{12y^3}\right)$

190.
ⓐ $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$
ⓑ $\left(-\sqrt[4]{6b^3}\right)\left(3\sqrt[4]{8b^3}\right)$

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

191.
ⓐ $\sqrt{7}(5+2\sqrt{7})$
ⓑ $\sqrt[3]{6}\left(4+\sqrt[3]{18}\right)$

192.
ⓐ $\sqrt{11}(8+4\sqrt{11})$
ⓑ $\sqrt[3]{3}\left(\sqrt[3]{9}+\sqrt[3]{18}\right)$

193.
ⓐ $\sqrt{11}(-3+4\sqrt{11})$
ⓑ $\sqrt[4]{3}\left(\sqrt[4]{54}+\sqrt[4]{18}\right)$

194.
ⓐ $\sqrt{2}(-5+9\sqrt{2})$
ⓑ $\sqrt[4]{2}\left(\sqrt[4]{12}+\sqrt[4]{24}\right)$

195. $(7+\sqrt{3})(9-\sqrt{3})$

196. $(8-\sqrt{2})(3+\sqrt{2})$

197.
ⓐ $(9-3\sqrt{2})(6+4\sqrt{2})$
ⓑ $\left(\sqrt[3]{x}-3\right)\left(\sqrt[3]{x}+1\right)$

198.
ⓐ $(3-2\sqrt{7})(5-4\sqrt{7})$
ⓑ $\left(\sqrt[3]{x}-5\right)\left(\sqrt[3]{x}-3\right)$

199.
ⓐ $(1+3\sqrt{10})(5-2\sqrt{10})$
ⓑ $\left(2\sqrt[3]{x}+6\right)\left(\sqrt[3]{x}+1\right)$

200.
ⓐ $(7-2\sqrt{5})(4+9\sqrt{5})$
ⓑ $\left(3\sqrt[3]{x}+2\right)\left(\sqrt[3]{x}-2\right)$

201. $(\sqrt{3}+\sqrt{10})(\sqrt{3}+2\sqrt{10})$

202. $(\sqrt{11}+\sqrt{5})(\sqrt{11}+6\sqrt{5})$

203. $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$

204. $(4\sqrt{6}+7\sqrt{13})(8\sqrt{6}-3\sqrt{13})$

205. ⓐ $(3+\sqrt{5})^2$ ⓑ $(2-5\sqrt{3})^2$

206. ⓐ $(4+\sqrt{11})^2$ ⓑ $(3-2\sqrt{5})^2$

207. ⓐ $(9-\sqrt{6})^2$ ⓑ $(10+3\sqrt{7})^2$

208. ⓐ $(5-\sqrt{10})^2$ ⓑ $(8+3\sqrt{2})^2$

209. $(4+\sqrt{2})(4-\sqrt{2})$

210. $(7+\sqrt{10})(7-\sqrt{10})$

211. $(4+9\sqrt{3})(4-9\sqrt{3})$

212. $(1+8\sqrt{2})(1-8\sqrt{2})$

213. $(12-5\sqrt{5})(12+5\sqrt{5})$

214. $(9-4\sqrt{3})(9+4\sqrt{3})$

215. $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

216. $\left(\sqrt[3]{4x}+3\right)\left(\sqrt[3]{4x}-3\right)$

Mixed Practice

217. $\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$

218. $\sqrt{175k^4}-\sqrt{63k^4}$

219. $\frac{5}{6}\sqrt{162}+\frac{3}{16}\sqrt{128}$

220. $\sqrt[3]{24}+\sqrt[3]{81}$

221. $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$

222. $8\sqrt[4]{13}-4\sqrt[4]{13}-3\sqrt[4]{13}$

223. $5\sqrt{12c^4}-3\sqrt{27c^6}$

224. $\sqrt{80a^5}-\sqrt{45a^5}$

225. $\frac{3}{5}\sqrt{75}-\frac{1}{4}\sqrt{48}$

226. $21\sqrt[3]{9}-2\sqrt[3]{9}$

227. $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$

228. $11\sqrt{11}-10\sqrt{11}$

229. $\sqrt{3}\cdot\sqrt{21}$

230. $(4\sqrt{6})(-\sqrt{18})$

231. $\left(7\sqrt[3]{4}\right)\left(-3\sqrt[3]{18}\right)$

232. $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$

233. $(\sqrt{29})^2$

234. $(-4\sqrt{17})(-3\sqrt{17})$

235. $(-4+\sqrt{17})(-3+\sqrt{17})$

236. $\left(3\sqrt[4]{8a^2}\right)\left(\sqrt[4]{12a^3}\right)$

237. $(6-3\sqrt{2})^2$

238. $\sqrt{3}(4-3\sqrt{3})$

239. $\sqrt[3]{3}\left(2\sqrt[3]{9}+\sqrt[3]{18}\right)$

240. $(\sqrt{6}+\sqrt{3})(\sqrt{6}+6\sqrt{3})$

Writing Exercises

241. Explain the when a radical expression is in simplest form.

242. Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

243.

ⓐ Explain why $(-\sqrt{n})^2$ is always non-negative, for $n \geq 0$.

ⓑ Explain why $-(\sqrt{n})^2$ is always non-positive, for $n \geq 0$.

244. Use the binomial square pattern to simplify $(3+\sqrt{2})^2$. Explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
add and subtract radical expressions.			
multiply radical expressions.			
use polynomial multiplication to multiply radical expressions.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

8.5 Divide Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{30}{48}$.
 If you missed this problem, review **Example 1.24**.
2. Simplify: $x^2 \cdot x^4$.
 If you missed this problem, review **Example 5.12**.
3. Multiply: $(7+3x)(7-3x)$.
 If you missed this problem, review **Example 5.32**.

Divide Radical Expressions

We have used the Quotient Property of Radical Expressions to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals.

We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

EXAMPLE 8.47

Simplify: ⓐ $\frac{\sqrt{72x^3}}{\sqrt{162x}}$ ⓑ $\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$.

Solution

ⓐ

$$\frac{\sqrt{72x^3}}{\sqrt{162x}}$$

Rewrite using the quotient property, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.

$$\sqrt{\frac{72x^3}{162x}}$$

Remove common factors.

$$\sqrt{\frac{\cancel{18} \cdot 4 \cdot x^2 \cdot \cancel{x}}{\cancel{18} \cdot 9 \cdot \cancel{x}}}$$

Simplify.

$$\sqrt{\frac{4x^2}{9}}$$

Simplify the radical.

$$\frac{2x}{3}$$

ⓑ

$$\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$$

Rewrite using the quotient property, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.

$$\sqrt[3]{\frac{32x^2}{4x^5}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{\frac{8}{x^3}}$$

Simplify the radical.

$$\frac{2}{x}$$

TRY IT : : 8.93

Simplify: ⓐ $\frac{\sqrt{50s^3}}{\sqrt{128s}}$ ⓑ $\frac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$.

TRY IT : : 8.94

Simplify: ⓐ $\frac{\sqrt{75q^5}}{\sqrt{108q}}$ ⓑ $\frac{\sqrt[3]{72b^2}}{\sqrt[3]{9b^5}}$.

EXAMPLE 8.48

Simplify: ⓐ $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$ ⓑ $\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$.

Solution

ⓐ

$$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$$

Rewrite using the quotient property.

$$\sqrt{\frac{147ab^8}{3a^3b^4}}$$

Remove common factors in the fraction.

$$\sqrt{\frac{49b^4}{a^2}}$$

Simplify the radical.

$$\frac{7b^2}{a}$$

ⓑ

	$\frac{\sqrt[3]{-250m\,n^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$
Rewrite using the quotient property.	$\sqrt[3]{\frac{-250m\,n^{-2}}{2m^{-2}n^4}}$
Simplify the fraction under the radical.	$\sqrt[3]{\frac{-125m^3}{n^6}}$
Simplify the radical.	$-\frac{5m}{n^2}$

TRY IT : : 8.95 Simplify: ⓐ $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$ ⓑ $\frac{\sqrt[3]{-128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$.

TRY IT : : 8.96 Simplify: ⓐ $\frac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$ ⓑ $\frac{\sqrt[3]{-81pq^{-1}}}{\sqrt[3]{3p^{-2}q^5}}$.

EXAMPLE 8.49

Simplify: $\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$.

Solution

	$\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$
Rewrite using the quotient property.	$\sqrt{\frac{54x^5y^3}{3x^2y}}$
Remove common factors in the fraction.	$\sqrt{18x^3y^2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9x^2y^2\cdot 2x}$
Rewrite the radical as the product of two radicals.	$\sqrt{9x^2y^2}\cdot\sqrt{2x}$
Simplify.	$3xy\sqrt{2x}$

TRY IT : : 8.97 Simplify: $\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$.

TRY IT : : 8.98 Simplify: $\frac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$.

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a

rational number in the denominator.

This process is still used today, and is useful in other areas of mathematics, too.

Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a radical expression is not considered simplified if the radicand contains a fraction.

Simplified Radical Expressions

A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

EXAMPLE 8.50

Simplify: ⓐ $\frac{4}{\sqrt{3}}$ ⓑ $\sqrt{\frac{3}{20}}$ ⓒ $\frac{3}{\sqrt{6x}}$.

Solution

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt{\frac{3}{20}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{3}}{\sqrt{20}}$
Simplify the denominator.	$\frac{\sqrt{3}}{2\sqrt{5}}$
Multiply the numerator and denominator by $\sqrt{5}$.	$\frac{\sqrt{3} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}}$

Simplify.	$\frac{\sqrt{15}}{2 \cdot 5}$
Simplify.	$\frac{\sqrt{15}}{10}$

ⓒ

	$\frac{3}{\sqrt{6x}}$
Multiply the numerator and denominator by $\sqrt{6x}$.	$\frac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$
Simplify.	$\frac{3\sqrt{6x}}{6x}$
Simplify.	$\frac{\sqrt{6x}}{2x}$

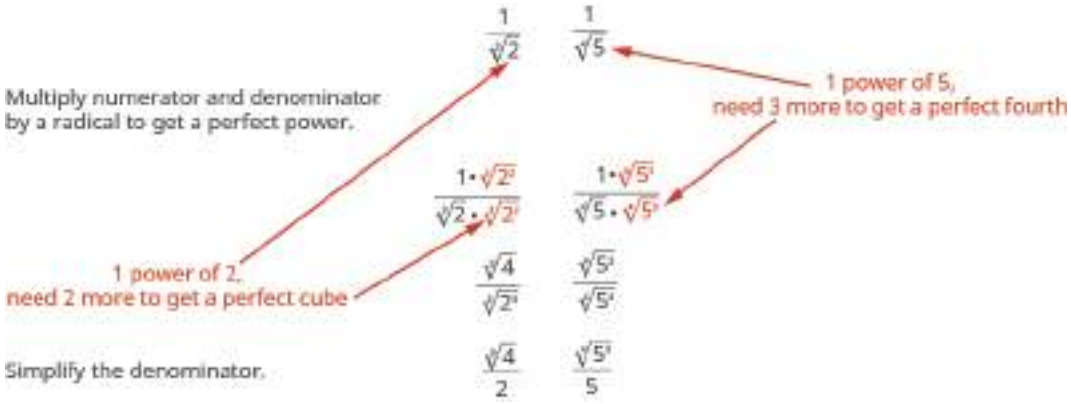
TRY IT : : 8.99 Simplify: ⓐ $\frac{5}{\sqrt{3}}$ ⓑ $\sqrt{\frac{3}{32}}$ ⓒ $\frac{2}{\sqrt{2x}}$.

TRY IT : : 8.100 Simplify: ⓐ $\frac{6}{\sqrt{5}}$ ⓑ $\sqrt{\frac{7}{18}}$ ⓒ $\frac{5}{\sqrt{5x}}$.

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,



We will use this technique in the next examples.

EXAMPLE 8.51

Simplify ⓐ $\frac{1}{\sqrt[3]{6}}$ ⓑ $\sqrt[3]{\frac{7}{24}}$ ⓒ $\frac{3}{\sqrt[3]{4x}}$.

Solution

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{1}{\sqrt[3]{6}}$
The radical in the denominator has one factor of 6. Multiply both the numerator and denominator by $\sqrt[3]{6^2}$, which gives us 2 more factors of 6.	$\frac{1 \cdot \sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \sqrt[3]{6^2}}$
Multiply. Notice the radicand in the denominator has 3 powers of 6.	$\frac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$
Simplify the cube root in the denominator.	$\frac{\sqrt[3]{36}}{6}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt{\frac{7}{24}}$
The fraction is not a perfect cube, so rewrite using the Quotient Property.	$\frac{\sqrt[3]{7}}{\sqrt[3]{24}}$
Simplify the denominator.	$\frac{\sqrt[3]{7}}{2\sqrt[3]{3}}$
Multiply the numerator and denominator by $\sqrt[3]{3^2}$. This will give us 3 factors of 3.	$\frac{\sqrt[3]{7} \cdot \sqrt[3]{3^2}}{2\sqrt[3]{3} \cdot \sqrt[3]{3^2}}$
Simplify.	$\frac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$
Remember, $\sqrt[3]{3^3} = 3$.	$\frac{\sqrt[3]{63}}{2 \cdot 3}$
Simplify.	$\frac{\sqrt[3]{63}}{6}$

ⓒ

	$\frac{3}{\sqrt[3]{4x}}$
Rewrite the radicand to show the factors.	$\frac{3}{\sqrt[3]{2^2 \cdot x}}$
Multiply the numerator and denominator by $\sqrt[3]{2 \cdot x^2}$. This will get us 3 factors of 2 and 3 factors of x.	$\frac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$
Simplify.	$\frac{3\sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$
Simplify the radical in the denominator.	$\frac{3\sqrt[3]{2x^2}}{2x}$

TRY IT : : 8.101 Simplify: ⓐ $\frac{1}{\sqrt[3]{7}}$ ⓑ $\sqrt[3]{\frac{5}{12}}$ ⓒ $\frac{5}{\sqrt[3]{9y}}$.

TRY IT : : 8.102 Simplify: ⓐ $\frac{1}{\sqrt[3]{2}}$ ⓑ $\sqrt[3]{\frac{3}{20}}$ ⓒ $\frac{2}{\sqrt[3]{25n}}$.

EXAMPLE 8.52

Simplify: ⓐ $\frac{1}{\sqrt[4]{2}}$ ⓑ $\sqrt[4]{\frac{5}{64}}$ ⓒ $\frac{2}{\sqrt[4]{8x}}$.

Solution

To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{1}{\sqrt[4]{2}}$
The radical in the denominator has one factor of 2. Multiply both the numerator and denominator by $\sqrt[4]{2^3}$, which gives us 3 more factors of 2.	$\frac{1 \cdot \sqrt[4]{2^3}}{\sqrt[4]{2} \cdot \sqrt[4]{2^3}}$
Multiply. Notice the radicand in the denominator has 4 powers of 2.	$\frac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$
Simplify the fourth root in the denominator.	$\frac{\sqrt[4]{8}}{2}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt[4]{\frac{5}{64}}$

The fraction is not a perfect fourth power, so rewrite using the Quotient Property.	$\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$
Rewrite the radicand in the denominator to show the factors.	$\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$
Simplify the denominator.	$\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$
Multiply the numerator and denominator by $\sqrt[4]{2^2}$. This will give us 4 factors of 2.	$\frac{\sqrt[4]{5}\cdot\sqrt[4]{2^2}}{2\sqrt[4]{2^2}\cdot\sqrt[4]{2^2}}$
Simplify.	$\frac{\sqrt[4]{5}\cdot\sqrt[4]{4}}{2\sqrt[4]{2^4}}$
Remember, $\sqrt[4]{2^4}=2$.	$\frac{\sqrt[4]{20}}{2\cdot 2}$
Simplify.	$\frac{\sqrt[4]{20}}{4}$

ⓒ

	$\frac{2}{\sqrt[4]{8x}}$
Rewrite the radicand to show the factors.	$\frac{2}{\sqrt[4]{2^3\cdot x}}$
Multiply the numerator and denominator by $\sqrt[4]{2\cdot x^3}$. This will get us 4 factors of 2 and 4 factors of x.	$\frac{2\cdot\sqrt[4]{2\cdot x^3}}{\sqrt[4]{2^3x}\cdot\sqrt[4]{2\cdot x^3}}$
Simplify.	$\frac{2\sqrt[4]{2x^3}}{\sqrt[4]{2^4x^4}}$
Simplify the radical in the denominator.	$\frac{2\sqrt[4]{2x^3}}{2x}$
Simplify the fraction.	$\frac{\sqrt[4]{2x^3}}{x}$

TRY IT : : 8.103 Simplify: ⓐ $\frac{1}{\sqrt[4]{3}}$ ⓑ $\sqrt[4]{\frac{3}{64}}$ ⓒ $\frac{3}{\sqrt[4]{125x}}$.

TRY IT : : 8.104 Simplify: ⓐ $\frac{1}{\sqrt[4]{5}}$ ⓑ $\sqrt[4]{\frac{7}{128}}$ ⓒ $\frac{4}{\sqrt[4]{4x}}$

Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.

$$(a-b)(a+b) \qquad (2-\sqrt{5})(2+\sqrt{5})$$
$$a^2-b^2 \qquad 2^2-(\sqrt{5})^2$$
$$4-5$$
$$-1$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

EXAMPLE 8.53

Simplify: $\frac{5}{2-\sqrt{3}}$.

Solution

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

TRY IT : : 8.105 Simplify: $\frac{3}{1-\sqrt{5}}$.

TRY IT : : 8.106 Simplify: $\frac{2}{4-\sqrt{6}}$.

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

EXAMPLE 8.54

Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$
Multiply the conjugates in the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$

Simplify the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

TRY IT : : 8.107 Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

TRY IT : : 8.108 Simplify: $\frac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

EXAMPLE 8.55

Simplify: $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution

	$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

TRY IT : : 8.109 Simplify: $\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

TRY IT : : 8.110 Simplify: $\frac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$.

MEDIA : :

Access these online resources for additional instruction and practice with dividing radical expressions.

- **Rationalize the Denominator (https://openstax.org/l/37RatDenom1)**
- **Dividing Radical Expressions and Rationalizing the Denominator (https://openstax.org/l/37RatDenom2)**
- **Simplifying a Radical Expression with a Conjugate (https://openstax.org/l/37RatDenom3)**
- **Rationalize the Denominator of a Radical Expression (https://openstax.org/l/37RatDenom4)**

8.5 EXERCISES

Practice Makes Perfect

Divide Square Roots

In the following exercises, simplify.

245. ⓐ $\frac{\sqrt{128}}{\sqrt{72}}$ ⓑ $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$

246. ⓐ $\frac{\sqrt{48}}{\sqrt{75}}$ ⓑ $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$

247. ⓐ $\frac{\sqrt{200m^5}}{\sqrt{98m}}$ ⓑ $\frac{\sqrt[3]{54y^2}}{\sqrt[3]{2y^5}}$

248. ⓐ $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$ ⓑ $\frac{\sqrt[3]{54y}}{\sqrt[3]{16y^4}}$

249. ⓐ $\frac{\sqrt{75r^3}}{\sqrt{108r^7}}$ ⓑ $\frac{\sqrt[3]{24x^7}}{\sqrt[3]{81x^4}}$

250. ⓐ $\frac{\sqrt{196q}}{\sqrt{484q^5}}$ ⓑ $\frac{\sqrt[3]{16m^4}}{\sqrt[3]{54m}}$

251. ⓐ $\frac{\sqrt{108p^5q^2}}{\sqrt{3p^3q^6}}$ ⓑ $\frac{\sqrt[3]{-16a^4b^{-2}}}{\sqrt[3]{2a^{-2}b}}$

252. ⓐ $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$ ⓑ $\frac{\sqrt[3]{-375y^4z^{-2}}}{\sqrt[3]{3y^{-2}z^4}}$

253. ⓐ $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$ ⓑ $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

254. ⓐ $\frac{\sqrt{810c^{-3}d^7}}{\sqrt{1000cd^{-1}}}$ ⓑ $\frac{\sqrt[3]{24a^7b^{-1}}}{\sqrt[3]{-81a^{-2}b^2}}$

255. $\frac{\sqrt{56x^5y^4}}{\sqrt{2xy^3}}$

256. $\frac{\sqrt{72a^3b^6}}{\sqrt{3ab^3}}$

257. $\frac{\sqrt[3]{48a^3b^6}}{\sqrt[3]{3a^{-1}b^3}}$

258. $\frac{\sqrt[3]{162x^{-3}y^6}}{\sqrt[3]{2x^3y^{-2}}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

259. ⓐ $\frac{10}{\sqrt{6}}$ ⓑ $\sqrt{\frac{4}{27}}$ ⓒ $\frac{10}{\sqrt{5x}}$

260. ⓐ $\frac{8}{\sqrt{3}}$ ⓑ $\sqrt{\frac{7}{40}}$ ⓒ $\frac{8}{\sqrt{2y}}$

261. ⓐ $\frac{6}{\sqrt{7}}$ ⓑ $\sqrt{\frac{8}{45}}$ ⓒ $\frac{12}{\sqrt{3p}}$

262. ⓐ $\frac{4}{\sqrt{5}}$ ⓑ $\sqrt{\frac{27}{80}}$ ⓒ $\frac{18}{\sqrt{6q}}$

263. ⓐ $\frac{1}{\sqrt[3]{5}}$ ⓑ $\sqrt[3]{\frac{5}{24}}$ ⓒ $\frac{4}{\sqrt[3]{36a}}$

264. ⓐ $\frac{1}{\sqrt[3]{3}}$ ⓑ $\sqrt[3]{\frac{5}{32}}$ ⓒ $\frac{7}{\sqrt[3]{49b}}$

265. ⓐ $\frac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\frac{7}{54}}$ ⓒ $\frac{3}{\sqrt[3]{3x^2}}$

266. ⓐ $\frac{1}{\sqrt[3]{13}}$ ⓑ $\sqrt[3]{\frac{3}{128}}$ ⓒ $\frac{3}{\sqrt[3]{6y^2}}$

267. ⓐ $\frac{1}{\sqrt[4]{7}}$ ⓑ $\sqrt[4]{\frac{5}{32}}$ ⓒ $\frac{4}{\sqrt[4]{4x^2}}$

268. ⓐ $\frac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\frac{9}{32}}$ ⓒ $\frac{6}{\sqrt[4]{9x^3}}$

269. ⓐ $\frac{1}{\sqrt[4]{9}}$ ⓑ $\sqrt[4]{\frac{25}{128}}$ ⓒ $\frac{6}{\sqrt[4]{27a}}$

270. ⓐ $\frac{1}{\sqrt[4]{8}}$ ⓑ $\sqrt[4]{\frac{27}{128}}$ ⓒ $\frac{16}{\sqrt[4]{64b^2}}$

Rationalize a Two Term Denominator

In the following exercises, simplify.

271. $\frac{8}{1-\sqrt{5}}$

272. $\frac{7}{2-\sqrt{6}}$

273. $\frac{6}{3-\sqrt{7}}$

274. $\frac{5}{4-\sqrt{11}}$

275. $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

276. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

277. $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

278. $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

279. $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

280. $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

281. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

282. $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Writing Exercises

283.

ⓐ Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

ⓑ Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

ⓒ Why are the two methods of simplifying square roots different?

284. Explain what is meant by the word rationalize in the phrase, "rationalize a denominator."

285. Explain why multiplying $\sqrt{2x}-3$ by its conjugate results in an expression with no radicals.

286. Explain why multiplying $\frac{7}{\sqrt[3]{x}}$ by $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$ does not rationalize the denominator.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No–I don't get it!
divide radical expressions.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

8.6 Solve Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
 If you missed this problem, review **Example 5.31**.
2. Solve: $2x-5=0$.
 If you missed this problem, review **Example 2.2**.
3. Solve $n^2-6n+8=0$.
 If you missed this problem, review **Example 6.45**.

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the n^{th} power. This will eliminate the radical.

$$\text{For } a \geq 0,\ (\sqrt[n]{a})^n = a.$$

EXAMPLE 8.56 HOW TO SOLVE A RADICAL EQUATION

Solve: $\sqrt{5n-4}-9=0$.

Solution

Step 1. Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Step 2. Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$(\sqrt{5n-4})^2=(9)^2$
Step 3. Solve the new equation.	Remember, $(\sqrt{a})^2=a$.	$5n-4=81$ $5n=85$ $n=17$

Step 4. Check the answer in the original equation.		Check the answer. $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9\stackrel{?}{=}0$ $0=0\checkmark$ The solution is $n=17$.

TRY IT : : 8.111 Solve: $\sqrt{3m+2}-5=0.$

TRY IT : : 8.112 Solve: $\sqrt{10z+1}-2=0.$

HOW TO : : SOLVE A RADICAL EQUATION WITH ONE RADICAL.

Step 1. Isolate the radical on one side of the equation.
Step 2. Raise both sides of the equation to the power of the index.
Step 3. Solve the new equation.
Step 4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

EXAMPLE 8.57

Solve: $\sqrt{9k-2}+1=0.$

Solution

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

TRY IT : : 8.113 Solve: $\sqrt{2r-3}+5=0.$

TRY IT : : 8.114 Solve: $\sqrt{7s-3}+2=0.$

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Binomial Squares

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

Don't forget the middle term!

EXAMPLE 8.58

Solve: $\sqrt{p-1}+1=p$.

Solution

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right. Then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the Zero Product Property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	

$p=1$: $\sqrt{p-1}+1=p$; $\sqrt{1-1}+1\stackrel{?}{=}1$; $\sqrt{0}+1\stackrel{?}{=}1$; $1=1$ ✓

$p=2$: $\sqrt{p-1}+1=p$; $\sqrt{2-1}+1\stackrel{?}{=}2$; $\sqrt{1}+1\stackrel{?}{=}2$; $2=2$ ✓

The solutions are $p=1,\quad p=2$.

TRY IT : : 8.115 Solve: $\sqrt{x-2}+2=x$.

TRY IT : : 8.116 Solve: $\sqrt{y-5}+5=y$.

When the index of the radical is 3, we cube both sides to remove the radical.

$$\left(\sqrt[3]{a}\right)^3 = a$$

EXAMPLE 8.59

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution

	$\sqrt[3]{5x+1}+8=4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1}=-4$
Cube both sides of the equation.	$\left(\sqrt[3]{5x+1}\right)^3=(-4)^3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	

$x=-13$

$\sqrt[3]{5x+1}+8=4$

$\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$

$\sqrt[3]{-64}+8\stackrel{?}{=}4$

$-4+8\stackrel{?}{=}4$

$4=4\checkmark$

The solution is $x=-13$.

TRY IT : : 8.117 Solve: $\sqrt[3]{4x-3}+8=5$

TRY IT : : 8.118 Solve: $\sqrt[3]{6x-10}+1=-3$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n=a^{m\cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2=x,\quad \left(x^{\frac{1}{3}}\right)^3=x$$

Remember, $x^{\frac{1}{2}}=\sqrt{x}$ and $x^{\frac{1}{3}}=\sqrt[3]{x}$.

EXAMPLE 8.60

Solve: $(3x-2)^{\frac{1}{4}}+3=5$.

Solution

$(3x-2)^{\frac{1}{4}}+3=5$

To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}}=2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4=(2)^4$
Simplify.	$3x-2=16$
Solve the equation.	$3x=18$
	$x=6$
Check the answer.	
$x=6$ $(3x-2)^{\frac{1}{4}}+3=5$ $(3\cdot 6-2)^{\frac{1}{4}}+3\stackrel{?}{=}5$ $(16)^{\frac{1}{4}}+3\stackrel{?}{=}5$ $2+3\stackrel{?}{=}5$ $5=5$ ✓	
	The solution is $x=6$.

TRY IT : : 8.119

Solve: $(9x+9)^{\frac{1}{4}}-2=1.$

TRY IT : : 8.120

Solve: $(4x-8)^{\frac{1}{4}}+5=7.$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an extraneous solution!

EXAMPLE 8.61

Solve: $\sqrt{r+4}-r+2=0.$

Solution

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$\left(\sqrt{r+4}\right)^2=(r-2)^2$
Simplify and then solve the equation	$r+4=r^2-4r+4$
It is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r \quad 0=r-5$

Solve the equation.	$r=0 \quad r=5$
Check your answer.	

$r=0$, $\sqrt{r+4}-r+2=0$; $\sqrt{0+4}-0+2\stackrel{?}{=}0$; $\sqrt{4}+2\stackrel{?}{=}0$; $4\neq 0$

$r=5$, $\sqrt{r+4}-r+2=0$; $\sqrt{5+4}-5+2\stackrel{?}{=}0$; $\sqrt{9}-3\stackrel{?}{=}0$; $0=0$ ✓

The solution is $r = 5$.

$r=0$ is an extraneous solution.

TRY IT : : 8.121 Solve: $\sqrt{m+9}-m+3=0$.

TRY IT : : 8.122 Solve: $\sqrt{n+1}-n+1=0$.

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

EXAMPLE 8.62

Solve: $3\sqrt{3x-5}-8=4$.

Solution

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$
Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$(\sqrt{3x-5})^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	

$x=7$, $3\sqrt{3x-5}-8=4$; $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$; $3\sqrt{21-5}-8\stackrel{?}{=}4$; $3\sqrt{16}-8\stackrel{?}{=}4$; $3(4)-8\stackrel{?}{=}4$; $4=4$ ✓

The solution is $x = 7$.

TRY IT : : 8.123 Solve: $2\sqrt{4a+4}-16=16$.

TRY IT : : 8.124 Solve: $3\sqrt{2b+3}-25=50$.

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

EXAMPLE 8.63

Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$.

Solution

The radical terms are isolated.	$\sqrt[3]{4x-3} = \sqrt[3]{3x+2}$
Since the index is 3, cube both sides of the equation.	$\left(\sqrt[3]{4x-3}\right)^3 = \left(\sqrt[3]{3x+2}\right)^3$
Simplify, then solve the new equation.	$4x-3 = 3x+2$
	$x-3 = 2$
	$x = 5$
	The solution is $x = 5$.

Check the answer.
We leave it to you to show that 5 checks!

TRY IT : : 8.125 Solve: $\sqrt[3]{5x-4}=\sqrt[3]{2x+5}$.

TRY IT : : 8.126 Solve: $\sqrt[3]{7x+1}=\sqrt[3]{2x-5}$.

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

EXAMPLE 8.64 HOW TO SOLVE A RADICAL EQUATION

Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Solution

Step 1. Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2. Raise both sides of the equation to the power of the index.	We square both sides.	$(\sqrt{m}+1)^2=(\sqrt{m+9})^2$
	Simplify—be very careful as you multiply!	
Step 3. Are there any more radicals?	There is still a radical in the equation.	$m+2\sqrt{m}+1=m+9$
If yes, repeat Step 1 and Step 2 again.	So we must repeat the previous steps. Isolate the radical term.	
	Here, we can easily isolate the radical by dividing both sides by 2.	$2\sqrt{m}=8$
If no, solve the new equation.	Square both sides.	$\sqrt{m}=4$
		$(\sqrt{m})^2=(4)^2$
		$m=16$

Step 4. Check the answer in the original equation.		$\sqrt{m}+1=\sqrt{m+9}$ $\sqrt{16}+1\stackrel{?}{=}\sqrt{16+9}$ $4+1\stackrel{?}{=}5$ $5=5\checkmark$ The solution is $m=16$.

TRY IT : : 8.127 Solve: $3-\sqrt{x}=\sqrt{x-3}$.

TRY IT : : 8.128 Solve: $\sqrt{x}+2=\sqrt{x+16}$.

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

HOW TO : : SOLVE A RADICAL EQUATION.

Step 1. Isolate one of the radical terms on one side of the equation.
Step 2. Raise both sides of the equation to the power of the index.
Step 3. Are there any more radicals?
If yes, repeat Step 1 and Step 2 again.
If no, solve the new equation.
Step 4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern is $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

EXAMPLE 8.65

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$\left(6\sqrt{q-2}\right)^2=\left(3q-6\right)^2$ $6^2\left(\sqrt{q-2}\right)^2=(3q)^2-2\cdot 3q\cdot 6+6^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$

Distribute.	$36q - 72 = 9q^2 - 36q + 36$
It is a quadratic equation, so get zero on one side.	$0 = 9q^2 - 72q + 108$
Factor the right side.	$0 = 9(q^2 - 8q + 12)$ $0 = 9(q - 6)(q - 2)$
Use the Zero Product Property.	$q - 6 = 0 \quad q - 2 = 0$ $q = 6 \quad q = 2$
The checks are left to you.	The solutions are $q = 6$ and $q = 2$.

TRY IT : : 8.129 Solve: $\sqrt{x-1} + 2 = \sqrt{2x+6}$

TRY IT : : 8.130 Solve: $\sqrt{x} + 2 = \sqrt{3x+4}$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

HOW TO : : USE A PROBLEM SOLVING STRATEGY FOR APPLICATIONS WITH FORMULAS.

Step 1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve the equation** using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Falling Objects

On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula

$$t = \frac{\sqrt{h}}{4}.$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

	$t=\frac{\sqrt{h}}{4}$
	$t=\frac{\sqrt{64}}{4}$
Take the square root of 64.	$t=\frac{8}{4}$
Simplify the fraction.	$t=2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

EXAMPLE 8.66

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for.	the time it takes for the sunglasses to reach the river
Step 3. Name what we are looking.	Let $t=$ time.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Step 5. Solve the equation.	$t=\frac{20}{4}$
	$t=5$
Step 6. Check the answer in the problem and make sure it makes sense.	$5\stackrel{?}{=}\frac{\sqrt{400}}{4}$ $5\stackrel{?}{=}\frac{20}{4}$ $5=5$ ✓
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7. Answer the question.	It will take 5 seconds for the sunglasses to reach the river.

TRY IT : : 8.131

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

TRY IT : : 8.132

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Skid Marks and Speed of a Car

If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

EXAMPLE 8.67

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution

Step 1. Read the problem	
Step 2. Identify what we are looking for.	the speed of a car
Step 3. Name what weare looking for,	Let $s =$ the speed.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5. Solve the equation.	$s = \sqrt{4,560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \overset{?}{\approx} \sqrt{24(190)}$ $67.5 \overset{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277...$ ✓
	The speed of the car before the brakes were applied was 67.5 miles per hour.

TRY IT : : 8.133

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

TRY IT : : 8.134

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

MEDIA : :

Access these online resources for additional instruction and practice with solving radical equations.

- **Solving an Equation Involving a Single Radical (https://openstax.org/l/37RadEquat1)**
- **Solving Equations with Radicals and Rational Exponents (https://openstax.org/l/37RadEquat2)**
- **Solving Radical Equations (https://openstax.org/l/37RadEquat3)**
- **Solve Radical Equations (https://openstax.org/l/37RadEquat4)**
- **Radical Equation Application (https://openstax.org/l/37RadEquat5)**

8.6 EXERCISES

Practice Makes Perfect

Solve Radical Equations

In the following exercises, solve.

287. $\sqrt{5x-6}=8$

288. $\sqrt{4x-3}=7$

289. $\sqrt{5x+1}=-3$

290. $\sqrt{3y-4}=-2$

291. $\sqrt[3]{2x}=-2$

292. $\sqrt[3]{4x-1}=3$

293. $\sqrt{2m-3}-5=0$

294. $\sqrt{2n-1}-3=0$

295. $\sqrt{6v-2}-10=0$

296. $\sqrt{12u+1}-11=0$

297. $\sqrt{4m+2}+2=6$

298. $\sqrt{6n+1}+4=8$

299. $\sqrt{2u-3}+2=0$

300. $\sqrt{5v-2}+5=0$

301. $\sqrt{u-3}-3=u$

302. $\sqrt{v-10}+10=v$

303. $\sqrt{r-1}=r-1$

304. $\sqrt{s-8}=s-8$

305. $\sqrt[3]{6x+4}=4$

306. $\sqrt[3]{11x+4}=5$

307. $\sqrt[3]{4x+5}-2=-5$

308. $\sqrt[3]{9x-1}-1=-5$

309. $(6x+1)^{\frac{1}{2}}-3=4$

310. $(3x-2)^{\frac{1}{2}}+1=6$

311. $(8x+5)^{\frac{1}{3}}+2=-1$

312. $(12x-5)^{\frac{1}{3}}+8=3$

313. $(12x-3)^{\frac{1}{4}}-5=-2$

314. $(5x-4)^{\frac{1}{4}}+7=9$

315. $\sqrt{x+1}-x+1=0$

316. $\sqrt{y+4}-y+2=0$

317. $\sqrt{z+100}-z=-10$

318. $\sqrt{w+25}-w=-5$

319. $3\sqrt{2x-3}-20=7$

320. $2\sqrt{5x+1}-8=0$

321. $2\sqrt{8r+1}-8=2$

322. $3\sqrt{7y+1}-10=8$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

323. $\sqrt{3u+7}=\sqrt{5u+1}$

324. $\sqrt{4v+1}=\sqrt{3v+3}$

325. $\sqrt{8+2r}=\sqrt{3r+10}$

326. $\sqrt{10+2c}=\sqrt{4c+16}$

327. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$

328. $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$

329. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

330. $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$

331. $\sqrt{a}+2=\sqrt{a+4}$

332. $\sqrt{r}+6=\sqrt{r+8}$

333. $\sqrt{u}+1=\sqrt{u+4}$

334. $\sqrt{x}+1=\sqrt{x+2}$

335. $\sqrt{a+5}-\sqrt{a}=1$

336. $-2=\sqrt{d-20}-\sqrt{d}$

337. $\sqrt{2x+1}=1+\sqrt{x}$

338. $\sqrt{3x+1}=1+\sqrt{2x-1}$

339. $\sqrt{2x-1}-\sqrt{x-1}=1$

340. $\sqrt{x+1}-\sqrt{x-2}=1$

341. $\sqrt{x+7}-\sqrt{x-5}=2$

342. $\sqrt{x+5}-\sqrt{x-3}=2$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

343. Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

344. Landscaping Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

345. Gravity A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.

346. Gravity A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

347. Accident investigation The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

348. Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Writing Exercises

349. Explain why an equation of the form $\sqrt{x}+1=0$ has no solution.

350.

ⓐ Solve the equation $\sqrt{r+4}-r+2=0$.

ⓑ Explain why one of the "solutions" that was found was not actually a solution to the equation.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No–I don't get it!
solve radical equations.			
solve radical equations with two radicals.			
use radicals in applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

8.7 Use Radicals in Functions

Learning Objectives

By the end of this section, you will be able to:

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $1-2x \geq 0$.
 If you missed this problem, review **Example 2.50**.
2. For $f(x)=3x-4$, evaluate $f(2)$, $f(-1)$, $f(0)$.
 If you missed this problem, review **Example 3.48**.
3. Graph $f(x)=\sqrt{x}$. State the domain and range of the function in interval notation.
 If you missed this problem, review **Example 3.56**.

Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

The square root function is $f(x)=\sqrt{x}$.

The cube root function is $f(x)=\sqrt[3]{x}$.

Radical Function

A **radical function** is a function that is defined by a radical expression.

To evaluate a radical function, we find the value of *f*(*x*) for a given value of *x* just as we did in our previous work with functions.

EXAMPLE 8.68

For the function $f(x)=\sqrt{2x-1}$, find ⓐ $f(5)$ ⓑ $f(-2)$.

Solution

ⓐ

	$f(x) = \sqrt{2x-1}$
To evaluate $f(5)$, substitute 5 for x.	$f(5) = \sqrt{2\cdot 5-1}$
Simplify.	$f(5) = \sqrt{9}$
Take the square root.	$f(5) = 3$

ⓑ

	$f(x) = \sqrt{2x-1}$
To evaluate $f(-2)$, substitute -2 for x.	$f(-2) = \sqrt{2(-2)-1}$
Simplify.	$f(-2) = \sqrt{-5}$

Since the square root of a negative number is not a real number, the function does not have a value at $x=-2$.

TRY IT : : 8.135 For the function $f(x)=\sqrt{3x-2}$, find ⓐ $f(6)$ ⓑ $f(0)$.

TRY IT : : 8.136 For the function $g(x) = \sqrt{5x+5}$, find ⓐ $g(4)$ ⓑ $g(-3)$.

We follow the same procedure to evaluate cube roots.

EXAMPLE 8.69

For the function $g(x) = \sqrt[3]{x-6}$, find ⓐ $g(14)$ ⓑ $g(-2)$.

Solution

ⓐ

$$g(x) = \sqrt[3]{x-6}$$

To evaluate $g(14)$, substitute 14 for x. $g(14) = \sqrt[3]{14-6}$

Simplify. $g(14) = \sqrt[3]{8}$

Take the cube root. $g(14) = 2$

ⓑ

$$g(x) = \sqrt[3]{x-6}$$

To evaluate $g(-2)$, substitute -2 for x. $g(-2) = \sqrt[3]{-2-6}$

Simplify. $g(-2) = \sqrt[3]{-8}$

Take the cube root. $g(-2) = -2$

TRY IT : : 8.137 For the function $g(x) = \sqrt[3]{3x-4}$, find ⓐ $g(4)$ ⓑ $g(1)$.

TRY IT : : 8.138 For the function $h(x) = \sqrt[3]{5x-2}$, find ⓐ $h(2)$ ⓑ $h(-5)$.

The next example has fourth roots.

EXAMPLE 8.70

For the function $f(x) = \sqrt[4]{5x-4}$, find ⓐ $f(4)$ ⓑ $f(-12)$

Solution

ⓐ

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(4)$, substitute 4 for x. $f(4) = \sqrt[4]{5\cdot 4-4}$

Simplify. $f(4) = \sqrt[4]{16}$

Take the fourth root. $f(4) = 2$

ⓑ

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(-12)$, substitute -12 for x. $f(-12) = \sqrt[4]{5(-12)-4}$

Simplify. $f(-12) = \sqrt[4]{-64}$

Since the fourth root of a negative number is not a real number, the function does not have a value at $x = -12$.

TRY IT : : 8.139 For the function $f(x)=\sqrt[4]{3x+4}$, find ⓐ $f(4)$ ⓑ $f(-1)$.

TRY IT : : 8.140 For the function $g(x)=\sqrt[4]{5x+1}$, find ⓐ $g(16)$ ⓑ $g(3)$.

Find the Domain of a Radical Function

To find the domain and range of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Properties of $\sqrt[n]{a}$

When n is an **even** number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

Domain of a Radical Function

When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
When the **index** of the radical is **odd**, the radicand can be any real number.

EXAMPLE 8.71

Find the domain of the function, $f(x)=\sqrt{3x-4}$. Write the domain in interval notation.

Solution

Since the function, $f(x)=\sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

$$\begin{array}{rrcl} & 3x-4 & \geq & 0 \\ \text{Solve.} & 3x & \geq & 4 \\ & x & \geq & \frac{4}{3} \end{array}$$

The domain of $f(x)=\sqrt{3x-4}$ is all values $x \geq \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

TRY IT : : 8.141 Find the domain of the function, $f(x)=\sqrt{6x-5}$. Write the domain in interval notation.

TRY IT : : 8.142 Find the domain of the function, $f(x)=\sqrt{4-5x}$. Write the domain in interval notation.

EXAMPLE 8.72

Find the domain of the function, $g(x)=\sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution

Since the function, $g(x)=\sqrt{\frac{6}{x-1}}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

The radicand cannot be zero since the numerator is not zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive since the numerator is positive. We know a positive divided by a positive is positive.

We set $x-1>0$ and solve.

$$\begin{array}{rrcl} & x-1 & > & 0 \\ \text{Solve.} & x & > & 1 \end{array}$$

Also, since the radicand is a fraction, we must realize that the denominator cannot be zero.

We solve $x-1=0$ to find the value that must be eliminated from the domain.

$$\begin{array}{rrcl} & x-1 & = & 0 \\ \text{Solve.} & x & = & 1 \text{ so } x \neq 1 \text{ in the domain.} \end{array}$$

Putting this together we get the domain is $x>1$ and we write it as $(1, \infty)$.

TRY IT : : 8.143 Find the domain of the function, $f(x)=\sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

TRY IT : : 8.144 Find the domain of the function, $h(x)=\sqrt{\frac{9}{x-5}}$. Write the domain in interval notation.

The next example involves a cube root and so will require different thinking.

EXAMPLE 8.73

Find the domain of the function, $f(x)=\sqrt[3]{2x^2+3}$. Write the domain in interval notation.

Solution

Since the function, $f(x)=\sqrt[3]{2x^2+3}$ has a radical with an index of 3, which is odd, we know the radicand can be any real number. This tells us the domain is any real number. In interval notation, we write $(-\infty, \infty)$.

The domain of $f(x)=\sqrt[3]{2x^2+3}$ is all real numbers and we write it in interval notation as $(-\infty, \infty)$.

TRY IT : : 8.145 Find the domain of the function, $f(x)=\sqrt[3]{3x^2-1}$. Write the domain in interval notation.

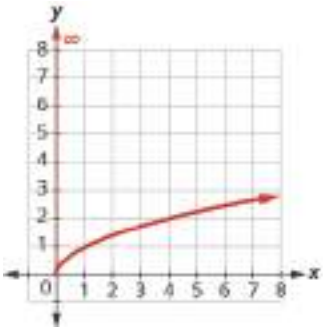
TRY IT : : 8.146 Find the domain of the function, $g(x)=\sqrt[3]{5x-4}$. Write the domain in interval notation.

Graph Radical Functions

Before we graph any radical function, we first find the domain of the function. For the function, $f(x)=\sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x \geq 0$ and we write this in interval notation as $[0, \infty)$.

Previously we used point plotting to graph the function, $f(x)=\sqrt{x}$. We chose *x*-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.



x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Once we see the graph, we can find the range of the function. The *y*-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

EXAMPLE 8.74

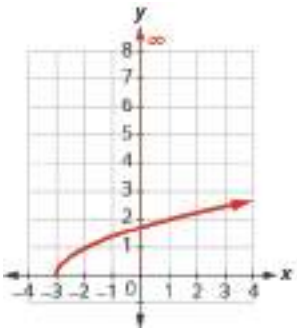
For the function $f(x) = \sqrt{x+3}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution

ⓐ Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x + 3 \geq 0$, then $x \geq -3$. This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty)$.

ⓑ To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.



x	$f(x)=\sqrt{x+3}$	$(x, f(x))$
−3	0	(−3, 0)
−2	1	(−2, 1)
1	2	(1, 2)
6	3	(6, 3)

ⓒ Looking at the graph, we see the *y*-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

TRY IT : : 8.147

For the function $f(x) = \sqrt{x+2}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

TRY IT : : 8.148

For the function $f(x) = \sqrt{x-2}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

In our previous work graphing functions, we graphed $f(x) = x^3$ but we did not graph the function $f(x) = \sqrt[3]{x}$. We will do this now in the next example.

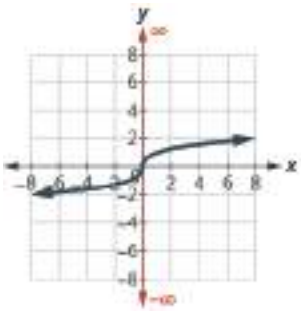
EXAMPLE 8.75

For the function $f(x) = \sqrt[3]{x}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution

ⓐ Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as $(-\infty, \infty)$

ⓑ To graph the function, we choose points in the interval $(-\infty, \infty)$ that will also give us a radicand which will be easy to take the cube root.



x	$f(x) = \sqrt[3]{x}$	$(x, f(x))$
−8	−2	(−8, −2)
−1	−1	(−1, −1)
0	0	(0, 0)
1	1	(1, 1)
8	2	(8, 2)

ⓒ Looking at the graph, we see the *y*-values of the function are all real numbers. The range then is $(-\infty, \infty)$.

TRY IT : : 8.149 For the function $f(x) = -\sqrt[3]{x}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

TRY IT : : 8.150 For the function $f(x) = \sqrt[3]{x-2}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

MEDIA : :

Access these online resources for additional instruction and practice with radical functions.

- **Domain of a Radical Function (https://openstax.org/l/37RadFuncDom1)**
- **Domain of a Radical Function 2 (https://openstax.org/l/37RadFuncDom2)**
- **Finding Domain of a Radical Function (https://openstax.org/l/37RadFuncDom3)**

8.7 EXERCISES

Practice Makes Perfect

Evaluate a Radical Function

In the following exercises, evaluate each function.

351. $f(x)=\sqrt{4x-4}$, find
ⓐ $f(5)$
ⓑ $f(0)$.

352. $f(x)=\sqrt{6x-5}$, find
ⓐ $f(5)$
ⓑ $f(-1)$.

353. $g(x)=\sqrt{6x+1}$, find
ⓐ $g(4)$
ⓑ $g(8)$.

354. $g(x)=\sqrt{3x+1}$, find
ⓐ $g(8)$
ⓑ $g(5)$.

355. $F(x)=\sqrt{3-2x}$, find
ⓐ $F(1)$
ⓑ $F(-11)$.

356. $F(x)=\sqrt{8-4x}$, find
ⓐ $F(1)$
ⓑ $F(-2)$.

357. $G(x)=\sqrt{5x-1}$, find
ⓐ $G(5)$
ⓑ $G(2)$.

358. $G(x)=\sqrt{4x+1}$, find
ⓐ $G(11)$
ⓑ $G(2)$.

359. $g(x)=\sqrt[3]{2x-4}$, find
ⓐ $g(6)$
ⓑ $g(-2)$.

360. $g(x)=\sqrt[3]{7x-1}$, find
ⓐ $g(4)$
ⓑ $g(-1)$.

361. $h(x)=\sqrt[3]{x^2-4}$, find
ⓐ $h(-2)$
ⓑ $h(6)$.

362. $h(x)=\sqrt[3]{x^2+4}$, find
ⓐ $h(-2)$
ⓑ $h(6)$.

363. For the function $f(x)=\sqrt[4]{2x^3}$, find
ⓐ $f(0)$
ⓑ $f(2)$.

364. For the function $f(x)=\sqrt[4]{3x^3}$, find
ⓐ $f(0)$
ⓑ $f(3)$.

365. For the function $g(x)=\sqrt[4]{4-4x}$, find
ⓐ $g(1)$
ⓑ $g(-3)$.

366. For the function $g(x)=\sqrt[4]{8-4x}$, find
ⓐ $g(-6)$
ⓑ $g(2)$.

Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

367. $f(x)=\sqrt{3x-1}$

368. $f(x)=\sqrt{4x-2}$

369. $g(x)=\sqrt{2-3x}$

370. $g(x)=\sqrt{8-x}$

371. $h(x)=\sqrt{\frac{5}{x-2}}$

372. $h(x)=\sqrt{\frac{6}{x+3}}$

373. $f(x)=\sqrt{\frac{x+3}{x-2}}$

374. $f(x)=\sqrt{\frac{x-1}{x+4}}$

375. $g(x)=\sqrt[3]{8x-1}$

376. $g(x)=\sqrt[3]{6x+5}$

377. $f(x)=\sqrt[3]{4x^2-16}$

378. $f(x)=\sqrt[3]{6x^2-25}$

379. $F(x) = \sqrt[4]{8x+3}$ **380.** $F(x) = \sqrt[4]{10-7x}$ **381.** $G(x) = \sqrt[5]{2x-1}$

382. $G(x) = \sqrt[5]{6x-3}$

Graph Radical Functions

In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

383. $f(x) = \sqrt{x+1}$ **384.** $f(x) = \sqrt{x-1}$ **385.** $g(x) = \sqrt{x+4}$

386. $g(x) = \sqrt{x-4}$ **387.** $f(x) = \sqrt{x}+2$ **388.** $f(x) = \sqrt{x}-2$

389. $g(x) = 2\sqrt{x}$ **390.** $g(x) = 3\sqrt{x}$ **391.** $f(x) = \sqrt{3-x}$

392. $f(x) = \sqrt{4-x}$ **393.** $g(x) = -\sqrt{x}$ **394.** $g(x) = -\sqrt{x}+1$

395. $f(x) = \sqrt[3]{x+1}$ **396.** $f(x) = \sqrt[3]{x-1}$ **397.** $g(x) = \sqrt[3]{x+2}$

398. $g(x) = \sqrt[3]{x-2}$ **399.** $f(x) = \sqrt[3]{x}+3$ **400.** $f(x) = \sqrt[3]{x}-3$

401. $g(x) = \sqrt[3]{x}$ **402.** $g(x) = -\sqrt[3]{x}$ **403.** $f(x) = 2\sqrt[3]{x}$

404. $f(x) = -2\sqrt[3]{x}$

Writing Exercises

405. Explain how to find the domain of a fourth root function.

406. Explain how to find the domain of a fifth root function.

407. Explain why $y = \sqrt[3]{x}$ is a function.

408. Explain why the process of finding the domain of a radical function with an even index is different from the process when the index is odd.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No–I don't get it!
evaluate a radical function.			
find the domain of a radical function.			
graph a radical function.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

8.8 Use the Complex Number System

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Be Prepared!

Before you get started, take this readiness quiz.

1. Given the numbers $-4,\ -\sqrt{7},\ 0.\overline{5}\ ,\ \frac{7}{3},\ 3,\ \sqrt{81},$ list the ⓐ rational numbers, ⓑ irrational numbers, ⓒ real numbers.
 If you missed this problem, review **Example 1.42**.
2. Multiply: $(x-3)(2x+5)$.
 If you missed this problem, review **Example 5.28**.
3. Rationalize the denominator: $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$.
 If you missed this problem, review **Example 5.32**.

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1},$ we are looking for a real number x so that $x^2 = -1$. Since all real numbers squared are positive numbers, there is no real number that equals –1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is –1.

Imaginary Unit

The **imaginary unit** i is the number whose square is –1.

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

We will use the imaginary unit to simplify the square roots of negative numbers.

Square Root of a Negative Number

If b is a positive real number, then

$$\sqrt{-b} = \sqrt{b}\,i$$

We will use this definition in the next example. Be careful that it is clear that the i is not under the radical. Sometimes you will see this written as $\sqrt{-b} = i\sqrt{b}$ to emphasize the i is not under the radical. But the $\sqrt{-b} = \sqrt{b}\,i$ is considered standard form.

EXAMPLE 8.76

Write each expression in terms of i and simplify if possible:

ⓐ $\sqrt{-25}$ ⓑ $\sqrt{-7}$ ⓒ $\sqrt{-12}$.

Solution

ⓐ

	$\sqrt{-25}$
Use the definition of he square root of negative numbers.	$\sqrt{25}\,i$
Simplify.	$5i$

ⓑ

	$\sqrt{-7}$
Use the definition of he square root of negative numbers.	$\sqrt{7}i$
Simplify.	Be careful that it is clear that i is not under the radical sign.

ⓒ

	$\sqrt{-12}$
Use the definition of he square root of negative numbers.	$\sqrt{12}\,i$
Simplify $\sqrt{12}$.	$2\sqrt{3}\,i$

TRY IT : : 8.151 Write each expression in terms of i and simplify if possible:

ⓐ $\sqrt{-81}$ ⓑ $\sqrt{-5}$ ⓒ $\sqrt{-18}$.

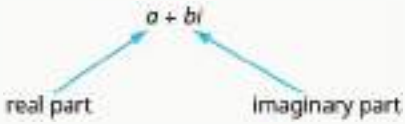
TRY IT : : 8.152 Write each expression in terms of i and simplify if possible:

ⓐ $\sqrt{-36}$ ⓑ $\sqrt{-3}$ ⓒ $\sqrt{-27}$.

Now that we are familiar with the imaginary number i, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form $a + bi$, where a, b are real numbers. We call a the real part and b the imaginary part.

Complex Number

A **complex number** is of the form $a + bi$, where a and b are real numbers.



A complex number is in standard form when written as $a + bi$, where a and b are real numbers.

If $b = 0$, then $a + bi$ becomes $a + 0 \cdot i = a$, and is a real number.

If $b \neq 0$, then $a + bi$ is an imaginary number.

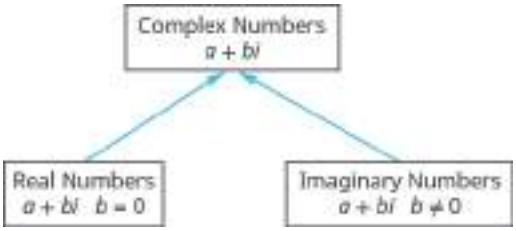
If $a = 0$, then $a + bi$ becomes $0 + bi = bi$, and is called a pure imaginary number.

We summarize this here.

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bi$ bi	Pure imaginary number

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $\sqrt{-b}=\sqrt{b}i$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.



Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

EXAMPLE 8.77

Add: $\sqrt{-12}+\sqrt{-27}$.

Solution

	$\sqrt{-12}+\sqrt{-27}$
Use the definition of he square root of negative numbers.	$\sqrt{12}\,i+\sqrt{27}\,i$
Simplify the square roots.	$2\sqrt{3}\,i+3\sqrt{3}\,i$
Add.	$5\sqrt{3}\,i$

TRY IT : : 8.153 Add: $\sqrt{-8}+\sqrt{-32}$.

TRY IT : : 8.154 Add: $\sqrt{-27}+\sqrt{-48}$.

Remember to add both the real parts and the imaginary parts in this next example.

EXAMPLE 8.78

Simplify: ⓐ $(4-3i)+(5+6i)$ ⓑ $(2-5i)-(5-2i)$.

Solution

ⓐ

	$(4-3i)+(5+6i)$
Use the Associative Property to put the real parts and the imaginary parts together.	$(4+5)+(-3i+6i)$
Simplify.	$9+3i$

ⓑ

	$(2-5i)-(5-2i)$
Distribute.	$2-5i-5+2i$
Use the Associative Property to put the real parts and the imaginary parts together.	$2-5-5i+2i$
Simplify.	$-3-3i$

TRY IT : : 8.155 Simplify: ⓐ $(2+7i)+(4-2i)$ ⓑ $(8-4i)-(2-i)$.

TRY IT : : 8.156 Simplify: ⓐ $(3-2i)+(-5-4i)$ ⓑ $(4+3i)-(2-6i)$.

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

EXAMPLE 8.79

Multiply: $2i(7-5i)$.

Solution

	$2i(7-5i)$
Distribute.	$14i-10i^2$
Simplify i^2.	$14i-10(-1)$
Multiply.	$14i+10$
Write in standard form.	$10+14i$

TRY IT : : 8.157 Multiply: $4i(5-3i)$.

TRY IT : : 8.158 Multiply: $-3i(2+4i)$.

In the next example, we multiply the binomials using the Distributive Property or FOIL.

EXAMPLE 8.80

Multiply: $(3+2i)(4-3i)$.

Solution

	$(3+2i)(4-3i)$
Use FOIL.	$12-9i+8i-6i^2$
Simplify i^2 and combine like terms.	$12-i-6(-1)$
Multiply.	$12-i+6$
Combine the real parts.	$18-i$

TRY IT : : 8.159 Multiply: $(5-3i)(-1-2i)$.

TRY IT : : 8.160 Multiply: $(-4-3i)(2+i)$.

In the next example, we could use FOIL or the Product of Binomial Squares Pattern.

EXAMPLE 8.81

Multiply: $(3+2i)^2$

Solution

	$(a+b)^2$ $(3+2i)^2$
Use the Product of Binomial Squares Pattern, $(a+b)^2=a^2+2ab+b^2$.	$3^2+2\cdot3\cdot2i+(2i)^2$
Simplify.	$9+12i+4i^2$
Simplify i^2.	$9+12i+4(-1)$
Simplify.	$5+12i$

TRY IT : : 8.161 Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

TRY IT : : 8.162 Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b}=\sqrt{b}i$. This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

EXAMPLE 8.82

Multiply: $\sqrt{-36}\cdot\sqrt{-4}$.

Solution

To multiply square roots of negative numbers, we first write them as complex numbers.

	$\sqrt{-36}\cdot\sqrt{-4}$
Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.	$\sqrt{36}\,i\cdot\sqrt{4}\,i$
Simplify.	$6i\cdot 2i$
Multiply.	$12i^2$
Simplify i^2 and multiply.	-12

TRY IT : : 8.163 Multiply: $\sqrt{-49}\cdot\sqrt{-4}$.

TRY IT : : 8.164 Multiply: $\sqrt{-36}\cdot\sqrt{-81}$.

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

EXAMPLE 8.83

Multiply: $(3-\sqrt{-12})(5+\sqrt{-27})$.

Solution

To multiply square roots of negative numbers, we first write them as complex numbers.

	$(3-\sqrt{-12})(5+\sqrt{-27})$
Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.	$(3-2\sqrt{3}\,i)(5+3\sqrt{3}\,i)$
Use FOIL.	$15+9\sqrt{3}\,i-10\sqrt{3}\,i-6\cdot 3i^2$
Combine like terms and simplify i^2.	$15-\sqrt{3}\,i-6\cdot(-3)$
Multiply and combine like terms.	$33-\sqrt{3}\,i$

TRY IT : : 8.165 Multiply: $(4-\sqrt{-12})(3-\sqrt{-48})$.

TRY IT : : 8.166 Multiply: $(-2+\sqrt{-8})(3-\sqrt{-18})$.

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b)$, $(a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Complex Conjugate Pair

A **complex conjugate pair** is of the form $a+bi$, $a-bi$.

We will multiply a complex conjugate pair in the next example.

EXAMPLE 8.84

Multiply: $(3-2i)(3+2i)$.

Solution

	$(3-2i)(3+2i)$
Use FOIL.	$9+6i-6i-4i^2$
Combine like terms and simplify i^2.	$9-4(-1)$
Multiply and combine like terms.	13

TRY IT : : 8.167 Multiply: $(4-3i)\cdot(4+3i)$.

TRY IT : : 8.168 Multiply: $(-2+5i)\cdot(-2-5i)$.

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$. The result is called a difference of squares. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the Product of Conjugates Pattern.

$$(3-2i)(3+2i)$$

$$3^2-(2i)^2$$

$$9-4i^2$$

$$9-4(-1)$$

$$13$$

Notice this is the same result we found in **Example 8.84**.

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)(a+bi)=a^2+b^2$

Product of Complex Conjugates

If a and b are real numbers, then

$$(a-bi)(a+bi)=a^2+b^2$$

EXAMPLE 8.85

Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

Solution

	$(a-b)(a+b)$ $(8-2i)(8+2i)$
Use the Product of Complex Conjugates Pattern, $(a-bi)(a+bi)=a^2+b^2$.	a^2+b^2 8^2+2^2
Simplify the squares.	$64+4$
Add.	68

TRY IT : : 8.169 Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

TRY IT : : 8.170 Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

EXAMPLE 8.86 HOW TO DIVIDE COMPLEX NUMBERS

Divide: $\frac{4+3i}{3-4i}$.

Solution

Step 1. Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$
Step 3. Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator.	$\frac{12+16i+9i+12i^2}{9+16}$
	Combine like terms.	$\frac{12+25i-12}{25}$
	Simplify.	$\frac{25i}{25}$
	Write the result in standard form.	i

TRY IT : : 8.171 Divide: $\frac{2+5i}{5-2i}$.

TRY IT : : 8.172 Divide: $\frac{1+6i}{6-i}$.

We summarize the steps here.

HOW TO : : HOW TO DIVIDE COMPLEX NUMBERS.

Step 1. Write both the numerator and denominator in standard form.
Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.
Step 3. Simplify and write the result in standard form.

EXAMPLE 8.87

Divide, writing the answer in standard form: $\frac{-3}{5+2i}$.

Solution

$$\frac{-3}{5+2i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator. $\frac{-3(5-2i)}{(5+2i)(5-2i)}$

Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator. $\frac{-15+6i}{5^2+2^2}$

Simplify. $\frac{-15+6i}{29}$

Write in standard form. $-\frac{15}{29}+\frac{6}{29}i$

TRY IT : : 8.173 Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

TRY IT : : 8.174 Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Be careful as you find the conjugate of the denominator.

EXAMPLE 8.88

Divide: $\frac{5+3i}{4i}$.

Solution

	$\frac{5+3i}{4i}$
Write the denominator in standard form.	$\frac{5+3i}{0+4i}$
Multiply the numerator and denominator by the complex conjugate of the denominator.	$\frac{(5+3i)(0-4i)}{(0+4i)(0-4i)}$
Simplify.	$\frac{(5+3i)(-4i)}{(4i)(-4i)}$
Multiply.	$\frac{-20i-12i^2}{-16i^2}$
Simplify the i^2.	$\frac{-20i+12}{16}$
Rewrite in standard form.	$\frac{12}{16}-\frac{20}{16}i$
Simplify the fractions.	$\frac{3}{4}-\frac{5}{4}i$

TRY IT : : 8.175 Divide: $\frac{3+3i}{2i}$.

TRY IT : : 8.176 Divide: $\frac{2+4i}{5i}$.

Simplify Powers of *i*

The powers of i make an interesting pattern that will help us simplify higher powers of *i*. Let's evaluate the powers of i to see the pattern.

i^1	i^2	i^3	i^4
i	-1	$i^2 \cdot i$	$i^2 \cdot i^2$
		$-1 \cdot i$	$(-1)(-1)$
		$-i$	1

i^5	i^6	i^7	i^8
$i^4 \cdot i$	$i^4 \cdot i^2$	$i^4 \cdot i^3$	$i^4 \cdot i^4$
$1 \cdot i$	$1 \cdot i^2$	$1 \cdot i^3$	$1 \cdot 1$
i	i^2	i^3	1
	-1	$-i$	

We summarize this now.

$$\begin{aligned} i^1 &= i & i^5 &= i \\ i^2 &= -1 & i^6 &= -1 \\ i^3 &= -i & i^7 &= -i \\ i^4 &= 1 & i^8 &= 1 \end{aligned}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of i. Since $i^4 = 1$, we rewrite each power, i^n, as a product using i^4 to a power and another power of i.

We rewrite it in the form $i^n = \left(i^4\right)^q \cdot i^r$, where the exponent, q, is the quotient of n divided by 4 and the exponent, r, is the remainder from this division. For example, to simplify i^{57}, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, $57 = 4 \cdot 14 + 1$. So we write $i^{57} = \left(1^4\right)^{14} \cdot i^1$ and then simplify from there.

$$\begin{array}{r} 14 \\ 4\overline{)57} \\ \underline{4} \\ 17 \\ \underline{16} \\ 1 \end{array}$$

$$\begin{array}{l} i^r \\ \left(i^4\right)^q \cdot i^r \\ 1 \cdot i \\ i \end{array}$$

EXAMPLE 8.89

Simplify: i^{86}.

Solution

	i^{86}
Divide 86 by 4 and rewrite i^{86} in the $i^n = \left(i^4\right)^q \cdot i^r$ form.	$\left(1^4\right)^{21} \cdot i^2$

$$\begin{array}{r} 21 \\ 4\overline{)86} \\ \underline{8} \\ 6 \\ \underline{4} \\ 2 \end{array}$$

Simplify.	$(1)^{21} \cdot (-1)$
Simplify.	-1

TRY IT : : 8.177 Simplify: i^{75}.

TRY IT : : 8.178 Simplify: i^{92}.

MEDIA : :

Access these online resources for additional instruction and practice with the complex number system.

- Expressing Square Roots of Negative Numbers with i (https://openstax.org/l/37CompNumb1)
- Subtract and Multiply Complex Numbers (https://openstax.org/l/37CompNumb2)
- Dividing Complex Numbers (https://openstax.org/l/37CompNumb3)
- Rewriting Powers of i (https://openstax.org/l/37CompNumb4)

8.8 EXERCISES

Practice Makes Perfect

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

409.
ⓐ $\sqrt{-16}$
ⓑ $\sqrt{-11}$
ⓒ $\sqrt{-8}$

410.
ⓐ $\sqrt{-121}$
ⓑ $\sqrt{-1}$
ⓒ $\sqrt{-20}$

411.
ⓐ $\sqrt{-100}$
ⓑ $\sqrt{-13}$
ⓒ $\sqrt{-45}$

412.
ⓐ $\sqrt{-49}$
ⓑ $\sqrt{-15}$
ⓒ $\sqrt{-75}$

Add or Subtract Complex Numbers

In the following exercises, add or subtract.

413. $\sqrt{-75}+\sqrt{-48}$

414. $\sqrt{-12}+\sqrt{-75}$

415. $\sqrt{-50}+\sqrt{-18}$

416. $\sqrt{-72}+\sqrt{-8}$

417. $(1+3i)+(7+4i)$

418. $(6+2i)+(3-4i)$

419. $(8-i)+(6+3i)$

420. $(7-4i)+(-2-6i)$

421. $(1-4i)-(3-6i)$

422. $(8-4i)-(3+7i)$

423. $(6+i)-(-2-4i)$

424. $(-2+5i)-(-5+6i)$

425. $\left(5-\sqrt{-36}\right)+\left(2-\sqrt{-49}\right)$

426. $\left(-3+\sqrt{-64}\right)+\left(5-\sqrt{-16}\right)$

427. $\left(-7-\sqrt{-50}\right)-\left(-32-\sqrt{-18}\right)$

428. $\left(-5+\sqrt{-27}\right)-\left(-4-\sqrt{-48}\right)$

Multiply Complex Numbers

In the following exercises, multiply.

429. $4i(5-3i)$

430. $2i(-3+4i)$

431. $-6i(-3-2i)$

432. $-i(6+5i)$

433. $(4+3i)(-5+6i)$

434. $(-2-5i)(-4+3i)$

435. $(-3+3i)(-2-7i)$

436. $(-6-2i)(-3-5i)$

In the following exercises, multiply using the Product of Binomial Squares Pattern.

437. $(3+4i)^2$

438. $(-1+5i)^2$

439. $(-2-3i)^2$

440. $(-6-5i)^2$

In the following exercises, multiply.

441. $\sqrt{-25}\cdot\sqrt{-36}$

442. $\sqrt{-4}\cdot\sqrt{-16}$

443. $\sqrt{-9}\cdot\sqrt{-100}$

444. $\sqrt{-64} \cdot \sqrt{-9}$

445. $(-2 - \sqrt{-27})(4 - \sqrt{-48})$

446. $(5 - \sqrt{-12})(-3 + \sqrt{-75})$

447. $(2 + \sqrt{-8})(-4 + \sqrt{-18})$

448. $(5 + \sqrt{-18})(-2 - \sqrt{-50})$

449. $(2 - i)(2 + i)$

450. $(4 - 5i)(4 + 5i)$

451. $(7 - 2i)(7 + 2i)$

452. $(-3 - 8i)(-3 + 8i)$

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

453. $(7 - i)(7 + i)$

454. $(6 - 5i)(6 + 5i)$

455. $(9 - 2i)(9 + 2i)$

456. $(-3 - 4i)(-3 + 4i)$

Divide Complex Numbers

In the following exercises, divide.

457. $\frac{3 + 4i}{4 - 3i}$

458. $\frac{5 - 2i}{2 + 5i}$

459. $\frac{2 + i}{3 - 4i}$

460. $\frac{3 - 2i}{6 + i}$

461. $\frac{3}{2 - 3i}$

462. $\frac{2}{4 - 5i}$

463. $\frac{-4}{3 - 2i}$

464. $\frac{-1}{3 + 2i}$

465. $\frac{1 + 4i}{3i}$

466. $\frac{4 + 3i}{7i}$

467. $\frac{-2 - 3i}{4i}$

468. $\frac{-3 - 5i}{2i}$

Simplify Powers of *i*

In the following exercises, simplify.

469. i^{41}

470. i^{39}

471. i^{66}

472. i^{48}

473. i^{128}

474. i^{162}

475. i^{137}

476. i^{255}

Writing Exercises

477. Explain the relationship between real numbers and complex numbers.

478. Aniket multiplied as follows and he got the wrong answer. What is wrong with his reasoning?

$$\sqrt{-7} \cdot \sqrt{-7}$$
$$\sqrt{49}$$
$$7$$

479. Why is $\sqrt{-64} = 8i$ but $\sqrt[3]{-64} = -4$.

480. Explain how dividing complex numbers is similar to rationalizing a denominator.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No–I don't get it!
evaluate the square root of a negative number.			
add or subtract complex numbers.			
multiply complex numbers.			
divide complex numbers.			
simplify powers of i.			

ⓑ *On a scale of* $1-10$, *how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

CHAPTER 8 REVIEW

KEY TERMS

complex conjugate pair A complex conjugate pair is of the form $a + bi$, $a - bi$.

complex number A complex number is of the form $a + bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit The imaginary unit i is the number whose square is –1. $i^2 = -1$ or $i = \sqrt{-1}$.

like radicals Like radicals are radical expressions with the same index and the same radicand.

radical equation An equation in which a variable is in the radicand of a radical expression is called a radical equation.

radical function A radical function is a function that is defined by a radical expression.

rationalizing the denominator Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

square of a number If $n^2 = m$, then m is the square of n.

square root of a number If $n^2 = m$, then n is a square root of m.

standard form A complex number is in standard form when written as $a + bi$, where a, b are real numbers.

KEY CONCEPTS

8.1 Simplify Expressions with Roots

- **Square Root Notation**
 - $\sqrt{m}$ is read 'the square root of m'
 - If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.



 - The square root of m, $\sqrt{m}$, is a positive number whose square is m.
- **n^{th} Root of a Number**
 - If $b^n = a$, then b is an n^{th} root of a.
 - The principal n^{th} root of a is written $\sqrt[n]{a}$.
 - n is called the *index* of the radical.
- **Properties of $\sqrt[n]{a}$**
 - When n is an even number and
 - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
 - $a < 0$, then $\sqrt[n]{a}$ is not a real number
 - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Simplifying Odd and Even Roots**
 - For any integer $n \geq 2$,
 - when n is odd $\sqrt[n]{a^n} = a$
 - when n is even $\sqrt[n]{a^n} = |a|$
 - We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

8.2 Simplify Radical Expressions

- **Simplified Radical Expression**

- For real numbers a, m and $n \geq 2$
 $\sqrt[n]{a}$ is considered simplified if a has no factors of m^n

- **Product Property of nth Roots**
 - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$
 $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

- **How to simplify a radical expression using the Product Property**

 Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.

 Step 2. Use the product rule to rewrite the radical as the product of two radicals.

 Step 3. Simplify the root of the perfect power.

- **Quotient Property of Radical Expressions**
 - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,
 $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

- **How to simplify a radical expression using the Quotient Property.**

 Step 1. Simplify the fraction in the radicand, if possible.

 Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.

 Step 3. Simplify the radicals in the numerator and the denominator.

8.3 Simplify Rational Exponents

- **Rational Exponent** $a^{\frac{1}{n}}$
 - If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.

- **Rational Exponent** $a^{\frac{m}{n}}$
 - For any positive integers m and n,
 $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

- **Properties of Exponents**
 - If a, b are real numbers and m, n are rational numbers, then
 - **Product Property** $a^m \cdot a^n = a^{m+n}$
 - **Power Property** $(a^m)^n = a^{m \cdot n}$
 - **Product to a Power** $(ab)^m = a^m b^m$
 - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n}, \ a \neq 0$
 - **Zero Exponent Definition** $a^0 = 1, \quad a \neq 0$
 - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \ b \neq 0$
 - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n}, \ a \neq 0$

8.4 Add, Subtract, and Multiply Radical Expressions

- **Product Property of Roots**
 - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

- **Special Products**

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a+b)(a-b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

8.5 Divide Radical Expressions

- **Quotient Property of Radical Expressions**
 - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

 $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- **Simplified Radical Expressions**
 - A radical expression is considered simplified if there are:
 - no factors in the radicand that have perfect powers of the index
 - no fractions in the radicand
 - no radicals in the denominator of a fraction

8.6 Solve Radical Equations

- **Binomial Squares**

 $(a+b)^2 = a^2 + 2ab + b^2$

 $(a-b)^2 = a^2 - 2ab + b^2$
- **Solve a Radical Equation**

 Step 1. Isolate one of the radical terms on one side of the equation.

 Step 2. Raise both sides of the equation to the power of the index.

 Step 3. Are there any more radicals?
 If yes, repeat Step 1 and Step 2 again.
 If no, solve the new equation.

 Step 4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**

 Step 1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.

 Step 2. Identify what we are looking for.

 Step 3. Name what we are looking for by choosing a variable to represent it.

 Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

 Step 5. Solve the equation using good algebra techniques.

 Step 6. Check the answer in the problem and make sure it makes sense.

 Step 7. Answer the question with a complete sentence.
- **Falling Objects**
 - On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
 - If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

8.7 Use Radicals in Functions

- **Properties of $\sqrt[n]{a}$**
 - When n is an **even** number and:
 $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
 $a < 0$, then $\sqrt[n]{a}$ is not a real number.
 - When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**
 - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
 - When the **index** of the radical is **odd**, the radicand can be any real number.

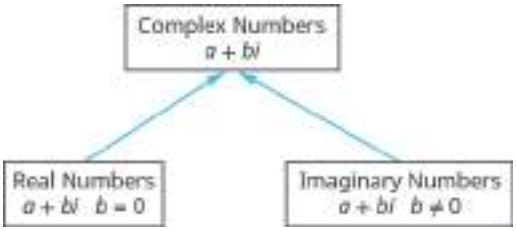
8.8 Use the Complex Number System

- **Square Root of a Negative Number**
 - If b is a positive real number, then $\sqrt{-b} = \sqrt{b}i$

	$a + bi$	
$b = 0$	$a + 0 \cdot i$ a	Real number
$b \neq 0$	$a + bi$	Imaginary number
$a = 0$	$0 + bi$ bi	Pure imaginary number

Table 8.32

 - A complex number is in **standard form** when written as $a + bi$, where a, b are real numbers.



- **Product of Complex Conjugates**
 - If a, b are real numbers, then
 $(a - bi)(a + bi) = a^2 + b^2$
- **How to Divide Complex Numbers**

Step 1. Write both the numerator and denominator in standard form.

Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.

Step 3. Simplify and write the result in standard form.

REVIEW EXERCISES

8.1 Simplify Expressions with Roots

Simplify Expressions with Roots

In the following exercises, simplify.

481. ⓐ $\sqrt{225}$ ⓑ $-\sqrt{16}$

482. ⓐ $-\sqrt{169}$ ⓑ $\sqrt{-8}$

483. ⓐ $\sqrt[3]{8}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{243}$

484. ⓐ $\sqrt[3]{-512}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1}$

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

485. ⓐ $\sqrt{68}$ ⓑ $\sqrt[3]{84}$

In the following exercises, approximate each root and round to two decimal places.

486. ⓐ $\sqrt{37}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{125}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

487.
ⓐ $\sqrt[3]{a^3}$
ⓑ $\sqrt[7]{b^7}$

488.
ⓐ $\sqrt{a^{14}}$
ⓑ $\sqrt{w^{24}}$

489.
ⓐ $\sqrt[4]{m^8}$
ⓑ $\sqrt[5]{n^{20}}$

490.
ⓐ $\sqrt{121m^{20}}$
ⓑ $-\sqrt{64a^2}$

491.
ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

492.
ⓐ $\sqrt{144x^2y^2}$
ⓑ $\sqrt{169w^8y^{10}}$
ⓒ $\sqrt[3]{8a^{51}b^6}$

8.2 Simplify Radical Expressions

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

493. $\sqrt{125}$

494. $\sqrt{675}$

495. ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

In the following exercises, simplify using absolute value signs as needed.

496.
ⓐ $\sqrt{a^{23}}$
ⓑ $\sqrt[3]{b^8}$
ⓒ $\sqrt[8]{c^{13}}$

497.
ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
ⓒ $\sqrt[6]{128b^7}$

498.
ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
ⓒ $\sqrt[4]{80x^8y^9}$

499.
ⓐ $\sqrt[5]{-32}$
ⓑ $\sqrt[8]{-1}$

500.
ⓐ $8+\sqrt{96}$
ⓑ $\frac{2+\sqrt{40}}{2}$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

501. ⓐ $\sqrt{\frac{72}{98}}$ ⓑ $\sqrt[3]{\frac{24}{81}}$ ⓒ $\sqrt[4]{\frac{6}{96}}$

502. ⓐ $\sqrt{\frac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

503. $\sqrt{\frac{300m^5}{64}}$

504.

ⓐ $\sqrt{\frac{28p^7}{q^2}}$

ⓑ $\sqrt[3]{\frac{81s^8}{t^3}}$

ⓒ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

505.

ⓐ $\sqrt{\frac{27p^2q}{108p^4q^3}}$

ⓑ $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$

ⓒ $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

506.

ⓐ $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

ⓑ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

ⓒ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

8.3 Simplify Rational Exponents

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

507. ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

In the following exercises, write with a rational exponent.

508. ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $\sqrt[6]{36r}$

In the following exercises, simplify.

509.

ⓐ $625^{\frac{1}{4}}$

ⓑ $243^{\frac{1}{5}}$

ⓒ $32^{\frac{1}{5}}$

510.

ⓐ $(-1{,}000)^{\frac{1}{3}}$

ⓑ $-1{,}000^{\frac{1}{3}}$

ⓒ $(1{,}000)^{-\frac{1}{3}}$

511.

ⓐ $(-32)^{\frac{1}{5}}$

ⓑ $(243)^{-\frac{1}{5}}$

ⓒ $-125^{\frac{1}{3}}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

512.

ⓐ $\sqrt[4]{r^7}$

ⓑ $\left(\sqrt[5]{2pq}\right)^3$

ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

In the following exercises, simplify.

513.

ⓐ $25^{\frac{3}{2}}$

ⓑ $9^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{2}{3}}$

514.

ⓐ $-64^{\frac{3}{2}}$

ⓑ $-64^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{3}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

515.

ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$

ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$

ⓒ $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

516.

ⓐ $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

ⓑ $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

8.4 Add, Subtract and Multiply Radical Expressions

Add and Subtract Radical Expressions

In the following exercises, simplify.

517.

ⓐ $7\sqrt{2} - 3\sqrt{2}$

ⓑ $7\sqrt[3]{p} + 2\sqrt[3]{p}$

ⓒ $5\sqrt[3]{x} - 3\sqrt[3]{x}$

518.

ⓐ $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$

ⓑ $8\sqrt[4]{11cd} + 5\sqrt[4]{11cd} - 9\sqrt[4]{11cd}$

519.

ⓐ $\sqrt{48} + \sqrt{27}$

ⓑ $\sqrt[3]{54} + \sqrt[3]{128}$

ⓒ $6\sqrt[4]{5} - \frac{3}{2}\sqrt[4]{320}$

520.

ⓐ $\sqrt{80c^7} - \sqrt{20c^7}$

ⓑ $2\sqrt[4]{162r^{10}} + 4\sqrt[4]{32r^{10}}$

521. $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

Multiply Radical Expressions

In the following exercises, simplify.

522.

ⓐ $(5\sqrt{6})(-\sqrt{12})$

ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

523.

ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$

ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

524.

ⓐ $\sqrt{11}\left(8 + 4\sqrt{11}\right)$

ⓑ $\sqrt[3]{3}\left(\sqrt[3]{9} + \sqrt[3]{18}\right)$

525.

ⓐ $(3 - 2\sqrt{7})(5 - 4\sqrt{7})$

ⓑ $\left(\sqrt[3]{x} - 5\right)\left(\sqrt[3]{x} - 3\right)$

526. $\left(2\sqrt{7} - 5\sqrt{11}\right)\left(4\sqrt{7} + 9\sqrt{11}\right)$

527.
ⓐ $(4+\sqrt{11})^2$
ⓑ $(3-2\sqrt{5})^2$

528. $(7+\sqrt{10})(7-\sqrt{10})$

529. $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

8.5 Divide Radical Expressions

Divide Square Roots

In the following exercises, simplify.

530.
ⓐ $\frac{\sqrt{48}}{\sqrt{75}}$
ⓑ $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$

531.
ⓐ $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$
ⓑ $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

532. ⓐ $\frac{8}{\sqrt{3}}$ ⓑ $\sqrt{\frac{7}{40}}$ ⓒ $\frac{8}{\sqrt{2y}}$

533. ⓐ $\frac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\frac{7}{54}}$ ⓒ $\frac{3}{\sqrt[3]{3x^2}}$

534. ⓐ $\frac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\frac{9}{32}}$ ⓒ $\frac{6}{\sqrt[4]{9x^3}}$

Rationalize a Two Term Denominator

In the following exercises, simplify.

535. $\frac{7}{2-\sqrt{6}}$

536. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

537. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

8.6 Solve Radical Equations

Solve Radical Equations

In the following exercises, solve.

538. $\sqrt{4x-3}=7$

539. $\sqrt{5x+1}=-3$

540. $\sqrt[3]{4x-1}=3$

541. $\sqrt{u-3}+3=u$

542. $\sqrt[3]{4x+5}-2=-5$

543. $(8x+5)^{\frac{1}{3}}+2=-1$

544. $\sqrt{y+4}-y+2=0$

545. $2\sqrt{8r+1}-8=2$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

546. $\sqrt{10+2c}=\sqrt{4c+16}$

547. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

548. $\sqrt{r}+6=\sqrt{r+8}$

549. $\sqrt{x+1}-\sqrt{x-2}=1$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

550. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

551. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

8.7 Use Radicals in Functions

Evaluate a Radical Function

In the following exercises, evaluate each function.

552. $g(x) = \sqrt{6x + 1}$, find

ⓐ $g(4)$

ⓑ $g(8)$

553. $G(x) = \sqrt{5x - 1}$, find

ⓐ $G(5)$

ⓑ $G(2)$

554. $h(x) = \sqrt[3]{x^2 - 4}$, find

ⓐ $h(-2)$

ⓑ $h(6)$

555. For the function $g(x) = \sqrt[4]{4 - 4x}$, find

ⓐ $g(1)$

ⓑ $g(-3)$

Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

556. $g(x) = \sqrt{2 - 3x}$

557. $F(x) = \sqrt{\frac{x + 3}{x - 2}}$

558. $f(x) = \sqrt[3]{4x^2 - 16}$

559. $F(x) = \sqrt[4]{10 - 7x}$

Graph Radical Functions

In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

560. $g(x) = \sqrt{x + 4}$

561. $g(x) = 2\sqrt{x}$

562. $f(x) = \sqrt[3]{x - 1}$

563. $f(x) = \sqrt[3]{x} + 3$

8.8 Use the Complex Number System

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

564.

ⓐ $\sqrt{-100}$

ⓑ $\sqrt{-13}$

ⓒ $\sqrt{-45}$

Add or Subtract Complex Numbers

In the following exercises, add or subtract.

565. $\sqrt{-50}+\sqrt{-18}$

566. $(8-i)+(6+3i)$

567. $(6+i)-(-2-4i)$

568. $\left(-7-\sqrt{-50}\right)-\left(-32-\sqrt{-18}\right)$

Multiply Complex Numbers

In the following exercises, multiply.

569. $(-2-5i)(-4+3i)$

570. $-6i(-3-2i)$

571. $\sqrt{-4}\cdot\sqrt{-16}$

572. $\left(5-\sqrt{-12}\right)\left(-3+\sqrt{-75}\right)$

In the following exercises, multiply using the Product of Binomial Squares Pattern.

573. $(-2-3i)^2$

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

574. $(9-2i)(9+2i)$

Divide Complex Numbers

In the following exercises, divide.

575. $\frac{2+i}{3-4i}$

576. $\frac{-4}{3-2i}$

Simplify Powers of *i*

In the following exercises, simplify.

577. i^{48}

578. i^{255}

PRACTICE TEST

In the following exercises, simplify using absolute values as necessary.

579. $\sqrt[3]{125x^9}$

580. $\sqrt{169x^8y^6}$

581. $\sqrt[3]{72x^8y^4}$

582. $\sqrt{\frac{45x^3y^4}{180x^5y^2}}$

In the following exercises, simplify. Assume all variables are positive.

583. ⓐ $216^{-\frac{1}{4}}$ ⓑ $-49^{\frac{3}{2}}$

584. $\sqrt{-45}$

585. $\frac{x^{-\frac{1}{4}} \cdot x^{\frac{5}{4}}}{x^{-\frac{3}{4}}}$

586. $\left(\frac{8x^{\frac{2}{3}}y^{-\frac{5}{2}}}{x^{-\frac{7}{3}}y^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

587. $\sqrt{48x^5} - \sqrt{75x^5}$

588. $\sqrt{27x^2} - 4x\sqrt{12} + \sqrt{108x^2}$

589. $2\sqrt{12x^5} \cdot 3\sqrt{6x^3}$

590. $\sqrt[3]{4}\left(\sqrt[3]{16} - \sqrt[3]{6}\right)$

591. $(4 - 3\sqrt{3})(5 + 2\sqrt{3})$

592. $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$

593. $\frac{\sqrt{245xy^{-4}}}{\sqrt{45x^{-4}y^3}}$

594. $\frac{1}{\sqrt[3]{5}}$

595. $\frac{3}{2 + \sqrt{3}}$

596. $\sqrt{-4} \cdot \sqrt{-9}$

597. $-4i(-2 - 3i)$

598. $\frac{4 + i}{3 - 2i}$

599. i^{172}

In the following exercises, solve.

600. $\sqrt{2x + 5} + 8 = 6$

601. $\sqrt{x + 5} + 1 = x$

602. $\sqrt[3]{2x^2 - 6x - 23} = \sqrt[3]{x^2 - 3x + 5}$

In the following exercise, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

603. $g(x) = \sqrt{x + 2}$



9 QUADRATIC EQUATIONS AND FUNCTIONS

Figure 9.1 Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology. (credit: "intographics"/Pixabay)

Chapter Outline

9.1 Solve Quadratic Equations Using the Square Root Property
9.2 Solve Quadratic Equations by Completing the Square
9.3 Solve Quadratic Equations Using the Quadratic Formula
9.4 Solve Quadratic Equations in Quadratic Form
9.5 Solve Applications of Quadratic Equations
9.6 Graph Quadratic Functions Using Properties
9.7 Graph Quadratic Functions Using Transformations
9.8 Solve Quadratic Inequalities

Introduction

Blink your eyes. You've taken a photo. That's what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

9.1 Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{128}$.
 If you missed this problem, review Example 8.13.
2. Simplify: $\sqrt{\frac{32}{5}}$.

If you missed this problem, review **Example 8.50**.

3. Factor: $9x^2 - 12x + 4$.
If you missed this problem, review **Example 6.23**.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the diffe ence of squares.	$(x-3)(x+3) = 0$
Use the Zero Product Property.	$x-3 = 0 \quad x-3 = 0$
Solve each equation.	$x = 3 \quad x = -3$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4, x = -4$ and $x = 5, x = -5$.

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so −13 is also a square root of 169. Therefore, both 13 and −13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}.$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

	$x^2 = 9$
Use the Square Root Property.	$x = \pm\sqrt{9}$
	$x = \pm 3$
	So $x = 3$ or $x = -3$.

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

	$x^2 = 7$
Use the Square Root Property.	$x = \sqrt{7}, \quad x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

EXAMPLE 9.1 HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $AX^2 = K$ USING THE SQUARE ROOT PROPERTY

Solve: $x^2 - 50 = 0$.

Solution

Step 1. Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$
Step 2. Use Square Root Property.	Remember to write the $\pm$ symbol.	$x = \pm\sqrt{50}$
Step 3. Simplify the radical.	Rewrite to show two solutions.	$x = \pm\sqrt{25}\cdot\sqrt{2}$ $x = \pm 5\sqrt{2}$ $x = 5\sqrt{2}, x = -5\sqrt{2}$
Step 4. Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x^2 - 50 = 0$ $(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$ ✓

TRY IT : : 9.1 Solve: $x^2 - 48 = 0.$

TRY IT : : 9.2 Solve: $y^2 - 27 = 0.$

The steps to take to use the Square Root Property to solve a quadratic equation are listed here.

HOW TO : : SOLVE A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY.

Step 1. Isolate the quadratic term and make its coefficient one.
Step 2. Use Square Root Property.
Step 3. Simplify the radical.
Step 4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

EXAMPLE 9.2

Solve: $3z^2 = 108.$

Solution

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6, \quad z = -6$
Check the solutions:	

$3z^2 = 108$ | $3z^2 = 108$
$3(6)^2 \stackrel{?}{=} 108$ | $3(-6)^2 \stackrel{?}{=} 108$
$3(36) \stackrel{?}{=} 108$ | $3(36) \stackrel{?}{=} 108$
$108 = 108$ ✓ | $108 = 108$ ✓

TRY IT : : 9.3 Solve: $2x^2 = 98.$

TRY IT : : 9.4 Solve: $5m^2 = 80.$

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

EXAMPLE 9.3

Solve: $x^2 + 72 = 0$.

Solution

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm\sqrt{-72}$
Simplify using complex numbers.	$x = \pm\sqrt{72}\,i$
Simplify the radical.	$x = \pm 6\sqrt{2}\,i$
Rewrite to show two solutions.	$x = 6\sqrt{2}\,i, \quad x = -6\sqrt{2}\,i$

Check the solutions:

$x^2+72=0$
$(6\sqrt{2}\,i)^2+72\stackrel{?}{=}0$
$6^2(\sqrt{2})^2i^2+72\stackrel{?}{=}0$
$36\cdot 2\cdot(-1)+72\stackrel{?}{=}0$
$0=0$ ✓

$x^2+72=0$
$(6\sqrt{2}\,i)^2+72\stackrel{?}{=}0$
$(-6)^2(\sqrt{2})^2i^2+72\stackrel{?}{=}0$
$36\cdot 2\cdot(-1)+72\stackrel{?}{=}0$
$0=0$ ✓

TRY IT : : 9.5 Solve: $c^2+12=0.$

TRY IT : : 9.6 Solve: $q^2+24=0.$

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

EXAMPLE 9.4

Solve: $\frac{2}{3}u^2+5=17.$

Solution

	$\frac{2}{3}u^2+5=17$
Isolate the quadratic term.	$\frac{2}{3}u^2=12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2}\cdot\frac{2}{3}u^2=\frac{3}{2}\cdot 12$
Simplify.	$u^2=18$
Use the Square Root Property.	$u=\pm\sqrt{18}$
Simplify the radical.	$u=\pm\sqrt{9\cdot 2}$
Simplify.	$u=\pm 3\sqrt{2}$
Rewrite to show two solutions.	$u=3\sqrt{2},\quad u=-3\sqrt{2}$

Check:

$\frac{2}{3}u^2+5=17$
$\frac{2}{3}(3\sqrt{2})^2+5\stackrel{?}{=}17$
$\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$
$12+5\stackrel{?}{=}17$
$17=17$ ✓

$\frac{2}{3}u^2+5=17$
$\frac{2}{3}(-3\sqrt{2})^2+5\stackrel{?}{=}17$
$\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$
$12+5\stackrel{?}{=}17$
$17=17$ ✓

TRY IT : : 9.7 Solve: $\frac{1}{2}x^2+4=24.$

TRY IT : : 9.8 Solve: $\frac{3}{4}y^2 - 3 = 18.$

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

EXAMPLE 9.5

Solve: $2x^2 - 8 = 41.$

Solution

	$2x^2 - 8 = 41$
Isolate the quadratic term.	$2x^2 = 49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2} = \frac{49}{2}$
Simplify.	$x^2 = \frac{49}{2}$
Use the Square Root Property.	$x = \pm\sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x = \pm\frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$x = \pm\frac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$
Simplify.	$x = \pm\frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$x = \frac{7\sqrt{2}}{2}, \quad x = -\frac{7\sqrt{2}}{2}$
Check: We leave the check for you.	

TRY IT : : 9.9 Solve: $5r^2 - 2 = 34.$

TRY IT : : 9.10 Solve: $3t^2 + 6 = 70.$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation of the form $a(x - h)^2 = k$ as well. Notice that the quadratic term, x, in the original form $ax^2 = k$ is replaced with $(x - h)$.

$$ax^2 = k \qquad a(x - h)^2 = k$$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x - h)^2$.

EXAMPLE 9.6

Solve: $4(y - 7)^2 = 48.$

Solution

	$4(y-7)^2=48$
Divide both sides by the coefficient 4.	$(y-7)^2=12$
Use the Square Root Property on the binomial	$y-7=\pm\sqrt{12}$
Simplify the radical.	$y-7=\pm 2\sqrt{3}$
Solve for y.	$y=7\pm 2\sqrt{3}$
Rewrite to show two solutions.	$y=7+2\sqrt{3},\quad y=7-2\sqrt{3}$
Check:	

$$\begin{aligned} 4(y-7)^2&=48\\ 4(7+2\sqrt{3}-7)^2&\stackrel{?}{=}48\\ 4(2\sqrt{3})^2&\stackrel{?}{=}48\\ 4(12)&\stackrel{?}{=}48\\ 48&=48\ \checkmark \end{aligned}\qquad \begin{aligned} 4(y-7)^2&=48\\ 4(7-2\sqrt{3}-7)^2&\stackrel{?}{=}48\\ 4(-2\sqrt{3})^2&\stackrel{?}{=}48\\ 4(12)&\stackrel{?}{=}48\\ 48&=48\ \checkmark \end{aligned}$$

TRY IT : : 9.11 Solve: $3(a-3)^2=54$.

TRY IT : : 9.12 Solve: $2(b+2)^2=80$.

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

EXAMPLE 9.7

Solve: $\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$.

Solution

$$\left(x-\frac{1}{3}\right)^2 = \frac{5}{9}$$

Use the Square Root Property. $x-\frac{1}{3} = \pm\sqrt{\frac{5}{9}}$

Rewrite the radical as a fraction of square roots. $x-\frac{1}{3} = \pm\frac{\sqrt{5}}{\sqrt{9}}$

Simplify the radical. $x-\frac{1}{3} = \pm\frac{\sqrt{5}}{3}$

Solve for x. $x = \frac{1}{3}\pm\frac{\sqrt{5}}{3}$

Rewrite to show two solutions. $x=\frac{1}{3}+\frac{\sqrt{5}}{3},\quad x=\frac{1}{3}-\frac{\sqrt{5}}{3}$

Check:
We leave the check for you.

TRY IT : : 9.13 Solve: $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

TRY IT : : 9.14 Solve: $\left(y+\frac{3}{4}\right)^2=\frac{7}{16}$.

We will start the solution to the next example by isolating the binomial term.

EXAMPLE 9.8

Solve: $2(x-2)^2+3=57$.

Solution

Subtract 3 from both sides to isolate the binomial term. $2(x-2)^2+3 = 57$

Divide both sides by 2. $2(x-2)^2 = 54$

Use the Square Root Property. $(x-2)^2 = 27$

Simplify the radical. $x-2 = \pm\sqrt{27}$

Solve for x. $x-2 = \pm 3\sqrt{3}$

Rewrite to show two solutions. $x = 2\pm 3\sqrt{3}$

$x=2+3\sqrt{3},\quad x=2-3\sqrt{3}$

Check:
We leave the check for you.

TRY IT : : 9.15 Solve: $5(a-5)^2+4=104$.

TRY IT : : 9.16 Solve: $3(b+3)^2-8=88$.

Sometimes the solutions are complex numbers.

EXAMPLE 9.9

Solve: $(2x-3)^2=-12$.

Solution

	$(2x-3)^2 = -12$
Use the Square Root Property.	$2x-3 = \pm\sqrt{-12}$
Simplify the radical.	$2x-3 = \pm 2\sqrt{3}\,i$
Add 3 to both sides.	$2x = 3 \pm 2\sqrt{3}\,i$
Divide both sides by 2.	$x = \frac{3 \pm 2\sqrt{3}\,i}{2}$
Rewrite in standard form.	$x = \frac{3}{2} \pm \frac{2\sqrt{3}\,i}{2}$
Simplify.	$x = \frac{3}{2} \pm \sqrt{3}\,i$
Rewrite to show two solutions.	$x = \frac{3}{2} + \sqrt{3}\,i, \quad x = \frac{3}{2} - \sqrt{3}\,i$

Check:
We leave the check for you.

TRY IT : : 9.17 Solve: $(3r+4)^2=-8$.

TRY IT : : 9.18 Solve: $(2t-8)^2=-10$.

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

EXAMPLE 9.10

Solve: $4n^2+4n+1=16$.

Solution

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2+4n+1=16$
Factor the perfect square trinomial.	$(2n+1)^2=16$
Use the Square Root Property.	$2n+1 = \pm\sqrt{16}$
Simplify the radical.	$2n+1 = \pm 4$
Solve for n.	$2n = -1 \pm 4$
Divide each side by 2.	$\frac{2n}{2} = \frac{-1 \pm 4}{2}$ $n = \frac{-1 \pm 4}{2}$
Rewrite to show two solutions.	$n = \frac{-1+4}{2}, \quad n = \frac{-1-4}{2}$
Simplify each equation.	$n = \frac{3}{2}, \qquad n = -\frac{5}{2}$

Check:

$$4n^2+4n+1=16$$
$$4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$$
$$4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$$
$$9+6+1\stackrel{?}{=}16$$
$$16=16\checkmark$$

$$4n^2+4n+1=16$$
$$4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$$
$$4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$$
$$25-10+1\stackrel{?}{=}16$$
$$16=16\checkmark$$

TRY IT : : 9.19 Solve: $9m^2-12m+4=25.$

TRY IT : : 9.20 Solve: $16n^2+40n+25=4.$

MEDIA : :

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- **Solving Quadratic Equations: The Square Root Property (https://openstax.org/l/37SqRtProp1)**
- **Using the Square Root Property to Solve Quadratic Equations (https://openstax.org/l/37SqRtProp2)**

9.1 EXERCISES

Practice Makes Perfect

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

1. $a^2 = 49$

2. $b^2 = 144$

3. $r^2 - 24 = 0$

4. $t^2 - 75 = 0$

5. $u^2 - 300 = 0$

6. $v^2 - 80 = 0$

7. $4m^2 = 36$

8. $3n^2 = 48$

9. $\frac{4}{3}x^2 = 48$

10. $\frac{5}{3}y^2 = 60$

11. $x^2 + 25 = 0$

12. $y^2 + 64 = 0$

13. $x^2 + 63 = 0$

14. $y^2 + 45 = 0$

15. $\frac{4}{3}x^2 + 2 = 110$

16. $\frac{2}{3}y^2 - 8 = -2$

17. $\frac{2}{5}a^2 + 3 = 11$

18. $\frac{3}{2}b^2 - 7 = 41$

19. $7p^2 + 10 = 26$

20. $2q^2 + 5 = 30$

21. $5y^2 - 7 = 25$

22. $3x^2 - 8 = 46$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

23. $(u - 6)^2 = 64$

24. $(v + 10)^2 = 121$

25. $(m - 6)^2 = 20$

26. $(n + 5)^2 = 32$

27. $\left(r - \frac{1}{2}\right)^2 = \frac{3}{4}$

28. $\left(x + \frac{1}{5}\right)^2 = \frac{7}{25}$

29. $\left(y + \frac{2}{3}\right)^2 = \frac{8}{81}$

30. $\left(t - \frac{5}{6}\right)^2 = \frac{11}{25}$

31. $(a - 7)^2 + 5 = 55$

32. $(b - 1)^2 - 9 = 39$

33. $4(x + 3)^2 - 5 = 27$

34. $5(x + 3)^2 - 7 = 68$

35. $(5c + 1)^2 = -27$

36. $(8d - 6)^2 = -24$

37. $(4x - 3)^2 + 11 = -17$

38. $(2y + 1)^2 - 5 = -23$

39. $m^2 - 4m + 4 = 8$

40. $n^2 + 8n + 16 = 27$

41. $x^2 - 6x + 9 = 12$

42. $y^2 + 12y + 36 = 32$

43. $25x^2 - 30x + 9 = 36$

44. $9y^2 + 12y + 4 = 9$

45. $36x^2 - 24x + 4 = 81$

46. $64x^2 + 144x + 81 = 25$

Mixed Practice

In the following exercises, solve using the Square Root Property.

47. $2r^2 = 32$

48. $4t^2 = 16$

49. $(a-4)^2 = 28$

50. $(b+7)^2 = 8$

51. $9w^2 - 24w + 16 = 1$

52. $4z^2 + 4z + 1 = 49$

53. $a^2 - 18 = 0$

54. $b^2 - 108 = 0$

55. $\left(p - \frac{1}{3}\right)^2 = \frac{7}{9}$

56. $\left(q - \frac{3}{5}\right)^2 = \frac{3}{4}$

57. $m^2 + 12 = 0$

58. $n^2 + 48 = 0.$

59. $u^2 - 14u + 49 = 72$

60. $v^2 + 18v + 81 = 50$

61. $(m-4)^2 + 3 = 15$

62. $(n-7)^2 - 8 = 64$

63. $(x+5)^2 = 4$

64. $(y-4)^2 = 64$

65. $6c^2 + 4 = 29$

66. $2d^2 - 4 = 77$

67. $(x-6)^2 + 7 = 3$

68. $(y-4)^2 + 10 = 9$

Writing Exercises

69. In your own words, explain the Square Root Property.

70. In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)^2 = 16$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x-h)^2 = k$ using the square root property.			

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be*

overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

9.2 Solve Quadratic Equations by Completing the Square

Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square
- Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square

Be Prepared!

Before you get started, take this readiness quiz.

1. Expand: $(x + 9)^2$.
 If you missed this problem, review **Example 5.32**.
2. Factor $y^2 - 14y + 49$.
 If you missed this problem, review **Example 6.9**.
3. Factor $5n^2 + 40n + 80$.
 If you missed this problem, review **Example 6.14**.

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y - 7)^2 = 12$ because the left side was a perfect square.

$$\begin{aligned} (y-7)^2 &= 12 \\ y-7 &= \pm\sqrt{12} \\ y-7 &= \pm 2\sqrt{3} \\ y &= 7 \pm 2\sqrt{3} \end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x - k)^2$ in order to use the Square Root Property.

$$\begin{aligned} x^2 - 10x + 25 &= 18 \\ (x-5)^2 &= 18 \end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

$(x + 9)^2$	$(y - 7)^2$
$(x + 9)(x + 9)$	$(y - 7)(y - 7)$
$x^2 + 9x + 9x + 81$	$y^2 - 7y - 7y + 49$
$x^2 + 18x + 81$	$y^2 - 14y + 49$

We restate the patterns here for reference.

Binomial Squares Pattern

If *a* and *b* are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$$

We can use this pattern to "make" a perfect square.

We will start with the expression $x^2 + 6x$. Since there is a plus sign between the two terms, we will use the $(a + b)^2$ pattern, $a^2 + 2ab + b^2 = (a + b)^2$.

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 6x + __ \end{array}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of $x^2 + 6x$ is a square, x^2. This tells us that $a = x$.

$$\begin{array}{l} a^2 + \quad 2ab \quad + b^2 \\ x^2 + 2 \cdot x \cdot b + b^2 \end{array}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b = 3$.

$$\begin{array}{l} a^2 + \quad 2ab \quad + b^2 \\ x^2 + 2 \cdot 3 \cdot x + __ \end{array}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2 = 9$.

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 6x + 9 \end{array}$$

We can now factor.

$$\begin{array}{l} (a+b)^2 \\ (x+3)^2 \end{array}$$

So we found that adding 9 to $x^2 + 6x$ 'completes the square', and we write it as $(x + 3)^2$.



HOW TO : : COMPLETE A SQUARE OF $x^2 + bx$.

Step 1. Identify b, the coefficient of x.

Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

Step 4. Factor the perfect square trinomial, writing it as a binomial squared.

EXAMPLE 9.11

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $x^2 - 26x$ ⓑ $y^2 - 9y$ ⓒ $n^2 + \frac{1}{2}n$

Solution

ⓐ

$x^2 - 26x$

The coefficient of x is −26.

Find $\left(\frac{1}{2}b\right)^2$.

$\left(\frac{1}{2}\cdot(-26)\right)^2$

$(13)^2$

169

Add 169 to the binomial to complete the square. $x^2 - 26x + 169$

Factor the perfect square trinomial, writing it as a binomial squared. $(x-13)^2$

ⓑ

$y^2 - 9y$

The coefficient of y is -9.

Find $\left(\frac{1}{2}b\right)^2$.

$\left(\frac{1}{2}\cdot(-9)\right)^2$

$\left(-\frac{9}{2}\right)^2$

$\frac{81}{4}$

Add $\frac{81}{4}$ to the binomial to complete the square. $y^2 - 9y + \frac{81}{4}$

Factor the perfect square trinomial, writing it as a binomial squared. $\left(y-\frac{9}{2}\right)^2$

ⓒ

$n^2 + \frac{1}{2}n$

The coefficient of n is $\frac{1}{2}$.

Find $\left(\frac{1}{2}b\right)^2$.

$\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$

$\left(\frac{1}{4}\right)^2$

$\frac{1}{16}$

Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2+\frac{1}{2}n+\frac{1}{16}$
Rewrite as a binomial square.	$\left(n+\frac{1}{4}\right)^2$

TRY IT : : 9.21

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ a^2-20a ⓑ m^2-5m ⓒ $p^2+\frac{1}{4}p$

TRY IT : : 9.22

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ b^2-4b ⓑ n^2+13n ⓒ $q^2-\frac{2}{3}q$

Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x = 40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

	$x^2+6x=40$
	$x^2+6x+__=40+__$
	$x^2+6x+9=40+9$
Add 9 to both sides to complete the square.	$(x+3)^2=49$

Now the equation is in the form to solve using the Square Root Property! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

EXAMPLE 9.12 HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $x^2+bx+c=0$ BY COMPLETING THE SQUARE

Solve by completing the square: $x^2+8x=48$.

Solution

Step 1. Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$x^2 + 8x = 48$
Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2 = 16$ Add 16 to BOTH sides of the equation.	$x^2 + 8x + \underset{\left(\frac{1}{2} \cdot 8\right)^2}{___} = 48$ $x^2 + 8x + 16 = 48 + 16$
Step 3. Factor the perfect square trinomial as a binomial square.	$x^2 + 8x + 16 = (x + 4)^2$ Add the terms on the right.	$(x + 4)^2 = 64$
Step 4. Use the Square Root Property.		$x + 4 = \pm\sqrt{64}$
Step 5. Simplify the radical and then solve the two resulting equations.		$x + 4 = \pm 8$ $x + 4 = 8 \quad x + 4 = -8$ $x = 4 \quad x = -12$
Step 6. Check the solutions.	Put each answer in the original equation to check. Substitute $x = 4$.	$x^2 + 8x = 48$ $(4)^2 + 8(4) \stackrel{?}{=} 48$ $16 + 32 \stackrel{?}{=} 48$ $48 = 48$ ✓
	Substitute $x = -12$.	$x^2 + 8x = 48$ $(-12)^2 + 8(-12) \stackrel{?}{=} 48$ $144 - 96 \stackrel{?}{=} 48$ $48 = 48$ ✓

TRY IT : : 9.23 Solve by completing the square: $x^2 + 4x = 5$.

TRY IT : : 9.24 Solve by completing the square: $y^2 - 10y = -9$.

The steps to solve a quadratic equation by completing the square are listed here.

HOW TO : : SOLVE A QUADRATIC EQUATION OF THE FORM $x^2 + bx + c = 0$ BY COMPLETING THE SQUARE.

Step 1. Isolate the variable terms on one side and the constant terms on the other.

Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

Step 3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right

Step 4. Use the Square Root Property.

Step 5. Simplify the radical and then solve the two resulting equations.

Step 6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

EXAMPLE 9.13

Solve by completing the square: $x^2 + 4x = -21$.

Solution

	$x^2 + 4x = -21$
The variable terms are on the left side. Take half of 4 and square it.	$x^2 + 4x + ____ = -21$, $\left(\frac{1}{2} \cdot 4\right)^2$
$\left(\frac{1}{2}(4)\right)^2 = 4$	
Add 4 to both sides.	$x^2 + 4x + 4 = -21 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+2)^2 = -17$
Use the Square Root Property.	$x + 2 = \pm\sqrt{-17}$
Simplify using complex numbers.	$x + 2 = \pm\sqrt{17}\,i$
Subtract 2 from each side.	$x = -2 \pm \sqrt{17}\,i$
Rewrite to show two solutions.	$x = -2 + \sqrt{17}\,i$, $x = -2 - \sqrt{17}\,i$
We leave the check to you.	

TRY IT : : 9.25 Solve by completing the square: $y^2 - 10y = -35$.

TRY IT : : 9.26 Solve by completing the square: $z^2 + 8z = -19$.

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

EXAMPLE 9.14

Solve by completing the square: $y^2 - 18y = -6$.

Solution

	$y^2 - 18y = -6$
The variable terms are on the left side. Take half of -18 and square it.	
$\left(\frac{1}{2}(-18)\right)^2 = 81$	$y^2 - 18y + ____ = -6$, $\left(\frac{1}{2} \cdot [-18]\right)^2$
Add 81 to both sides.	$y^2 - 18y + 81 = -6 + 81$
Factor the perfect square trinomial, writing it as a binomial squared.	$(y-9)^2 = 75$

Use the Square Root Property.	$y-9=\pm\sqrt{75}$
Simplify the radical.	$y-9=\pm 5\sqrt{3}$
Solve for y.	$y=9\pm 5\sqrt{3}$

Check.

$$y^2-18y=-6 \qquad y^2-18y=-6$$
$$(9+5\sqrt{3})^2-18(9+5\sqrt{3}) \stackrel{?}{=} -6 \qquad (9-5\sqrt{3})^2-18(9-5\sqrt{3}) \stackrel{?}{=} -6$$
$$81+90\sqrt{3}+75-162+90\sqrt{3} \stackrel{?}{=} -6 \qquad 81+90\sqrt{3}+75-162+90\sqrt{3} \stackrel{?}{=} -6$$
$$-6=-6\checkmark \qquad -6=-6\checkmark$$

Another way to check this would be to use a calculator. Evaluate y^2-18y for both of the solutions. The answer should be -6.

TRY IT : : 9.27 Solve by completing the square: $x^2-16x=-16$.

TRY IT : : 9.28 Solve by completing the square: $y^2+8y=11$.

We will start the next example by isolating the variable terms on the left side of the equation.

EXAMPLE 9.15

Solve by completing the square: $x^2+10x+4=15$.

Solution

	$x^2+10x+4=15$
Isolate the variable terms on the left side. Subtract 4 to get the constant terms on the right side.	$x^2+10x=11$
Take half of 10 and square it.	
$\left(\frac{1}{2}(10)\right)^2=25$	$x^2+10x+____=11$ $\left(\frac{1}{2}\cdot(10)\right)^2$
Add 25 to both sides.	$x^2+10x+25=11+25$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+5)^2=36$
Use the Square Root Property.	$x+5=\pm\sqrt{36}$
Simplify the radical.	$x+5=\pm 6$
Solve for x.	$x=-5\pm 6$
Rewrite to show two solutions.	$x=-5+6, \quad x=-5-6$
Solve the equations.	$x=1, \quad x=-11$

Check:

$$x^2+10x+4=15 \qquad x^2+10x+4=15$$
$$(1)^2+10(1)+4\stackrel{?}{=}15 \qquad (-11)^2+10(-11)+4\stackrel{?}{=}15$$
$$1+10+4\stackrel{?}{=}15 \qquad 121+110+4\stackrel{?}{=}15$$
$$15=15\checkmark \qquad 15=15\checkmark$$

TRY IT : : 9.29 Solve by completing the square: $a^2+4a+9=30$.

TRY IT : : 9.30 Solve by completing the square: $b^2+8b-4=16$.

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

EXAMPLE 9.16

Solve by completing the square: $n^2=3n+11$.

Solution

	$n^2=3n+11$
Subtract $3n$ to get the variable terms on the left side.	$n^2-3n=11$
Take half of -3 and square it.	
$\left(\frac{1}{2}(-3)\right)^2=\frac{9}{4}$	$n^2-3n+\underline{\quad}=11$
Add $\frac{9}{4}$ to both sides.	$n^2-3n+\frac{9}{4}=11+\frac{9}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(n-\frac{3}{2}\right)^2=\frac{44}{4}+\frac{9}{4}$
Add the fractions on the right side.	$\left(n-\frac{3}{2}\right)^2=\frac{53}{4}$
Use the Square Root Property.	$n-\frac{3}{2}=\pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n-\frac{3}{2}=\pm\frac{\sqrt{53}}{2}$
Solve for *n*.	$n=\frac{3}{2}\pm\frac{\sqrt{53}}{2}$
Rewrite to show two solutions.	$n=\frac{3}{2}+\frac{\sqrt{53}}{2}, \quad n=\frac{3}{2}-\frac{\sqrt{53}}{2}$
Check: We leave the check for you!	

TRY IT : : 9.31 Solve by completing the square: $p^2=5p+9$.

TRY IT : : 9.32 Solve by completing the square: $q^2 = 7q - 3.$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the Zero Product Property since it says "If $a \cdot b = 0,$ then $a = 0$ or $b = 0$." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

EXAMPLE 9.17

Solve by completing the square: $(x - 3)(x + 5) = 9.$

Solution

	$(x-3)(x+5)=9$
We multiply the binomials on the left.	$x^2+2x-15=9$
Add 15 to isolate the constant terms on the right.	$x^2+2x \quad =24$
Take half of 2 and square it.	
$\left(\frac{1}{2}\cdot(2)\right)^2=1$	$x^2+2x+\underset{\left(\frac{1}{2}\cdot(2)\right)^2}{____}=24$
Add 1 to both sides.	$x^2+2x+1=24+1$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+1)^2=25$
Use the Square Root Property.	$x+1=\pm\sqrt{25}$
Solve for x.	$x=-1\pm5$
Rewrite to show two solutions.	$x=-1+5, x=-1-5$
Simplify.	$x=4, \quad x=-6$
Check: We leave the check for you!	

TRY IT : : 9.33 Solve by completing the square: $(c - 2)(c + 8) = 11.$

TRY IT : : 9.34 Solve by completing the square: $(d - 7)(d + 3) = 56.$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of completing the square works best when the coefficient of x^2 is 1, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

EXAMPLE 9.18

Solve by completing the square: $3x^2 - 12x - 15 = 0.$

Solution

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2-12x-15=0$
Factor out the greatest common factor.	$3(x^2-4x-5)=0$
Divide both sides by 3 to isolate the trinomial with coefficient 1.	$\frac{3(x^2-4x-5)}{3}=\frac{0}{3}$
Simplify.	$x^2-4x-5=0$
Add 5 to get the constant terms on the right side.	$x^2-4x=5$
Take half of 4 and square it.	
$\left(\frac{1}{2}(-4)\right)^2=4$	$x^2-4x+\underline{\quad}=5$ $\left(\frac{1}{2}\cdot(-4)\right)$
Add 4 to both sides.	$x^2-4x+4=5+4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x-2)^2=9$
Use the Square Root Property.	$x-2=\pm\sqrt{9}$
Solve for x.	$x-2=\pm 3$
Rewrite to show two solutions.	$x=2+3,\ x=2-3$
Simplify.	$x=5,\quad x=-1$
Check:	

$x=5$

$3x^2-12x-15=0$

$3(5)^2-12(5)-15\stackrel{?}{=}0$

$75-60-15\stackrel{?}{=}0$

$0=0$ ✓

$x=-1$

$3x^2-12x-15=0$

$3(-1)^2-12(-1)-15\stackrel{?}{=}0$

$3+12-15\stackrel{?}{=}0$

$0=0$ ✓

TRY IT : : 9.35 Solve by completing the square: $2m^2+16m+14=0$.

TRY IT : : 9.36 Solve by completing the square: $4n^2-24n-56=8$.

To complete the square, the coefficient of the x^2 must be 1. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

EXAMPLE 9.19

Solve by completing the square: $2x^2-3x=20$.

Solution

To complete the square we need the coefficient of x^2 to be one. We will divide both sides of the equation by the coefficient of x^2. Then we can continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it.	
$\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underset{\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2}{____} = 10$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for *x*.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show two solutions.	$x = \frac{3}{4} + \frac{13}{4}, \quad x = \frac{3}{4} - \frac{13}{4}$
Simplify.	$x = 4, \quad x = -\frac{5}{2}$
Check: We leave the check for you!	

TRY IT : : 9.37 Solve by completing the square: $3r^2 - 2r = 21.$

TRY IT : : 9.38 Solve by completing the square: $4t^2 + 2t = 20.$

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a quadratic equation by completing the square to include equations of the form $ax^2 + bx + c = 0$.

HOW TO : : SOLVE A QUADRATIC EQUATION OF THE FORM $ax^2 + bx + c = 0$ BY COMPLETING THE SQUARE.

Step 1. Divide by a to make the coefficient of x^2 term 1.

Step 2. Isolate the variable terms on one side and the constant terms on the other.

Step 3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

Step 4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right

Step 5. Use the Square Root Property.

Step 6. Simplify the radical and then solve the two resulting equations.

Step 7. Check the solutions.

EXAMPLE 9.20

Solve by completing the square: $3x^2 + 2x = 4$.

Solution

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2 + 2x = 4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2 + 2x}{3} = \frac{4}{3}$
Simplify.	$x^2 + \frac{2}{3}x = \frac{4}{3}$
Take half of $\frac{2}{3}$ and square it.	
$\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2 = \frac{1}{9}$	$x^2 + \frac{2}{3}x + \underset{\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2}{___} = \frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x + \frac{1}{3}\right)^2 = \frac{12}{9} + \frac{1}{9}$
Use the Square Root Property.	$x + \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x + \frac{1}{3} = \pm\frac{\sqrt{13}}{3}$
Solve for x.	$x = -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$
Rewrite to show two solutions.	$x = -\frac{1}{3} + \frac{\sqrt{13}}{3}, \quad x = -\frac{1}{3} - \frac{\sqrt{13}}{3}$
Check: We leave the check for you!	

TRY IT : : 9.39 Solve by completing the square: $4x^2 + 3x = 2$.

TRY IT : : 9.40 Solve by completing the square: $3y^2 - 10y = -5$.

MEDIA : :

Access these online resources for additional instruction and practice with completing the square.

- **Completing Perfect Square Trinomials (https://openstax.org/l/37CompTheSq1)**
- **Completing the Square 1 (https://openstax.org/l/37CompTheSq2)**
- **Completing the Square to Solve Quadratic Equations (https://openstax.org/l/37CompTheSq3)**
- **Completing the Square to Solve Quadratic Equations: More Examples (https://openstax.org/l/37CompTheSq4)**
- **Completing the Square 4 (https://openstax.org/l/37CompTheSq5)**

9.2 EXERCISES

Practice Makes Perfect

Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

71.
ⓐ $m^2 - 24m$
ⓑ $x^2 - 11x$
ⓒ $p^2 - \frac{1}{3}p$

72.
ⓐ $n^2 - 16n$
ⓑ $y^2 + 15y$
ⓒ $q^2 + \frac{3}{4}q$

73.
ⓐ $p^2 - 22p$
ⓑ $y^2 + 5y$
ⓒ $m^2 + \frac{2}{5}m$

74.
ⓐ $q^2 - 6q$
ⓑ $x^2 - 7x$
ⓒ $n^2 - \frac{2}{3}n$

Solve Quadratic Equations of the form $x^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

75. 5. $u^2 + 2u = 3$

76. $z^2 + 12z = -11$

77. $x^2 - 20x = 21$

78. $y^2 - 2y = 8$

79. $m^2 + 4m = -44$

80. $n^2 - 2n = -3$

81. $r^2 + 6r = -11$

82. $t^2 - 14t = -50$

83. $a^2 - 10a = -5$

84. $b^2 + 6b = 41$

85. $x^2 + 5x = 2$

86. $y^2 - 3y = 2$

87. $u^2 - 14u + 12 = 1$

88. $z^2 + 2z - 5 = 2$

89. $r^2 - 4r - 3 = 9$

90. $t^2 - 10t - 6 = 5$

91. $v^2 = 9v + 2$

92. $w^2 = 5w - 1$

93. $x^2 - 5 = 10x$

94. $y^2 - 14 = 6y$

95. $(x + 6)(x - 2) = 9$

96. $(y + 9)(y + 7) = 80$

97. $(x + 2)(x + 4) = 3$

98. $(x - 2)(x - 6) = 5$

Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

99. $3m^2 + 30m - 27 = 6$

100. $2x^2 - 14x + 12 = 0$

101. $2n^2 + 4n = 26$

102. $5x^2 + 20x = 15$

103. $2c^2 + c = 6$

104. $3d^2 - 4d = 15$

105. $2x^2 + 7x - 15 = 0$

106. $3x^2 - 14x + 8 = 0$

107. $2p^2 + 7p = 14$

108. $3q^2 - 5q = 9$

109. $5x^2 - 3x = -10$

110. $7x^2 + 4x = -3$

Writing Exercises

111. Solve the equation $x^2 + 10x = -25$

ⓐ by using the Square Root Property

ⓑ by Completing the Square

ⓒ Which method do you prefer? Why?

112. Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

9.3 Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $b^2 - 4ab$ when $a = 3$ and $b = -2$.
 If you missed this problem, review **Example 1.21**.
2. Simplify: $\sqrt{108}$.
 If you missed this problem, review **Example 8.13**.
3. Simplify: $\sqrt{50}$.
 If you missed this problem, review **Example 8.76**.

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for *x*.

We start with the standard form of a quadratic equation and solve it for *x* by completing the square.

	$ax^2 + bx + c = 0 \quad a \neq 0$
Isolate the variable terms on one side.	$ax^2 + bx = -c$
Make the coefficient of x^2 equal to 1, by dividing by *a*.	$\frac{ax^2}{a} + \frac{b}{a}x = -\frac{c}{a}$
Simplify.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$	$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + = -\frac{c}{a} + \frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c \cdot 4a}{a \cdot 4a}$

Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
	This equation is the Quadratic Formula.

Quadratic Formula

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the formula is an equation. Make sure you use both sides of the equation.

EXAMPLE 9.21 HOW TO SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

Solve by using the Quadratic Formula: $2x^2 + 9x - 5 = 0$.

Solution

Step 1. Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$
Step 2. Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $a=2, b=9, c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4\cdot2\cdot(-5)}}{2\cdot2}$

Step 3. Simplify the fraction, and solve for x.		$x=\frac{-9\pm\sqrt{81-(-40)}}{4}$ $x=\frac{-9\pm\sqrt{121}}{4}$ $x=\frac{-9\pm 11}{4}$ $x=\frac{-9+11}{4}$ $\quad x=\frac{-9-11}{4}$ $x=\frac{2}{4}$ $\quad x=\frac{-20}{4}$ $x=\frac{1}{2}$ $\quad x=-5$
Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x=\frac{1}{2}$.	$2x^2+9x-5=0$ $2\left(\frac{1}{2}\right)^2+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $\frac{1}{2}+\frac{9}{2}-5\stackrel{?}{=}0$ $\frac{10}{2}-5\stackrel{?}{=}0$ $5-5\stackrel{?}{=}0$ $0=0$ ✓
	Substitute $x=-5$.	$2x^2+9x-5=0$ $2(-5)^2+9(-5)-5\stackrel{?}{=}0$ $2\cdot 25-45-5\stackrel{?}{=}0$ $50-45-5\stackrel{?}{=}0$ $0=0$ ✓

TRY IT : : 9.41 Solve by using the Quadratic Formula: $3y^2-5y+2=0$.

TRY IT : : 9.42 Solve by using the Quadratic Formula: $4z^2+2z-6=0$.

HOW TO : : SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA.

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, and c.

Step 2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.

Step 3. Simplify.

Step 4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic

Formula is an EQUATION. Be sure you start with "$x =$".

EXAMPLE 9.22

Solve by using the Quadratic Formula: $x^2 - 6x = -5$.

Solution

	$x^2 - 6x = -5$
Write the equation in standard form by adding 5 to each side.	$x^2 - 6x + 5 = 0$
This equation is now in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the values of a, b, c.	$a = 1, b = -6, c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$ $x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6+4}{2}, \quad x = \frac{6-4}{2}$
Simplify.	$x = \frac{10}{2}, \quad x = \frac{2}{2}$
	$x = 5, \quad x = 1$
Check:	

$x^2 - 6x + 5 = 0$
$5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0$
$25 - 30 + 5 \stackrel{?}{=} 0$
$0 = 0$ ✓

$x^2 - 6x + 5 = 0$
$1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0$
$1 - 6 + 5 \stackrel{?}{=} 0$
$0 = 0$ ✓

TRY IT : : 9.43 Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

TRY IT : : 9.44 Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

EXAMPLE 9.23

Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

Solution

	$2x^2+10x+11=0$
This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+10x+11=0$
Identify the values of a, b, and c.	$a=2,\ b=10,\ c=11$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x=\frac{-(10)\pm\sqrt{(10)^2-4\cdot 2\cdot(11)}}{2\cdot 2}$
Simplify.	$x=\frac{-10\pm\sqrt{100-88}}{4}$
	$x=\frac{-10\pm\sqrt{12}}{4}$
Simplify the radical.	$x=\frac{-10\pm 2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x=\frac{2(-5\pm\sqrt{3})}{4}$
Remove the common factors.	$x=\frac{-5\pm\sqrt{3}}{2}$
Rewrite to show two solutions.	$x=\frac{-5+\sqrt{3}}{2},\quad x=\frac{-5-\sqrt{3}}{2}$
Check: We leave the check for you!	

TRY IT : : 9.45 Solve by using the Quadratic Formula: $3m^2+12m+7=0$.

TRY IT : : 9.46 Solve by using the Quadratic Formula: $5n^2+4n-4=0$.

When we substitute a, b, and c into the Quadratic Formula and the radicand is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

EXAMPLE 9.24

Solve by using the Quadratic Formula: $3p^2+2p+9=0$.

Solution

	$3p^2+2p+9=0$
This equation is in standard form	$ax^2+bx+c=0$ $3p^2+2p+9=0$
Identify the values of a, b, c.	$a=3,\ b=2,\ c=9$
Write the Quadratic Formula.	$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$p=\frac{-(2)\pm\sqrt{(2)^2-4\cdot 3\cdot(9)}}{2\cdot 3}$

Simplify.	$p=\frac{-2\pm\sqrt{4-108}}{6}$
	$p=\frac{-2\pm\sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p=\frac{-2\pm\sqrt{104}\,i}{6}$
Simplify the radical.	$p=\frac{-2\pm2\sqrt{26}\,i}{6}$
Factor the common factor in the numerator.	$p=\frac{2(-1\pm\sqrt{26}\,i)}{6}$
Remove the common factors.	$p=\frac{-1\pm\sqrt{26}\,i}{3}$
Rewrite in standard $a+bi$ form.	$p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$
Write as two solutions.	$p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$

TRY IT : : 9.47 Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

TRY IT : : 9.48 Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

EXAMPLE 9.25

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

Solution

Our first step is to get the equation in standard form.

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$
This equation is now in standard form	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot1\cdot(4)}}{2\cdot1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$
	$x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm2\sqrt{5}}{2}$

Factor the common factor in the numerator.	$x=\frac{2(-3\pm2\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm2\sqrt{5}$
Write as two solutions.	$x=-3+2\sqrt{5},\ x=-3-2\sqrt{5}$
Check: We leave the check for you!	

TRY IT : : 9.49 Solve by using the Quadratic Formula: $x(x+2)-5=0$.

TRY IT : : 9.50 Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

EXAMPLE 9.26

Solve by using the Quadratic Formula: $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$.

Solution

Our first step is to clear the fractions.

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$
Multiply.	$3u^2+4u=2$
Subtract 2 to get the equation in standard form.	$ax^2+bx+c=0$ $3u^2+4u-2=0$
Identify the values of a, b, and c.	$a=3,\ b=4,\ c=-2$
Write the Quadratic Formula.	$u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, and c.	$u=\frac{-(4)\pm\sqrt{(4)^2-4\cdot3\cdot(-2)}}{2\cdot3}$
Simplify.	$u=\frac{-4\pm\sqrt{16+24}}{6}$
	$u=\frac{-4\pm\sqrt{40}}{6}$
Simplify the radical.	$u=\frac{-4\pm2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u=\frac{2(-2\pm\sqrt{10})}{6}$
Remove the common factors.	$u=\frac{-2\pm\sqrt{10}}{3}$
Rewrite to show two solutions.	$u=\frac{-2+\sqrt{10}}{3},\quad u=\frac{-2-\sqrt{10}}{3}$

Check:
We leave the check for you!

TRY IT : : 9.51 Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

TRY IT : : 9.52 Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Think about the equation $(x - 3)^2 = 0$. We know from the Zero Product Property that this equation has only one solution, $x = 3$.

We will see in the next example how using the Quadratic Formula to solve an equation whose standard form is a perfect square trinomial equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0 , which leads to only one solution.

EXAMPLE 9.27

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$ax^2 + bx + c = 0$ $4x^2 - 20x + 25 = 0$
Identify the values of a, b, and c.	$a = 4, b = -20, c = 25$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$
	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x - 5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

TRY IT : : 9.53 Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

TRY IT : : 9.54 Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation

without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,
the quantity $b^2 - 4ac$ is called the discriminant.

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Quadratic Equation (in standard form)	Discriminant $b^2 - 4ac$	Value of the Discriminant	Number and Type of solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5)$ 121	+	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25$ 0	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9$ -104	–	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions.** $x = \frac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution.** $x = \frac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions.** $x = \frac{-b \pm \sqrt{-}}{2a}$

Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

EXAMPLE 9.28

Determine the number of solutions to each quadratic equation.

ⓐ $3x^2 + 7x - 9 = 0$ ⓑ $5n^2 + n + 4 = 0$ ⓒ $9y^2 - 6y + 1 = 0$.

Solution

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

	$3x^2+7x-9=0$
The equation is in standard form, identify a, b, and c.	$a=3,\quad b=7,\ c=-9$
Write the discriminant.	b^2-4ac
Substitute in the values of a, b, and c.	$(7)^2-4\cdot 3\cdot(-9)$
Simplify.	$49+108$
	157

Since the discriminant is positive, there are 2 real solutions to the equation.

ⓑ

	$5n^2+n+4=0$
The equation is in standard form, identify a, b, and c.	$a=5,\quad b=1,\quad c=4$
Write the discriminant.	b^2-4ac
Substitute in the values of a, b, and c.	$(1)^2-4\cdot 5\cdot 4$
Simplify.	$1-80$
	-79

Since the discriminant is negative, there are 2 complex solutions to the equation.

ⓒ

	$9y^2-6y+1=0$
The equation is in standard form, identify a, b, and c.	$a=9,\ b=-6,\ c=1$
Write the discriminant.	b^2-4ac
Substitute in the values of a, b, and c.	$(-6)^2-4\cdot 9\cdot 1$
Simplify.	$36-36$
	0

Since the discriminant is 0, there is 1 real solution to the equation.

TRY IT : : 9.55 Determine the numberand type of solutions to each quadratic equation.

ⓐ $8m^2-3m+6=0$ ⓑ $5z^2+6z-2=0$ ⓒ $9w^2+24w+16=0.$

TRY IT : : 9.56 Determine the number and type of solutions to each quadratic equation.

ⓐ $b^2+7b-13=0$ ⓑ $5a^2-6a+10=0$ ⓒ $4r^2-20r+25=0.$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use

it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.



HOW TO : : IDENTIFY THE MOST APPROPRIATE METHOD TO SOLVE A QUADRATIC EQUATION.

Step 1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.

Step 2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.

Step 3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

EXAMPLE 9.29

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $5z^2 = 17$ ⓑ $4x^2 - 12x + 9 = 0$ ⓒ $8u^2 + 6u = 11.$

Solution

ⓐ

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k,$ the most appropriate method is to use the Square Root Property.

ⓑ

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

ⓒ

$$8u^2 + 6u = 11$$

Put the equation in standard form. $8u^2 + 6u - 11 = 0$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

TRY IT : : 9.57 Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $x^2 + 6x + 8 = 0$ ⓑ $(n-3)^2 = 16$ ⓒ $5p^2 - 6p = 9.$

TRY IT : : 9.58 Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $8a^2 + 3a - 9 = 0$ ⓑ $4b^2 + 4b + 1 = 0$ ⓒ $5c^2 = 125.$

MEDIA : :

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- **Using the Quadratic Formula (https://openstax.org/l/37QuadForm1)**
- **Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions (https://openstax.org/l/37QuadForm2)**
- **Discriminant in Quadratic Formula (https://openstax.org/l/37QuadForm3)**

9.3 EXERCISES

Practice Makes Perfect

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

113. $4m^2 + m - 3 = 0$

114. $4n^2 - 9n + 5 = 0$

115. $2p^2 - 7p + 3 = 0$

116. $3q^2 + 8q - 3 = 0$

117. $p^2 + 7p + 12 = 0$

118. $q^2 + 3q - 18 = 0$

119. $r^2 - 8r = 33$

120. $t^2 + 13t = -40$

121. $3u^2 + 7u - 2 = 0$

122. $2p^2 + 8p + 5 = 0$

123. $2a^2 - 6a + 3 = 0$

124. $5b^2 + 2b - 4 = 0$

125. $x^2 + 8x - 4 = 0$

126. $y^2 + 4y - 4 = 0$

127. $3y^2 + 5y - 2 = 0$

128. $6x^2 + 2x - 20 = 0$

129. $2x^2 + 3x + 3 = 0$

130. $2x^2 - x + 1 = 0$

131. $8x^2 - 6x + 2 = 0$

132. $8x^2 - 4x + 1 = 0$

133. $(v + 1)(v - 5) - 4 = 0$

134. $(x + 1)(x - 3) = 2$

135. $(y + 4)(y - 7) = 18$

136. $(x + 2)(x + 6) = 21$

137. $\frac{1}{3}m^2 + \frac{1}{12}m = \frac{1}{4}$

138. $\frac{1}{3}n^2 + n = -\frac{1}{2}$

139. $\frac{3}{4}b^2 + \frac{1}{2}b = \frac{3}{8}$

140. $\frac{1}{9}c^2 + \frac{2}{3}c = 3$

141. $16c^2 + 24c + 9 = 0$

142. $25d^2 - 60d + 36 = 0$

143. $25q^2 + 30q + 9 = 0$

144. $16y^2 + 8y + 1 = 0$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

145.

ⓐ $4x^2 - 5x + 16 = 0$

ⓑ $36y^2 + 36y + 9 = 0$

ⓒ $6m^2 + 3m - 5 = 0$

146.

ⓐ $9v^2 - 15v + 25 = 0$

ⓑ $100w^2 + 60w + 9 = 0$

ⓒ $5c^2 + 7c - 10 = 0$

147.

ⓐ $r^2 + 12r + 36 = 0$

ⓑ $8t^2 - 11t + 5 = 0$

ⓒ $3v^2 - 5v - 1 = 0$

148.

ⓐ $25p^2 + 10p + 1 = 0$

ⓑ $7q^2 - 3q - 6 = 0$

ⓒ $7y^2 + 2y + 8 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

149.

ⓐ $x^2 - 5x - 24 = 0$

ⓑ $(y + 5)^2 = 12$

ⓒ $14m^2 + 3m = 11$

150.

ⓐ $(8v + 3)^2 = 81$

ⓑ $w^2 - 9w - 22 = 0$

ⓒ $4n^2 - 10 = 6$

151.

ⓐ $6a^2 + 14 = 20$

ⓑ $\left(x - \frac{1}{4}\right)^2 = \frac{5}{16}$

ⓒ $y^2 - 2y = 8$

152.

ⓐ $8b^2 + 15b = 4$

ⓑ $\frac{5}{9}v^2 - \frac{2}{3}v = 1$

ⓒ $\left(w + \frac{4}{3}\right)^2 = \frac{2}{9}$

Writing Exercises

153. Solve the equation $x^2 + 10x = 120$

ⓐ by completing the square

ⓑ using the Quadratic Formula

ⓒ Which method do you prefer? Why?

154. Solve the equation $12y^2 + 23y = 24$

ⓐ by completing the square

ⓑ using the Quadratic Formula

ⓒ Which method do you prefer? Why?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

9.4 Solve Quadratic Equations in Quadratic Form

Learning Objectives

By the end of this section, you will be able to:

- Solve equations in quadratic form

Be Prepared!

Before you get started, take this readiness quiz.

1. Factor by substitution: $y^4 - y^2 - 20$.
 If you missed this problem, review **Example 6.21**.
2. Factor by substitution: $(y-4)^2 + 8(y-4) + 15$.
 If you missed this problem, review **Example 6.22**.
3. Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$ ⓑ $\left(x^{\frac{1}{3}}\right)^2$ ⓒ $\left(x^{-1}\right)^2$.
 If you missed this problem, review **Example 8.33**.

Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard u for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$.) So we let $u = x^2$ and factored.

	$x^4 - 4x^2 - 5$
	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$u^2 - 4u - 5$
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$(x^2+1)(x^2-5)$

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard u to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax^2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

EXAMPLE 9.30 HOW TO SOLVE EQUATIONS IN QUADRATIC FORM

Solve: $6x^4 - 7x^2 + 2 = 0$

Solution

Step 1. Identify a substitution that will put the equation in quadratic form.	Since $(x^2)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$
Step 2. Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6(x^2)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3. Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u - 1)(3u - 2) = 0$ $2u - 1 = 0,\ 3u - 2 = 0$ $2u = 1,\ 3u = 2$ $u = \frac{1}{2}\quad u = \frac{2}{3}$
Step 4. Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2}\qquad x^2 = \frac{2}{3}$
Step 5. Solve for the original variable.	Solve for x, using the Square Root Property.	$x = \pm\sqrt{\frac{1}{2}}\qquad x = \pm\sqrt{\frac{2}{3}}$ $x = \pm\frac{\sqrt{2}}{2}\qquad x = \pm\frac{\sqrt{6}}{3}$ There are four solutions. $x = \frac{\sqrt{2}}{2}\qquad x = \frac{\sqrt{6}}{3}$ $x = -\frac{\sqrt{2}}{2}\qquad x = -\frac{\sqrt{6}}{3}$
Step 6. Check the solutions.	Check all four solutions. We will show one check here.	$x = \frac{\sqrt{2}}{2}$ $6x^4 - 7x^2 + 2 = 0$ $6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$ $6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right) + 2 \stackrel{?}{=} 0$ $\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$ $0 = 0$ ✓ We leave the other checks to you!

TRY IT : : 9.59 Solve: $x^4 - 6x^2 + 8 = 0$.

TRY IT : : 9.60 Solve: $x^4 - 11x^2 + 28 = 0$.

We summarize the steps to solve an equation in quadratic form.

HOW TO : : SOLVE EQUATIONS IN QUADRATIC FORM.

Step 1. Identify a substitution that will put the equation in quadratic form.
Step 2. Rewrite the equation with the substitution to put it in quadratic form.
Step 3. Solve the quadratic equation for u.
Step 4. Substitute the original variable back into the results, using the substitution.
Step 5. Solve for the original variable.
Step 6. Check the solutions.

In the next example, the binomial in the middle term, $(x - 2)$ is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

EXAMPLE 9.31

Solve: $(x-2)^2 + 7(x-2) + 12 = 0$.

Solution

	$(x-2)^2 + 7(x-2) + 12 = 0$
Prepare for the substitution.	$(x-2)^2 + 7(x-2) + 12 = 0$
Let $u = x - 2$ and substitute.	$u^2 + 7u + 12 = 0$
Solve by factoring.	$(u+3)(u+4) = 0$ $u+3=0, \quad u+4=0$ $u=-3, \quad u=-4$
Replace u with $x - 2$.	$x-2=-3, \quad x-2=-4$
Solve for x.	$x=-1, \quad x=-2$

Check:

$x = -1$

$(x-2)^2 + 7(x-2) + 12 = 0$
$(-1-2)^2 + 7(-1-2) + 12 \stackrel{?}{=} 0$
$(-3)^2 + 7(-3) + 12 \stackrel{?}{=} 0$
$9 - 21 + 12 \stackrel{?}{=} 0$
$0 = 0$ ✓

$x = -2$

$(x-2)^2 + 7(x-2) + 12 = 0$
$(-2-2)^2 + 7(-2-2) + 12 \stackrel{?}{=} 0$
$(-4)^2 + 7(-4) + 12 \stackrel{?}{=} 0$
$16 - 28 + 12 \stackrel{?}{=} 0$
$0 = 0$ ✓

TRY IT : : 9.61 Solve: $(x-5)^2 + 6(x-5) + 8 = 0$.

TRY IT : : 9.62 Solve: $(y-4)^2 + 8(y-4) + 15 = 0$.

In the next example, we notice that $(\sqrt{x})^2 = x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

EXAMPLE 9.32

Solve: $x - 3\sqrt{x} + 2 = 0$.

Solution

The $\sqrt{x}$ in the middle term, is squared in the first term $(\sqrt{x})^2 = x$. If we let $u = \sqrt{x}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$x-3\sqrt{x}+2=0$
Rewrite the trinomial to prepare for the substitution.	$(\sqrt{x})^2-3\sqrt{x}+2=0$
Let $u = \sqrt{x}$ and substitute.	$u^2-3u+2=0$
Solve by factoring.	$(u-2)(u-1)=0$ $u-2=0,\quad u-1=0$ $u=2,\quad u=1$
Replace u with $\sqrt{x}$.	$\sqrt{x}=2,\quad \sqrt{x}=1$
Solve for x, by squaring both sides.	$x=4,\quad x=1$
Check:	

$$\begin{array}{rr} x=4 & x=1 \\ x-3\sqrt{x}+2=0 & x-3\sqrt{x}+2=0 \\ 4-3\sqrt{4}+2\stackrel{?}{=}0 & 1-3\sqrt{1}+2\stackrel{?}{=}0 \\ 4-6+2\stackrel{?}{=}0 & 1-3+2\stackrel{?}{=}0 \\ 0=0\checkmark & 0=0\checkmark \end{array}$$

TRY IT : : 9.63 Solve: $x - 7\sqrt{x} + 12 = 0$.

TRY IT : : 9.64 Solve: $x - 6\sqrt{x} + 8 = 0$.

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

EXAMPLE 9.33

Solve: $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$.

Solution

The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$. If we let $u = x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$
Rewrite the trinomial to prepare for the substitution.	$\left(x^{\frac{1}{3}}\right)^2-2\left(x^{\frac{1}{3}}\right)-24=0$
Let $u = x^{\frac{1}{3}}$ and substitute.	$u^2-2u-24=0$

Solve by factoring.	$(u-6)(u+4)=0$
	$u-6=0, \quad u+4=0$
	$u=6, \quad u=-4$
Replace u with $x^{\frac{1}{3}}$.	$x^{\frac{1}{3}}=6, \quad x^{\frac{1}{3}}=-4$
Solve for x by cubing both sides.	$(x^{\frac{1}{3}})^3=(6)^3, \quad (x^{\frac{1}{3}})^3=(-4)^3$
	$x=216, \quad x=-64$

Check:

$x=216$

$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$

$(216)^{\frac{2}{3}}-2(216)^{\frac{1}{3}}-24\stackrel{?}{=}0$

$36-12-24\stackrel{?}{=}0$

$0=0$ ✓

$x=-64$

$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$

$(-64)^{\frac{2}{3}}-2(-64)^{\frac{1}{3}}-24\stackrel{?}{=}0$

$16+8-24\stackrel{?}{=}0$

$0=0$ ✓

TRY IT : : 9.65

Solve: $x^{\frac{2}{3}}-5x^{\frac{1}{3}}-14=0.$

TRY IT : : 9.66

Solve: $x^{\frac{1}{2}}+8x^{\frac{1}{4}}+15=0.$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

EXAMPLE 9.34

Solve: $3x^{-2}-7x^{-1}+2=0.$

Solution

The x^{-1} in the middle term is squared in the first term $\left(x^{-1}\right)^2=x^{-2}$. If we let $u=x^{-1}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$3x^{-2}-7x^{-1}+2=0$
Rewrite the trinomial to prepare for the substitution.	$3(x^{-1})^2-7(x^{-1})+2=0$
Let $u=x^{-1}$ and substitute.	$3u^2-7u+2=0$
Solve by factoring.	$(3u-1)(u-2)=0$
	$3u-1=0, \quad u-2=0$
	$u=\frac{1}{3}, \quad u=2$
Replace u with x^{-1}.	$x^{-1}=\frac{1}{3}, \quad x^{-1}=2$
Solve for x by taking the reciprocal since $x^{-1}=\frac{1}{x}$.	$x=3, \quad x=\frac{1}{2}$

Check:

$x = 3$

$3x^{-2} - 7x^{-1} + 2 = 0$

$3(3)^{-2} - 7(3)^{-1} + 2 \stackrel{?}{=} 0$

$3\left(\frac{1}{9}\right) - 7\left(\frac{1}{3}\right) + 2 \stackrel{?}{=} 0$

$\frac{1}{3} - \left(\frac{7}{3}\right) + \frac{6}{3} \stackrel{?}{=} 0$

$0 = 0$ ✓

$x = \frac{1}{2}$

$3x^{-2} - 7x^{-1} + 2 = 0$

$3\left(\frac{1}{2}\right)^{-2} - 7\left(\frac{1}{2}\right)^{-1} + 2 \stackrel{?}{=} 0$

$3(4) - 7(2) + 2 \stackrel{?}{=} 0$

$12 - 14 + 2 \stackrel{?}{=} 0$

$0 = 0$ ✓

TRY IT : : 9.67 Solve: $8x^{-2} - 10x^{-1} + 3 = 0.$

TRY IT : : 9.68 Solve: $6x^{-2} - 23x^{-1} + 20 = 0.$

MEDIA : :

Access this online resource for additional instruction and practice with solving quadratic equations.

- **Solving Equations in Quadratic Form (https://openstax.org/l/37QuadForm4)**

9.4 EXERCISES

Practice Makes Perfect

Solve Equations in Quadratic Form

In the following exercises, solve.

155. $x^4 - 7x^2 + 12 = 0$

156. $x^4 - 9x^2 + 18 = 0$

157. $x^4 - 13x^2 - 30 = 0$

158. $x^4 + 5x^2 - 36 = 0$

159. $2x^4 - 5x^2 + 3 = 0$

160. $4x^4 - 5x^2 + 1 = 0$

161. $2x^4 - 7x^2 + 3 = 0$

162. $3x^4 - 14x^2 + 8 = 0$

163. $(x-3)^2 - 5(x-3) - 36 = 0$

164. $(x+2)^2 - 3(x+2) - 54 = 0$

165. $(3y+2)^2 + (3y+2) - 6 = 0$

166. $(5y-1)^2 + 3(5y-1) - 28 = 0$

167. $\left(x^2+1\right)^2 - 5\left(x^2+1\right) + 4 = 0$

168. $\left(x^2-4\right)^2 - 4\left(x^2-4\right) + 3 = 0$

169. $2\left(x^2-5\right)^2 - 5\left(x^2-5\right) + 2 = 0$

170. $2\left(x^2-5\right)^2 - 7\left(x^2-5\right) + 6 = 0$

171. $x - \sqrt{x} - 20 = 0$

172. $x - 8\sqrt{x} + 15 = 0$

173. $x + 6\sqrt{x} - 16 = 0$

174. $x + 4\sqrt{x} - 21 = 0$

175. $6x + \sqrt{x} - 2 = 0$

176. $6x + \sqrt{x} - 1 = 0$

177. $10x - 17\sqrt{x} + 3 = 0$

178. $12x + 5\sqrt{x} - 3 = 0$

179. $x^{\frac{2}{3}} + 9x^{\frac{1}{3}} + 8 = 0$

180. $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} = 28$

181. $x^{\frac{2}{3}} + 4x^{\frac{1}{3}} = 12$

182. $x^{\frac{2}{3}} - 11x^{\frac{1}{3}} + 30 = 0$

183. $6x^{\frac{2}{3}} - x^{\frac{1}{3}} = 12$

184. $3x^{\frac{2}{3}} - 10x^{\frac{1}{3}} = 8$

185. $8x^{\frac{2}{3}} - 43x^{\frac{1}{3}} + 15 = 0$

186. $20x^{\frac{2}{3}} - 23x^{\frac{1}{3}} + 6 = 0$

187. $x + 8x^{\frac{1}{2}} + 7 = 0$

188. $2x - 7x^{\frac{1}{2}} = 15$

189. $6x^{-2} + 13x^{-1} + 5 = 0$

190. $15x^{-2} - 26x^{-1} + 8 = 0$

191. $8x^{-2} - 2x^{-1} - 3 = 0$

192. $15x^{-2} - 4x^{-1} - 4 = 0$

Writing Exercises

193. Explain how to recognize an equation in quadratic form.

194. Explain the procedure for solving an equation in quadratic form.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve equations in quadratic form.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

9.5 Solve Applications of Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by quadratic equations

Be Prepared!

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is −100. Find the numbers.
 If you missed this problem, review Example 2.18.
2. Solve: $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.
 If you missed this problem, review Example 7.35.
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
 If you missed this problem, review Example 2.34.

Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let's first summarize the methods we now have to solve quadratic equations.

Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As you solve each equation, choose the method that is most convenient for you to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.



HOW TO : : USE A PROBLEM-SOLVING STRATEGY.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.

Step 5. **Solve** the equation using algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence

We have solved number applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n + 2$. The next one would be $n + 2 + 2$ or $n + 4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

	Consecutive even integers		**Consecutive odd integers**
	64, 66, 68		77, 79, 81
n	1^{st} even integer	n	1^{st} odd integer
$n+2$	2^{nd} consecutive even integer	$n+2$	2^{nd} consecutive odd integer
$n+4$	3^{rd} consecutive even integer	$n+4$	3^{rd} consecutive odd integer

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

EXAMPLE 9.35

The product of two consecutive odd integers is 195. Find the integers.

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. We are looking for two consecutive odd integers.

Step 3. Name what we are looking for.
Let $n =$ the first odd integer.
$n+2 =$ the next odd integer

Step 4. Translate into an equation. State the problem in one sentence.
"The product of two consecutive odd integers is 195."

The product of the first odd integer and the second odd integer is 195.

Translate into an equation.

$$n(n+2) = 195$$

Step 5. Solve the equation. Distribute.

$$n^2+2n = 195$$

Write the equation in standard form.

$$n^2+2n-195 = 0$$

Factor.

$$(n+15)(n-13) = 0$$

Use the Zero Product Property.

$$n+15=0 \quad n-13=0$$

Solve each equation.

$$n=-15, \quad n=13$$

There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.

First odd integer $n=13$	First odd integer $n=-15$
next odd integer $n+2$	next odd integer $n+2$
$13+2$	$-15+2$
15	-13

Step 6. Check the answer.
Do these pairs work?
Are they consecutive odd integers?

$13,\ 15$ yes
$-13,\ -15$ yes

Is their product 195?

$13 \cdot 15 = 195$ yes
$-13(-15) = 195$ yes

Step 7. Answer the question. Two consecutive odd integers whose product is 195 are 13, 15 and $-13,\ -15$.

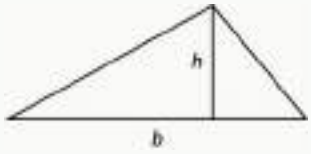
TRY IT : : 9.69 The product of two consecutive odd integers is 99. Find the integers.

TRY IT : : 9.70 The product of two consecutive even integers is 168. Find the integers.

We will use the formula for the area of a triangle to solve the next example.

Area of a Triangle

For a triangle with base, *b*, and height, *h*, the area, *A*, is given by the formula $A = \frac{1}{2}bh$.

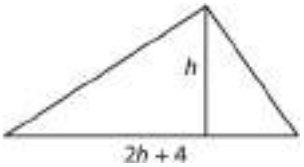


Recall that when we solve geometric applications, it is helpful to draw the figure.

EXAMPLE 9.36

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the base and height.
Step 3. Name what we are looking for.	Let *h* = the height of the triangle. $2h + 4$ = the base of the triangle
Step 4. Translate into an equation. We know the area. Write the formula for the area of a triangle.	$A = \frac{1}{2}bh$
Step 5. Solve the equation. Substitute in the values.	$120 = \frac{1}{2}(2h + 4)h$
Distribute.	$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in standard form.	$h^2 + 2h - 120 = 0$
Factor.	$(h - 10)(h + 12) = 0$
Use the Zero Product Property.	$h - 10 = 0 \quad h + 12 = 0$
Simplify.	$h = 10, \quad \cancel{h = -12}$
Since *h* is the height of a window, a value of *h* = −12 does not make sense.	
The height of the triangle $h = 10$.	

The base of the triangle $2h + 4$.

$$2 \cdot 10 + 4$$
$$24$$

Step 6. Check the answer. Does a triangle with height 10 and base 24 have area 120? Yes.	
Step 7. Answer the question.	The height of the triangular window is 10 feet and the base is 24 feet.

TRY IT : : 9.71

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

TRY IT : : 9.72

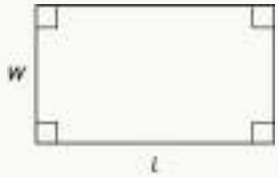
If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

Area of a Rectangle

For a rectangle with length, *L*, and width, *W*, the area, *A*, is given by the formula $A = LW$.



EXAMPLE 9.37

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution

Step 1. **Read** the problem. Draw a picture.	
Step 2. **Identify** what we are looking for.	We are looking for the length and width.
Step 3. **Name** what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle

Step 4. **Translate** into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5. **Solve** the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$
This is a quadratic equation; rewrite it in standard form. Solve the equation using the Quadratic Formula.	$0 = 3w^2 - w - 150$
Identify the a, b, c values.	$a = 3, b = -1, c = -150$
Write the Quadratic Formula.	$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$w = \frac{1 \pm \sqrt{1 + 1800}}{6}$ $w = \frac{1 \pm \sqrt{1801}}{6}$
Rewrite to show two solutions.	$w = \frac{1 + \sqrt{1801}}{6}, \quad w = \frac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w \approx 7.2, \quad w \approx -6.9$ Width $w \approx 7.2$ Length $\approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Step 6. **Check** the answer. Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150.	
Step 7. **Answer** the question.	The width of the rectangle is approximately 7.2 feet and the length is approximately 20.6 feet.

TRY IT : : 9.73

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

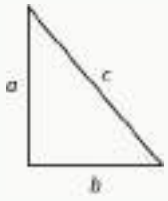
TRY IT : : 9.74

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot.?

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Pythagorean Theorem

In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.

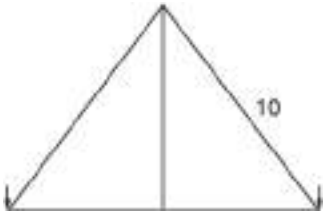


EXAMPLE 9.38

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution

Step 1. Read the problem. Draw a picture.



Step 2. Identify what we are looking for.

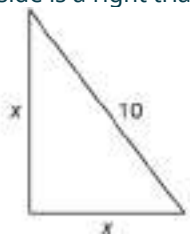
We are looking for the height of the pole.

Step 3. Name what we are looking for.

The distance from the base of the pole to either stake is the same as the height of the pole.

Let $x =$ the height of the pole.
$x =$ the distance from pole to stake

Each side is a right triangle. We draw a picture of one of them.



Step 4. Translate into an equation. We can use the Pythagorean Theorem to solve for x. Write the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$

Step 5. Solve the equation. Substitute.

$$x^2 + x^2 = 10^2$$

Simplify.

$$2x^2 = 100$$

Divide by 2 to isolate the variable.

$$\frac{2x^2}{2} = \frac{100}{2}$$

Simplify.

$$x^2 = 50$$

Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}, \quad x = -5\sqrt{2}$
	If we approximate this number to the nearest tenth with a calculator, we find $x \approx 7.1$.
Step 6. Check the answer. Check on your own in the Pythagorean Theorem.	
Step 7. Answer the question.	The pole should be about 7.1 feet tall.

TRY IT : : 9.75

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

TRY IT : : 9.76

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Projectile motion

The height in feet, h , of an object shot upwards into the air with initial velocity, v_0 , after t seconds is given by the formula

$$h = -16t^2 + v_0 t$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

EXAMPLE 9.39

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of seconds, which is time.
Step 3. Name what we are looking for.	Let $t =$ the number of seconds.
Step 4. Translate into an equation. Use the formula.	$h = -16t^2 + v_0 t$
Step 5. Solve the equation. We know the velocity v_0 is 130 feet per second. The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$

This is a quadratic equation, rewrite it in standard form. Solve the equation using the Quadratic Formula.	$16t^2 - 130t + 260 = 0$
Identify the values of a, b, c.	$a = 16, b = -130, c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32}, \quad t = \frac{130 - \sqrt{260}}{32}$
Approximate the answer with a calculator.	$t \approx 4.6$ seconds, $t \approx 3.6$ seconds
Step 6. Check the answer. The check is left to you.	
Step 7. Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

TRY IT : : 9.77

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h = -16t^2 + v_0t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

TRY IT : : 9.78

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h = -16t^2 + v_0t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate	• Time	= Distance

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

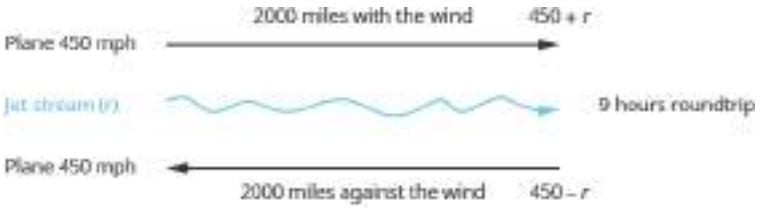
Some uniform motion problems are also modeled by quadratic equations.

EXAMPLE 9.40

Professor Smith just returned from a conference that was 2,000 miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is 450 + *r*.

When the plane flies against the wind, the wind decreases its speed and the rate is 450 – *r*.

Write in the rates.
Write in the distances.
Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$.
We divide the distance by the rate in each row, and place the expression in the time column.

Type	Rate	·	Time	=	Distance
Headwind	$450 - r$		$\frac{2000}{450 - r}$		2000
Tailwind	$450 + r$		$\frac{2000}{450 + r}$		2000
			9		

We know the times add to 9 and so we write our equation.

$$\frac{2000}{450 - r} + \frac{2000}{450 + r} = 9$$

We multiply both sides by the LCD.

$$(450 - r)(450 + r)\left(\frac{2000}{450 - r} + \frac{2000}{450 + r}\right) = 9(450 - r)(450 + r)$$

Simplify.

$$2000(450 + r) + 2000(450 - r) = 9(450 - r)(450 + r)$$

Factor the 2,000.

$$2000(450 + r + 450 - r) = 9\left(450^2 - r^2\right)$$

Solve.

$$2000(900) = 9\left(450^2 - r^2\right)$$

Divide by 9.

$$2000(100) = 450^2 - r^2$$

Simplify.

$$200000 = 202500 - r^2$$
$$-2500 = -r^2$$
$$50 = r \quad \text{The speed of the jet stream.}$$

Check:
Is 50 mph a reasonable speed for the jet stream? Yes.
If the plane is traveling 450 mph and the wind is 50 mph,
Tailwind

$450 + 50 = 500$ mph $\quad \frac{2000}{500} = 4$ hours

Headwind

$450 - 50 = 400$ mph $\quad \frac{2000}{400} = 5$ hours

The times add to 9 hours, so it checks.

The speed of the jet stream was 50 mph.

TRY IT : : 9.79

MaryAnne just returned from a visit with her grandchildren back east . The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

TRY IT : : 9.80

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We'll use a similar scenario now.

EXAMPLE 9.41

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

Solution

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

Let $x =$ the number of hours for Press #2 to complete the job.
Enter the hours per job for Press #1, Press #2, and when they work together.

	Number of hours needed to complete the job.	Part of job completed/hour
Press #1	$x+12$	$\frac{1}{x+12}$
Press #2	x	$\frac{1}{x}$
Together	8	$\frac{1}{8}$

The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.
Translate to an equation.

Work completed by
Press #1 + Press #2 = Together

$$\frac{1}{x+12}+\frac{1}{x}=\frac{1}{8}$$

Solve.

$$\frac{1}{x+12}+\frac{1}{x}=\frac{1}{8}$$

Multiply by the LCD, $8x(x+12)$.

$$8x(x+12)\left(\frac{1}{x+12}+\frac{1}{x}\right)=\left(\frac{1}{8}\right)8x(x+12)$$

Simplify.

$$8x+8(x+12)=x(x+12)$$

$$8x+8x+96=x^2+12x$$

$$0=x^2-4x-96$$

Solve.

$$0=(x-12)(x+8)$$

$$x-12=0,\ x+8=0$$

$$x=12,\ x=-8 \text{ hours}$$

Since the idea of negative hours does not make sense, we use the value $x=12$.

$12+12$	12
24 hours	12 hours

Write our sentence answer.

Press #1 would take 24 hours and
Press #2 would take 12 hours to do the job alone.

TRY IT : : 9.81

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

TRY IT : : 9.82

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

MEDIA : :

Access these online resources for additional instruction and practice with solving applications modeled by quadratic equations.

- **Word Problems Involving Quadratic Equations (https://openstax.org/l/37QuadForm5)**
- **Quadratic Equation Word Problems (https://openstax.org/l/37QuadForm6)**
- **Applying the Quadratic Formula (https://openstax.org/l/37QuadForm7)**

9.5 EXERCISES

Practice Makes Pefect

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method.

195. The product of two consecutive odd numbers is 255. Find the numbers.

196. The product of two consecutive even numbers is 360. Find the numbers.

197. The product of two consecutive even numbers is 624. Find the numbers.

198. The product of two consecutive odd numbers is 1,023. Find the numbers.

199. The product of two consecutive odd numbers is 483. Find the numbers.

200. The product of two consecutive even numbers is 528. Find the numbers.

In the following exercises, solve using any method. Round your answers to the nearest tenth, if needed.

201. A triangle with area 45 square inches has a height that is two less than four times the base Find the base and height of the triangle.

202. The base of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the base and height of the triangle.

203. The area of a triangular flower bed in the park has an area of 120 square feet. The base is 4 feet longer that twice the height. What are the base and height of the triangle?

204. A triangular banner for the basketball championship hangs in the gym. It has an area of 75 square feet. What is the length of the base and height , if the base is two-thirds of the height?

205. The length of a rectangular driveway is five feet more than three times the width. The area is 50 square feet. Find the length and width of the driveway.

206. A rectangular lawn has area 140 square yards. Its width that is six less than twice the length. What are the length and width of the lawn?

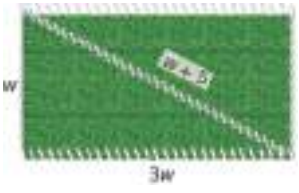
207. A rectangular table for the dining room has a surface area of 24 square feet. The length is two more feet than twice the width of the table. Find the length and width of the table.

208. The new computer has a surface area of 168 square inches. If the the width is 5.5 inches less that the length, what are the dimensions of the computer?

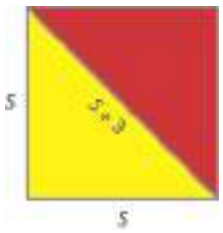
209. The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.

210. The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times as the length of the other leg . Round to the nearest tenth. Find the lengths of the three sides of the triangle.

211. A rectangular garden will be divided into two plots by fencing it on the diagonal. The diagonal distance from one corner of the garden to the opposite corner is five yards longer than the width of the garden. The length of the garden is three times the width. Find the length of the diagonal of the garden.



212. Nautical flags are used to represent letters of the alphabet. The flag for the letter, O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The hypotenuse of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.



213. Gerry plans to place a 25-foot ladder against the side of his house to clean his gutters. The bottom of the ladder will be 5 feet from the house.How for up the side of the house will the ladder reach?

214. John has a 10-foot piece of rope that he wants to use to support his 8-foot tree. How far from the base of the tree should he secure the rope?

215. A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0t$ to determine when the height of the firework rocket will be 1200 feet.

216. An arrow is shot vertically upward at a rate of 220 feet per second. Use the projectile formula $h = -16t^2 + v_0t$, to determine when height of the arrow will be 400 feet.

217. A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

218. A stone is dropped from a 196-foot platform. Use the formula $h = -16t^2 + v_0t + 196$ to determine how many seconds it will take for the stone to hit the ground. (Since the stone is dropped, $v_0 = 0$.)

219. The businessman took a small airplane for a quick flight up the coast for a lunch meeting and then returned home. The plane flew a total of 4 hours and each way the trip was 200 miles. What was the speed of the wind that affected the plane which was flying at a speed of 120 mph?

220. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 300 miles. If the plane was flying at 125 mph, what was the speed of the wind that affected the plane?

221. Roy kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

222. Rick paddled up the river, spent the night camping, and and then paddled back. He spent 10 hours paddling and the campground was 24 miles away. If Rick kayaked at a speed of 5 mph, what was the speed of the current?

223. Two painters can paint a room in 2 hours if they work together. The less experienced painter takes 3 hours more than the more experienced painter to finish the job. How long does it take for each painter to paint the room individually?

224. Two gardeners can do the weekly yard maintenance in 8 minutes if they work together. The older gardener takes 12 minutes more than the younger gardener to finish the job by himself. How long does it take for each gardener to do the weekly yard maintainence individually?

225. It takes two hours for two machines to manufacture 10,000 parts. If Machine #1 can do the job alone in one hour less than Machine #2 can do the job, how long does it take for each machine to manufacture 10,000 parts alone?

226. Sully is having a party and wants to fill his swimming pool. If he only uses his hose it takes 2 hours more than if he only uses his neighbor's hose. If he uses both hoses together, the pool fills in 4 hours. How long does it take for each hose to fill the hot tub?

Writing Exercises

227. Make up a problem involving the product of two consecutive odd integers.

ⓐ Start by choosing two consecutive odd integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where p is the product you found in part (b).

ⓓ Did you get the numbers you started with?

228. Make up a problem involving the product of two consecutive even integers.

ⓐ Start by choosing two consecutive even integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where p is the product you found in part (b).

ⓓ Did you get the numbers you started with?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

9.6 Graph Quadratic Functions Using Properties

Learning Objectives

By the end of this section, you will be able to:

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
 If you missed this problem, review **Example 3.54**.
2. Solve: $2x^2 + 3x - 2 = 0$.
 If you missed this problem, review **Example 6.45**.
3. Evaluate $-\frac{b}{2a}$ when $a = 3$ and $b = -6$.
 If you missed this problem, review **Example 1.21**.

Recognize the Graph of a Quadratic Function

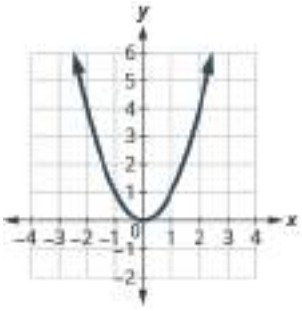
Previously we very briefly looked at the function $f(x) = x^2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f(x) = ax^2 + bx + c$ if $a \neq 0$. We call this kind of function a quadratic function.

Quadratic Function

A **quadratic function**, where a, b, and c are real numbers and $a \neq 0$, is a function of the form

$$f(x) = ax^2 + bx + c$$

We graphed the quadratic function $f(x) = x^2$ by plotting points.



x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Every quadratic function has a graph that looks like this. We call this figure a **parabola**.
Let's practice graphing a parabola by plotting a few points.

EXAMPLE 9.42

Graph $f(x) = x^2 - 1$.

Solution

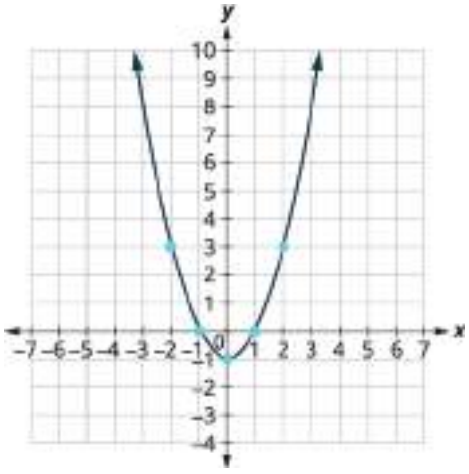
We will graph the function by plotting points.

Choose integer values for x, substitute them into the equation and simplify to find $f(x)$.

Record the values of the ordered pairs in the chart.

$f(x) = x^2 - 1$	
x	$f(x)$
0	−1
1	0
−1	0
2	3
−2	3

Plot the points, and then connect them with a smooth curve. The result will be the graph of the function $f(x) = x^2 - 1$.



TRY IT : : 9.83 Graph $f(x) = -x^2$. .

TRY IT : : 9.84 Graph $f(x) = x^2 + 1$.

All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward. See **Figure 9.2**.

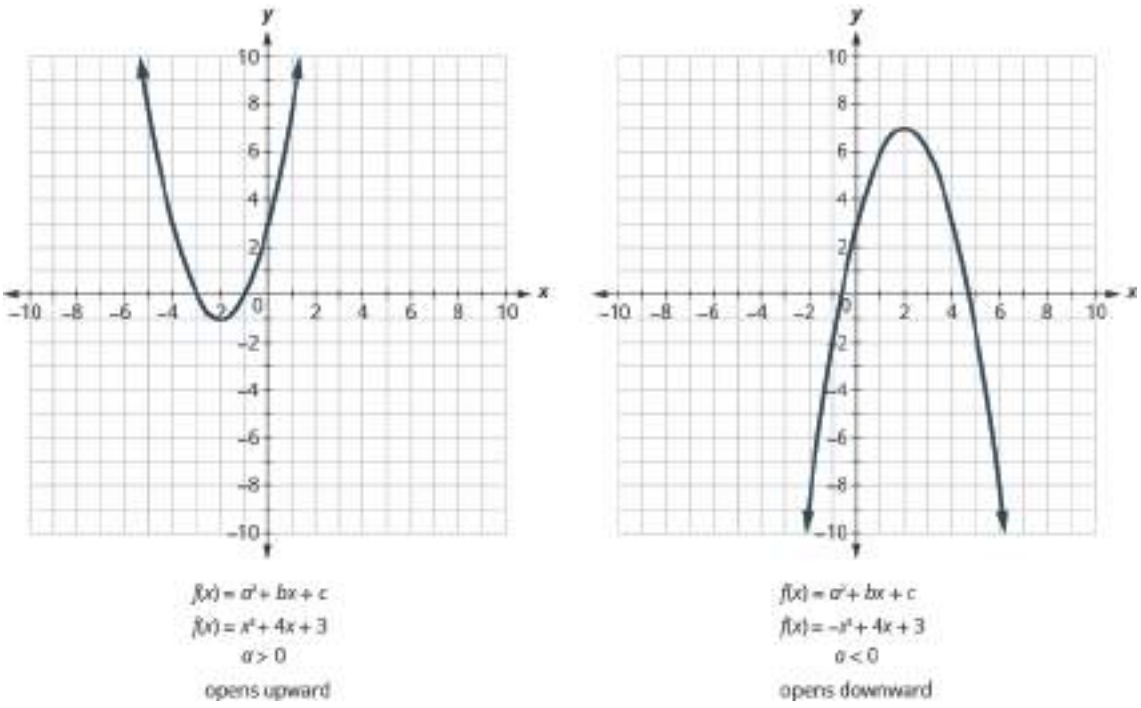


Figure 9.2

Notice that the only difference in the two functions is the negative sign before the quadratic term (x^2 in the equation of the graph in Figure 9.2). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Parabola Orientation

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward
- $a < 0$, the parabola opens downward

EXAMPLE 9.43

Determine whether each parabola opens upward or downward:

ⓐ $f(x) = -3x^2 + 2x - 4$ ⓑ $f(x) = 6x^2 + 7x - 9$.

Solution

ⓐ

Find the value of "a". $f(x) = ax^2 + bx + c$

$f(x) = -3x^2 + 2x - 4$

$a = -3$

Since the "a" is negative, the parabola will open downward.

ⓑ

Find the value of "a".	$f(x) = ax^2 + bx + c$ $f(x) = 6x^2 + 7x - 9$ $a = 6$
	Since the "a" is positive, the parabola will open upward.

TRY IT : : 9.85

Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ $f(x) = 2x^2 + 5x - 2$ ⓑ $f(x) = -3x^2 - 4x + 7$.

TRY IT : : 9.86

Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ $f(x) = -2x^2 - 2x - 3$ ⓑ $f(x) = 5x^2 - 2x - 1$.

Find the Axis of Symmetry and Vertex of a Parabola

Look again at Figure 9.2. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry. See Figure 9.3.

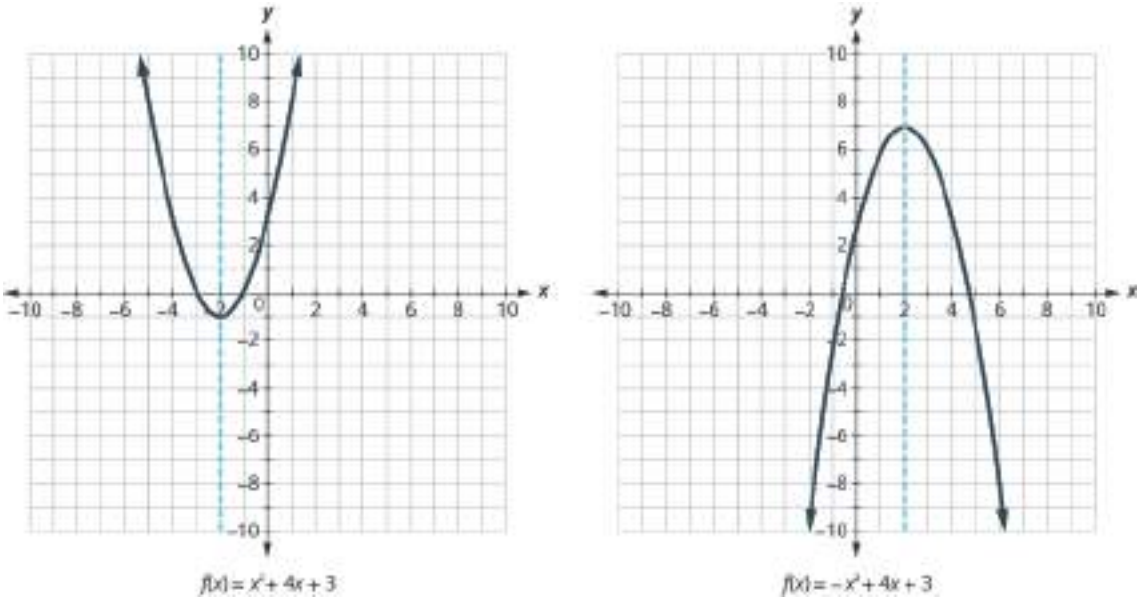


Figure 9.3

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$f(x) = ax^2 + bx + c$

$f(x) = x^2 + 4x + 3$

axis of symmetry

$x = -\frac{b}{2a}$

$x = -\frac{4}{2 \cdot 1}$

$x = -2$

$f(x) = ax^2 + bx + c$

$f(x) = -x^2 + 4x + 3$

axis of symmetry

$x = -\frac{b}{2a}$

$x = -\frac{4}{2(-1)}$

$x = 2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex we substitute the value of the x-coordinate into the quadratic function.

$f(x) = x^2 + 4x + 3$

axis of symmetry is $x = -2$

vertex is $(-2, __)$

$f(x) = x^2 + 4x + 3$

$f(x) = (-2)^2 + 4(-2) + 3$

$f(x) = -1$

vertex is $(-2, -1)$

$f(x) = -x^2 + 4x + 3$

axis of symmetry is $x = 2$

vertex is $(2, __)$

$f(x) = -x^2 + 4x + 3$

$f(x) = -(2)^2 + 4(2) + 3$

$f(x) = 7$

vertex is $(2, 7)$

Axis of Symmetry and Vertex of a Parabola

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$.
- the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

EXAMPLE 9.44

For the graph of $f(x) = 3x^2 - 6x + 2$ find:

ⓐ the axis of symmetry ⓑ the vertex.

Solution

ⓐ

$f(x) = ax^2 + bx + c$

$f(x) = 3x^2 - 6x + 2$

The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.	
Substitute the values of a, b into the equation.	$x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$

The axis of symmetry is the line $x = 1$.

ⓑ

	$f(x) = 3x^2 - 6x + 2$
The vertex is a point on the line of symmetry, so its *x*-coordinate will be $x = 1$. Find $f(1)$.	$f(1) = 3(1)^2 - 6(1) + 2$
Simplify.	$f(1) = 3 \cdot 1 - 6 + 2$
The result is the *y*-coordinate.	$f(1) = -1$
	The vertex is $(1, -1)$.

TRY IT : : 9.87 For the graph of $f(x) = 2x^2 - 8x + 1$ find:

ⓐ the axis of symmetry ⓑ the vertex.

TRY IT : : 9.88 For the graph of $f(x) = 2x^2 - 4x - 3$ find:

ⓐ the axis of symmetry ⓑ the vertex.

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the *x*- and *y*-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the *y*-intercept the value of *x* is zero. So to find the *y*-intercept, we substitute *x* = 0 into the function.

Let's find the *y*-intercepts of the two parabolas shown in **Figure 9.4**.

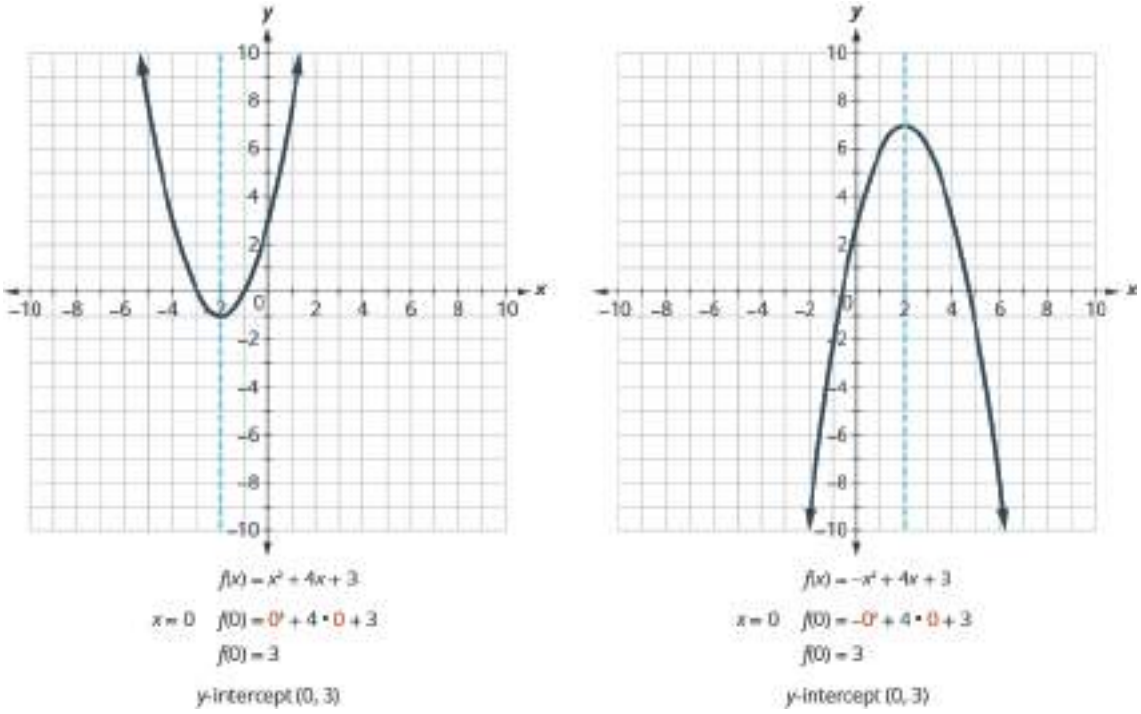


Figure 9.4

An *x*-intercept results when the value of $f(x)$ is zero. To find an *x*-intercept, we let $f(x) = 0$. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for *x*.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ 0 &= ax^2 + bx + c \end{aligned}$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the *x*-intercepts of the two parabolas we looked at. First we will find the *x*-intercepts of the parabola whose function is $f(x) = x^2 + 4x + 3$.

$$f(x) = x^2 + 4x + 3$$

Let $f(x) = 0$.	$0 = x^2 + 4x + 3$
Factor.	$0 = (x + 1)(x + 3)$
Use the Zero Product Property.	$x + 1 = 0 \quad x + 3 = 0$
Solve.	$x = -1 \quad x = -3$
	The *x*-intercepts are $(-1, 0)$ and $(-3, 0)$.

Now we will find the *x*-intercepts of the parabola whose function is $f(x) = -x^2 + 4x + 3$.

	$f(x) = -x^2 + 4x + 3$
Let $f(x) = 0$.	$0 = -x^2 + 4x + 3$
This quadratic does not factor, so we use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$a = -1,\ b = 4,\ c = 3$	$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(3)}}{2(-1)}$
Simplify.	$x = \frac{-4 \pm \sqrt{28}}{-2}$
	$x = \frac{-4 \pm 2\sqrt{7}}{-2}$
	$x = \frac{-2(2 \pm \sqrt{7})}{-2}$
	$x = 2 \pm \sqrt{7}$
	The x-intercepts are $(2 + \sqrt{7},\ 0)$ and $(2 - \sqrt{7},\ 0)$.

We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph,

$$(2 + \sqrt{7},\ 0) \approx (4.6,\ 0) \qquad (2 - \sqrt{7},\ 0) \approx (-0.6,\ 0)$$

Do these results agree with our graphs? See **Figure 9.5**.

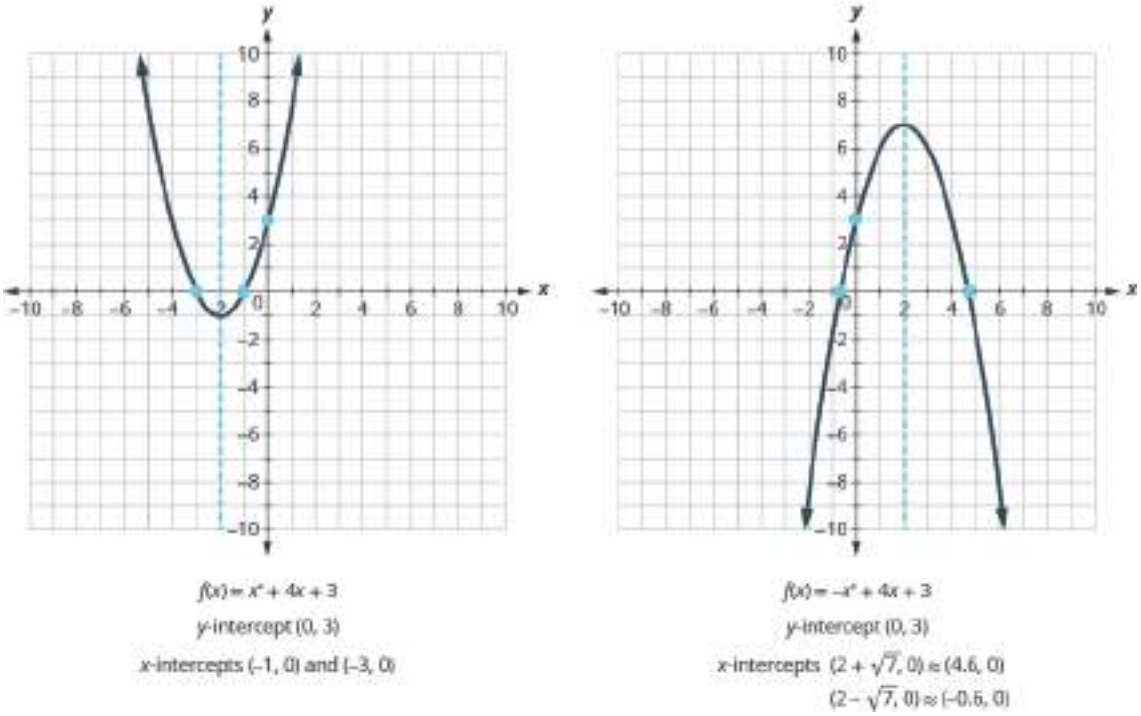


Figure 9.5

Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

***y*-intercept**	***x*-intercepts**
Let $x = 0$ and solve for $f(x)$.	Let $f(x) = 0$ and solve for x.

EXAMPLE 9.45

Find the intercepts of the parabola whose function is $f(x) = x^2 - 2x - 8$.

Solution

To find the ***y*-intercept**, let $x = 0$ and solve for $f(x)$.	$f(x) = x^2 - 2x - 8$
	$f(0) = 0^2 - 2 \cdot 0 - 8$
	$f(0) = -8$
	When $x = 0$, then $f(0) = -8$. The *y*-intercept is the point $(0, -8)$.
To find the ***x*-intercept**, let $f(x) = 0$ and solve for x.	$f(x) = x^2 - 2x - 8$
	$0 = x^2 - 2x - 8$
Solve by factoring.	$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$

$4 = x \quad -2 = x$

When $f(x) = 0$, then $x = 4$ or $x = -2$.
The *x*-intercepts are the points $(4, 0)$ and $(-2, 0)$.

TRY IT : : 9.89 Find the intercepts of the parabola whose function is $f(x) = x^2 + 2x - 8$.

TRY IT : : 9.90 Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 12$.

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic functions of the form $f(x) = ax^2 + bx + c$. The graphs of these functions are parabolas. The *x*-intercepts of the parabolas occur where $f(x) = 0$.

For example:

Quadratic equation

$$\begin{aligned} x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \\ x - 5 = 0 \quad x + 3 &= 0 \\ x = 5 \qquad x &= -3 \end{aligned}$$

Let $f(x) = 0$.

Quadratic function

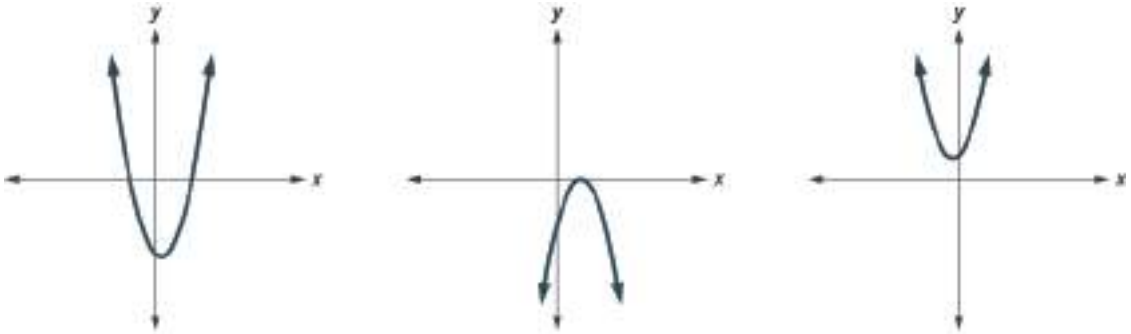
$$\begin{aligned} f(x) &= x^2 - 2x - 15 \\ 0 &= x^2 - 2x - 15 \\ 0 &= (x-5)(x+3) \\ x - 5 &= 0 \quad x + 3 = 0 \\ x &= 5 \qquad x = -3 \end{aligned}$$

$(5, 0)$ and $(-3, 0)$
x-intercepts

The solutions of the quadratic function are the *x* values of the *x*-intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the *x*-intercepts of the graphs, the number of *x*-intercepts is the same as the number of solutions.

Previously, we used the discriminant to determine the number of solutions of a quadratic function of the form $ax^2 + bx + c = 0$. Now we can use the discriminant to tell us how many *x*-intercepts there are on the graph.



$b^2 - 4ac > 0$
Two solutions
Two *x*-intercepts

$b^2 - 4ac = 0$
One solution
One *x*-intercept

$b^2 - 4ac < 0$
No real solution
No *x*-intercept

Before you to find the values of the *x*-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

EXAMPLE 9.46

Find the intercepts of the parabola for the function $f(x) = 5x^2 + x + 4$.

Solution

	$f(x) = 5x^2 + x + 4$
To find the *y*-intercept, let $x = 0$ and solve for $f(x)$.	$f(0) = 5 \cdot 0^2 + 0 + 4$
	$f(0) = 4$
	When $x = 0$, then $f(0) = 4$. The *y*-intercept is the point $(0, 4)$.
To find the *x*-intercept, let $f(x) = 0$ and solve for x.	$f(x) = 5x^2 + x + 4$
	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions which is also the number of *x*-intercepts.	
$b^2 - 4ac$ $1^2 - 4 \cdot 5 \cdot 4$ $1 - 80$ -79	
	Since the value of the discriminant is negative, there is no real solution to the equation. There are no *x*-intercepts.

TRY IT : : 9.91 Find the intercepts of the parabola whose function is $f(x) = 3x^2 + 4x + 4$.

TRY IT : : 9.92 Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 5$.

Graph Quadratic Functions Using Properties

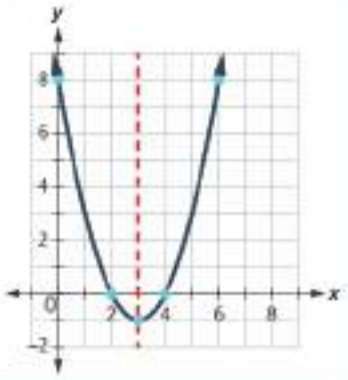
Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

EXAMPLE 9.47 HOW TO GRAPH A QUADRATIC FUNCTION USING PROPERTIES

Graph $f(x) = x^2 - 6x + 8$ by using its properties.

Solution

Step 1. Determine whether the parabola opens upward or downward.	Look at a in the equation. $f(x) = x^2 - 6x + 8$ Since a is positive, the parabola opens upward.	$f(x) = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.**

Step 2. Find the axis of symmetry.	$f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
Step 3. Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is (3, −1).**
Step 4. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.	We find $f(0)$. We use the axis of symmetry to find a point symmetric to the *y*-intercept. The *y*-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	*y*-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The *y*-intercept is (0, 8).** Point symmetric to *y*-intercept: **The point is (6, 8).**
Step 5. Find the *x*-intercepts. Find additional points if needed.	We solve $f(x) = 0$. We can solve this quadratic equation by factoring.	*x*-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The *x*-intercepts are (2, 0) and (4, 0).**
Step 6. Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the *y*-intercept. We connect these 5 points to sketch the parabola.	

TRY IT : : 9.93 Graph $f(x) = x^2 + 2x - 8$ by using its properties.

TRY IT : : 9.94 Graph $f(x) = x^2 - 8x + 12$ by using its properties.

We list the steps to take in order to graph a quadratic function here.

HOW TO : : TO GRAPH A QUADRATIC FUNCTION USING PROPERTIES.

Step 1. Determine whether the parabola opens upward or downward.
Step 2. Find the equation of the axis of symmetry.
Step 3. Find the vertex.
Step 4. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
Step 5. Find the *x*-intercepts. Find additional points if needed.
Step 6. Graph the parabola.

We were able to find the *x*-intercepts in the last example by factoring. We find the *x*-intercepts in the next example by factoring, too.

EXAMPLE 9.48

Graph $f(x) = x^2 + 6x - 9$ by using its properties.

Solution

$f(x) = ax^2 + bx + c$
$f(x) = -x^2 + 6x - 9$

Since a is -1, the parabola opens downward.



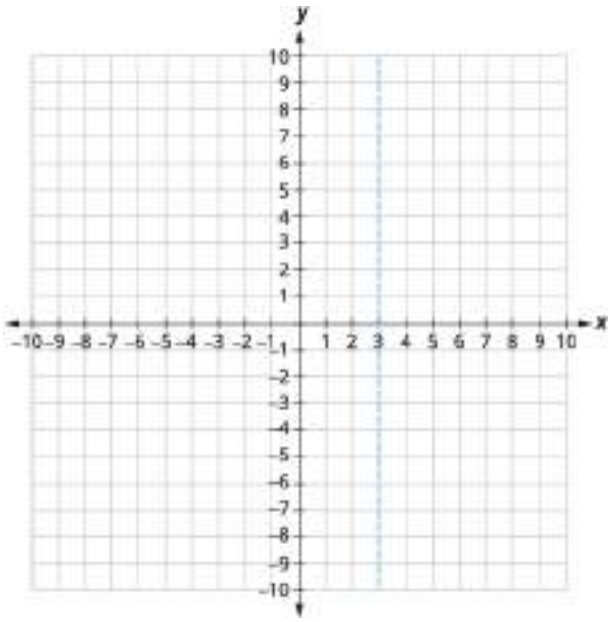
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.

$x = -\frac{b}{2a}$

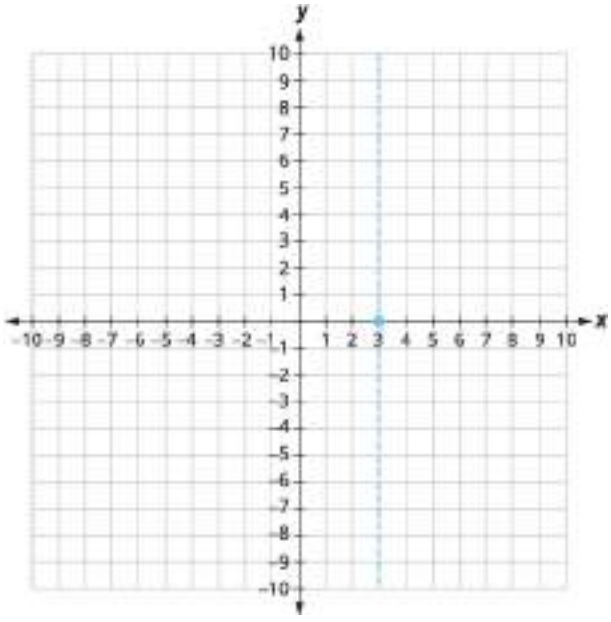
$x = -\frac{6}{2(-1)}$

$x = 3$

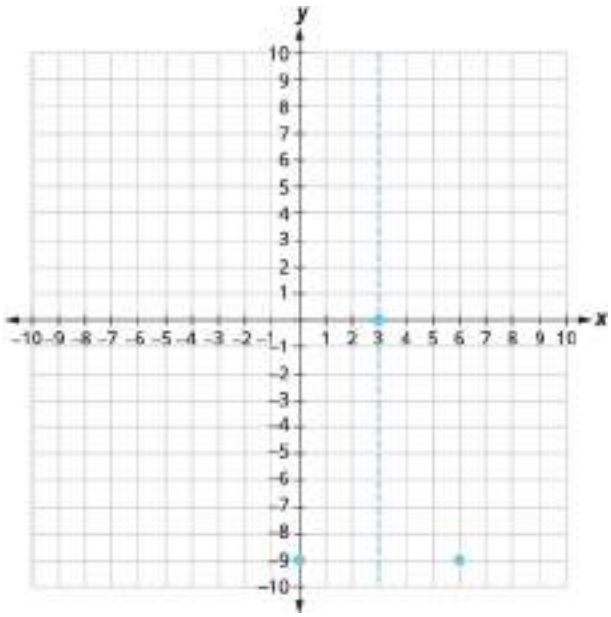
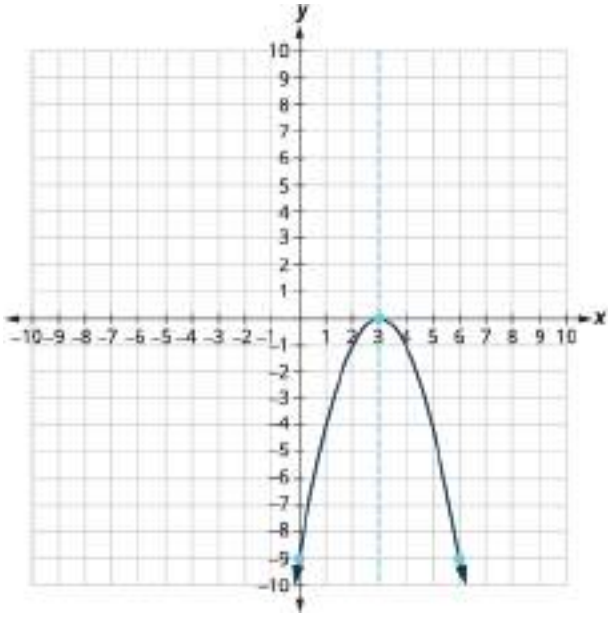
The axis of symmetry is $x = 3$.
The vertex is on the line $x = 3$.



Find $f(3)$.	$f(x) = -x^2 + 6x - 9$
	$f(3) = -3^2 + 6 \cdot 3 - 9$
	$f(3) = -9 + 18 - 9$
	$f(3) = 0$
	The vertex is $(3, 0)$.



The *y*-intercept occurs when $x = 0$. Find $f(0)$.	$f(x) = -x^2 + 6x - 9$
Substitute $x = 0$.	$f(0) = -0^2 + 6 \cdot 0 - 9$
Simplify.	$f(0) = -9$

	The *y*-intercept is $(0, -9)$.
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.	
	Point symmetric to the *y*-intercept is $(6, -9)$
The *x*-intercept occurs when $f(x) = 0$.	$f(x) = -x^2 + 6x - 9$
Find $f(x) = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for *x*.	$0 = 3$
Connect the points to graph the parabola.	

TRY IT : : 9.95 Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.

TRY IT : : 9.96 Graph $f(x) = 4x^2 + 24x + 36$ by using its properties.

For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

How many x-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

EXAMPLE 9.49

Graph $f(x) = x^2 + 4x + 5$ by using its properties.

Solution

$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 5$

Since a is 1, the parabola opens upward.



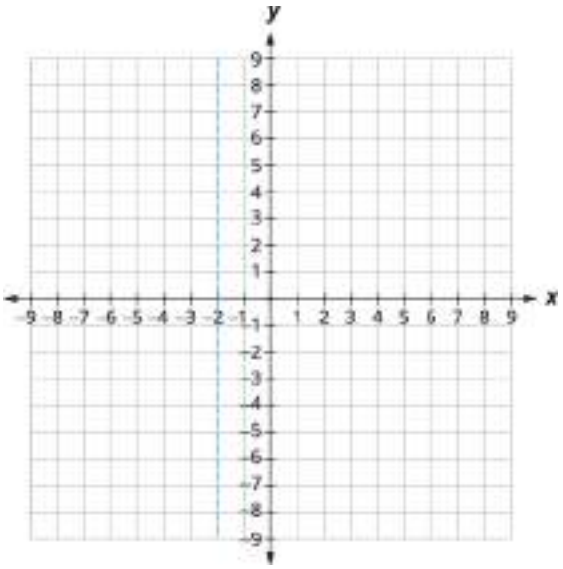
To find the axis of symmetry, find $x = -\frac{b}{2a}$.

$x = -\frac{b}{2a}$

$x = -\frac{4}{(2)1}$

$x = -2$

The equation of the axis of symmetry is $x = -2$.



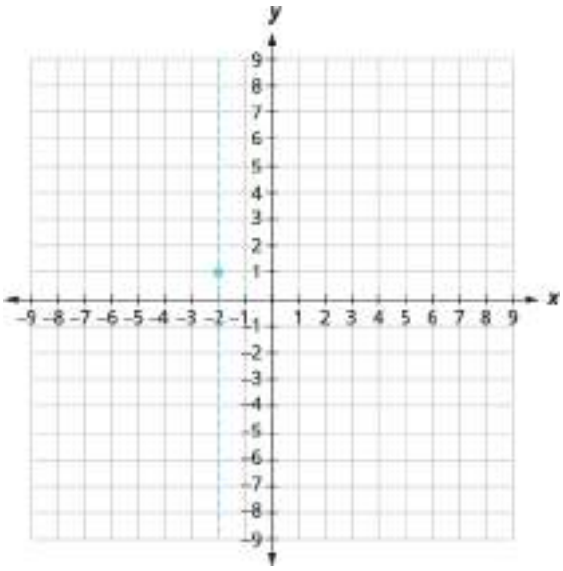
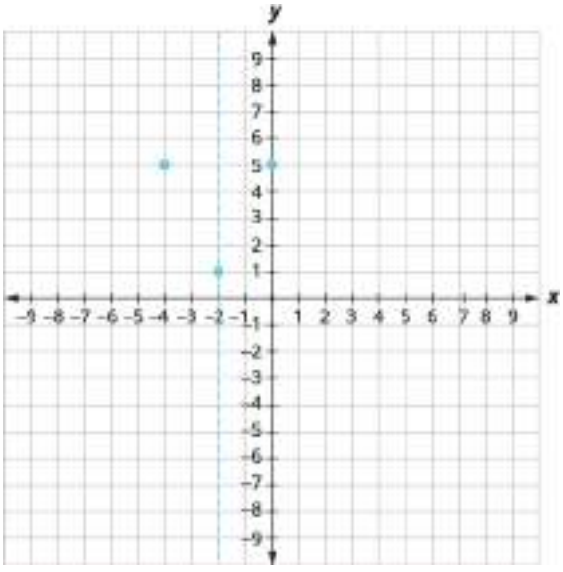
The vertex is on the line $x = -2$.

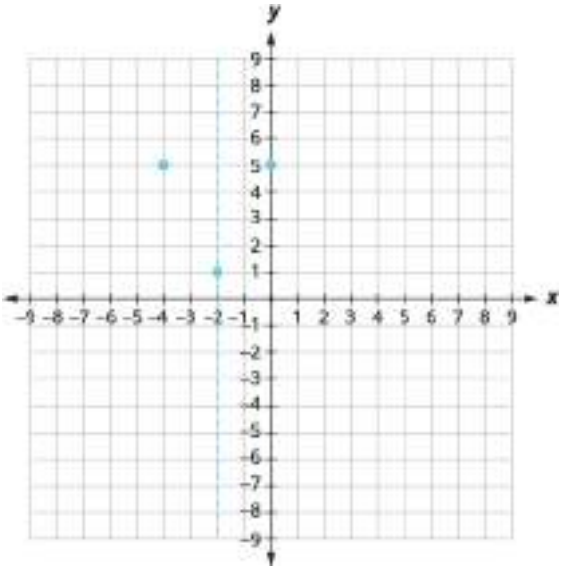
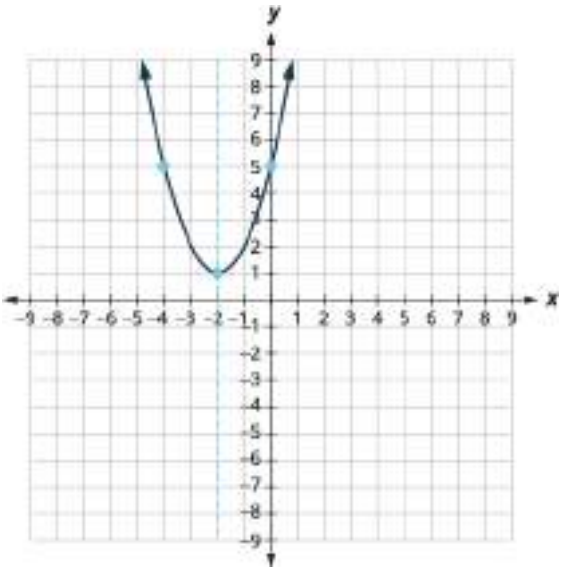
Find $f(x)$ when $x = -2$.

$f(x) = x^2 + 4x + 5$

$f(-2) = (-2)^2 + 4(-2) + 5$

$f(-2) = 4 - 8 + 5$

	$f(-2) = 1$
	The vertex is $(-2, 1)$.
	
The *y*-intercept occurs when $x = 0$.	$f(x) = x^2 + 4x - 5$
Find $f(0)$.	$f(0) = 5$
Simplify.	$f(0) = 5$
	The *y*-intercept is $(0, 5)$.
The point $(-4, 5)$ is two units to the left of the line of symmetry. The point two units to the right of the line of symmetry is $(0, 5)$.	
	Point symmetric to the *y*-intercept is $(-4, 5)$.

The x-intercept occurs when $f(x) = 0$.	
Find $f(x) = 0$.	$0 = x^2 + 4x + 5$
Test the discriminant.	
	$b^2 - 4ac$
	$4^2 - 4 \cdot 1 \cdot 5$
	$16 - 20$
	-4
Since the value of the discriminant is negative, there is no real solution and so no x-intercept.	
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.	

TRY IT : : 9.97 Graph $f(x) = x^2 - 2x + 3$ by using its properties.

TRY IT : : 9.98 Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.

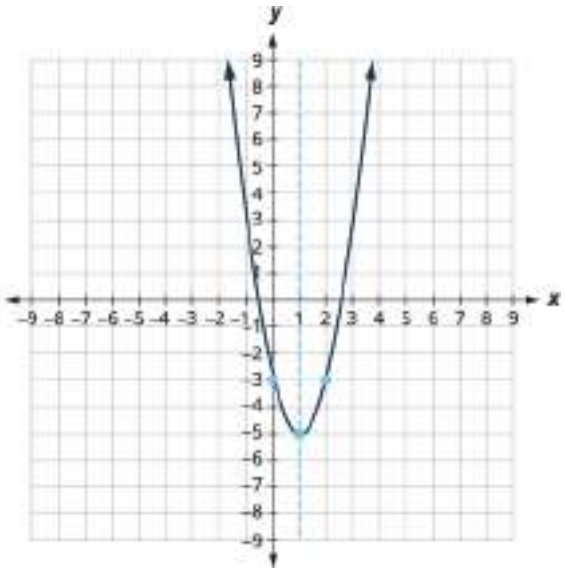
Finding the *y*-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the Quadratic Formula to find the *x*-intercepts.

EXAMPLE 9.50

Graph $f(x) = 2x^2 - 4x - 3$ by using its properties.

Solution

	$f(x) = ax^2 + bx + c$ $f(x) = 2x^2 - 4x - 3$
Since a is 2, the parabola opens upward. 	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{-4}{2 \cdot 2}$
	$x = 1$
	The equation of the axis of symmetry is $x = 1$.
The vertex is on the line $x = 1$.	$f(x) = 2x^2 - 4x - 3$
Find $f(1)$.	$f(x) = 2(1)^2 - 4(1) - 3$
	$f(1) = 2 - 4 - 3$
	$f(1) = -5$
	The vertex is $(1, -5)$.
The *y*-intercept occurs when $x = 0$.	$f(x) = 2x^2 - 4x - 3$
Find $f(0)$.	$f(0) = 2(0)^2 - 4(0) - 3$
Simplify.	$f(0) = -3$
	The *y*-intercept is $(0, -3)$.
The point $(0, -3)$ is one unit to the left of the line of symmetry.	**Point symmetric to the *y*-intercept is $(2, -3)$**
The point one unit to the right of the line of symmetry is $(2, -3)$.	
The *x*-intercept occurs when $y = 0$.	$f(x) = 2x^2 - 4x - 3$
Find $f(x) = 0$.	$0 = 2x^2 - 4x - 3$
Use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute in the values of a, b, and c.	$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(3)}}{2(2)}$
Simplify.	$x = \frac{-4 \pm \sqrt{16 + 24}}{4}$

Simplify inside the radical.	$x=\frac{4\pm\sqrt{40}}{4}$
Simplify the radical.	$x=\frac{4\pm2\sqrt{10}}{4}$
Factor the GCF.	$x=\frac{2(2\pm\sqrt{10})}{4}$
Remove common factors.	$x=\frac{2\pm\sqrt{10}}{2}$
Write as two equations.	$x=\frac{2+\sqrt{10}}{2},\quad x=\frac{2-\sqrt{10}}{2}$
Approximate the values.	$x\approx2.5,\quad x\approx-0.6$
	The approximate values of the *x*-intercepts are $(2.5,\ 0)$ and $(-0.6,\ 0)$.
Graph the parabola using the points found.	

TRY IT : : 9.99 Graph $f(x) = 5x^2 + 10x + 3$ by using its properties.

TRY IT : : 9.100 Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.

Solve Maximum and Minimum Applications

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The *y*-coordinate of the vertex is the minimum value of a parabola that opens upward. It is the maximum value of a parabola that opens downward. See Figure 9.6.

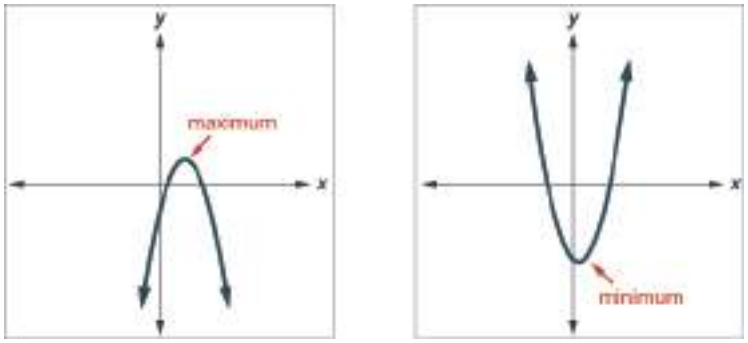


Figure 9.6

Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

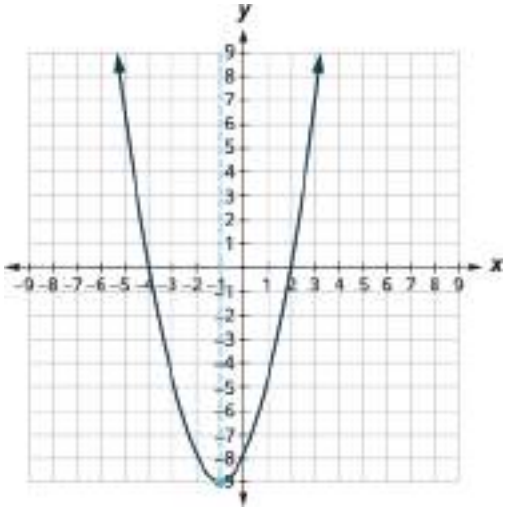
- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

EXAMPLE 9.51

Find the minimum or maximum value of the quadratic function $f(x) = x^2 + 2x - 8$.

Solution

	$f(x) = x^2 + 2x - 8$
Since a is positive, the parabola opens upward. The quadratic equation has a minimum.	
Find the equation of the axis of symmetry.	$x = -\frac{b}{2a}$
	$x = -\frac{2}{2 \times 1}$
	$x = -1$
	The equation of the axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$f(x) = x^2 + 2x - 8$
Find $f(-1)$.	$f(-1) = (-1)^2 + 2(-1) - 8$
	$f(-1) = 1 - 2 - 8$
	$f(-1) = -9$
	The vertex is $(-1, -9)$.
Since the parabola has a minimum, the y-coordinate of the vertex is the minimum y-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x = -1$.	



Show the graph to verify the result.

TRY IT : : 9.101 Find the maximum or minimum value of the quadratic function $f(x) = x^2 - 8x + 12$.

TRY IT : : 9.102 Find the maximum or minimum value of the quadratic function $f(x) = -4x^2 + 16x - 11$.

We have used the formula

$$h(t) = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, h , of an object shot upwards into the air with initial velocity, v_0, after t seconds .

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex (t, h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

EXAMPLE 9.52

The quadratic equation $h(x) = -16t^2 + 176t + 4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

ⓐ How many seconds will it take the volleyball to reach its maximum height? ⓑ Find the maximum height of the volleyball.

Solution

$$h(t) = -16t^2 + 176t + 4$$

Since a is negative, the parabola opens downward.
The quadratic function has a maximum.

ⓐ

Find the equation of the axis of symmetry.	$t = -\frac{b}{2a}$
	$t = -\frac{176}{2(-16)}$
	$t = 5.5$
	The equation of the axis of symmetry is $t = 5.5$.
The vertex is on the line $t = 5.5$.	The maximum occurs when $t = 5.5$ seconds.

ⓑ

Find $h(5.5)$.	$h(t) = -16t^2 + 176t + 4$
	$h(t) = -16(5.5)^2 + 176(5.5) + 4$
Use a calculator to simplify.	$h(t) = 488$
	The vertex is $(5.5, 488)$.

Since the parabola has a maximum, the h-coordinate of the vertex is the maximum value of the quadratic function.
The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.
After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

TRY IT : : 9.103

Solve, rounding answers to the nearest tenth.

The quadratic function $h(x) = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

TRY IT : : 9.104

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of. $h(x) = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height?

MEDIA : :

Access these online resources for additional instruction and practice with graphing quadratic functions using properties.

- **Quadratic Functions: Axis of Symmetry and Vertex (https://openstax.org/l/37QuadFunct1)**
- **Finding x- and y-intercepts of a Quadratic Function (https://openstax.org/l/37QuadFunct2)**
- **Graphing Quadratic Functions (https://openstax.org/l/37QuadFunct3)**
- **Solve Maxiumum or Minimum Applications (https://openstax.org/l/37QuadFunct4)**
- **Quadratic Applications: Minimum and Maximum (https://openstax.org/l/37QuadFunct5)**

9.6 EXERCISES

Practice Makes Perfect

Recognize the Graph of a Quadratic Function

In the following exercises, graph the functions by plotting points.

229. $f(x) = x^2 + 3$

230. $f(x) = x^2 - 3$

231. $y = -x^2 + 1$

232. $f(x) = -x^2 - 1$

For each of the following exercises, determine if the parabola opens up or down.

233.

ⓐ $f(x) = -2x^2 - 6x - 7$

ⓑ $f(x) = 6x^2 + 2x + 3$

234.

ⓐ $f(x) = 4x^2 + x - 4$

ⓑ $f(x) = -9x^2 - 24x - 16$

235.

ⓐ $f(x) = -3x^2 + 5x - 1$

ⓑ $f(x) = 2x^2 - 4x + 5$

236.

ⓐ $f(x) = x^2 + 3x - 4$

ⓑ $f(x) = -4x^2 - 12x - 9$

Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find ⓐ the equation of the axis of symmetry and ⓑ the vertex of its graph.

237. $f(x) = x^2 + 8x - 1$

238. $f(x) = x^2 + 10x + 25$

239. $f(x) = -x^2 + 2x + 5$

240. $f(x) = -2x^2 - 8x - 3$

Find the Intercepts of a Parabola

In the following exercises, find the intercepts of the parabola whose function is given.

241. $f(x) = x^2 + 7x + 6$

242. $f(x) = x^2 + 10x - 11$

243. $f(x) = x^2 + 8x + 12$

244. $f(x) = x^2 + 5x + 6$

245. $f(x) = -x^2 + 8x - 19$

246. $f(x) = -3x^2 + x - 1$

247. $f(x) = x^2 + 6x + 13$

248. $f(x) = x^2 + 8x + 12$

249. $f(x) = 4x^2 - 20x + 25$

250. $f(x) = -x^2 - 14x - 49$

251. $f(x) = -x^2 - 6x - 9$

252. $f(x) = 4x^2 + 4x + 1$

Graph Quadratic Functions Using Properties

In the following exercises, graph the function by using its properties.

253. $f(x) = x^2 + 6x + 5$

254. $f(x) = x^2 + 4x - 12$

255. $f(x) = x^2 + 4x + 3$

256. $f(x) = x^2 - 6x + 8$

257. $f(x) = 9x^2 + 12x + 4$

258. $f(x) = -x^2 + 8x - 16$

259. $f(x) = -x^2 + 2x - 7$

260. $f(x) = 5x^2 + 2$

261. $f(x) = 2x^2 - 4x + 1$

262. $f(x) = 3x^2 - 6x - 1$

263. $f(x) = 2x^2 - 4x + 2$

264. $f(x) = -4x^2 - 6x - 2$

265. $f(x) = -x^2 - 4x + 2$

266. $f(x) = x^2 + 6x + 8$

267. $f(x) = 5x^2 - 10x + 8$

268. $f(x) = -16x^2 + 24x - 9$

269. $f(x) = 3x^2 + 18x + 20$

270. $f(x) = -2x^2 + 8x - 10$

Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value of each function.

271. $f(x) = 2x^2 + x - 1$

272. $y = -4x^2 + 12x - 5$

273. $y = x^2 - 6x + 15$

274. $y = -x^2 + 4x - 5$

275. $y = -9x^2 + 16$

276. $y = 4x^2 - 49$

In the following exercises, solve. Round answers to the nearest tenth.

277. An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

278. A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic function $h(t) = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

279. A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

280. A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic function $h(t) = -16t^2 + 122t + 0$ to find how long it will take for the ball to reach its maxiumum height, and then find the maximum height.

281. A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

282. A retailer who sells backpacks estimates that by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic function $R(x) = -x^2 + 100x$ is used to find the R, received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

283. A retailer who sells fashion boots estimates that by selling them for x dollars each, he will be able to sell $70 - x$ boots a week. Use the quadratic function $R(x) = -x^2 + 70x$ to find the revenue received when the average selling price of a pair of fashion boots is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

284. A cell phone company estimates that by charging x dollars each for a certain cell phone, they can sell $8 - x$ cell phones per day. Use the quadratic function $R(x) = -x^2 + 8x$ to find the revenue received when the selling price of a cell phone is x. Find the selling price that will give them the maximum revenue, and then find the amount of the maximum revenue.

285. A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A(x) = x(240 - 2x)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

286. A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic function $A(x) = x(100 - 2x)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

287. A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the "walls." He wants to maximize the area using 80 feet of fencing. The quadratic function $A(x) = x(80 - 2x)$ gives the area of the patio, where x is the width of one side. Find the maximum area of the patio.

288. A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic function $A(x) = x(300 - 2x)$ to determine the maximum area of the fenced in yard.

Writing Exercise

289. How do the graphs of the functions $f(x) = x^2$ and $f(x) = x^2 - 1$ differ? We graphed them at the start of this section. What is the difference between their graphs? How are their graphs the same?

290. Explain the process of finding the vertex of a parabola.

291. Explain how to find the intercepts of a parabola.

292. How can you use the discriminant when you are graphing a quadratic function?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

9.7 Graph Quadratic Functions Using Transformations

Learning Objectives

By the end of this section, you will be able to:

- Graph quadratic functions of the form $f(x) = x^2 + k$
- Graph quadratic functions of the form $f(x) = (x - h)^2$
- Graph quadratic functions of the form $f(x) = ax^2$
- Graph quadratic functions using transformations
- Find a quadratic function from its graph

Be Prepared!

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
 If you missed this problem, review **Example 3.54**.
2. Factor completely: $y^2 - 14y + 49$.
 If you missed this problem, review **Example 6.24**.
3. Factor completely: $2x^2 - 16x + 32$.
 If you missed this problem, review **Example 6.26**.

Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the last section, we learned how to graph quadratic functions using their properties. Another method involves starting with the basic graph of $f(x) = x^2$ and 'moving' it according to information given in the function equation. We call this graphing quadratic functions using transformations.

In the first example, we will graph the quadratic function $f(x) = x^2$ by plotting points. Then we will see what effect adding a constant, *k*, to the equation will have on the graph of the new function $f(x) = x^2 + k$.

EXAMPLE 9.53

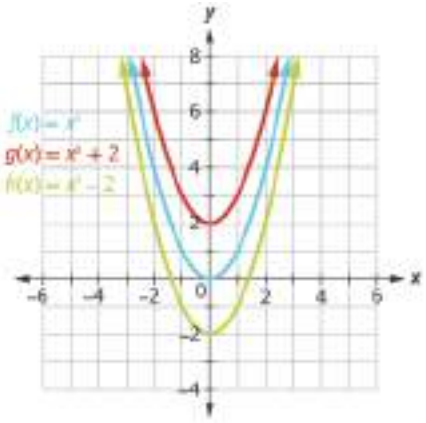
Graph $f(x) = x^2$, $g(x) = x^2 + 2$, and $h(x) = x^2 - 2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

Solution

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = x^2 + 2$	$(x, g(x))$	$h(x) = x^2 - 2$	$(x, h(x))$
−3	9	(−3, 9)	9 + 2	(−3, 11)	9 − 2	(−3, 7)
−2	4	(−2, 4)	4 + 2	(−2, 6)	4 − 2	(−2, 2)
−1	1	(−1, 1)	1 + 2	(−1, 3)	1 − 2	(−1, 1)
0	0	(0, 0)	0 + 2	(0, 2)	0 − 2	(0, −2)
1	1	(1, 1)	1 + 2	(1, 3)	1 − 2	(1, −1)
2	4	(2, 4)	4 + 2	(2, 6)	4 − 2	(2, 2)
3	9	(3, 9)	9 + 2	(3, 11)	9 − 2	(3, 7)

The $g(x)$ values are two more than the $f(x)$ values. Also, the $h(x)$ values are two less than the $f(x)$ values. Now we will graph all three functions on the same rectangular coordinate system.



The graph of $g(x) = x^2 + 2$ is the same as the graph of $f(x) = x^2$ but shifted up 2 units.

The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ but shifted down 2 units.

TRY IT : : 9.105

ⓐ Graph $f(x) = x^2,\ g(x) = x^2 + 1,$ and $h(x) = x^2 - 1$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

TRY IT : : 9.106

ⓐ Graph $f(x) = x^2,\ g(x) = x^2 + 6,$ and $h(x) = x^2 - 6$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

The last example shows us that to graph a quadratic function of the form $f(x) = x^2 + k,$ we take the basic parabola graph of $f(x) = x^2$ and vertically shift it up $(k > 0)$ or shift it down $(k < 0)$.

This transformation is called a vertical shift.

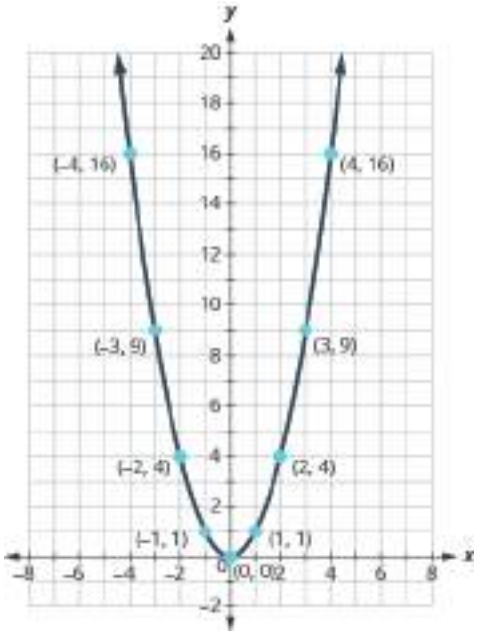
Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift

The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.

- If $k > 0$, shift the parabola vertically up k units.
- If $k < 0$, shift the parabola vertically down $|k|$ units.

Now that we have seen the effect of the constant, k, it is easy to graph functions of the form $f(x) = x^2 + k.$ We just start with the basic parabola of $f(x) = x^2$ and then shift it up or down.

It may be helpful to practice sketching $f(x) = x^2$ quickly. We know the values and can sketch the graph from there.



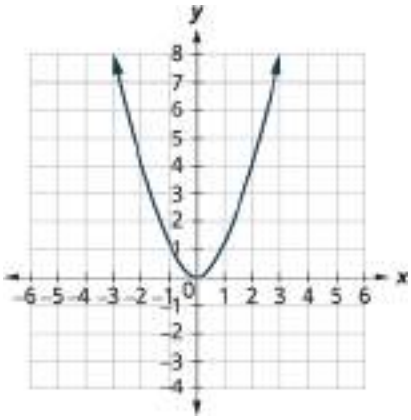
Once we know this parabola, it will be easy to apply the transformations. The next example will require a vertical shift.

EXAMPLE 9.54

Graph $f(x) = x^2 - 3$ using a vertical shift.

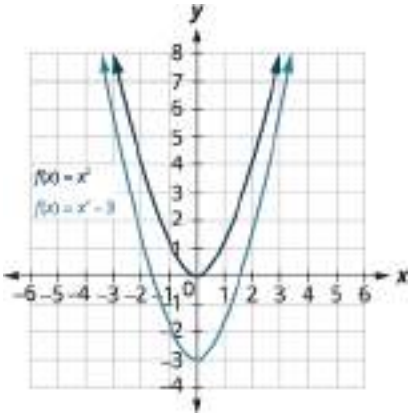
Solution

We first draw the graph of $f(x) = x^2$ on the grid.



Determine k.	$f(x) = x^2 + k$ $f(x) = x^2 - 3$
	$k = -3$

Shift the graph $f(x) = x^2$ down 3.



TRY IT : : 9.107 Graph $f(x) = x^2 - 5$ using a vertical shift.

TRY IT : : 9.108 Graph $f(x) = x^2 + 7$ using a vertical shift.

Graph Quadratic Functions of the form $f(x) = (x - h)^2$

In the first example, we graphed the quadratic function $f(x) = x^2$ by plotting points and then saw the effect of adding a constant *k* to the function had on the resulting graph of the new function $f(x) = x^2 + k$.

We will now explore the effect of subtracting a constant, *h*, from *x* has on the resulting graph of the new function $f(x) = (x - h)^2$.

EXAMPLE 9.55

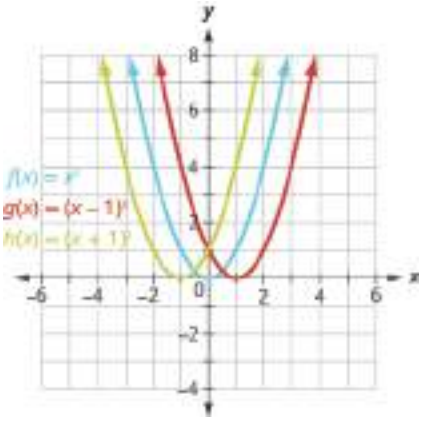
Graph $f(x) = x^2$, $g(x) = (x - 1)^2$, and $h(x) = (x + 1)^2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

Solution

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = (x-1)^2$	$(x, g(x))$	$h(x) = (x+1)^2$	$(x, h(x))$
−3	9	(−3, 9)	16	(−3, 16)	4	(−3, 4)
−2	4	(−2, 4)	9	(−2, 9)	1	(−2, 1)
−1	1	(−1, 1)	4	(−1, 4)	0	(−1, 0)
0	0	(0, 0)	1	(0, 1)	1	(0, 1)
1	1	(1, 1)	0	(1, 0)	4	(1, 4)
2	4	(2, 4)	1	(2, 1)	9	(2, 9)
3	9	(3, 9)	4	(3, 4)	16	(3, 16)

The *g*(*x*) values and the *h*(*x*) values share the common numbers 0, 1, 4, 9, and 16, but are shifted.



The graph of $g(x) = (x - 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 1 unit.

The graph of $h(x) = (x + 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 1 unit.



TRY IT : : 9.109

ⓐ Graph $f(x) = x^2$, $g(x) = (x + 2)^2$, and $h(x) = (x - 2)^2$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

TRY IT : : 9.110

ⓐ Graph $f(x) = x^2$, $g(x) = x^2 + 5$, and $h(x) = x^2 - 5$ on the same rectangular coordinate system.

ⓑ Describe what effect adding a constant to the function has on the basic parabola.

The last example shows us that to graph a quadratic function of the form $f(x) = (x - h)^2$, we take the basic parabola graph of $f(x) = x^2$ and shift it left ($h > 0$) or shift it right ($h < 0$).

This transformation is called a horizontal shift.

Graph a Quadratic Function of the form $f(x) = (x - h)^2$ Using a Horizontal Shift

The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.

- If $h > 0$, shift the parabola horizontally left h units.
- If $h < 0$, shift the parabola horizontally right $|h|$ units.

Now that we have seen the effect of the constant, h, it is easy to graph functions of the form $f(x) = (x - h)^2$. We just start with the basic parabola of $f(x) = x^2$ and then shift it left or right.

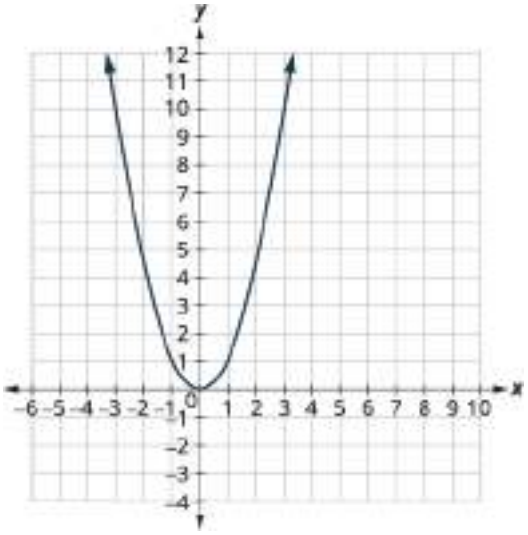
The next example will require a horizontal shift.

EXAMPLE 9.56

Graph $f(x) = (x - 6)^2$ using a horizontal shift.

Solution

We first draw the graph of $f(x) = x^2$ on the grid.



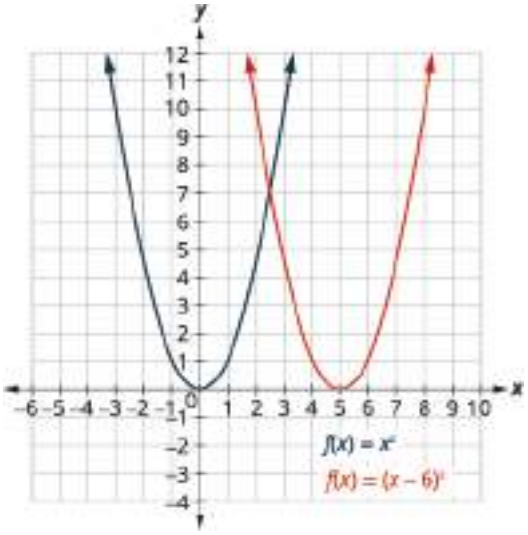
Determine *h*.

$f(x) = (x-h)^2$

$f(x) = (x-6)^2$

$h = 6$

Shift the graph $f(x) = x^2$ to the right 6 units.



TRY IT : : 9.111 Graph $f(x) = (x-4)^2$ using a horizontal shift.

TRY IT : : 9.112 Graph $f(x) = (x+6)^2$ using a horizontal shift.

Now that we know the effect of the constants *h* and *k*, we will graph a quadratic function of the form $f(x) = (x-h)^2 + k$ by first drawing the basic parabola and then making a horizontal shift followed by a vertical shift. We could do the vertical shift followed by the horizontal shift, but most students prefer the horizontal shift followed by the vertical.

EXAMPLE 9.57

Graph $f(x) = (x+1)^2 - 2$ using transformations.

Solution

This function will involve two transformations and we need a plan.

Let's first identify the constants *h*, *k*.

$$f(x) = (x+1)^2 - 2$$

$$f(x) = (x-h)^2 + k$$

$$f(x) = (x-(-1))^2 + (-2)$$

$$h = -1 \quad k = -2$$

The *h* constant gives us a horizontal shift and the *k* gives us a vertical shift.

$f(x) = x^2 \longrightarrow f(x) = (x+1)^2 \longrightarrow f(x) = (x+1)^2 - 2$

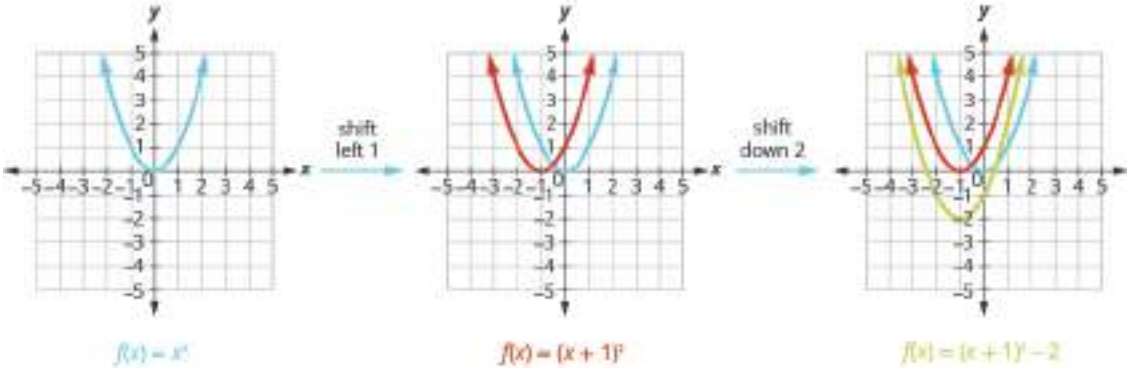
$h = -1$ Shift left 1 unit

$k = -2$ Shift down 2 units

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = (x+1)^2$, shift the graph $f(x) = x^2$ to the left 1 unit.

To graph $f(x) = (x+1)^2 - 2$, shift the graph $f(x) = (x+1)^2$ down 2 units.



TRY IT : : 9.113 Graph $f(x) = (x+2)^2 - 3$ using transformations.

TRY IT : : 9.114 Graph $f(x) = (x-3)^2 + 1$ using transformations.

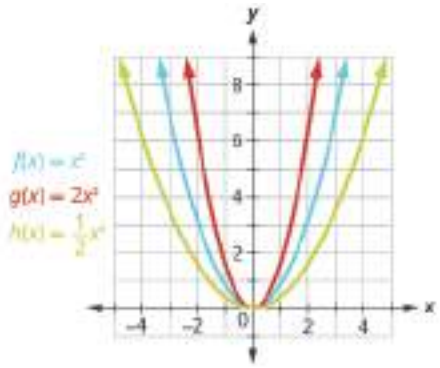
Graph Quadratic Functions of the Form $f(x) = ax^2$

So far we graphed the quadratic function $f(x) = x^2$ and then saw the effect of including a constant *h* or *k* in the equation had on the resulting graph of the new function. We will now explore the effect of the coefficient *a* on the resulting graph of the new function $f(x) = ax^2$.

Let's look at the quadratic functions $f(x) = x^2$, $g(x) = 2x^2$ and $h(x) = \frac{1}{2}x^2$.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = 2x^2$	$(x, g(x))$	$h(x) = \frac{1}{2}x^2$	$(x, h(x))$
−2	4	(−2, 4)	2 • 4	(−2, 8)	$\frac{1}{2}$ • 4	(−2, 2)
−1	1	(−1, 1)	2 • 1	(−1, 2)	$\frac{1}{2}$ • 1	$\left(-1, \frac{1}{2}\right)$
0	0	(0, 0)	2 • 0	(0, 0)	$\frac{1}{2}$ • 0	(0, 0)
1	1	(1, 1)	2 • 1	(1, 2)	$\frac{1}{2}$ • 1	$\left(1, \frac{1}{2}\right)$
2	4	(2, 4)	2 • 4	(2, 8)	$\frac{1}{2}$ • 4	(2, 2)

If we graph these functions, we can see the effect of the constant a, assuming $a > 0$.



The graph of the function $g(x) = 2x^2$ is "skinnier" than the graph of $f(x) = x^2$.
The graph of the function $h(x) = \frac{1}{2}x^2$ is "wider" than the graph of $f(x) = x^2$.

To graph a function with constant a it is easiest to choose a few points on $f(x) = x^2$ and multiply the y-values by a.

Graph of a Quadratic Function of the form $f(x) = ax^2$

The coefficient a in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.

- If $0 < |a| < 1$, the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
- If $|a| > 1$, the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.

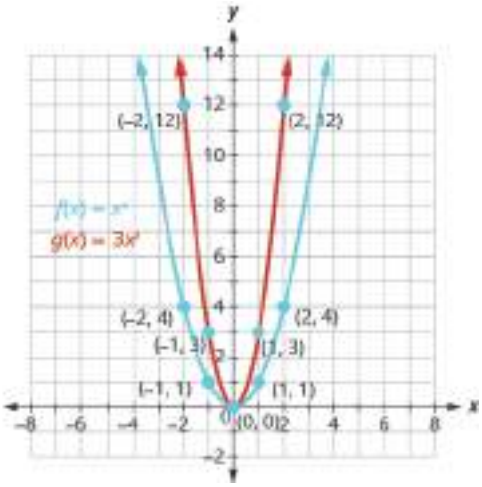
EXAMPLE 9.58

Graph $f(x) = 3x^2$.

Solution

We will graph the functions $f(x) = x^2$ and $g(x) = 3x^2$ on the same grid. We will choose a few points on $f(x) = x^2$ and then multiply the y-values by 3 to get the points for $g(x) = 3x^2$.

	$f(x) = x^2$	$g(x) = 3x^2$	
x	$(x, f(x))$	$(x, g(x))$	
−2	(−2, 4)	(−2, 12)	$3 \cdot 4 = 12$
−1	(−1, 1)	(−1, 3)	$3 \cdot 1 = 3$
0	(0, 0)	(0, 0)	$3 \cdot 0 = 0$
1	(1, 1)	(1, 3)	$3 \cdot 1 = 3$
2	(2, 4)	(2, 12)	$3 \cdot 4 = 12$



TRY IT : : 9.115 Graph $f(x) = -3x^2$.

TRY IT : : 9.116 Graph $f(x) = 2x^2$.

Graph Quadratic Functions Using Transformations

We have learned how the constants *a*, *h*, and *k* in the functions, $f(x) = x^2 + k$, $f(x) = (x - h)^2$, and $f(x) = ax^2$ affect their graphs. We can now put this together and graph quadratic functions $f(x) = ax^2 + bx + c$ by first putting them into the form $f(x) = a(x - h)^2 + k$ by completing the square. This form is sometimes known as the vertex form or standard form.

We must be careful to both add and subtract the number to the SAME side of the function to complete the square. We cannot add the number to both sides as we did when we completed the square with quadratic equations.

Quadratic Equation	Quadratic Function
$x^2 + 8x + 6 = 0$	$f(x) = x^2 + 8x + 6$
$x^2 + 8x = -6$	$f(x) = x^2 + 8x + 6$
$x^2 + 8x + 16 = -6 + 16$	$f(x) = x^2 + 8x + 16 + 6 - 16$
$(x + 4)^2 = 10$	$f(x) = (x + 4)^2 - 10$
Add 16 to both sides	Add and subtract 16 from the same side

When we complete the square in a function with a coefficient of x^2 that is not one, we have to factor that coefficient from just the *x*-terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to focus only on the *x*-terms.

Once we get the constant we want to complete the square, we must remember to multiply it by that coefficient before we then subtract it.

EXAMPLE 9.59

Rewrite $f(x) = -3x^2 - 6x - 1$ in the $f(x) = a(x - h)^2 + k$ form by completing the square.

Solution

	$f(x) = -3x^2 - 6x - 1$
Separate the x terms from the constant.	$f(x) = -3x^2 - 6x \quad - 1$
Factor the coefficient of x^2, -3.	$f(x) = -3(x^2 + 2x) \quad - 1$
Prepare to complete the square.	$f(x) = -3(x^2 + 2x \quad) \quad - 1$
Take half of 2 and then square it to complete the square. $\left(\frac{1}{2} \cdot 2\right)^2 = 1$	
The constant 1 completes the square in the parentheses, but the parentheses is multiplied by -3. So we are really adding -3 We must then add 3 to not change the value of the function.	$f(x) = -3(x^2 + 2x + 1) - 1 + 3$ $-3 \cdot 1 = -3$ so add 3
Rewrite the trinomial as a square and subtract the constants.	$f(x) = -3(x + 1)^2 + 2$
The function is now in the $f(x) = a(x - h)^2 + k$ form.	$f(x) = a(x - h)^2 + k$ $f(x) = -3(x + 1)^2 + 2$

TRY IT : : 9.117

Rewrite $f(x) = -4x^2 - 8x + 1$ in the $f(x) = a(x - h)^2 + k$ form by completing the square.

TRY IT : : 9.118 Rewrite $f(x) = 2x^2 - 8x + 3$ in the $f(x) = a(x - h)^2 + k$ form by completing the square.

Once we put the function into the $f(x) = (x - h)^2 + k$ form, we can then use the transformations as we did in the last few problems. The next example will show us how to do this.

EXAMPLE 9.60

Graph $f(x) = x^2 + 6x + 5$ by using transformations.

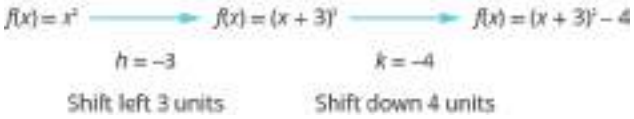
Solution

Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ vertex form by completing the square.

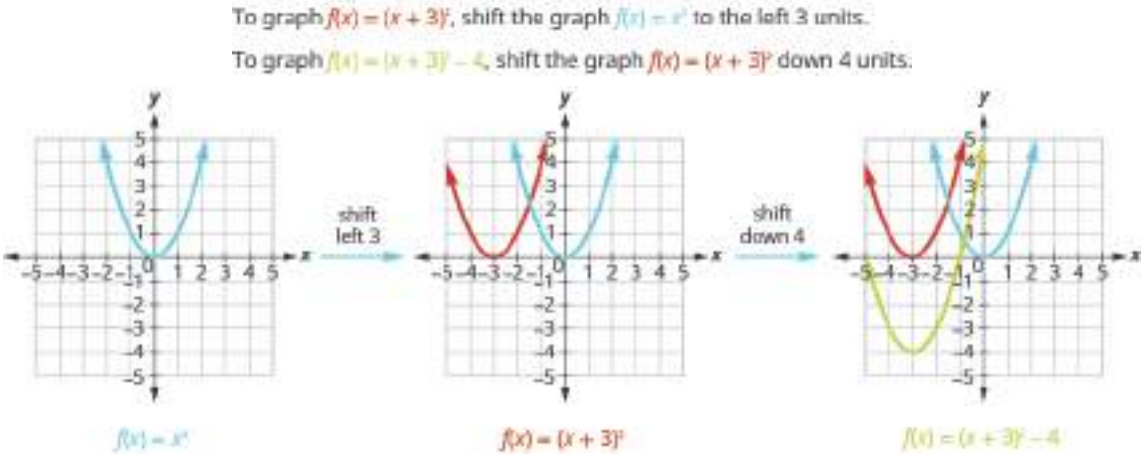
	$f(x)=x^2+6x+5$
Separate the x terms from the constant.	$f(x)=x^2+6x+5$
Take half of 6 and then square it to complete the square. $\left(\frac{1}{2}\cdot 6\right)^2=9$	
We both add 9 and subtract 9 to not change the value of the function.	$f(x)=x^2+6x+9+5-9$
Rewrite the trinomial as a square and subtract the constants.	$f(x)=(x+3)^2-4$
The function is now in the $f(x)=(x-h)^2+k$ form.	$f(x)=(x-h)^2+k$ $f(x)=(x+3)^2-4$

Step 2: Graph the function using transformations.

Looking at the h, k values, we see the graph will take the graph of $f(x)=x^2$ and shift it to the left 3 units and down 4 units.



We first draw the graph of $f(x)=x^2$ on the grid.



TRY IT : : 9.119 Graph $f(x)=x^2+2x-3$ by using transformations.

TRY IT : : 9.120 Graph $f(x)=x^2-8x+12$ by using transformations.

We list the steps to take to graph a quadratic function using transformations here.

HOW TO : : GRAPH A QUADRATIC FUNCTION USING TRANSFORMATIONS.

Step 1. Rewrite the function in $f(x)=a(x-h)^2+k$ form by completing the square.

Step 2. Graph the function using transformations.

EXAMPLE 9.61

Graph $f(x) = -2x^2 - 4x + 2$ by using transformations.

Solution

Step 1. Rewrite the function in $f(x) = a(x-h)^2 + k$ vertex form by completing the square.

	$f(x) = -2x^2 - 4x + 2$
Separate the x terms from the constant.	$f(x) = -2x^2 - 4x + 2$
We need the coefficient of x^2 to be one. We factor -2 from the x-terms.	$f(x) = -2(x^2 + 2x) + 2$
Take half of 2 and then square it to complete the square. $\left(\frac{1}{2} \cdot 2\right)^2 = 1$	
We add 1 to complete the square in the parentheses, but the parentheses is multiplied by -2. Se we are really adding -2. To not change the value of the function we add 2.	$f(x) = -2(x^2 + 2x + 1) + 2 + 2$
Rewrite the trinomial as a square and subtract the constants.	$f(x) = -2(x+1)^2 + 4$
The function is now in the $f(x) = a(x-h)^2 + k$ form.	$f(x) = a(x-h)^2 + k$ $f(x) = -2(x+1)^2 + 4$

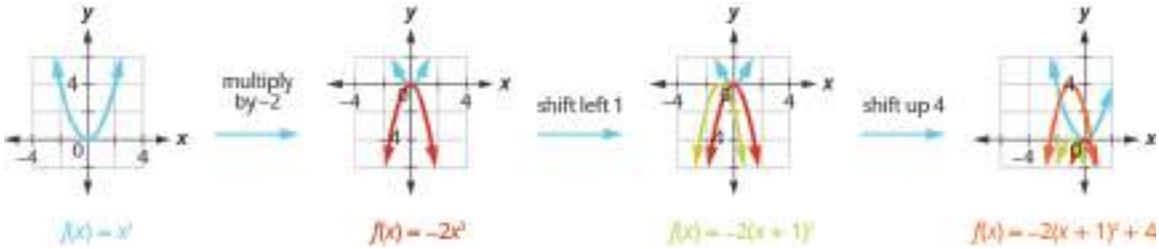
Step 2. Graph the function using transformations.

$f(x) = x^2 \longrightarrow f(x) = -2x^2 \longrightarrow f(x) = -2(x+1)^2 \longrightarrow f(x) = -2(x+1)^2 + 4$

$a = -2$ Multiply y-values by -2 | $h = -1$ Shift left 1 unit | $k = 4$ Shift up 4 units

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = -2x^2$, multiply the y-values in parabola of $f(x) = x^2$ by -2.
To graph $f(x) = -2(x+1)^2$, shift the graph $f(x) = -2x^2$ to the left 1 unit.
To graph $f(x) = -2(x+1)^2 + 4$, shift the graph $f(x) = (x+1)^2$ up 4 units.

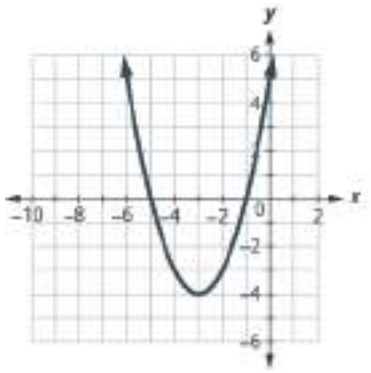


TRY IT : : 9.121 Graph $f(x) = -3x^2 + 12x - 4$ by using transformations.

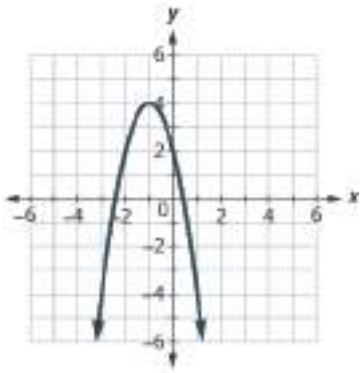
TRY IT : : 9.122 Graph $f(x) = -2x^2 + 12x - 9$ by using transformations.

Now that we have completed the square to put a quadratic function into $f(x) = a(x-h)^2 + k$ form, we can also use this technique to graph the function using its properties as in the previous section.

If we look back at the last few examples, we see that the vertex is related to the constants *h* and *k*.



$f(x) = (x+3)^2 - 4$
$h = -3 \quad k = -4$
Vertex = (−3, −4)



$f(x) = -2(x+1)^2 + 4$
$h = -1 \quad k = 4$
Vertex = (−1, 4)

In each case, the vertex is (*h*, *k*). Also the axis of symmetry is the line $x = h$.

We rewrite our steps for graphing a quadratic function using properties for when the function is in $f(x) = a(x-h)^2 + k$ form.

HOW TO : : GRAPH A QUADRATIC FUNCTION IN THE FORM $f(x) = a(x-h)^2 + k$ USING PROPERTIES.

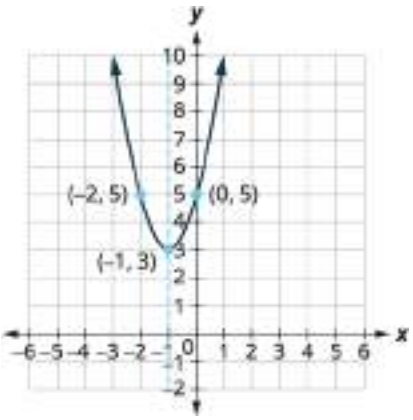
Step 1. Rewrite the function in $f(x) = a(x-h)^2 + k$ form.
Step 2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
Step 3. Find the axis of symmetry, $x = h$.
Step 4. Find the vertex, (*h*, *k*).
Step 5. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
Step 6. Find the *x*-intercepts.
Step 7. Graph the parabola.

EXAMPLE 9.62

ⓐ Rewrite $f(x) = 2x^2 + 4x + 5$ in $f(x) = a(x-h)^2 + k$ form and ⓑ graph the function using properties.

Solution

Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square.	$f(x) = 2x^2 + 4x + 5$
	$f(x) = 2(x^2 + 2x) + 5$
	$f(x) = 2(x^2 + 2x + 1) + 5 - 2$
	$f(x) = 2(x+1)^2 + 3$
Identify the constants a, h, k.	$a = 2 \quad h = -1 \quad k = 3$

Since $a = 2$, the parabola opens upward.	
The axis of symmetry is $x = h$.	The axis of symmetry is $x = -1$.
The vertex is (h, k).	The vertex is $(-1, 3)$.
Find the *y*-intercept by finding $f(0)$.	$f(0) = 2 \cdot 0^2 + 4 \cdot 0 + 5$
	$f(0) = 5$
	y-intercept $(0, 5)$
Find the point symmetric to $(0, 5)$ across the axis of symmetry.	$(-2, 5)$
Find the *x*-intercepts.	The discriminant negative, so there are no *x*-intercepts. Graph the parabola.
	

TRY IT : : 9.123

ⓐ Rewrite $f(x) = 3x^2 - 6x + 5$ in $f(x) = a(x - h)^2 + k$ form and ⓑ graph the function using properties.

TRY IT : : 9.124

ⓐ Rewrite $f(x) = -2x^2 + 8x - 7$ in $f(x) = a(x - h)^2 + k$ form and ⓑ graph the function using properties.

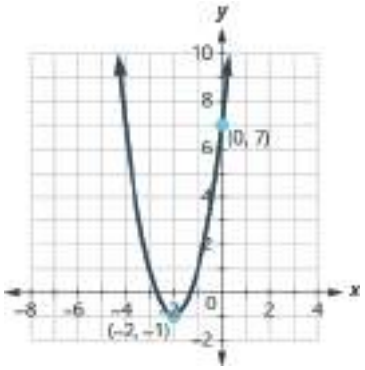
Find a Quadratic Function from its Graph

So far we have started with a function and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the function.

EXAMPLE 9.63

Determine the quadratic function whose graph is shown.

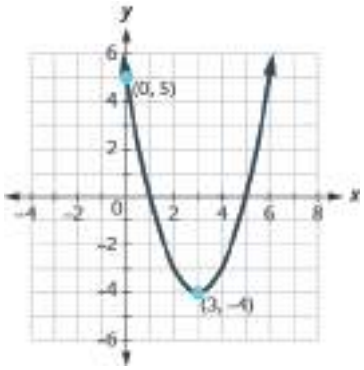


Solution

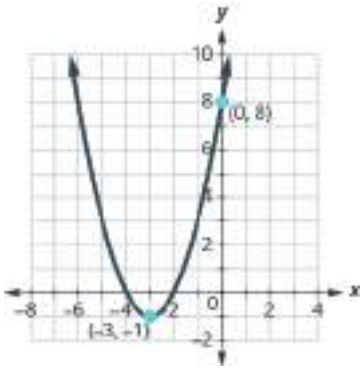
Since it is quadratic, we start with the $f(x) = a(x-h)^2 + k$ form.

The vertex, (h, k), is $(-2, -1)$ so $h = -2$ and $k = -1$.	$f(x) = a(x-(-2))^2 - 1$
To fin a, we use the y-intercept, $(0, 7)$.	
So $f(0) = 7$.	$7 = a(0+2)^2 - 1$
Solve for a.	$7 = 4a - 1$
	$8 = 4a$
	$2 = a$
Write the function.	$f(x) = a(x-h)^2 + k$
Substitute in $h = -2$, $k = -1$ and $a = 2$.	$f(x) = 2(x+2)^2 - 1$

TRY IT : : 9.125 Write the quadratic function in $f(x) = a(x-h)^2 + k$ form whose graph is shown.



TRY IT : : 9.126 Determine the quadratic function whose graph is shown.



MEDIA : :

Access these online resources for additional instruction and practice with graphing quadratic functions using transformations.

- **Function Shift Rules Applied to Quadratic Functions (https://openstax.org/l/37QuadFuncTran1)**
- **Changing a Quadratic from Standard Form to Vertex Form (https://openstax.org/l/37QuadFuncTran2)**
- **Using Transformations to Graph Quadratic Functions (https://openstax.org/l/37QuadFuncTran3)**
- **Finding Quadratic Equation in Vertex Form from Graph (https://openstax.org/l/37QuadFuncTran4)**

9.7 EXERCISES

Practice Makes Perfect

Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the following exercises, ⓐ graph the quadratic functions on the same rectangular coordinate system and ⓑ describe what effect adding a constant, k, to the function has on the basic parabola.

293. $f(x) = x^2$, $g(x) = x^2 + 4$, and $h(x) = x^2 - 4$.

294. $f(x) = x^2$, $g(x) = x^2 + 7$, and $h(x) = x^2 - 7$.

In the following exercises, graph each function using a vertical shift.

295. $f(x) = x^2 + 3$

296. $f(x) = x^2 - 7$

297. $g(x) = x^2 + 2$

298. $g(x) = x^2 + 5$

299. $h(x) = x^2 - 4$

300. $h(x) = x^2 - 5$

Graph Quadratic Functions of the form $f(x) = (x - h)^2$

In the following exercises, ⓐ graph the quadratic functions on the same rectangular coordinate system and ⓑ describe what effect adding a constant, h, to the function has on the basic parabola.

301. $f(x) = x^2$, $g(x) = (x - 3)^2$, and $h(x) = (x + 3)^2$.

302. $f(x) = x^2$, $g(x) = (x + 4)^2$, and $h(x) = (x - 4)^2$.

In the following exercises, graph each function using a horizontal shift.

303. $f(x) = (x - 2)^2$

304. $f(x) = (x - 1)^2$

305. $f(x) = (x + 5)^2$

306. $f(x) = (x + 3)^2$

307. $f(x) = (x - 5)^2$

308. $f(x) = (x + 2)^2$

In the following exercises, graph each function using transformations.

309. $f(x) = (x + 2)^2 + 1$

310. $f(x) = (x + 4)^2 + 2$

311. $f(x) = (x - 1)^2 + 5$

312. $f(x) = (x - 3)^2 + 4$

313. $f(x) = (x + 3)^2 - 1$

314. $f(x) = (x + 5)^2 - 2$

315. $f(x) = (x - 4)^2 - 3$

316. $f(x) = (x - 6)^2 - 2$

Graph Quadratic Functions of the form $f(x) = ax^2$

In the following exercises, graph each function.

317. $f(x) = -2x^2$

318. $f(x) = 4x^2$

319. $f(x) = -4x^2$

320. $f(x) = -x^2$

321. $f(x) = \frac{1}{2}x^2$

322. $f(x) = \frac{1}{3}x^2$

323. $f(x) = \frac{1}{4}x^2$

324. $f(x) = -\frac{1}{2}x^2$

Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x-h)^2 + k$ form by completing the square.

325. $f(x) = -3x^2 - 12x - 5$

326. $f(x) = 2x^2 - 12x + 7$

327. $f(x) = 3x^2 + 6x - 1$

328. $f(x) = -4x^2 - 16x - 9$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)^2 + k$ form and ⓑ graph it by using transformations.

329. $f(x) = x^2 + 6x + 5$

330. $f(x) = x^2 + 4x - 12$

331. $f(x) = x^2 + 4x - 12$

332. $f(x) = x^2 - 6x + 8$

333. $f(x) = x^2 - 6x + 15$

334. $f(x) = x^2 + 8x + 10$

335. $f(x) = -x^2 + 8x - 16$

336. $f(x) = -x^2 + 2x - 7$

337. $f(x) = -x^2 - 4x + 2$

338. $f(x) = -x^2 + 4x - 5$

339. $f(x) = 5x^2 - 10x + 8$

340. $f(x) = 3x^2 + 18x + 20$

341. $f(x) = 2x^2 - 4x + 1$

342. $f(x) = 3x^2 - 6x - 1$

343. $f(x) = -2x^2 + 8x - 10$

344. $f(x) = -3x^2 + 6x + 1$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)^2 + k$ form and ⓑ graph it using properties.

345. $f(x) = 2x^2 + 4x + 6$

346. $f(x) = 3x^2 - 12x + 7$

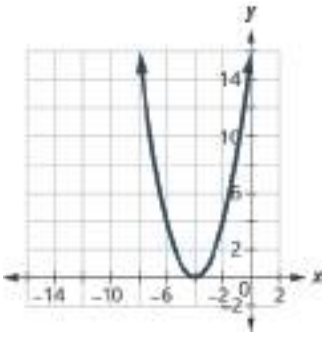
347. $f(x) = -x^2 + 2x - 4$

348. $f(x) = -2x^2 - 4x - 5$

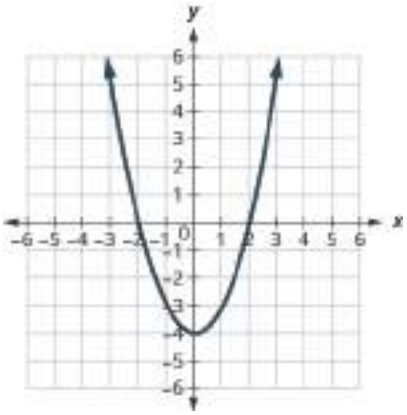
Matching

In the following exercises, match the graphs to one of the following functions: ⓐ $f(x) = x^2 + 4$ ⓑ $f(x) = x^2 - 4$ ⓒ $f(x) = (x+4)^2$ ⓓ $f(x) = (x-4)^2$ ⓔ $f(x) = (x+4)^2 - 4$ ⓕ $f(x) = (x+4)^2 + 4$ ⓖ $f(x) = (x-4)^2 - 4$ ⓗ $f(x) = (x-4)^2 + 4$

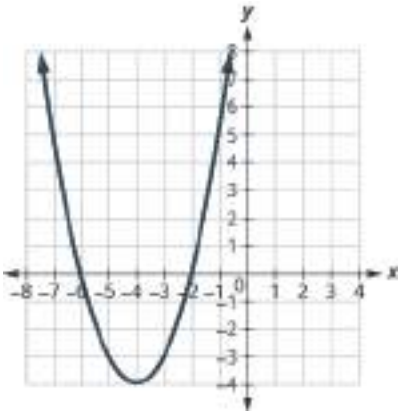
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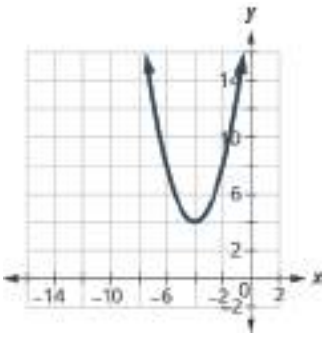
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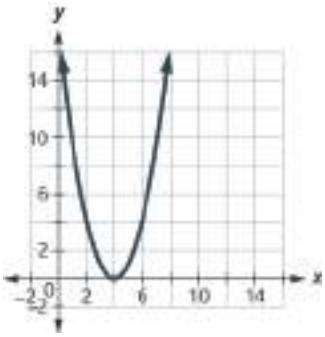
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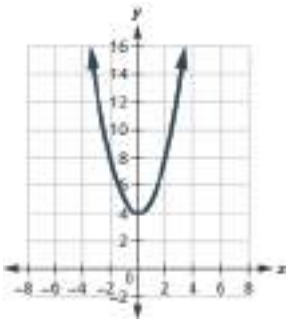
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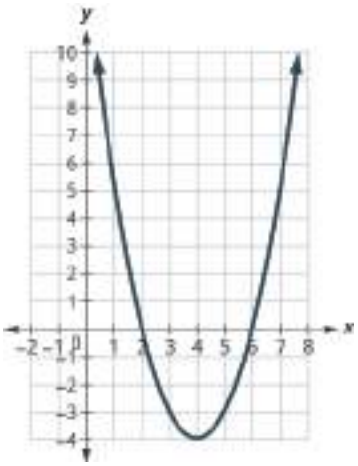
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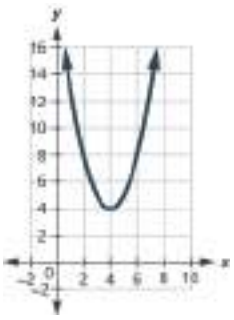
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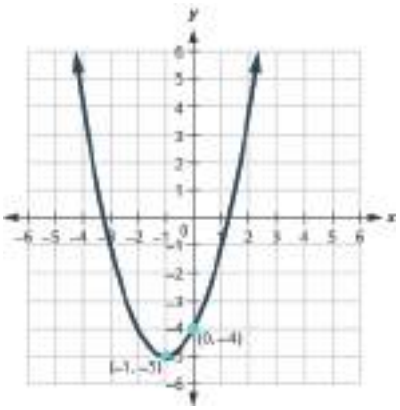
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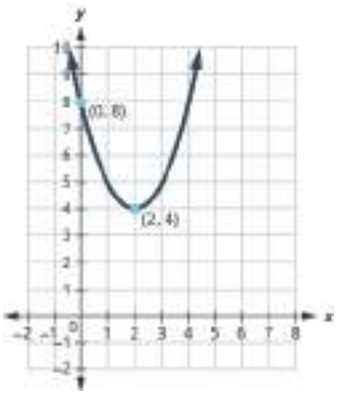
Find a Quadratic Function from its Graph

In the following exercises, write the quadratic function in $f(x) = a(x - h)^2 + k$ *form whose graph is shown.*

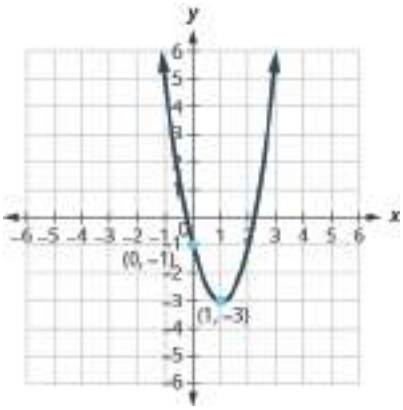
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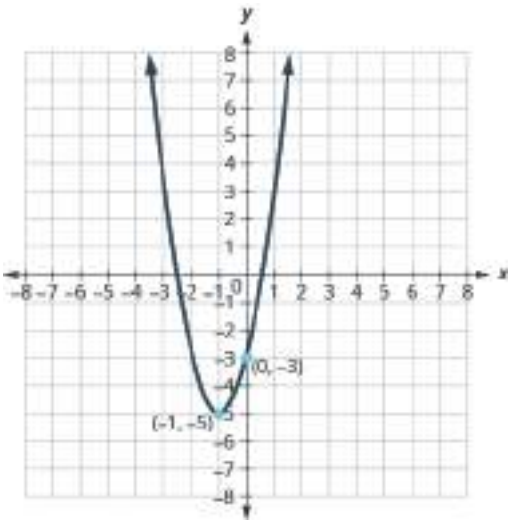
358.



359.



360.



Writing Exercise

361. Graph the quadratic function $f(x) = x^2 + 4x + 5$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

362. Graph the quadratic function $f(x) = 2x^2 - 4x - 3$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No I don't get it!
graph Quadratic Functions of the form $f(x) = x^2 + k$.			
graph Quadratic Functions of the form $f(x) = (x - h)^2$.			
graph Quadratic Functions of the form $f(x) = ax^2$.			
graph Quadratic Functions Using Transformations.			
find a Quadratic Function from its Graph.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

9.8 Solve Quadratic Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic inequalities graphically
- Solve quadratic inequalities algebraically

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $2x - 3 = 0$.
 If you missed this problem, review **Example 2.2**.
2. Solve: $2y^2 + y = 15$.
 If you missed this problem, review **Example 6.45**.
3. Solve $\frac{1}{x^2 + 2x - 8} > 0$
 If you missed this problem, review **Example 7.56**.

We have learned how to solve linear inequalities and rational inequalities previously. Some of the techniques we used to solve them were the same and some were different.

We will now learn to solve inequalities that have a quadratic expression. We will use some of the techniques from solving linear and rational inequalities as well as quadratic equations.

We will solve quadratic inequalities two ways—both graphically and algebraically.

Solve Quadratic Inequalities Graphically

A quadratic equation is in standard form when written as $ax^2 + bx + c = 0$. If we replace the equal sign with an inequality sign, we have a **quadratic inequality** in standard form.

Quadratic Inequality

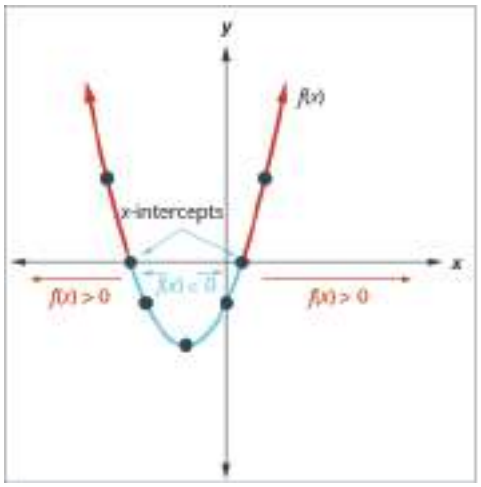
A **quadratic inequality** is an inequality that contains a quadratic expression.

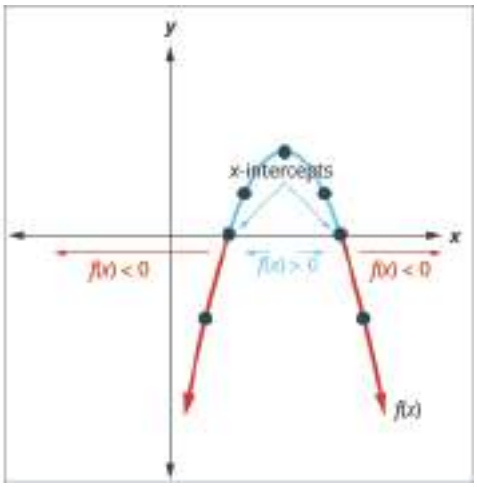
The standard form of a quadratic inequality is written:

$$ax^2 + bx + c < 0 \qquad ax^2 + bx + c \leq 0$$
$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c \geq 0$$

The graph of a quadratic function $f(x) = ax^2 + bx + c = 0$ is a parabola. When we ask when is $ax^2 + bx + c < 0$, we are asking when is $f(x) < 0$. We want to know when the parabola is below the x-axis.

When we ask when is $ax^2 + bx + c > 0$, we are asking when is $f(x) > 0$. We want to know when the parabola is above the y-axis.



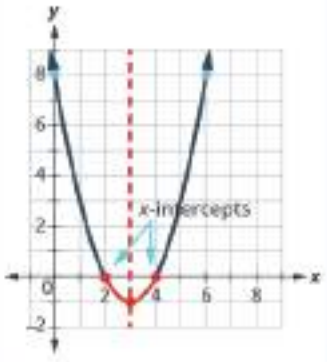


EXAMPLE 9.64 HOW TO SOLVE A QUADRATIC INEQUALITY GRAPHICALLY

Solve $x^2 - 6x + 8 < 0$ graphically. Write the solution in interval notation.

Solution

Step 1. Write the quadratic inequality in standard form.	The inequality is in standard form	$x^2 - 6x + 8 < 0$

Step 2. Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations.	We will graph using the properties. Look at a in the equation. $f(x) = x^2 - 6x + 8$ Since a is positive, the parabola opens upward.  $f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	$f(x) = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.** Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is (3, –1).**
	We find $f(0)$	y-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The y-intercept is (0, 8).**
	We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	Point symmetric to y-intercept **The point is (6, 8).**
	We solve $f(x) = 0$. We can solve this quadratic equation by factoring.	x-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The x-intercepts are (2, 0) and (4, 0).**
	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Step 3. Determine the solution from the graph.	$x^2 - 6x + 8 < 0$ The inequality asks for the values of x which make the function less than 0. Which values of x make the parabola below the x-axis. We do not include the values 2, 4 as the inequality is less than only.	The solution, in interval notation, is $(2, 4)$.

TRY IT : : 9.127 ⓐ Solve $x^2 + 2x - 8 < 0$ graphically and ⓑ write the solution in interval notation.

TRY IT : : 9.128 ⓐ Solve $x^2 - 8x + 12 \geq 0$ graphically and ⓑ write the solution in interval notation.

We list the steps to take to solve a quadratic inequality graphically.

HOW TO : : SOLVE A QUADRATIC INEQUALITY GRAPHICALLY.

Step 1. Write the quadratic inequality in standard form.

Step 2. Graph the function $f(x) = ax^2 + bx + c$.

Step 3. Determine the solution from the graph.

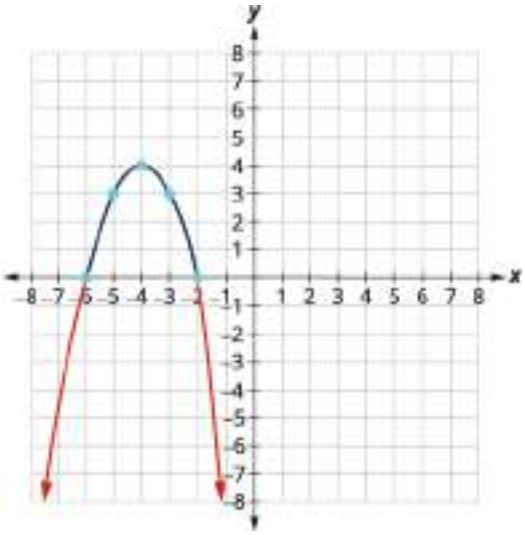
In the last example, the parabola opened upward and in the next example, it opens downward. In both cases, we are looking for the part of the parabola that is below the x-axis but note how the position of the parabola affects the solution.

EXAMPLE 9.65

Solve $-x^2 - 8x - 12 \leq 0$ graphically. Write the solution in interval notation.

Solution

The quadratic inequality in standard form.	$-x^2 - 8x - 12 \leq 0$
Graph the function $f(x) = -x^2 - 8x - 12$.	The parabola opens downward. 
Find the line of symmetry.	$x = -\frac{b}{2a}$ $x = -\frac{-8}{2(-1)}$ $x = -4$
Find the vertex.	$f(x) = -x^2 - 8x - 12$ $f(-4) = -(-4)^2 - 8(-4) - 12$ $f(-4) = -16 + 32 - 12$ $f(-4) = 4$ Vertex $(-4, 4)$

Find the x-intercepts. Let $f(x) = 0$.	$f(x) = -x^2 - 8x - 12$ $0 = -x^2 - 8x - 12$
Factor. Use the Zero Product Property.	$0 = -1(x + 6)(x + 2)$ $x = -6 \quad x = -2$
Graph the parabola.	x-intercepts $(-6, 0), (-2, 0)$ 
Determine the solution from the graph. We include the x-intercepts as the inequality is "less than or equal to."	$(-\infty, -6] \cup [-2, \infty)$

TRY IT : : 9.129 ⓐ Solve $-x^2 - 6x - 5 > 0$ graphically and ⓑ write the solution in interval notation.

TRY IT : : 9.130 ⓐ Solve $-x^2 + 10x - 16 \leq 0$ graphically and ⓑ write the solution in interval notation.

Solve Quadratic Inequalities Algebraically

The algebraic method we will use is very similar to the method we used to solve rational inequalities. We will find the critical points for the inequality, which will be the solutions to the related quadratic equation. Remember a polynomial expression can change signs only where the expression is zero.

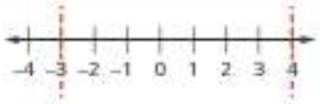
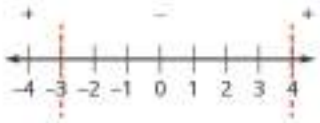
We will use the critical points to divide the number line into intervals and then determine whether the quadratic expression willl be postive or negative in the interval. We then determine the solution for the inequality.

EXAMPLE 9.66 HOW TO SOLVE QUADRATIC INEQUALITIES ALGEBRAICALLY

Solve $x^2 - x - 12 \geq 0$ algebraically. Write the solution in interval notation.

Solution

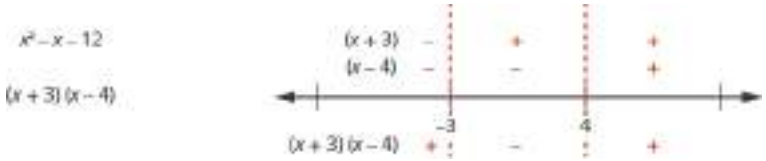
Step 1. Write the quadratic inequality in standard form.	The inequality is in standard form	$x^2 - x - 12 \geq 0$

Step 2. Determine the critical points—the solutions to the related quadratic equation.	Change the inequality sign to an equal sign and then solve the equation.	$x^2 - x - 12 = 0$ $(x + 3)(x - 4) = 0$ $x + 3 = 0 \quad x - 4 = 0$ $x = -3 \quad x = 4$
Step 3. Use the critical points to divide the number line into intervals.	Use -3 and 4 to divide the number line into intervals	
Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.	Test: $x = -5$ $x = 0$ $x = 5$	$x^2 - x - 12$ $\quad$ $x^2 - x - 12$ $\quad$ $x^2 - x - 12$ $(-5)^2 - (-5) - 12$ $\quad$ $0^2 - 0 - 12$ $\quad$ $5^2 - 5 - 12$ 18 $\quad$ -12 $\quad$ 8 
Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.	$x^2 - x - 12 \geq 0$ The inequality is positive in the first and last intervals and equals 0 at the points -4, 3.	The solution, in interval notation, is $(-\infty, -3] \cup [4, \infty)$.

TRY IT : : 9.131 Solve $x^2 + 2x - 8 \geq 0$ algebraically. Write the solution in interval notation.

TRY IT : : 9.132 Solve $x^2 - 2x - 15 \leq 0$ algebraically. Write the solution in interval notation.

In this example, since the expression $x^2 - x - 12$ factors nicely, we can also find the sign in each interval much like we did when we solved rational inequalities. We find the sign of each of the factors, and then the sign of the product. Our number line would like this:



The result is the same as we found using the other method.

We summarize the steps here.

HOW TO : : SOLVE A QUADRATIC INEQUALITY ALGEBRAICALLY.

Step 1. Write the quadratic inequality in standard form.

Step 2. Determine the critical points—the solutions to the related quadratic equation.

Step 3. Use the critical points to divide the number line into intervals.

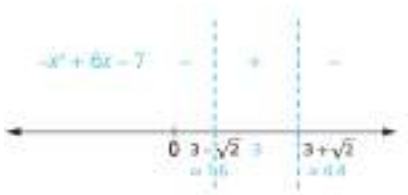
Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.

Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

EXAMPLE 9.67

Solve $x^2 + 6x - 7 \geq 0$ algebraically. Write the solution in interval notation.

Solution

Write the quadratic inequality in standard form.	$-x^2 + 6x - 7 \geq 0$
Multiply both sides of the inequality by -1. Remember to reverse the inequality sign.	$x^2 - 6x + 7 \leq 0$
Determine the critical points by solving the related quadratic equation.	$x^2 - 6x + 7 = 0$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (7)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{8}}{2}$
Simplify the radical.	$x = \frac{6 \pm 2\sqrt{2}}{2}$
Remove the common factor, 2.	$x = \frac{2(3 \pm \sqrt{2})}{2}$ $x = 3 \pm \sqrt{2}$ $x = 3 + \sqrt{2}$ $\quad x = 3 - \sqrt{2}$ $x \approx 1.6$ $\quad x \approx 4.4$
Use the critical points to divide the number line into intervals. Test numbers from each interval in the original inequality.	
Determine the intervals where the inequality is correct. Write the solution in interval notation.	$-x^2 + 6x - 7 \geq 0$ in the middle interval $[3 - \sqrt{2}, \quad 3 + \sqrt{2}]$

TRY IT : : 9.133 Solve $-x^2 + 2x + 1 \geq 0$ algebraically. Write the solution in interval notation.

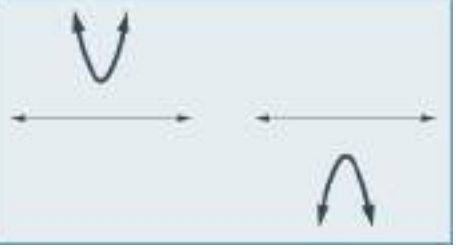
TRY IT : : 9.134 Solve $-x^2 + 8x - 14 < 0$ algebraically. Write the solution in interval notation.

The solutions of the quadratic inequalities in each of the previous examples, were either an interval or the union of two intervals. This resulted from the fact that, in each case we found two solutions to the corresponding quadratic equation $ax^2 + bx + c = 0$. These two solutions then gave us either the two x-intercepts for the graph or the two critical points to divide the number line into intervals.

This correlates to our previous discussion of the number and type of solutions to a quadratic equation using the

discriminant.

For a quadratic equation of the form $ax^2 + bx + c = 0,\ a \neq 0$.

Discriminant	Number/Type of solution	Typical Graph
$b^2 - 4ac > 0$	2 real solutions 2 x-intercepts on graph	
$b^2 - 4ac = 0$	1 real solution 1 x-intercept on graph	
$b^2 - 4ac < 0$	2 complex solutions No x-intercept	

The last row of the table shows us when the parabolas never intersect the x-axis. Using the Quadratic Formula to solve the quadratic equation, the radicand is a negative. We get two complex solutions.

In the next example, the quadratic inequality solutions will result from the solution of the quadratic equation being complex.

EXAMPLE 9.68

Solve, writing any solution in interval notation:

ⓐ $x^2 - 3x + 4 > 0$ ⓑ $x^2 - 3x + 4 \leq 0$

Solution

ⓐ

Write the quadratic inequality in standard form.	$-x^2 - 3x + 4 > 0$
Determine the critical points by solving the related quadratic equation.	$x^2 - 3x + 4 = 0$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$
Simplify.	$x = \frac{3 \pm \sqrt{-7}}{2}$
Simplify the radicand.	$x = \frac{3 \pm \sqrt{7}i}{2}$

The complex solutions tell us the parabola does not intercept the *x*-axis. Also, the parabola opens upward. This tells us that the parabola is completely above the *x*-axis.

Complex solutions



We are to find the solution to $x^2 - 3x + 4 > 0$. Since for all values of x the graph is above the *x*-axis, all values of *x* make the inequality true. In interval notation we write $(-\infty, \infty)$.

ⓑ

Write the quadratic inequality in standard form.	$x^2 - 3x + 4 \leq 0$
Determine the critical points by solving the related quadratic equation	$x^2 - 3x + 4 = 0$

Since the corresponding quadratic equation is the same as in part (a), the parabola will be the same. The parabola opens upward and is completely above the *x*-axis—no part of it is below the *x*-axis.

We are to find the solution to $x^2 - 3x + 4 \leq 0$. Since for all values of *x* the graph is never below the *x*-axis, no values of *x* make the inequality true. There is no solution to the inequality.

TRY IT : : 9.135 Solve and write any solution in interval notation:
ⓐ $-x^2 + 2x - 4 \leq 0$ ⓑ $-x^2 + 2x - 4 \geq 0$

TRY IT : : 9.136 Solve and write any solution in interval notation:
ⓐ $x^2 + 3x + 3 < 0$ ⓑ $x^2 + 3x + 3 > 0$

9.8 EXERCISES

Practice Makes Perfect

Solve Quadratic Inequalities Graphically

In the following exercises, ⓐ solve graphically and ⓑ write the solution in interval notation.

363. $x^2+6x+5>0$

364. $x^2+4x-12<0$

365. $x^2+4x+3\leq 0$

366. $x^2-6x+8\geq 0$

367. $-x^2-3x+18\leq 0$

368. $-x^2+2x+24<0$

369. $-x^2+x+12\geq 0$

370. $-x^2+2x+15>0$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

371. $x^2+3x-4\geq 0$

372. $x^2+x-6\leq 0$

373. $x^2-7x+10<0$

374. $x^2-4x+3>0$

375. $x^2+8x>-15$

376. $x^2+8x<-12$

377. $x^2-4x+2\leq 0$

378. $-x^2+8x-11<0$

379. $x^2-10x>-19$

380. $x^2+6x<-3$

381. $-6x^2+19x-10\geq 0$

382. $-3x^2-4x+4\leq 0$

383. $-2x^2+7x+4\geq 0$

384. $2x^2+5x-12>0$

385. $x^2+3x+5>0$

386. $x^2-3x+6\leq 0$

387. $-x^2+x-7>0$

388. $-x^2-4x-5<0$

389. $-2x^2+8x-10<0$

390. $-x^2+2x-7\geq 0$

Writing Exercises

391. Explain critical points and how they are used to solve quadratic inequalities algebraically.

392. Solve $x^2+2x\geq 8$ both graphically and algebraically. Which method do you prefer, and why?

393. Describe the steps needed to solve a quadratic inequality graphically.

394. Describe the steps needed to solve a quadratic inequality algebraically.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic inequalities graphically.			
solve quadratic inequalities algebraically.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

CHAPTER 9 REVIEW

KEY TERMS

discriminant In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

quadratic function A quadratic function, where a, b, and c are real numbers and $a \neq 0$, is a function of the form $f(x) = ax^2 + bx + c$.

quadratic inequality A quadratic inequality is an inequality that contains a quadratic expression.

KEY CONCEPTS

9.1 Solve Quadratic Equations Using the Square Root Property

- Square Root Property
 - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$ or $x = \pm\sqrt{k}$

How to solve a quadratic equation using the square root property.

Step 1. Isolate the quadratic term and make its coefficient one.

Step 2. Use Square Root Property.

Step 3. Simplify the radical.

Step 4. Check the solutions.

9.2 Solve Quadratic Equations by Completing the Square

- Binomial Squares Pattern
 If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$$

- How to Complete a Square

Step 1. Identify b, the coefficient of x.

Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$

Step 4. Rewrite the trinomial as a binomial square

- How to solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square.

Step 1. Divide by a to make the coefficient of x^2 term 1.

Step 2. Isolate the variable terms on one side and the constant terms on the other.

Step 3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

Step 4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.

Step 5. Use the Square Root Property.

Step 6. Simplify the radical and then solve the two resulting equations.

Step 7. Check the solutions.

9.3 Solve Quadratic Equations Using the Quadratic Formula

- Quadratic Formula
 - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.

 Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, c.

 Step 2. Write the Quadratic Formula. Then substitute in the values of a, b, c.

 Step 3. Simplify.

 Step 4. Check the solutions.

- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
 - For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,
 - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
 - if $b^2 - 4ac = 0$, the equation has 1 real solution.
 - if $b^2 - 4ac < 0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
 - Factoring
 - Square Root Property
 - Completing the Square
 - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.

 Step 1. Try Factoring first. If the quadratic factors easily, this method is very quick.

 Step 2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.

 Step 3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

9.4 Solve Quadratic Equations in Quadratic Form

- How to solve equations in quadratic form.

 Step 1. Identify a substitution that will put the equation in quadratic form.

 Step 2. Rewrite the equation with the substitution to put it in quadratic form.

 Step 3. Solve the quadratic equation for u.

 Step 4. Substitute the original variable back into the results, using the substitution.

 Step 5. Solve for the original variable.

 Step 6. Check the solutions.

9.5 Solve Applications of Quadratic Equations

- Methods to Solve Quadratic Equations
 - Factoring
 - Square Root Property
 - Completing the Square
 - Quadratic Formula
- How to use a Problem-Solving Strategy.

 Step 1. **Read** the problem. Make sure all the words and ideas are understood.

 Step 2. **Identify** what we are looking for.

 Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

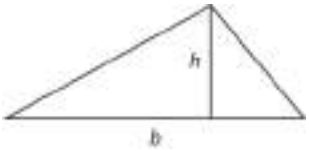
Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5. **Solve** the equation using good algebra techniques.

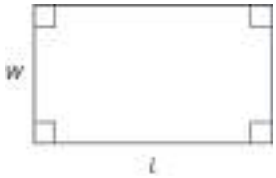
Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

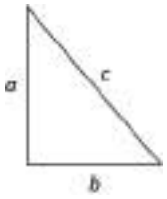
- Area of a Triangle
 - For a triangle with base, *b*, and height, *h*, the area, *A*, is given by the formula $A = \frac{1}{2}bh$.



- Area of a Rectangle
 - For a rectangle with length, *L*, and width, *W*, the area, *A*, is given by the formula $A = LW$.



- Pythagorean Theorem
 - In any right triangle, where *a* and *b* are the lengths of the legs, and *c* is the length of the hypotenuse, $a^2 + b^2 = c^2$.



- Projectile motion
 - The height in feet, *h*, of an object shot upwards into the air with initial velocity, v_0, after *t* seconds is given by the formula $h = -16t^2 + v_0t$.

9.6 Graph Quadratic Functions Using Properties

- Parabola Orientation
 - For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if
 - $a > 0$, the parabola opens upward.
 - $a < 0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:
 - the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
 - the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
 - the *y*-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
 - To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

***y*-intercept**	***x*-intercepts**
Let $x = 0$ and solve for $f(x)$.	Let $f(x) = 0$ and solve for x.

- How to graph a quadratic function using properties.
 Step 1. Determine whether the parabola opens upward or downward.
 Step 2. Find the equation of the axis of symmetry.
 Step 3. Find the vertex.
 Step 4. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
 Step 5. Find the *x*-intercepts. Find additional points if needed.
 Step 6. Graph the parabola.
- Minimum or Maximum Values of a Quadratic Equation
 - The *y*-coordinate of the vertex of the graph of a quadratic equation is the
 - *minimum* value of the quadratic equation if the parabola opens *upward*.
 - *maximum* value of the quadratic equation if the parabola opens *downward*.

9.7 Graph Quadratic Functions Using Transformations

- Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift
 - The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.
 - If $k > 0$, shift the parabola vertically up k units.
 - If $k < 0$, shift the parabola vertically down $|k|$ units.
- Graph a Quadratic Function of the form $f(x) = (x - h)^2$ Using a Horizontal Shift
 - The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.
 - If $h > 0$, shift the parabola horizontally left h units.
 - If $h < 0$, shift the parabola horizontally right $|h|$ units.
- Graph of a Quadratic Function of the form $f(x) = ax^2$
 - The coefficient a in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.
 If $0 < |a| < 1$, then the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
 If $|a| > 1$, then the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.
- How to graph a quadratic function using transformations
 Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form by completing the square.
 Step 2. Graph the function using transformations.
- Graph a quadratic function in the vertex form $f(x) = a(x - h)^2 + k$ using properties
 Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form.
 Step 2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
 Step 3. Find the axis of symmetry, $x = h$.
 Step 4. Find the vertex, (h, k).
 Step 5. Find they-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
 Step 6. Find the *x*-intercepts, if possible.
 Step 7. Graph the parabola.

9.8 Solve Quadratic Inequalities

- Solve a Quadratic Inequality Graphically

 Step 1. Write the quadratic inequality in standard form.

 Step 2. Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations.

 Step 3. Determine the solution from the graph.
- How to Solve a Quadratic Inequality Algebraically

 Step 1. Write the quadratic inequality in standard form.

 Step 2. Determine the critical points -- the solutions to the related quadratic equation.

 Step 3. Use the critical points to divide the number line into intervals.

 Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.

 Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

REVIEW EXERCISES

9.1 Solve Quadratic Equations Using the Square Root Property

Solve Quadratic Equations of the form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

395. $y^2 = 144$

396. $n^2 - 80 = 0$

397. $4a^2 = 100$

398. $2b^2 = 72$

399. $r^2 + 32 = 0$

400. $t^2 + 18 = 0$

401. $\frac{2}{3}w^2 - 20 = 30$

402. 11. $5c^2 + 3 = 19$

Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

403. $(p-5)^2 + 3 = 19$

404. $(u+1)^2 = 45$

405. $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$

406. $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$

407. $(n-4)^2 - 50 = 150$

408. $(4c-1)^2 = -18$

409. $n^2 + 10n + 25 = 12$

410. $64a^2 + 48a + 9 = 81$

9.2 Solve Quadratic Equations by Completing the Square

Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

411. $x^2 + 22x$

412. $m^2 - 8m$

413. $a^2 - 3a$

414. $b^2 + 13b$

In the following exercises, solve by completing the square.

415. $d^2 + 14d = -13$

416. $y^2 - 6y = 36$

417. $m^2 + 6m = -109$

418. $t^2 - 12t = -40$

419. $v^2 - 14v = -31$

420. $w^2 - 20w = 100$

421. $m^2 + 10m - 4 = -13$

422. $n^2 - 6n + 11 = 34$

423. $a^2 = 3a + 8$

424. $b^2 = 11b - 5$

425. $(u + 8)(u + 4) = 14$

426. $(z - 10)(z + 2) = 28$

Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

427. $3p^2 - 18p + 15 = 15$

428. $5q^2 + 70q + 20 = 0$

429. $4y^2 - 6y = 4$

430. $2x^2 + 2x = 4$

431. $3c^2 + 2c = 9$

432. $4d^2 - 2d = 8$

433. $2x^2 + 6x = -5$

434. $2x^2 + 4x = -5$

9.3 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

435. $4x^2 - 5x + 1 = 0$

436. $7y^2 + 4y - 3 = 0$

437. $r^2 - r - 42 = 0$

438. $t^2 + 13t + 22 = 0$

439. $4v^2 + v - 5 = 0$

440. $2w^2 + 9w + 2 = 0$

441. $3m^2 + 8m + 2 = 0$

442. $5n^2 + 2n - 1 = 0$

443. $6a^2 - 5a + 2 = 0$

444. $4b^2 - b + 8 = 0$

445. $u(u - 10) + 3 = 0$

446. $5z(z - 2) = 3$

447. $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$

448. $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$

449. $4c^2 + 4c + 1 = 0$

450. $9d^2 - 12d = -4$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

451.

ⓐ $9x^2 - 6x + 1 = 0$

ⓑ $3y^2 - 8y + 1 = 0$

ⓒ $7m^2 + 12m + 4 = 0$

ⓓ $5n^2 - n + 1 = 0$

452.

ⓐ $5x^2 - 7x - 8 = 0$

ⓑ $7x^2 - 10x + 5 = 0$

ⓒ $25x^2 - 90x + 81 = 0$

ⓓ $15x^2 - 8x + 4 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

453.

ⓐ $16r^2 - 8r + 1 = 0$

ⓑ $5t^2 - 8t + 3 = 9$

ⓒ $3(c + 2)^2 = 15$

454.

ⓐ $4d^2 + 10d - 5 = 21$

ⓑ $25x^2 - 60x + 36 = 0$

ⓒ $6(5v - 7)^2 = 150$

9.4 Solve Equations in Quadratic Form

Solve Equations in Quadratic Form

In the following exercises, solve.

455. $x^4 - 14x^2 + 24 = 0$

456. $x^4 + 4x^2 - 32 = 0$

457. $4x^4 - 5x^2 + 1 = 0$

458. $(2y + 3)^2 + 3(2y + 3) - 28 = 0$

459. $x + 3\sqrt{x} - 28 = 0$

460. $6x + 5\sqrt{x} - 6 = 0$

461. $x^{\frac{2}{3}} - 10x^{\frac{1}{3}} + 24 = 0$

462. $x + 7x^{\frac{1}{2}} + 6 = 0$

463. $8x^{-2} - 2x^{-1} - 3 = 0$

9.5 Solve Applications of Quadratic Equations

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.

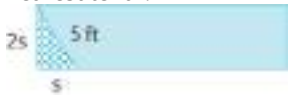
464. Find two consecutive odd numbers whose product is 323.

465. Find two consecutive even numbers whose product is 624.

466. A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.

467. Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.

468. A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.



469. A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

470. The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

471. For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

472. A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

473. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

474. Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

475. Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

9.6 Graph Quadratic Functions Using Properties

Recognize the Graph of a Quadratic Function

In the following exercises, graph by plotting point.

476. Graph $y = x^2 - 2$

477. Graph $y = -x^2 + 3$

In the following exercises, determine if the following parabolas open up or down.

478.

ⓐ $y = -3x^2 + 3x - 1$

ⓑ $y = 5x^2 + 6x + 3$

479.

ⓐ $y = x^2 + 8x - 1$

ⓑ $y = -4x^2 - 7x + 1$

Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find ⓐ the equation of the axis of symmetry and ⓑ the vertex.

480. $y = -x^2 + 6x + 8$

481. $y = 2x^2 - 8x + 1$

Find the Intercepts of a Parabola

In the following exercises, find the x- and y-intercepts.

482. $y = x^2 - 4x + 5$

483. $y = x^2 - 8x + 15$

484. $y = x^2 - 4x + 10$

485. $y = -5x^2 - 30x - 46$

486. $y = 16x^2 - 8x + 1$

487. $y = x^2 + 16x + 64$

Graph Quadratic Functions Using Properties

In the following exercises, graph by using its properties.

488. $y = x^2 + 8x + 15$

489. $y = x^2 - 2x - 3$

490. $y = -x^2 + 8x - 16$

491. $y = 4x^2 - 4x + 1$

492. $y = x^2 + 6x + 13$

493. $y = -2x^2 - 8x - 12$

Solve Maximum and Minimum Applications

In the following exercises, find the minimum or maximum value.

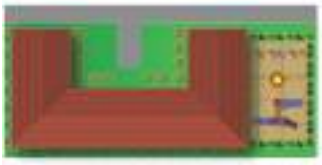
494. $y = 7x^2 + 14x + 6$

495. $y = -3x^2 + 12x - 10$

In the following exercises, solve. Rounding answers to the nearest tenth.

496. A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.

497. A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.



9.7 Graph Quadratic Functions Using Transformations

Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the following exercises, graph each function using a vertical shift.

498. $g(x) = x^2 + 4$

499. $h(x) = x^2 - 3$

In the following exercises, graph each function using a horizontal shift.

500. $f(x) = (x + 1)^2$

501. $g(x) = (x - 3)^2$

In the following exercises, graph each function using transformations.

502. $f(x) = (x + 2)^2 + 3$

503. $f(x) = (x + 3)^2 - 2$

504. $f(x) = (x - 1)^2 + 4$

505. $f(x) = (x - 4)^2 - 3$

Graph Quadratic Functions of the form $f(x) = ax^2$

In the following exercises, graph each function.

506. $f(x) = 2x^2$

507. $f(x) = -x^2$

508. $f(x) = \frac{1}{2}x^2$

Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x - h)^2 + k$ form by completing the square.

509. $f(x) = 2x^2 - 4x - 4$

510. $f(x) = 3x^2 + 12x + 8$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)^2 + k$ form and ⓑ graph it by using transformations.

511. $f(x) = 3x^2 - 6x - 1$ **512.** $f(x) = -2x^2 - 12x - 5$ **513.** $f(x) = 2x^2 + 4x + 6$

514. $f(x) = 3x^2 - 12x + 7$

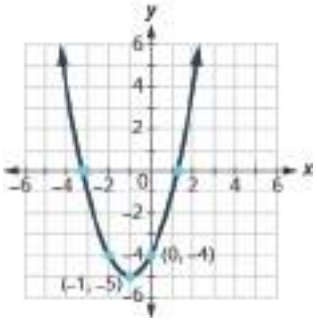
In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)^2 + k$ form and ⓑ graph it using properties.

515. $f(x) = -3x^2 - 12x - 5$ **516.** $f(x) = 2x^2 - 12x + 7$

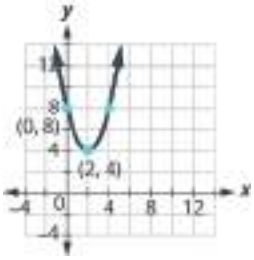
Find a Quadratic Function from its Graph

In the following exercises, write the quadratic function in $f(x) = a(x-h)^2 + k$ form.

517.



518.



9.8 Solve Quadratic Inequalities

Solve Quadratic Inequalities Graphically

In the following exercises, solve graphically and write the solution in interval notation.

519. $x^2 - x - 6 > 0$ **520.** $x^2 + 4x + 3 \leq 0$ **521.** $-x^2 - x + 2 \geq 0$

522. $-x^2 + 2x + 3 < 0$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

523. $x^2 - 6x + 8 < 0$ **524.** $x^2 + x > 12$ **525.** $x^2 - 6x + 4 \leq 0$

526. $2x^2 + 7x - 4 > 0$ **527.** $-x^2 + x - 6 > 0$ **528.** $x^2 - 2x + 4 \geq 0$

PRACTICE TEST

529. Use the Square Root Property to solve the quadratic equation $3(w+5)^2 = 27$.

530. Use Completing the Square to solve the quadratic equation $a^2 - 8a + 7 = 23$.

531. Use the Quadratic Formula to solve the quadratic equation $2m^2 - 5m + 3 = 0$.

Solve the following quadratic equations. Use any method.

532. $2x(3x-2) - 1 = 0$

533. $\frac{9}{4}y^2 - 3y + 1 = 0$

Use the discriminant to determine the number and type of solutions of each quadratic equation.

534. $6p^2 - 13p + 7 = 0$

535. $3q^2 - 10q + 12 = 0$

Solve each equation.

536. $4x^4 - 17x^2 + 4 = 0$

537. $y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - 3 = 0$

For each parabola, find ⓐ which direction it opens, ⓑ the equation of the axis of symmetry, ⓒ the vertex, ⓓ the x- and y-intercepts, and e) the maximum or minimum value.

538. $y = 3x^2 + 6x + 8$

539. $y = -x^2 - 8x + 16$

Graph each quadratic function using intercepts, the vertex, and the equation of the axis of symmetry.

540. $f(x) = x^2 + 6x + 9$

541. $f(x) = -2x^2 + 8x + 4$

In the following exercises, graph each function using transformations.

542. $f(x) = (x+3)^2 + 2$

543. $f(x) = x^2 - 4x - 1$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

544. $x^2 - 6x - 8 \leq 0$

545. $2x^2 + x - 10 > 0$

Model the situation with a quadratic equation and solve by any method.

546. Find two consecutive even numbers whose product is 360.

547. The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

548. A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.



10 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Figure 10.1 Hydroponic systems allow botanists to grow crops without land. (credit: "Izhamwong"/Wikimedia Commons)

Chapter Outline

10.1 Finding Composite and Inverse Functions
10.2 Evaluate and Graph Exponential Functions
10.3 Evaluate and Graph Logarithmic Functions
10.4 Use the Properties of Logarithms
10.5 Solve Exponential and Logarithmic Equations

Introduction

As the world population continues to grow, food supplies are becoming less able to meet the increasing demand. At the same time, available resources of fertile soil for growing plants is dwindling. One possible solution—grow plants without soil. Botanists around the world are expanding the potential of hydroponics, which is the process of growing plants without soil. To provide the plants with the nutrients they need, the botanists keep careful growth records. Some growth is described by the types of functions you will explore in this chapter—exponential and logarithmic. You will evaluate and graph these functions, and solve equations using them.

10.1 Finding Composite and Inverse Functions

Learning Objectives

By the end of this section, you will be able to:

- Find and evaluate composite functions
- Determine whether a function is one-to-one
- Find the inverse of a function

Be Prepared!

Before you get started, take this readiness quiz.

1. If $f(x) = 2x - 3$ and $g(x) = x^2 + 2x - 3$, find $f(4)$.
 If you missed this problem, review **Example 3.48**.
2. Solve for x, $3x + 2y = 12$.
 If you missed this problem, review **Example 2.31**.
3. Simplify: $5\frac{(x+4)}{5} - 4$.
 If you missed this problem, review **Example 1.25**.

In this chapter, we will introduce two new types of functions, exponential functions and logarithmic functions. These

functions are used extensively in business and the sciences as we will see.

Find and Evaluate Composite Functions

Before we introduce the functions, we need to look at another operation on functions called composition. In composition, the output of one function is the input of a second function. For functions f and g, the composition is written $f \circ g$ and is defined by $(f \circ g)(x) = f(g(x))$.

We read $f(g(x))$ as "f of g of x."



To do a composition, the output of the first function, $g(x)$, becomes the input of the second function, *f*, and so we must be sure that it is part of the domain of *f*.

Composition of Functions

The composition of functions *f* and *g* is written $f \cdot g$ and is defined by

$$(f \circ g)(x) = f(g(x))$$

We read $f(g(x))$ as f of g of *x*.

We have actually used composition without using the notation many times before. When we graphed quadratic functions using translations, we were composing functions. For example, if we first graphed $g(x) = x^2$ as a parabola and then shifted it down vertically four units, we were using the composition defined by $(f \circ g)(x) = f(g(x))$ where $f(x) = x - 4$.



The next example will demonstrate that $(f \circ g)(x)$, $(g \circ f)(x)$ and $(f \cdot g)(x)$ usually result in different outputs.

EXAMPLE 10.1

For functions $f(x) = 4x - 5$ and $g(x) = 2x + 3$, find: ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

Solution

ⓐ

Use the definition of $(f \circ g)(x)$.	$(f \circ g)(x) = f(g(x))$
Substitute $2x + 3$ for $g(x)$.	$(f \circ g)(x) = f(2x + 3)$
Find $f(2x + 3)$ where $f(x) = 4x - 5$.	$(f \circ g)(x) = 4(2x + 3) - 5$
Distribute.	$(f \circ g)(x) = 8x + 12 - 5$
Simplify.	$(f \circ g)(x) = 8x + 7$

ⓑ

Use the definition of $(f \circ g)(x)$.	$(g \circ f)(x) = g(f(x))$
Substitute $4x-5$ for $f(x)$.	$(g \circ f)(x) = g(4x-5)$
Find $g(4x-5)$ where $g(x) = 2x+3$.	$(g \circ f)(x) = 2(4x-5)+3$
Distribute.	$(g \circ f)(x) = 8x-10+3$
Simplify.	$(g \circ f)(x) = 8x-7$

Notice the difference in the result in part ⓐ and part ⓑ.

ⓒ Notice that $(f \cdot g)(x)$ is different than $(f \circ g)(x)$. In part ⓐ we did the composition of the functions. Now in part ⓒ we are not composing them, we are multiplying them.

Use the definition o $(f \cdot g)(x)$.	$(f \cdot g)(x) = f(x) \cdot g(x)$
Substitute $f(x) = 4x-5$ and $g(x) = 2x+3$.	$(f \cdot g)(x) = (4x-5) \cdot (2x+3)$
Multiply.	$(f \cdot g)(x) = 8x^2 + 2x - 15$

TRY IT : : 10.1 For functions $f(x) = 3x-2$ and $g(x) = 5x+1$, find ⓐ $(f \circ g)(x)$ ⓑ $(g \circ f)(x)$ ⓒ $(f \cdot g)(x)$.

TRY IT : : 10.2

For functions $f(x) = 4x-3$, and $g(x) = 6x-5$, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

In the next example we will evaluate a composition for a specific value.

EXAMPLE 10.2

For functions $f(x) = x^2 - 4$, and $g(x) = 3x+2$, find: ⓐ $(f \circ g)(-3)$, ⓑ $(g \circ f)(-1)$, and ⓒ $(f \circ f)(2)$.

Solution

ⓐ

Use the definition of $(f \circ g)(-3)$.	$(f \circ g)(-3) = f(g(-3))$
Find $g(-3)$ where $g(x) = 3x+2$.	$(f \circ g)(-3) = f(3 \cdot (-3)+2)$
Simplify.	$(f \circ g)(-3) = f(-7)$
Find $f(-7)$ where $f(x) = x^2-4$.	$(f \circ g)(-3) = (-7)^2 - 4$
Simplify.	$(f \circ g)(-3) = 45$

ⓑ

Use the definition of $(g \circ f)(-1)$.	$(g \circ f)(-1) = g(f(-1))$
Find $f(-1)$ where $f(x) = x^2 - 4$.	$(g \circ f)(-1) = g((-1)^2 - 4)$
Simplify.	$(g \circ f)(-1) = g(-3)$
Find $g(-3)$ where $g(x) = 3x + 2$.	$(g \circ f)(-1) = 3(-3) + 2$
Simplify.	$(g \circ f)(-1) = -7$

ⓒ

Use the definition of $(f \circ f)(2)$.	$(f \circ f)(2) = f(f(2))$
Find $f(2)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = f(2^2 - 4)$
Simplify.	$(f \circ f)(2) = f(0)$
Find $f(0)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = 0^2 - 4$
Simplify.	$(f \circ f)(2) = -4$

TRY IT : : 10.3

For functions $f(x) = x^2 - 9$, and $g(x) = 2x + 5$, find ⓐ $(f \circ g)(-2)$, ⓑ $(g \circ f)(-3)$, and ⓒ $(f \circ f)(4)$.

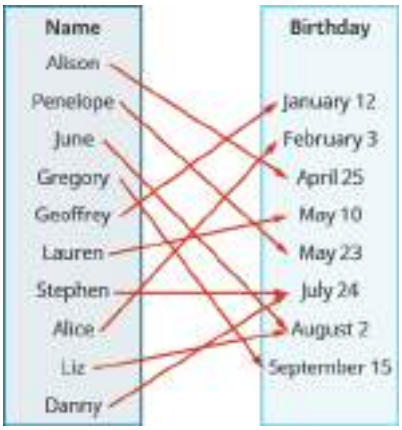
TRY IT : : 10.4

For functions $f(x) = x^2 + 1$, and $g(x) = 3x - 5$, find ⓐ $(f \circ g)(-1)$, ⓑ $(g \circ f)(2)$, and ⓒ $(f \circ f)(-1)$.

Determine Whether a Function is One-to-One

When we first introduced functions, we said a function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

We used the birthday example to help us understand the definition. Every person has a birthday, but no one has two birthdays and it is okay for two people to share a birthday. Since each person has exactly one birthday, that relation is a function.



A function is **one-to-one** if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

Our example of the birthday relation is not a one-to-one function. Two people can share the same birthday. The range value August 2 is the birthday of Liz and June, and so one range value has two domain values. Therefore, the function is

not one-to-one.

One-to-One Function

A function is **one-to-one** if each value in the range corresponds to one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value. There are no repeated *y*-values.

EXAMPLE 10.3

For each set of ordered pairs, determine if it represents a function and, if so, if the function is one-to-one.

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$ and ⓑ $\{(0, 0), (1, 1), (4, 2), (9, 3), (16, 4)\}$.

Solution

ⓐ

$$\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

But each *y*-value is not paired with only one *x*-value, $(-3, 27)$ and $(3, 27)$, for example. So this function is not one-to-one.

ⓑ

$$\{(0, 0), (1, 1), (4, 2), (9, 3), (16, 4)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

Since each *y*-value is paired with only one *x*-value, this function is one-to-one.

TRY IT : : 10.5

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$

ⓑ $\{(-4, 8), (-2, 4), (-1, 2), (0, 0), (1, 2), (2, 4), (4, 8)\}$

TRY IT : : 10.6

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(27, -3), (8, -2), (1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$

ⓑ $\{(7, -3), (-5, -4), (8, 0), (0, 0), (-6, 4), (-2, 2), (-1, 3)\}$

To help us determine whether a relation is a function, we use the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. Also, if any vertical line intersects the graph in more than one point, the graph does not represent a function.

The vertical line is representing an *x*-value and we check that it intersects the graph in only one *y*-value. Then it is a function.

To check if a function is one-to-one, we use a similar process. We use a horizontal line and check that each horizontal line intersects the graph in only one point. The horizontal line is representing a *y*-value and we check that it intersects the graph in only one *x*-value. If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function. This is the **horizontal line test**.

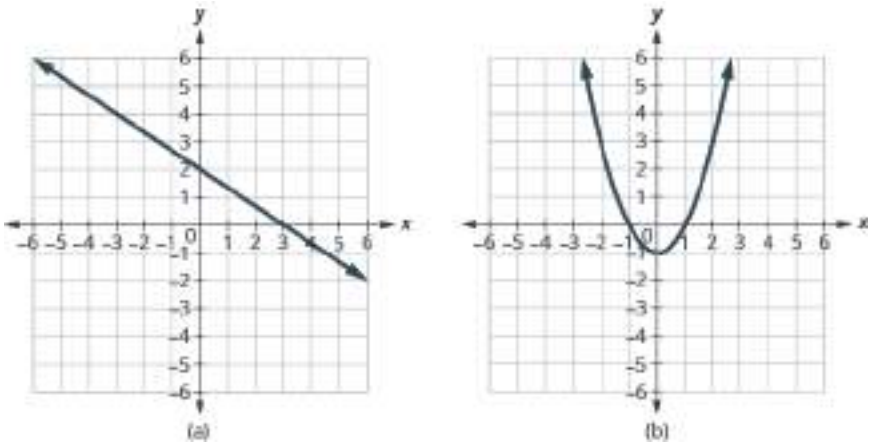
Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function.

We can test whether a graph of a relation is a function by using the vertical line test. We can then tell if the function is one-to-one by applying the horizontal line test.

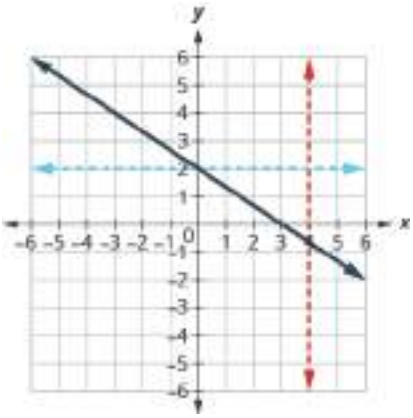
EXAMPLE 10.4

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



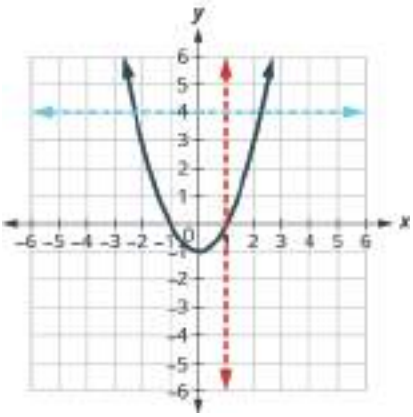
Solution

ⓐ



Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.

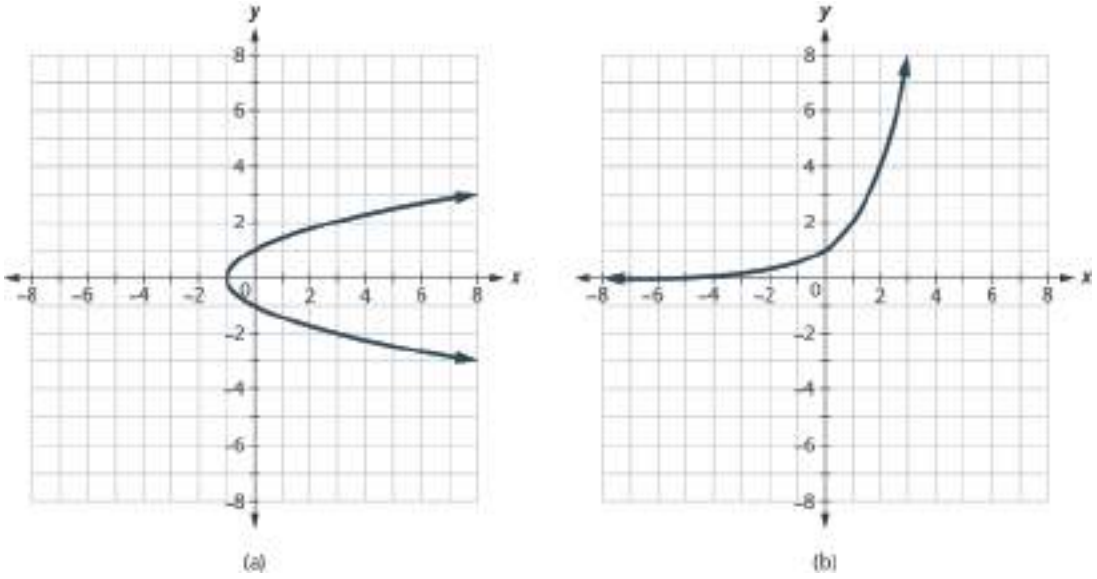
ⓑ



Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. The horizontal line shown on the graph intersects it in two points. This graph does not represent a one-to-one function.

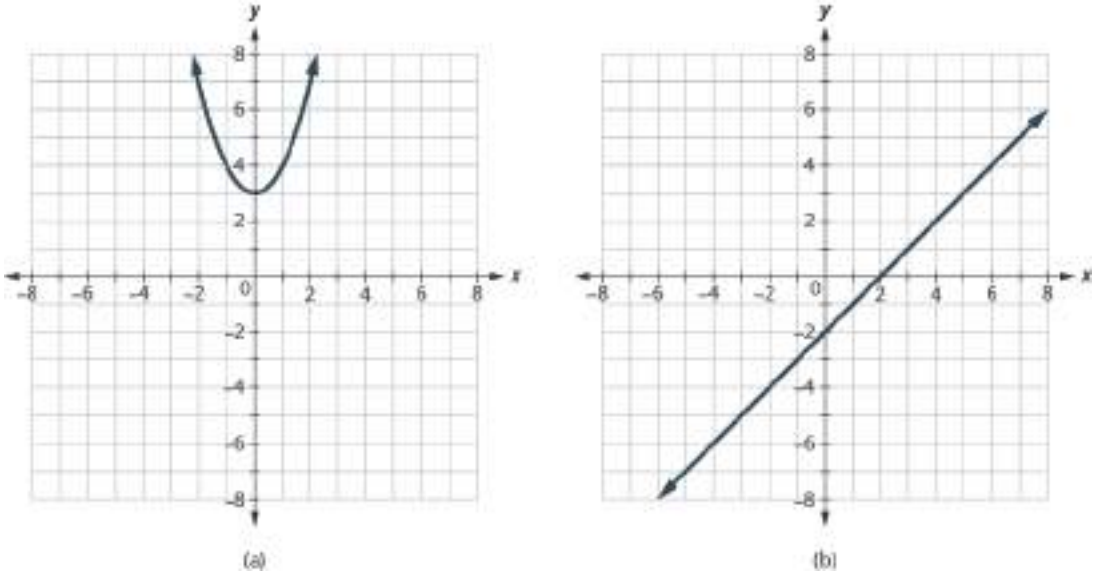
TRY IT : : 10.7

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



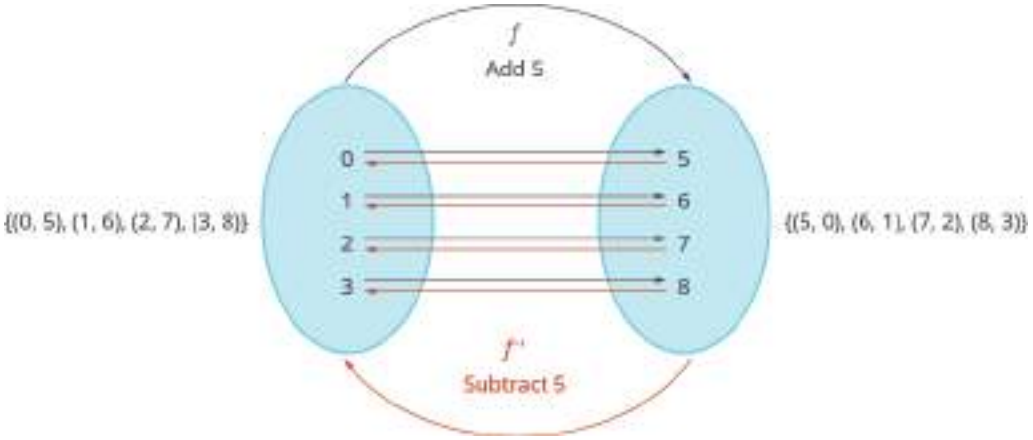
TRY IT : : 10.8

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



Find the Inverse of a Function

Let's look at a one-to one function, f, represented by the ordered pairs $\{(0, 5), (1, 6), (2, 7), (3, 8)\}$. For each x-value, f adds 5 to get the y-value. To 'undo' the addition of 5, we subtract 5 from each y-value and get back to the original x-value. We can call this "taking the inverse of f" and name the function f^{-1}.



Notice that that the ordered pairs of f and f^{-1} have their x-values and y-values reversed. The domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f.

Inverse of a Function Defined by Ordered Pairs

If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).

In the next example we will find the inverse of a function defined by ordered pairs.

EXAMPLE 10.5

Find the inverse of the function $\{(0, 3), (1, 5), (2, 7), (3, 9)\}$. Determine the domain and range of the inverse function.

Solution

This function is one-to-one since every x-value is paired with exactly one y-value.

To find the inverse we reverse the x-values and y-values in the ordered pairs of the function.

Function	$\{(0, 3), (1, 5), (2, 7), (3, 9)\}$
Inverse Function	$\{(3, 0), (5, 1), (7, 2), (9, 3)\}$
Domain of Inverse Function	$\{3, 5, 7, 9\}$
Range of Inverse Function	$\{0, 1, 2, 3\}$

TRY IT : : 10.9

Find the inverse of $\{(0, 4), (1, 7), (2, 10), (3, 13)\}$. Determine the domain and range of the inverse function.

TRY IT : : 10.10

Find the inverse of $\{(-1, 4), (-2, 1), (-3, 0), (-4, 2)\}$. Determine the domain and range of the inverse function.

We just noted that if $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).

So if a point (a, b) is on the graph of a function $f(x)$, then the ordered pair (b, a) is on the graph of $f^{-1}(x)$. See Figure 10.2.

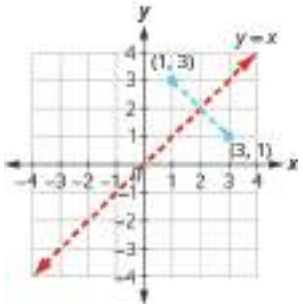


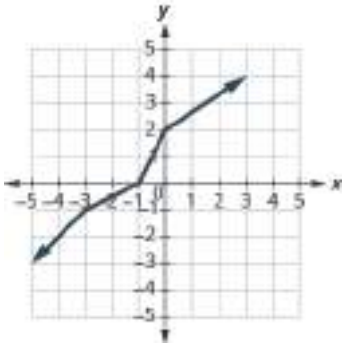
Figure 10.2

The distance between any two pairs (a, b) and (b, a) is cut in half by the line $y = x$. So we say the points are mirror images of each other through the line $y = x$.

Since every point on the graph of a function $f(x)$ is a mirror image of a point on the graph of $f^{-1}(x)$, we say the graphs are mirror images of each other through the line $y = x$. We will use this concept to graph the inverse of a function in the next example.

EXAMPLE 10.6

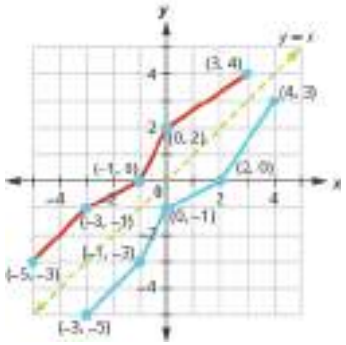
Graph, on the same coordinate system, the inverse of the one-to one function shown.



Solution

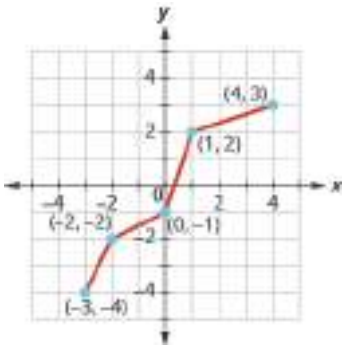
We can use points on the graph to find points on the inverse graph. Some points on the graph are: $(-5, -3), (-3, -1), (-1, 0), (0, 2), (3, 4)$.

So, the inverse function will contain the points: $(-3, -5), (-1, -3), (0, -1), (2, 0), (4, 3)$.

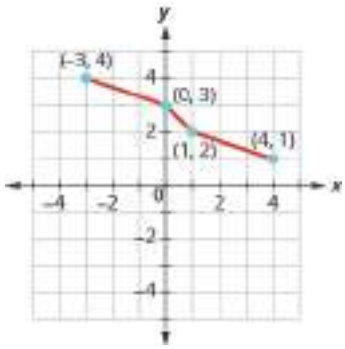


Notice how the graph of the original function and the graph of the inverse functions are mirror images through the line $y = x$.

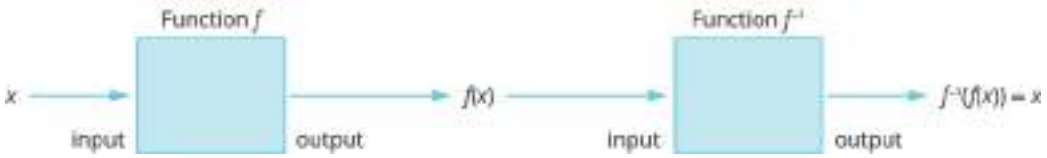
TRY IT : : 10.11 Graph, on the same coordinate system, the inverse of the one-to one function.



TRY IT : : 10.12 Graph, on the same coordinate system, the inverse of the one-to one function.



When we began our discussion of an inverse function, we talked about how the inverse function 'undoes' what the original function did to a value in its domain in order to get back to the original *x*-value.



Inverse Functions

$$f^{-1}(f(x)) = x, \text{ for all } x \text{ in the domain of } f$$
$$f\left(f^{-1}(x)\right) = x, \text{ for all } x \text{ in the domain of } f^{-1}$$

We can use this property to verify that two functions are inverses of each other.

EXAMPLE 10.7

Verify that $f(x) = 5x - 1$ and $g(x) = \frac{x+1}{5}$ are inverse functions.

Solution

The functions are inverses of each other if $g(f(x)) = x$ and $f(g(x)) = x$.

	$g(f(x)) \stackrel{?}{=} x$
Substitute $5x-1$ for $f(x)$.	$g(5x-1) \stackrel{?}{=} x$
Find $g(5x-1)$ where $g(x)=\frac{x+1}{5}$.	$\frac{(5x-1)+1}{5} \stackrel{?}{=} x$
Simplify.	$\frac{5x}{5} \stackrel{?}{=} x$
Simplify.	$x = x$ ✓

	$f(g(x)) \stackrel{?}{=} x$
Substitute $\frac{x+1}{5}$ for $g(x)$.	$f\left(\frac{x+1}{5}\right) \stackrel{?}{=} x$
Find $f\left(\frac{x+1}{5}\right)$ where $f(x)=5x-1$.	$5\left(\frac{x+1}{5}\right)-1 \stackrel{?}{=} x$
Simplify.	$x+1-1 \stackrel{?}{=} x$
Simplify.	$x = x$ ✓

Since both $g(f(x))=x$ and $f(g(x))=x$ are true, the functions $f(x)=5x-1$ and $g(x)=\frac{x+1}{5}$ are inverse functions. That is, they are inverses of each other.

TRY IT : : 10.13 Verify that the functions are inverse functions.

$f(x)=4x-3$ and $g(x)=\frac{x+3}{4}$.

TRY IT : : 10.14 Verify that the functions are inverse functions.

$f(x)=2x+6$ and $g(x)=\frac{x-6}{2}$.

We have found inverses of function defined by ordered pairs and from a graph. We will now look at how to find an inverse using an algebraic equation. The method uses the idea that if $f(x)$ is a one-to-one function with ordered pairs $(x,\ y)$, then its inverse function $f^{-1}(x)$ is the set of ordered pairs $(y,\ x)$.

If we reverse the *x* and *y* in the function and then solve for *y*, we get our inverse function.

EXAMPLE 10.8 HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

Find the inverse of $f(x)=4x+7$.

Solution

Step 1. Substitute *y* for $f(x)$.	Replace $f(x)$ with *y*.	$f(x)=4x+7$ $y=4x+7$
Step 2. Interchange the variables *x* and *y*.	Replace *x* with *y* and then *y* with *x*.	$x=4y+7$

Step 3. Solve for y.	Subtract 7 from each side.	$x-7=4y$
	Divide by 4.	$\frac{x-7}{4}=y$
Step 4. Substitute $f^{-1}(x)$ for y.	Replace y with $f^{-1}(x)$.	$\frac{x-7}{4}=f^{-1}(x)$
Step 5. Verify that the functions are inverses.	Show $f^{-1}(f(x))=x$	$f^{-1}(f(x)) \stackrel{?}{=} x$
	and $f(f^{-1}(x))=x$	$f^{-1}(4x+7) \stackrel{?}{=} x$
		$\frac{(4x+7)-7}{4} \stackrel{?}{=} x$
		$\frac{4x}{4} \stackrel{?}{=} x$
		$x=x$ ✓
		$f(f^{-1}(x)) \stackrel{?}{=} x$
		$f\left(\frac{x-7}{4}\right) \stackrel{?}{=} x$
		$4\left(\frac{x-7}{4}\right)+7 \stackrel{?}{=} x$
		$x-7+7 \stackrel{?}{=} x$
		$x=x$ ✓

TRY IT : : 10.15 Find the inverse of the function $f(x)=5x-3$.

TRY IT : : 10.16 Find the inverse of the function $f(x)=8x+5$.

We summarize the steps below.

HOW TO : : HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

Step 1. Substitute y for $f(x)$.

Step 2. Interchange the variables x and y.

Step 3. Solve for y.

Step 4. Substitute $f^{-1}(x)$ for y.

Step 5. Verify that the functions are inverses.

EXAMPLE 10.9 HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

Find the inverse of $f(x)=\sqrt[5]{2x-3}$.

Solution

$$f(x) = \sqrt[5]{2x-3}$$

Substitute y for $f(x)$. $\quad y = \sqrt[5]{2x-3}$

Interchange the variables x and y. $\quad x = \sqrt[5]{2y-3}$

Solve for y. $\quad (x)^5 = \left(\sqrt[5]{2y-3}\right)^5$

$$x^5 = 2y-3$$

$$x^5+3 = 2y$$

$$\frac{x^5+3}{2} = y$$

Substitute $f^{-1}(x)$ for y. $\quad f^{-1}(x) = \frac{x^5+3}{2}$

Verify that the functions are inverses.

$$f^{-1}(f(x)) \stackrel{?}{=} x \qquad f\left(f^{-1}(x)\right) \stackrel{?}{=} x$$

$$f^{-1}\left(\sqrt[5]{2x-3}\right) \stackrel{?}{=} x \qquad f\left(\frac{x^5+3}{2}\right) \stackrel{?}{=} x$$

$$\frac{\left(\sqrt[5]{2x-3}\right)^5+3}{2} \stackrel{?}{=} x \qquad \sqrt[5]{2\left(\frac{x^5+3}{2}\right)-3} \stackrel{?}{=} x$$

$$\frac{2x-3+3}{2} \stackrel{?}{=} x \qquad \sqrt[5]{x^5+3-3} \stackrel{?}{=} x$$

$$\frac{2x}{2} \stackrel{?}{=} x \qquad \sqrt[5]{x^5} \stackrel{?}{=} x$$

$$x = x \checkmark \qquad x = x \checkmark$$

TRY IT : : 10.17 Find the inverse of the function $f(x) = \sqrt[5]{3x-2}$.

TRY IT : : 10.18 Find the inverse of the function $f(x) = \sqrt[4]{6x-7}$.

10.1 EXERCISES

Practice Makes Perfect

Find and Evaluate Composite Functions

In the following exercises, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

1. $f(x) = 4x + 3$ and $g(x) = 2x + 5$

2. $f(x) = 3x - 1$ and $g(x) = 5x - 3$

3. $f(x) = 6x - 5$ and $g(x) = 4x + 1$

4. $f(x) = 2x + 7$ and $g(x) = 3x - 4$

5. $f(x) = 3x$ and $g(x) = 2x^2 - 3x$

6. $f(x) = 2x$ and $g(x) = 3x^2 - 1$

7. $f(x) = 2x - 1$ and $g(x) = x^2 + 2$

8. $f(x) = 4x + 3$ and $g(x) = x^2 - 4$

In the following exercises, find the values described.

9. For functions $f(x) = 2x^2 + 3$ and $g(x) = 5x - 1$, find

ⓐ $(f \circ g)(-2)$

ⓑ $(g \circ f)(-3)$

ⓒ $(f \circ f)(-1)$

10. For functions $f(x) = 5x^2 - 1$ and $g(x) = 4x - 1$, find

ⓐ $(f \circ g)(1)$

ⓑ $(g \circ f)(-1)$

ⓒ $(f \circ f)(2)$

11. For functions $f(x) = 2x^3$ and $g(x) = 3x^2 + 2$, find

ⓐ $(f \circ g)(-1)$

ⓑ $(g \circ f)(1)$

ⓒ $(g \circ g)(1)$

12. For functions $f(x) = 3x^3 + 1$ and $g(x) = 2x^2 - 3$, find

ⓐ $(f \circ g)(-2)$

ⓑ $(g \circ f)(-1)$

ⓒ $(g \circ g)(1)$

Determine Whether a Function is One-to-One

In the following exercises, determine if the set of ordered pairs represents a function and if so, is the function one-to-one.

13. $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$

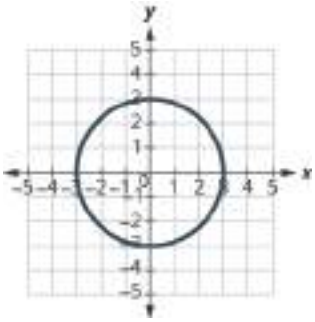
14. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

15. $\{(-3, -5), (-2, -3), (-1, -1), (0, 1), (1, 3), (2, 5), (3, 7)\}$

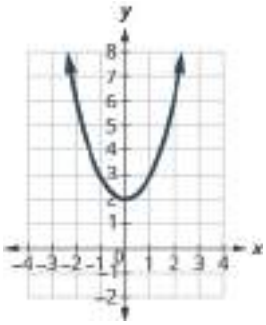
16. $\{(5, 3), (4, 2), (3, 1), (2, 0), (1, -1), (0, -2), (-1, -3)\}$

In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

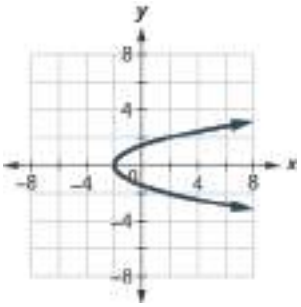
17. ⓐ



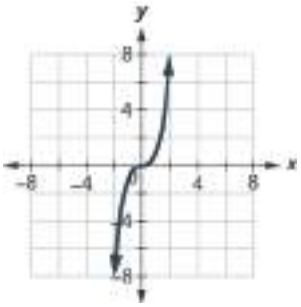
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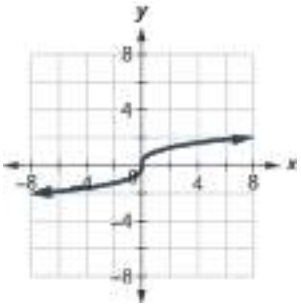
18. ⓐ



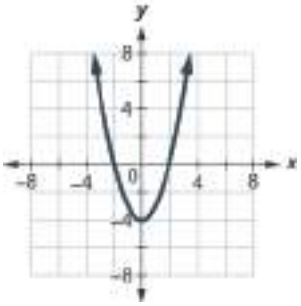
ⓑ



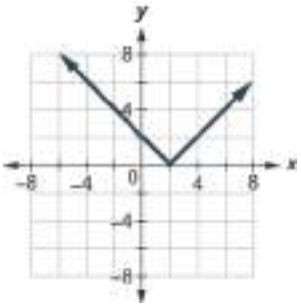
19. ⓐ



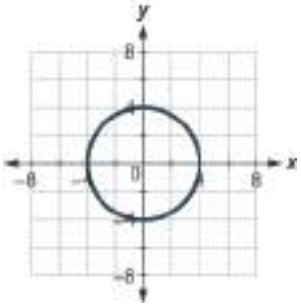
ⓑ



20. ⓐ



ⓑ



In the following exercises, find the inverse of each function. Determine the domain and range of the inverse function.

21. {(2, 1), (4, 2), (6, 3), (8, 4)}

22. {(6, 2), (9, 5), (12, 8), (15, 11)}

23. {(0, −2), (1, 3), (2, 7), (3, 12)}

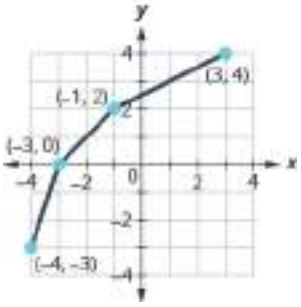
24. {(0, 0), (1, 1), (2, 4), (3, 9)}

25. {(−2, −3), (−1, −1), (0, 1), (1, 3)}

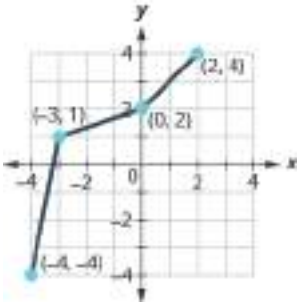
26. {(5, 3), (4, 2), (3, 1), (2, 0)}

In the following exercises, graph, on the same coordinate system, the inverse of the one-to-one function shown.

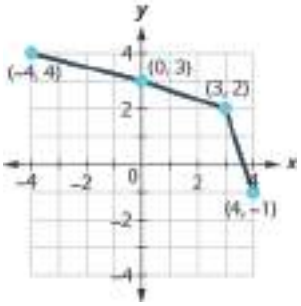
27.



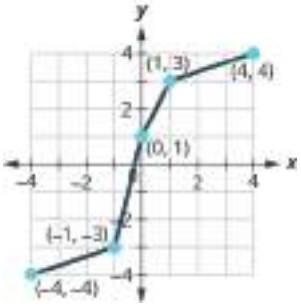
28.



29.



30.



In the following exercises, determine whether or not the given functions are inverses.

31. $f(x) = x + 8$ and $g(x) = x - 8$

32. $f(x) = x - 9$ and $g(x) = x + 9$

33. $f(x) = 7x$ and $g(x) = \frac{x}{7}$

34. $f(x) = \frac{x}{11}$ and $g(x) = 11x$

35. $f(x) = 7x + 3$ and $g(x) = \frac{x-3}{7}$

36. $f(x) = 5x - 4$ and $g(x) = \frac{x-4}{5}$

37. $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 2$

38. $f(x) = \sqrt[3]{x-4}$ and $g(x) = x^3 + 4$

In the following exercises, find the inverse of each function.

39. $f(x) = x - 12$

40. $f(x) = x + 17$

41. $f(x) = 9x$

42. $f(x) = 8x$

43. $f(x) = \frac{x}{6}$

44. $f(x) = \frac{x}{4}$

45. $f(x) = 6x - 7$

46. $f(x) = 7x - 1$

47. $f(x) = -2x + 5$

48. $f(x) = -5x - 4$

49. $f(x) = x^2 + 6, \quad x \geq 0$

50. $f(x) = x^2 - 9, \quad x \geq 0$

51. $f(x) = x^3 - 4$

52. $f(x) = x^3 + 6$

53. $f(x) = \frac{1}{x+2}$

54. $f(x) = \frac{1}{x-6}$

55. $f(x) = \sqrt{x-2}, \quad x \geq 2$

56. $f(x) = \sqrt{x+8}, \quad x \geq -8$

57. $f(x) = \sqrt[3]{x-3}$

58. $f(x) = \sqrt[3]{x+5}$

59. $f(x) = \sqrt[4]{9x-5}, \quad x \geq \frac{5}{9}$

60. $f(x) = \sqrt[4]{8x-3}, \quad x \geq \frac{3}{8}$

61. $f(x) = \sqrt[5]{-3x+5}$

62. $f(x) = \sqrt[5]{-4x-3}$

Writing Exercises

63. Explain how the graph of the inverse of a function is related to the graph of the function.

64. Explain how to find the inverse of a function from its equation. Use an example to demonstrate the steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find and evaluate composite functions.			
determine whether a function is one-to-one.			
find the inverse of a function.			

ⓑ *If most of your checks were:*

...confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

...with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

...no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

10.2 Evaluate and Graph Exponential Functions

Learning Objectives

By the end of this section, you will be able to:

- Graph exponential functions
- Solve Exponential equations
- Use exponential models in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\left(\frac{x^3}{x^2}\right)$.
 If you missed this problem, review **Example 5.13**.
2. Evaluate: ⓐ 2^0 ⓑ $\left(\frac{1}{3}\right)^0$.
 If you missed this problem, review **Example 5.14**.
3. Evaluate: ⓐ 2^{-1} ⓑ $\left(\frac{1}{3}\right)^{-1}$.
 If you missed this problem, review **Example 5.15**.

Graph Exponential Functions

The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

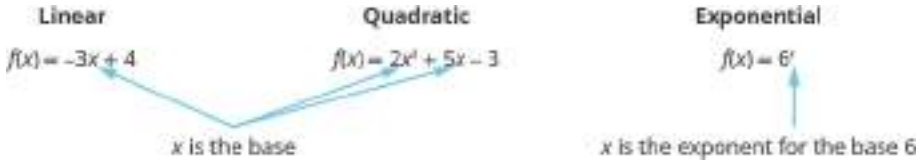
An **exponential function** is a function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$.

Exponential Function

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form

$$f(x) = a^x$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.



Our definition says $a \neq 1$. If we let $a = 1$, then $f(x) = a^x$ becomes $f(x) = 1^x$. Since $1^x = 1$ for all real numbers, $f(x) = 1$. This is the constant function.

Our definition also says $a > 0$. If we let a base be negative, say -4, then $f(x) = (-4)^x$ is not a real number when $x = \frac{1}{2}$.

$$\begin{aligned} f(x) &= (-4)^x \\ f\left(\tfrac{1}{2}\right) &= (-4)^{\frac{1}{2}} \\ f\left(\tfrac{1}{2}\right) &= \sqrt{-4} \quad \text{not a real number} \end{aligned}$$

In fact, $f(x) = (-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a > 0$.

By graphing a few exponential functions, we will be able to see their unique properties.

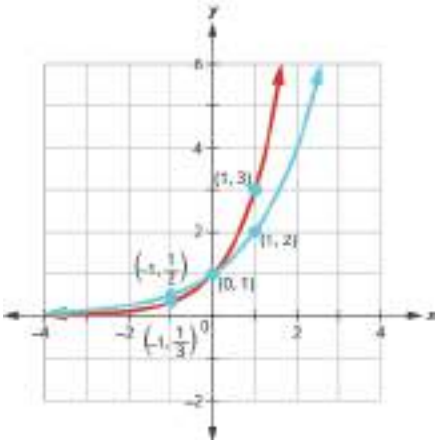
EXAMPLE 10.10

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

Solution

We will use point plotting to graph the functions.

x	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 3^x$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	$(0, 1)$	$3^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$	$3^1 = 3$	$(1, 3)$
2	$2^2 = 4$	$(2, 4)$	$3^2 = 9$	$(2, 9)$
3	$2^3 = 8$	$(3, 8)$	$3^3 = 27$	$(3, 27)$



TRY IT : : 10.19 Graph: $f(x) = 4^x$.

TRY IT : : 10.20 Graph: $g(x) = 5^x$.

If we look at the graphs from the previous Example and Try Its, we can identify some of the properties of exponential functions.

The graphs of $f(x) = 2^x$ and $g(x) = 3^x$, as well as the graphs of $f(x) = 4^x$ and $g(x) = 5^x$, all have the same basic shape. This is the shape we expect from an exponential function where $a > 1$.

We notice, that for each function, the graph contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any a.

The graph of each function, $f(x) = a^x$ also contains the point $(1, a)$. The graph of $f(x) = 2^x$ contained $(1, 2)$ and the graph of $g(x) = 3^x$ contained $(1, 3)$. This makes sense as $a^1 = a$.

Notice too, the graph of each function $f(x) = a^x$ also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x) = 2^x$ contained $\left(-1, \frac{1}{2}\right)$ and the graph of $g(x) = 3^x$ contained $\left(-1, \frac{1}{3}\right)$. This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty, \infty)$.

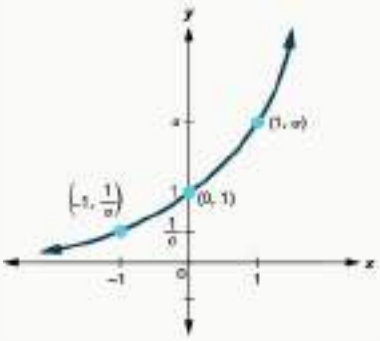
Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers.

We write the range in interval notation as $(0, \infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y = 0$, the *x*-axis, a horizontal asymptote.

Properties of the Graph of $f(x) = a^x$ when $a > 1$

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
x-intercept	None
y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$



Our definition of an exponential function $f(x) = a^x$ says $a > 0$, but the examples and discussion so far has been about functions where $a > 1$. What happens when $0 < a < 1$? The next example will explore this possibility.

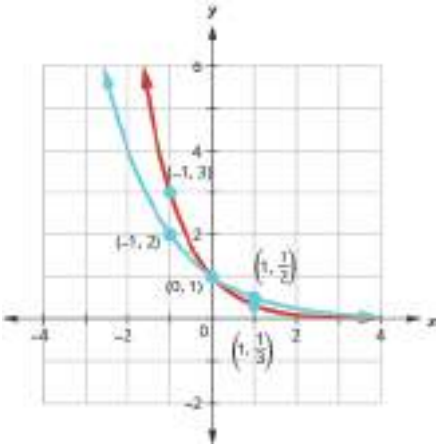
EXAMPLE 10.11

On the same coordinate system, graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

Solution

We will use point plotting to graph the functions.

x	$f(x)=\left(\frac{1}{2}\right)^x$	$(x, f(x))$	$g(x)=\left(\frac{1}{3}\right)^x$	$(x, g(x))$
−2	$\left(\frac{1}{2}\right)^{-2}=2^2=4$	(−2, 4)	$\left(\frac{1}{3}\right)^{-2}=3^2=9$	(−2, 9)
−1	$\left(\frac{1}{2}\right)^{-1}=2^1=2$	(−1, 2)	$\left(\frac{1}{3}\right)^{-1}=3^1=3$	(−1, 3)
0	$\left(\frac{1}{2}\right)^0=1$	(0, 1)	$\left(\frac{1}{3}\right)^0=1$	(0, 1)
1	$\left(\frac{1}{2}\right)^1=\frac{1}{2}$	$\left(1, \frac{1}{2}\right)$	$\left(\frac{1}{3}\right)^1=\frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$	$\left(2, \frac{1}{4}\right)$	$\left(\frac{1}{3}\right)^2=\frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^3=\frac{1}{8}$	$\left(3, \frac{1}{8}\right)$	$\left(\frac{1}{3}\right)^3=\frac{1}{27}$	$\left(3, \frac{1}{27}\right)$



TRY IT : : 10.21 Graph: $f(x)=\left(\frac{1}{4}\right)^x$.

TRY IT : : 10.22 Graph: $g(x)=\left(\frac{1}{5}\right)^x$.

Now let's look at the graphs from the previous Example and Try Its so we can now identify some of the properties of exponential functions where $0<a<1$.

The graphs of $f(x)=\left(\frac{1}{2}\right)^x$ and $g(x)=\left(\frac{1}{3}\right)^x$ as well as the graphs of $f(x)=\left(\frac{1}{4}\right)^x$ and $g(x)=\left(\frac{1}{5}\right)^x$ all have the same basic shape. While this is the shape we expect from an exponential function where $0<a<1$, the graphs go down from left to right while the previous graphs, when $a>1$, went from up from left to right.

We notice that for each function, the graph still contains the point (0, 1). This make sense because $a^0=1$ for any *a*.

As before, the graph of each function, $f(x)=a^x$, also contains the point $(1, a)$. The graph of $f(x)=\left(\frac{1}{2}\right)^x$ contained $\left(1, \frac{1}{2}\right)$ and the graph of $g(x)=\left(\frac{1}{3}\right)^x$ contained $\left(1, \frac{1}{3}\right)$. This makes sense as $a^1=a$.

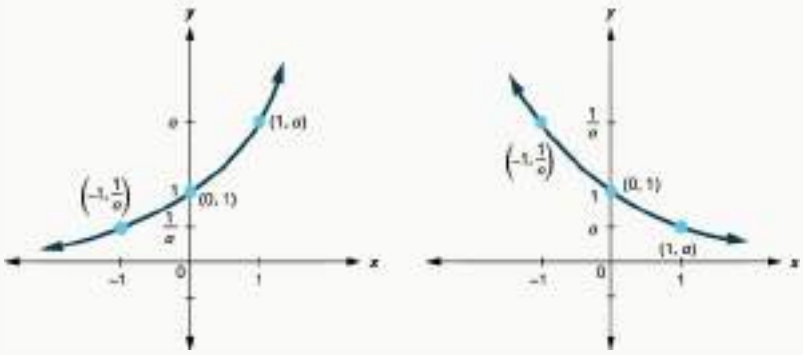
Notice too that the graph of each function, $f(x)=a^x$, also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x)=\left(\frac{1}{2}\right)^x$ contained $(-1, 2)$ and the graph of $g(x)=\left(\frac{1}{3}\right)^x$ contained $(-1, 3)$. This makes sense as $a^{-1}=\frac{1}{a}$.

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $(-\infty, \infty)$. Again, the graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Properties of the Graph of $f(x) = a^x$

when $a > 1$		when $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

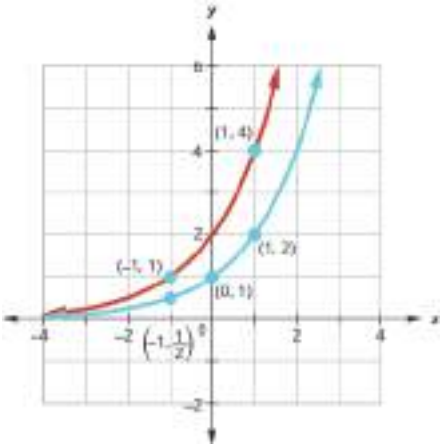
EXAMPLE 10.12

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 2^{x+1}$.

Solution

We will use point plotting to graph the functions.

x	$f(x)=2^x$	$(x, f(x))$	$g(x)=2^{x+1}$	$(x, g(x))$
-2	$2^{-2}=\frac{1}{2^2}=\frac{1}{4}$	$\left(-2,\frac{1}{4}\right)$	$2^{-2+1}=\frac{1}{2^1}=\frac{1}{2}$	$\left(-2,\frac{1}{2}\right)$
-1	$2^{-1}=\frac{1}{2^1}=\frac{1}{2}$	$\left(-1,\frac{1}{2}\right)$	$2^{-1+1}=2^0=1$	$(-1, 1)$
0	$2^0=1$	$(0, 1)$	$2^{0+1}=2^1=2$	$(0, 2)$
1	$2^1=2$	$(1, 2)$	$2^{1+1}=2^2=4$	$(1, 4)$
2	$2^2=4$	$(2, 4)$	$2^{2+1}=2^3=8$	$(2, 8)$
3	$2^3=8$	$(3, 8)$	$2^{3+1}=2^4=16$	$(3, 16)$



TRY IT : : 10.23 On the same coordinate system, graph: $f(x)=2^x$ and $g(x)=2^{x-1}$.

TRY IT : : 10.24 On the same coordinate system, graph: $f(x)=3^x$ and $g(x)=3^{x+1}$.

Looking at the graphs of the functions $f(x)=2^x$ and $g(x)=2^{x+1}$ in the last example, we see that adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

Let's now consider another situation that might be graphed more easily by translation, once we recognize the pattern.

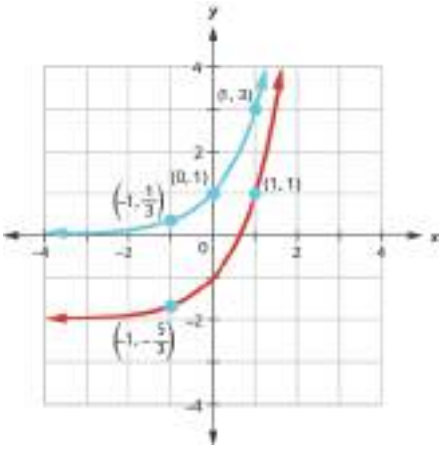
EXAMPLE 10.13

On the same coordinate system graph $f(x)=3^x$ and $g(x)=3^x-2$.

Solution

We will use point plotting to graph the functions.

x	$f(x)=3^x$	$(x, g(x))$	$g(x)=3^x-2$	$(x, g(x))$
-2	$3^{-2}=\frac{1}{9}$	$\left(-2,\frac{1}{9}\right)$	$3^{-2}-2=\frac{1}{9}-2=-\frac{17}{9}$	$\left(-2,-\frac{17}{9}\right)$
-1	$3^{-1}=\frac{1}{3}$	$\left(-1,\frac{1}{3}\right)$	$3^{-1}-2=\frac{1}{3}-2=-\frac{5}{3}$	$\left(-1,-\frac{5}{3}\right)$
0	$3^0=1$	$(0, 1)$	$3^0-2=1-2=-1$	$(0,-1)$
1	$3^1=3$	$(1, 3)$	$3^1-2=3-2=1$	$(1, 1)$
2	$3^2=9$	$(2, 9)$	$3^2-2=9-2=7$	$(2, 8)$



TRY IT : : 10.25 On the same coordinate system, graph: $f(x) = 3^x$ and $g(x) = 3^x + 2$.

TRY IT : : 10.26 On the same coordinate system, graph: $f(x) = 4^x$ and $g(x) = 4^x - 2$.

Looking at the graphs of the functions $f(x) = 3^x$ and $g(x) = 3^x - 2$ in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, *e*. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $\left(1 + \frac{1}{n}\right)^n$ as *n* gets larger and larger. We say, as *n* approaches infinity, or increases without bound. The table shows the value of $\left(1 + \frac{1}{n}\right)^n$ for several values of n.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829...
1,000	2.716923932...
10,000	2.718145927...
100,000	2.718268237...
1,000,000	2.718280469...
1,000,000,000	2.718281827...

Table 10.7

$$e \approx 2.718281827$$

The number *e* is like the number π in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number *e* is called the **natural base**.

Natural Base e

The number *e* is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as *n* increases without bound. We say, as *n* approaches infinity,

$$e \approx 2.718281827...$$

The exponential function whose base is e, $f(x)=e^x$ is called the **natural exponential function**.

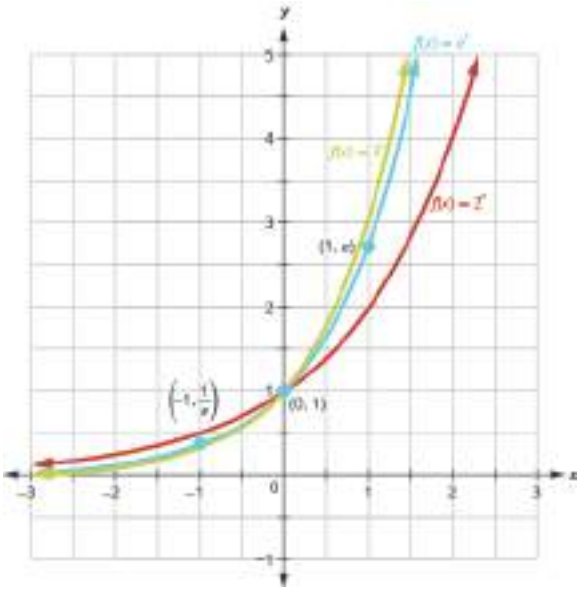
Natural Exponential Function

The natural exponential function is an exponential function whose base is e

$$f(x)=e^x$$

The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Let's graph the function $f(x)=e^x$ on the same coordinate system as $g(x)=2^x$ and $h(x)=3^x$.



Notice that the graph of $f(x)=e^x$ is "between" the graphs of $g(x)=2^x$ and $h(x)=3^x$. Does this make sense as $2<e<3$?

Solve Exponential Equations

Equations that include an exponential expression a^x are called exponential equations. To solve them we use a property that says as long as $a>0$ and $a\neq 1$, if $a^x=a^y$ then it is true that $x=y$. In other words, in an exponential equation, if the bases are equal then the exponents are equal.

One-to-One Property of Exponential Equations

For $a>0$ and $a\neq 1$,

$$\text{If } a^x=a^y, \text{ then } x=y.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

EXAMPLE 10.14 HOW TO SOLVE AN EXPONENTIAL EQUATION

Solve: $3^{2x-5}=27$.

Solution

Step 1. Write both sides of the equation with the same base.	Since the left side has base 3, we write the right side with base 3. $27 = 3^3$	$3^{2x-5} = 27$ $3^{2x-5} = 3^3$
Step 2. Write a new equation by setting the exponents equal.	Since the bases are the same, the exponents must be equal.	$2x - 5 = 3$
Step 3. Solve the equation.	Add 5 to each side. Divide by 2.	$2x = 8$ $x = 4$
Step 4. Check the solution.	Substitute $x = 4$ into the original equation.	$3^{2x-5} = 27$ $3^{2\cdot 4-5} \stackrel{?}{=} 27$ $3^3 \stackrel{?}{=} 27$ $27 = 27$ ✓

TRY IT : : 10.27 Solve: $3^{3x-2} = 81$.

TRY IT : : 10.28 Solve: $7^{x-3} = 7$.

The steps are summarized below.

HOW TO : : HOW TO SOLVE AN EXPONENTIAL EQUATION

Step 1. Write both sides of the equation with the same base, if possible.
Step 2. Write a new equation by setting the exponents equal.
Step 3. Solve the equation.
Step 4. Check the solution.

In the next example, we will use our properties on exponents.

EXAMPLE 10.15

Solve $\frac{e^{x^2}}{e^3} = e^{2x}$.

Solution

	$\frac{e^{x^2}}{e^3} = e^{2x}$
Use the Property of Exponents: $\frac{a^m}{a^n} = a^{m-n}$.	$e^{x^2-3} = e^{2x}$
Write a new equation by setting the exponents equal.	$x^2 - 3 = 2x$
Solve the equation.	$x^2 - 2x - 3 = 0$

$$(x-3)(x+1)=0$$

$$x=3,\quad x=-1$$

Check the solutions.

$x=3$

$\frac{e^{x^2}}{e^x} \stackrel{?}{=} e^{x}$

$\frac{e^{x^2}}{e^x} \stackrel{?}{=} e^{x-1}$

$\frac{e^{x^2}}{e^x} \stackrel{?}{=} e^{x}$

$e^x = e^x$ ✓

$x=-1$

$\frac{e^{x^2}}{e^x} \stackrel{?}{=} e^{x}$

$\frac{e^{x^2}}{e^x} \stackrel{?}{=} e^{x-1}$

$\frac{e^{x^2}}{e^x} \stackrel{?}{=} e^{x}$

$e^{-1} = e^{-1}$ ✓

TRY IT : : 10.29 Solve: $\frac{e^{x^2}}{e^x}=e^2$.

TRY IT : : 10.30 Solve: $\frac{e^{x^2}}{e^x}=e^6$.

Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, *P*, is invested at an interest rate, *r*, for *t* years, the new balance, *A*, will depend on how often the interest is compounded. If the interest is compounded *n* times a year we use the formula $A=P\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A=Pe^{rt}$. These are the formulas for **compound interest**.

Compound Interest

For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A*, is:

$A=P\left(1+\frac{r}{n}\right)^{nt}$ when compounded n times a year.

$A=Pe^{rt}$ when compounded continuously.

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

EXAMPLE 10.16

A total of $\$10{,}000$ was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$\begin{aligned} A &= ? \\ P &= \$10{,}000 \\ r &= 0.05 \\ t &= 18 \text{ years} \end{aligned}$$

ⓐ

For quarterly compounding, $n = 4$. There are 4 quarters in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10{,}000\left(1 + \frac{0.05}{4}\right)^{4 \cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24{,}459.20$$

ⓑ

For monthly compounding, $n = 12$. There are 12 months in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10{,}000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 18}$$

Compute the amount.

$$A = \$24{,}550.08$$

ⓒ

For compounding continuously,

$$A = Pe^{rt}$$

Substitute the values in the formula.

$$A = 10{,}000e^{0.05 \cdot 18}$$

Compute the amount.

$$A = \$24{,}596.03$$

TRY IT : : 10.31

Angela invested $\$15{,}000$ in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

TRY IT : : 10.32

Allan invested $10,000 in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0, as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:

$$A = A_0 e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

EXAMPLE 10.17

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution

Identify the values of each variable in the formulas.	$A = ?$
Be sure to put the percent in decimal form.	$A_0 = 100$
Be sure the units match—the rate is per hour and the time is in hours.	$r = 0.25/\text{hour}$ $t = 24 \text{ hours}$
Substitute the values in the formula: $A = A_0 e^{rt}$.	$A = 100e^{0.25 \cdot 24}$
Compute the amount.	$A = 40{,}342.88$
Round to the nearest whole virus.	$A = 40{,}343$
	The researcher will find 40,34 viruses.

TRY IT : : 10.33

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

TRY IT : : 10.34

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

MEDIA : :

Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

- **Graphing Exponential Functions (https://openstax.org/l/37Graphexponent)**
- **Solving Exponential Equations (https://openstax.org/l/37Solvelikebase)**
- **Applications of Exponential Functions (https://openstax.org/l/37Exponentapp)**
- **Continuously Compound Interest (https://openstax.org/l/37Compoundint)**
- **Radioactive Decay and Exponential Growth (https://openstax.org/l/37Exponentdecay)**

10.2 EXERCISES

Practice Makes Perfect

Graph Exponential Functions

In the following exercises, graph each exponential function.

65. $f(x) = 2^x$

66. $g(x) = 3^x$

67. $f(x) = 6^x$

68. $g(x) = 7^x$

69. $f(x) = (1.5)^x$

70. $g(x) = (2.5)^x$

71. $f(x) = \left(\frac{1}{2}\right)^x$

72. $g(x) = \left(\frac{1}{3}\right)^x$

73. $f(x) = \left(\frac{1}{6}\right)^x$

74. $g(x) = \left(\frac{1}{7}\right)^x$

75. $f(x) = (0.4)^x$

76. $g(x) = (0.6)^x$

In the following exercises, graph each function in the same coordinate system.

77. $f(x) = 4^x,\ g(x) = 4^{x-1}$

78. $f(x) = 3^x,\ g(x) = 3^{x-1}$

79. $f(x) = 2^x,\ g(x) = 2^{x-2}$

80. $f(x) = 2^x,\ g(x) = 2^{x+2}$

81. $f(x) = 3^x,\ g(x) = 3^x + 2$

82. $f(x) = 4^x,\ g(x) = 4^x + 2$

83. $f(x) = 2^x,\ g(x) = 2^x + 1$

84. $f(x) = 2^x,\ g(x) = 2^x - 1$

In the following exercises, graph each exponential function.

85. $f(x) = 3^{x+2}$

86. $f(x) = 3^{x-2}$

87. $f(x) = 2^x + 3$

88. $f(x) = 2^x - 3$

89. $f(x) = \left(\frac{1}{2}\right)^{x-4}$

90. $f(x) = \left(\frac{1}{2}\right)^x - 3$

91. $f(x) = e^x + 1$

92. $f(x) = e^{x-2}$

93. $f(x) = -2^x$

94. $f(x) = 3^x$

Solve Exponential Equations

In the following exercises, solve each equation.

95. $2^{3x-8} = 16$

96. $2^{2x-3} = 32$

97. $3^{x+3} = 9$

98. $3^{x^2} = 81$

99. $4^{x^2} = 4$

100. $4^x = 32$

101. $4^{x+2} = 64$

102. $4^{x+3} = 16$

103. $2^{x^2+2x} = \frac{1}{2}$

104. $3^{x^2-2x} = \frac{1}{3}$

105. $e^{3x} \cdot e^4 = e^{10}$

106. $e^{2x} \cdot e^3 = e^9$

107. $\frac{e^{x^2}}{e^2} = e^x$

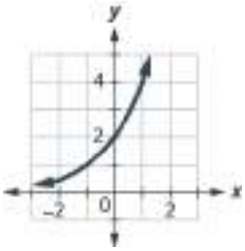
108. $\frac{e^{x^2}}{e^3} = e^{2x}$

In the following exercises, match the graphs to one of the following functions: ⓐ 2^x ⓑ 2^{x+1} ⓒ 2^{x-1} ⓓ $2^x + 2$ ⓔ $2^x - 2$ ⓕ 3^x

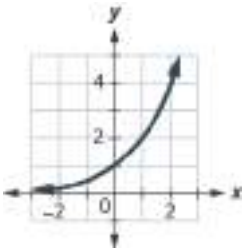
109.



110.



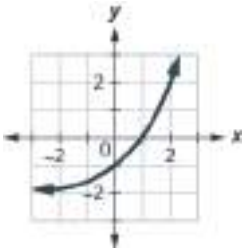
111.



112.



113.



114.



Use exponential models in applications

In the following exercises, use an exponential model to solve.

115. Edgar accumulated $\$5{,}000$ in credit card debt. If the interest rate is 20% per year, and he does not make any payments for 2 years, how much will he owe on this debt in 2 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

116. Cynthia invested $\$12{,}000$ in a savings account. If the interest rate is 6%, how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

117. Rochelle deposits $\$5{,}000$ in an IRA. What will be the value of her investment in 25 years if the investment is earning 8% per year and is compounded continuously?

118. Nazerhy deposits $\$8{,}000$ in a certificate of deposit. The annual interest rate is 6% and the interest will be compounded quarterly. How much will the certificate be worth in 10 years?

119. A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. He starts his experiment with 100 of the bacteria that grows at a rate of 6% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

120. A biologist is observing the growth pattern of a virus. She starts with 50 of the virus that grows at a rate of 20% per hour. She will check on the virus in 24 hours. How many viruses will she find?

121. In the last ten years the population of Indonesia has grown at a rate of 1.12% per year to 258,316,051. If this rate continues, what will be the population in 10 more years?

122. In the last ten years the population of Brazil has grown at a rate of 0.9% per year to 205,823,665. If this rate continues, what will be the population in 10 more years?

Writing Exercises

123. Explain how you can distinguish between exponential functions and polynomial functions.

124. Compare and contrast the graphs of $y = x^2$ and $y = 2^x$.

125. What happens to an exponential function as the values of x decreases? Will the graph ever cross the y-axis? Explain.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
graph exponential functions.			
solve exponential equations.			
use exponential models in applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

10.3 Evaluate and Graph Logarithmic Functions

Learning Objectives

By the end of this section, you will be able to:

- Convert between exponential and logarithmic form
- Evaluate logarithmic functions
- Graph Logarithmic functions
- Solve logarithmic equations
- Use logarithmic models in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 81$.
 If you missed this problem, review **Example 6.46**.
2. Evaluate: 3^{-2}.
 If you missed this problem, review **Example 5.15**.
3. Solve: $2^4 = 3x - 5$.
 If you missed this problem, review **Example 2.2**.

We have spent some time finding the inverse of many functions. It works well to 'undo' an operation with another operation. Subtracting 'undoes' addition, multiplication 'undoes' division, taking the square root 'undoes' squaring.

As we studied the exponential function, we saw that it is one-to-one as its graphs pass the horizontal line test. This means an exponential function does have an inverse. If we try our algebraic method for finding an inverse, we run into a problem.

	$f(x) = a^x$
Rewrite with $y = f(x)$.	$y = a^x$
Interchange the variables x and y.	$x = a^y$
Solve for y.	Oops! We have no way to solve for y!

To deal with this we define the logarithm function with base a to be the inverse of the exponential function $f(x) = a^x$.

We use the notation $f^{-1}(x) = \log_a x$ and say the inverse function of the exponential function is the logarithmic function.

Logarithmic Function

The function $f(x) = \log_a x$ is the **logarithmic function** with base a, where $a > 0$, $x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

Convert Between Exponential and Logarithmic Form

Since the equations $y = \log_a x$ and $x = a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See **Figure 10.3**. Notice the positions of the exponent and base.



Figure 10.3

If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, "base to the exponent give us the number."

EXAMPLE 10.18

Convert to logarithmic form: ⓐ $2^3 = 8$, ⓑ $5^{\frac{1}{2}} = \sqrt{5}$, and ⓒ $\left(\frac{1}{2}\right)^x = \frac{1}{16}$.

Solution

Identify the base and the exponent.

(a)	(b)	(c)
$2^3 = 8$	$5^{\frac{1}{2}} = \sqrt{5}$	$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$
$y = \log_2 x$	$y = \log_5 x$	$y = \log_{\frac{1}{2}} x$
$3 = \log_2 8$	$\frac{1}{2} = \log_5 \sqrt{5}$	$4 = \log_{\frac{1}{2}} \frac{1}{16}$
If $2^3 = 8$, then $3 = \log_2 8$.	If $5^{\frac{1}{2}} = \sqrt{5}$, then $\frac{1}{2} = \log_5 \sqrt{5}$.	If $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, then $4 = \log_{\frac{1}{2}} \frac{1}{16}$.

TRY IT : : 10.35 Convert to logarithmic form: ⓐ $3^2 = 9$ ⓑ $7^{\frac{1}{2}} = \sqrt{7}$ ⓒ $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

TRY IT : : 10.36 Convert to logarithmic form: ⓐ $4^3 = 64$ ⓑ $4^{\frac{1}{3}} = \sqrt[3]{4}$ ⓒ $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

In the next example we do the reverse—convert logarithmic form to exponential form.

EXAMPLE 10.19

Convert to exponential form: ⓐ $2 = \log_8 64$, ⓑ $0 = \log_4 1$, and ⓒ $-3 = \log_{10} \frac{1}{1000}$.

Solution

Identify the base and the exponent.

(a)	(b)	(c)
$2 = \log_8 64$	$0 = \log_4 1$	$-3 = \log_{10} \frac{1}{1000}$
$x = a^y$	$x = a^y$	$x = a^y$
$64 = 8^2$	$1 = 4^0$	$\frac{1}{1000} = 10^{-3}$
If $2 = \log_8 64$, then $64 = 8^2$.	If $0 = \log_4 1$, then $1 = 4^0$.	If $-3 = \log_{10} \frac{1}{1000}$, then $\frac{1}{1000} = 10^{-3}$.

TRY IT : : 10.37 Convert to exponential form: ⓐ $3 = \log_4 64$ ⓑ $0 = \log_x 1$ ⓒ $-2 = \log_{10} \frac{1}{100}$

TRY IT : : 10.38 Convert to exponential form: ⓐ $3 = \log_3 27$ ⓑ $0 = \log_x 1$ ⓒ $-1 = \log_{10} \frac{1}{10}$

Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

EXAMPLE 10.20

Find the value of x: ⓐ $\log_x 36 = 2$, ⓑ $\log_4 x = 3$, and ⓒ $\log_{\frac{1}{2}} \frac{1}{8} = x$.

Solution

ⓐ

	$\log_x 36 = 2$
Convert to exponential form.	$x^2 = 36$
Solve the quadratic.	$x = 6, \quad \cancel{x = -6}$
The base of a logarithmic function must be positive, so we eliminate $x = -6$.	$x = 6$ Therefore, $\log_6 36 = 2$.

ⓑ

	$\log_4 x = 3$
Convert to exponential form.	$4^3 = x$
Simplify.	$x = 64$ Therefore, $\log_4 64 = 3$.

ⓒ

	$\log_{\frac{1}{2}} \frac{1}{8} = x$
Convert to exponential form.	$\left(\frac{1}{2}\right)^x = \frac{1}{8}$
Rewrite $\frac{1}{8}$ as $\left(\frac{1}{2}\right)^3$.	$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$
With the same base, the exponents must be equal.	$x = 3$ Therefore, $\log_{\frac{1}{2}} \frac{1}{8} = 3$

TRY IT : : 10.39 Find the value of x: ⓐ $\log_x 64 = 2$ ⓑ $\log_5 x = 3$ ⓒ $\log_{\frac{1}{2}} \frac{1}{4} = x$

TRY IT : : 10.40 Find the value of x: ⓐ $\log_x 81 = 2$ ⓑ $\log_3 x = 5$ ⓒ $\log_{\frac{1}{3}} \frac{1}{27} = x$

When see an expression such as $\log_3 27$, we can find its exact value two ways. By inspection we realize it means "3 to what power will be 27"? Since $3^3 = 27$, we know $\log_3 27 = 3$. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

EXAMPLE 10.21

Find the exact value of each logarithm without using a calculator:

ⓐ $\log_5 25$,

ⓑ $\log_9 3$, and ⓒ $\log_2 \frac{1}{16}$.

Solution

ⓐ

	$\log_5 25$
5 to what power will be 25?	$\log_5 25 = 2$
Or	
Set the expression equal to x.	$\log_5 25 = x$
Change to exponential form.	$5^x = 25$
Rewrite 25 as 5^2.	$5^x = 5^2$
With the same base the exponents must be equal.	$x = 2$ Therefore, $\log_5 25 = 2$.

ⓑ

	$\log_9 3$
Set the expression equal to x.	$\log_9 3 = x$
Change to exponential form.	$9^x = 3$
Rewrite 9 as 3^2.	$\left(3^2\right)^x = 3^1$
Simplify the exponents.	$3^{2x} = 3^1$
With the same base the exponents must be equal.	$2x = 1$
Solve the equation.	$x = \frac{1}{2}$ Therefore, $\log_9 3 = \frac{1}{2}$.

ⓒ

	$\log_2 \frac{1}{16}$
Set the expression equal to x.	$\log_2 \frac{1}{16} = x$
Change to exponential form.	$2^x = \frac{1}{16}$
Rewrite 16 as 2^4.	$2^x = \frac{1}{2^4}$
	$2^x = 2^{-4}$
With the same base the exponents must be equal.	$x = -4$ Therefore, $\log_2 \frac{1}{16} = -4$.

TRY IT : : 10.41 Find the exact value of each logarithm without using a calculator:

ⓐ $\log_{12} 144$

ⓑ $\log_4 2$

ⓒ $\log_2 \frac{1}{32}$

TRY IT : : 10.42 Find the exact value of each logarithm without using a calculator:

ⓐ $\log_9 81$

ⓑ $\log_8 2$

ⓒ $\log_3 \frac{1}{9}$

Graph Logarithmic Functions

To graph a logarithmic function $y = \log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$. Generally, when we look for ordered pairs for the graph of a function, we usually choose an x-value and then determine its corresponding y-value. In this case you may find it easier to choose y-values and then determine its corresponding x-value.

EXAMPLE 10.22

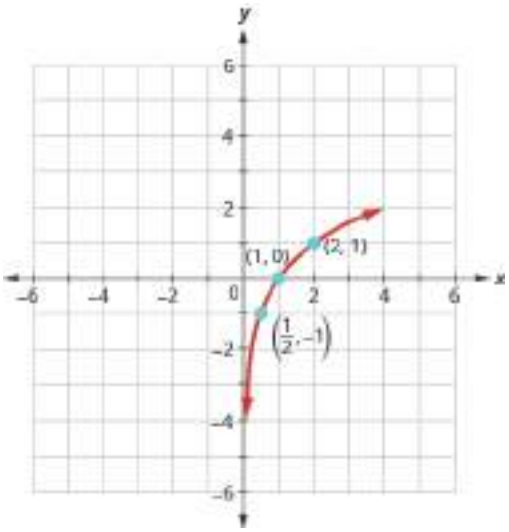
Graph $y = \log_2 x$.

Solution

To graph the function, we will first rewrite the logarithmic equation, $y = \log_2 x$, in exponential form, $2^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

y	$2^y = x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(\frac{1}{4}, 2\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(\frac{1}{2}, -1\right)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$



TRY IT : : 10.43 Graph: $y = \log_3 x$.

TRY IT : : 10.44 Graph: $y = \log_5 x$.

The graphs of $y = \log_2 x$, $y = \log_3 x$, and $y = \log_5 x$ are the shape we expect from a logarithmic function where $a > 1$.

We notice that for each function the graph contains the point $(1, 0)$. This make sense because $0 = \log_a 1$ means $a^0 = 1$ which is true for any *a*.

The graph of each function, also contains the point $(a, 1)$. This makes sense as $1 = \log_a a$ means $a^1 = a$. which is true for any *a*.

Notice too, the graph of each function $y = \log_a x$ also contains the point $\left(\frac{1}{a}, -1\right)$. This makes sense as $-1 = \log_a \frac{1}{a}$ means $a^{-1} = \frac{1}{a}$, which is true for any a.

Look at each graph again. Now we will see that many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

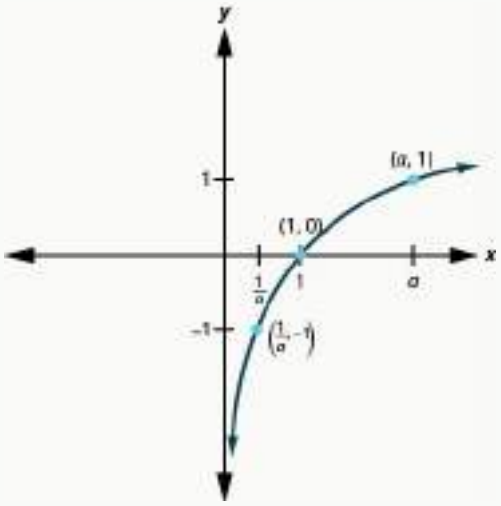
What is the domain of the function? The graph never hits the y-axis. The domain is all positive numbers. We write the domain in interval notation as $(0, \infty)$.

What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as $(-\infty, \infty)$.

When the graph approaches the y-axis so very closely but will never cross it, we call the line $x = 0$, the y-axis, a vertical asymptote.

Properties of the Graph of $y = \log_a x$ when $a > 1$

Domain	$(0, \infty)$
Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$
y-intercept	None
Contains	$(a, 1)$, $\left(\frac{1}{a}, -1\right)$
Asymptote	y-axis



Our next example looks at the graph of $y = \log_a x$ when $0 < a < 1$.

EXAMPLE 10.23

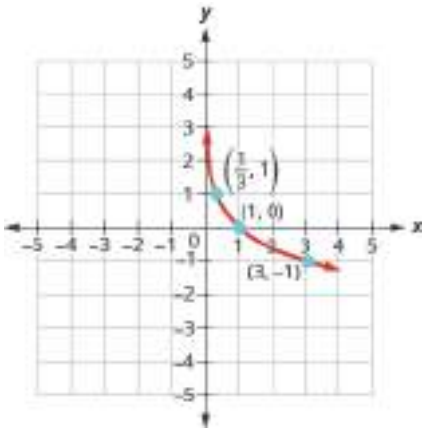
Graph $y = \log_{\frac{1}{3}} x$.

Solution

To graph the function, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}} x$, in exponential form, $\left(\frac{1}{3}\right)^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(9, -2)$
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(3, -1)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(1, 0)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(\frac{1}{9}, 2\right)$
3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(\frac{1}{27}, 3\right)$



TRY IT : : 10.45 Graph: $y = \log_{\frac{1}{2}} x$.

TRY IT : : 10.46 Graph: $y = \log_{\frac{1}{4}} x$.

Now, let's look at the graphs $y = \log_{\frac{1}{2}} x$, $y = \log_{\frac{1}{3}} x$ and $y = \log_{\frac{1}{4}} x$, so we can identify some of the properties of logarithmic functions where $0 < a < 1$.

The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where $0 < a < 1$.

We notice, that for each function again, the graph contains the points, $(1, 0)$, $(a, 1)$, $\left(\frac{1}{a}, -1\right)$. This make sense for the

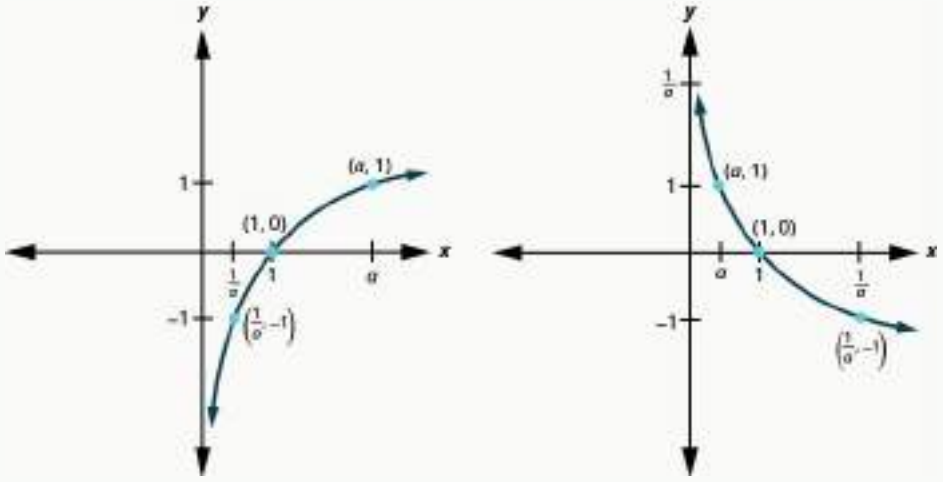
same reasons we argued above.

We notice the domain and range are also the same—the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. The y-axis is again the vertical asymptote.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Properties of the Graph of $y = \log_a x$

when $a > 1$		when $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
y-intercept	none	y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	y-axis	Asymptote	y-axis
Basic shape	increasing	Basic shape	Decreasing



We talked earlier about how the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$. The graphs in Figure 10.4 show both the exponential (blue) and logarithmic (red) functions on the same graph for both $a > 1$ and $0 < a < 1$.

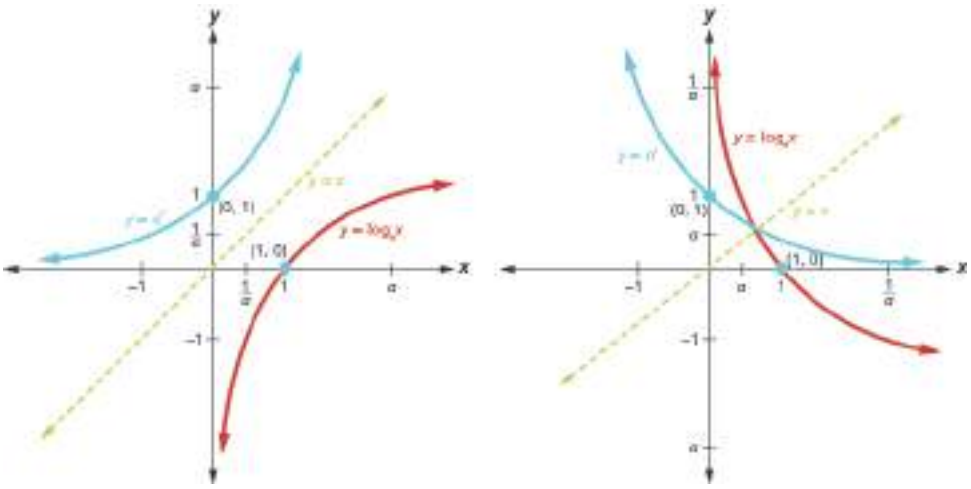


Figure 10.4

Notice how the graphs are reflections of each other through the line $y = x$. We know this is true of inverse functions.

Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the *x*-axis is the horizontal asymptote for the exponential functions and the *y*-axis is the vertical asymptote for the logarithmic functions.

Solve Logarithmic Equations

When we talked about exponential functions, we introduced the number *e*. Just as *e* was a base for an exponential function, it can be used a base for logarithmic functions too. The logarithmic function with base *e* is called the **natural logarithmic function**. The function $f(x) = \log_e x$ is generally written $f(x) = \ln x$ and we read it as "el en of x."

Natural Logarithmic Function

The function $f(x) = \ln x$ is the **natural logarithmic function** with base e, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

When the base of the logarithm function is 10, we call it the **common logarithmic function** and the base is not shown. If the base *a* of a logarithm is not shown, we assume it is 10.

Common Logarithmic Function

The function $f(x) = \log x$ is the **common logarithmic function** with base 10, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

It will be important for you to use your calculator to evaluate both common and natural logarithms.

Look for the log and ln keys on your calculator.

To solve logarithmic equations, one strategy is to change the equation to exponential form and then solve the exponential equation as we did before. As we solve logarithmic equations, $y = \log_a x$, we need to remember that for the base *a*, $a > 0$ and $a \neq 1$. Also, the domain is $x > 0$. Just as with radical equations, we must check our solutions to eliminate any extraneous solutions.

EXAMPLE 10.24

Solve: ⓐ $\log_a 49 = 2$ and ⓑ $\ln x = 3$.

Solution

ⓐ

$$\log_a 49 = 2$$

Rewrite in exponential form. $a^2 = 49$

Solve the equation using the square root property. $a = \pm 7$

The base cannot be negative, so we eliminate $a = -7$. $a = 7, \cancel{a = -7}$

Check.

$a = 7$ $\log_a 49 = 2$

$\log_7 49 \stackrel{?}{=} 2$

$7^2 \stackrel{?}{=} 49$

$49 = 49$ ✓

ⓑ

$$\ln x = 3$$

Rewrite in exponential form. $e^3 = x$

Check.

$x = e^3$ $\ln x = 3$

$\ln e^3 \stackrel{?}{=} 3$

$e^3 = e^3$ ✓

TRY IT : : 10.47 Solve: ⓐ $\log_a 121 = 2$ ⓑ $\ln x = 7$

TRY IT : : 10.48 Solve: ⓐ $\log_a 64 = 3$ ⓑ $\ln x = 9$

EXAMPLE 10.25

Solve: ⓐ $\log_2(3x-5) = 4$ and ⓑ $\ln e^{2x} = 4$.

Solution

ⓐ

$$\log_2(3x-5) = 4$$

Rewrite in exponential form. $2^4 = 3x-5$

Simplify. $16 = 3x-5$

Solve the equation. $21 = 3x$

$7 = x$

Check.

$x = 7$ $\log_2(3x-5) = 4$

$\log_2(3\cdot 7-5) \stackrel{?}{=} 4$

$\log_2(16) \stackrel{?}{=} 4$

$2^4 \stackrel{?}{=} 16$

$16 = 16$ ✓

ⓑ

	$\ln e^{2x} = 4$
Rewrite in exponential form.	$e^4 = e^{2x}$
Since the bases are the same the exponents are equal.	$4 = 2x$
Solve the equation.	$2 = x$

Check.

$x = 2$

$$\ln e^{2x} = 4$$
$$\ln e^{2\cdot 2} \stackrel{?}{=} 4$$
$$\ln e^4 \stackrel{?}{=} 4$$
$$e^4 = e^4 \checkmark$$

TRY IT : : 10.49 Solve: ⓐ $\log_2(5x-1)=6$ ⓑ $\ln e^{3x}=6$

TRY IT : : 10.50 Solve: ⓐ $\log_3(4x+3)=3$ ⓑ $\ln e^{4x}=4$

Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The 10^{-12} in the formula represents the intensity of sound that is barely audible.

Decibel Level of Sound

The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is

$$D = 10\log\left(\frac{I}{10^{-12}}\right)$$

EXAMPLE 10.26

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity 10^{-2} watts per square inch?

Solution

	$D = 10\log\left(\frac{I}{10^{-12}}\right)$
Substitute in the intensity level, I.	$D = 10\log\left(\frac{10^{-2}}{10^{-12}}\right)$
Simplify.	$D = 10\log(10^{10})$
Since $\log 10^{10} = 10$.	$D = 10 \cdot 10$
Multiply.	$D = 100$
	The decibel level of music coming through earphones is 100 dB.

TRY IT : : 10.51

What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

TRY IT : : 10.52 What is the decibel level heavy city traffic with intensity 10^{-3} watts per square inch?

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where I is the intensity of the shock wave. This model provides a way to measure earthquake intensity.

Earthquake Intensity

The magnitude *R* of an earthquake is measured by $R = \log I$, where *I* is the intensity of its shock wave.

EXAMPLE 10.27

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $108 million dollars of damage. Compare the intensities of the two earthquakes.

Solution

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

Convert the magnitudes to intensities.	$R = \log I$
1906 earthquake	$7.8 = \log I$
Convert to exponential form.	$I = 10^{7.8}$
2014 earthquake	$5.1 = \log I$
Convert to exponential form.	$I = 10^{5.1}$
Form a ratio of the intensities.	$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$
Substitute in the values.	$\frac{10^{7.8}}{10^{5.1}}$
Divide by subtracting the exponents.	$10^{2.7}$
Evaluate.	501

The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake.

TRY IT : : 10.53

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

TRY IT : : 10.54

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

MEDIA : :

Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

- **Re-writing logarithmic equations in exponential form (https://openstax.org/l/37logasexponent)**
- **Simplifying Logarithmic Expressions (https://openstax.org/l/37Simplifylog)**
- **Graphing logarithmic functions (https://openstax.org/l/37Graphlog)**
- **Using logarithms to calculate decibel levels (https://openstax.org/l/37Finddecibel)**

10.3 EXERCISES

Practice Makes Perfect

Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

126. $4^2 = 16$

127. $2^5 = 32$

128. $3^3 = 27$

129. $5^3 = 125$

130. $10^3 = 1000$

131. $10^{-2} = \frac{1}{100}$

132. $x^{\frac{1}{2}} = \sqrt{3}$

133. $x^{\frac{1}{3}} = \sqrt[3]{6}$

134. $32^x = \sqrt[4]{32}$

135. $17^x = \sqrt[5]{17}$

136. $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

137. $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

138. $3^{-2} = \frac{1}{9}$

139. $4^{-3} = \frac{1}{64}$

140. $e^x = 6$

141. $e^3 = x$

In the following exercises, convert each logarithmic equation to exponential form.

142. $3 = \log_4 64$

143. $6 = \log_2 64$

144. $4 = \log_x 81$

145. $5 = \log_x 32$

146. $0 = \log_{12} 1$

147. $0 = \log_7 1$

148. $1 = \log_3 3$

149. $1 = \log_9 9$

150. $-4 = \log_{10} \frac{1}{10{,}000}$

151. $3 = \log_{10} 1{,}000$

152. $5 = \log_e x$

153. $x = \log_e 43$

Evaluate Logarithmic Functions

In the following exercises, find the value of x in each logarithmic equation.

154. $\log_x 49 = 2$

155. $\log_x 121 = 2$

156. $\log_x 27 = 3$

157. $\log_x 64 = 3$

158. $\log_3 x = 4$

159. $\log_5 x = 3$

160. $\log_2 x = -6$

161. $\log_3 x = -5$

162. $\log_{\frac{1}{4}} \frac{1}{16} = x$

163. $\log_{\frac{1}{3}} \frac{1}{9} = x$

164. $\log_{\frac{1}{4}} 64 = x$

165. $\log_{\frac{1}{9}} 81 = x$

In the following exercises, find the exact value of each logarithm without using a calculator.

166. $\log_7 49$

167. $\log_6 36$

168. $\log_4 1$

169. $\log_5 1$

170. $\log_{16} 4$

171. $\log_{27} 3$

172. $\log_{\frac{1}{2}} 2$

173. $\log_{\frac{1}{2}} 4$

174. $\log_2 \frac{1}{16}$

175. $\log_3 \frac{1}{27}$

176. $\log_4 \frac{1}{16}$

177. $\log_9 \frac{1}{81}$

Graph Logarithmic Functions

In the following exercises, graph each logarithmic function.

178. $y = \log_2 x$

179. $y = \log_4 x$

180. $y = \log_6 x$

181. $y = \log_7 x$

182. $y = \log_{1.5} x$

183. $y = \log_{2.5} x$

184. $y = \log_{\frac{1}{3}} x$

185. $y = \log_{\frac{1}{5}} x$

186. $y = \log_{0.4} x$

187. $y = \log_{0.6} x$

Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

188. $\log_a 16 = 2$

189. $\log_a 81 = 2$

190. $\log_a 8 = 3$

191. $\log_a 27 = 3$

192. $\log_a 32 = 2$

193. $\log_a 24 = 3$

194. $\ln x = 5$

195. $\ln x = 4$

196. $\log_2 (5x + 1) = 4$

197. $\log_2 (6x + 2) = 5$

198. $\log_3 (4x - 3) = 2$

199. $\log_3 (5x - 4) = 4$

200. $\log_4 (5x + 6) = 3$

201. $\log_4 (3x - 2) = 2$

202. $\ln e^{4x} = 8$

203. $\ln e^{2x} = 6$

204. $\log x^2 = 2$

205. $\log(x^2 - 25) = 2$

206. $\log_2\left(x^2 - 4\right) = 5$

207. $\log_3\left(x^2 + 2\right) = 3$

Use Logarithmic Models in Applications

In the following exercises, use a logarithmic model to solve.

208. What is the decibel level of normal conversation with intensity 10^{-6} watts per square inch?

209. What is the decibel level of a whisper with intensity 10^{-10} watts per square inch?

210. What is the decibel level of the noise from a motorcycle with intensity 10^{-2} watts per square inch?

211. What is the decibel level of the sound of a garbage disposal with intensity 10^{-2} watts per square inch?

212. In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2010, Haiti also experienced an intense earthquake which measured 7.0 on the Richter scale. Compare the intensities of the two earthquakes.

213. The Los Angeles area experiences many earthquakes. In 1994, the Northridge earthquake measured magnitude of 6.7 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Writing Exercises

214. Explain how to change an equation from logarithmic form to exponential form.

215. Explain the difference between common logarithms and natural logarithms.

216. Explain why $\log_a a^x = x$.

217. Explain how to find the $\log_7 32$ on your calculator.

Self Check

ⓐ

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
convert between exponential and logarithmic form.			
evaluate logarithmic functions.			
graph logarithmic functions.			
solve logarithmic functions.			
use logarithmic models in applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

10.4 Use the Properties of Logarithms

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate: ⓐ a^0 ⓑ a^1.
 If you missed this problem, review **Example 5.14**.
2. Write with a rational exponent: $\sqrt[3]{x^2 y}$.
 If you missed this problem, review **Example 8.27**.
3. Round to three decimal places: 2.5646415.
 If you missed this problem, review **Example 1.34**.

Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0 = 1,$ we can convert this to logarithmic form and get $\log_a 1 = 0.$ Also, since $a^1 = a,$ we get $\log_a a = 1.$

Properties of Logarithms

$$\log_a 1 = 0 \qquad \log_a a = 1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

EXAMPLE 10.28

Evaluate using the properties of logarithms: ⓐ $\log_8 1$ and ⓑ $\log_6 6.$

Solution

ⓐ

	$\log_8 1$	
Use the property, $\log_a 1 = 0.$	0	$\log_8 1 = 0$

ⓑ

	$\log_6 6$	
Use the property, $\log_a a = 1.$	1	$\log_6 6 = 1$

TRY IT : : 10.55 Evaluate using the properties of logarithms: ⓐ $\log_{13} 1$ ⓑ $\log_9 9.$

TRY IT : : 10.56 Evaluate using the properties of logarithms: ⓐ $\log_5 1$ ⓑ $\log_7 7.$

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $a^{\log_a x} = x$ converts to the logarithmic equation $\log_a x = \log_a x,$ which is a true statement

for positive values for *x* only.

The logarithmic equation $\log_a a^x = x$ converts to the exponential equation $a^x = a^x$, which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power. These two properties show the composition of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

Inverse Properties of Logarithms

For $a > 0$, $x > 0$ and $a \neq 1$,

$$a^{\log_a x} = x \qquad \log_a a^x = x$$

In the next example, apply the inverse properties of logarithms.

EXAMPLE 10.29

Evaluate using the properties of logarithms: ⓐ $4^{\log_4 9}$ and ⓑ $\log_3 3^5$.

Solution

ⓐ

Use the property, $a^{\log_a x} = x$.

$4^{\log_4 9}$

9

$4^{\log_4 9} = 9$

ⓑ

Use the property, $a^{\log_a x} = x$.

$\log_3 3^5$

5

$\log_3 3^5 = 5$

TRY IT : : 10.57 Evaluate using the properties of logarithms: ⓐ $5^{\log_5 15}$ ⓑ $\log_7 7^4$.

TRY IT : : 10.58 Evaluate using the properties of logarithms: ⓐ $2^{\log_2 8}$ ⓑ $\log_2 2^{15}$.

There are three more properties of logarithms that will be useful in our work. We know exponential functions and logarithmic function are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $a^m \cdot a^n = a^{m+n}$, we see that to multiply the same base, we add the exponents. The **Product Property of Logarithms**, $\log_a M \cdot N = \log_a M + \log_a N$ tells us to take the log of a product, we add the log of the factors.

Product Property of Logarithms

If $M > 0$, $N > 0$, $a > 0$ and $a \neq 1$, then,

$$\log_a (M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

EXAMPLE 10.30

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible: ⓐ $\log_3 7x$ and ⓑ $\log_4 64xy$.

Solution

ⓐ

	$\log_3 7x$
Use the Product Property, $\log_a(M \cdot N) = \log_a M + \log_a N$.	$\log_3 7 + \log_3 x$
	$\log_3 7x = \log_3 7 + \log_3 x$

ⓑ

	$\log_4 64xy$
Use the Product Property, $\log_a(M \cdot N) = \log_a M + \log_a N$.	$\log_4 64 + \log_4 x + \log_4 y$
Simplify by evaluating $\log_4 64$.	$3 + \log_4 x + \log_4 y$
	$\log_4 64xy = 3 + \log_4 x + \log_4 y$

TRY IT : : 10.59

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

ⓐ $\log_3 3x$ ⓑ $\log_2 8xy$

TRY IT : : 10.60

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

ⓐ $\log_9 9x$ ⓑ $\log_3 27xy$

Similarly, in the Quotient Property of Exponents, $\frac{a^m}{a^n} = a^{m-n}$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $\log_a \frac{M}{N} = \log_a M - \log_a N$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Quotient Property of Logarithms

If $M > 0$, $N > 0$,a > 0 and a $\neq 1$, then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_a M - \log_a N \neq \log_a(M - N)$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

EXAMPLE 10.31

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_5 \frac{5}{7}$ and ⓑ $\log \frac{x}{100}$

Solution

ⓐ

	$\log_5 \frac{5}{7}$
Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.	$\log_5 5 - \log_5 7$
Simplify.	$1 - \log_5 7$
	$\log_5 \frac{5}{7} = 1 - \log_5 7$

ⓑ

	$\log \frac{x}{100}$
Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.	$\log x - \log 100$
Simplify.	$\log x - 2$
	$\log \frac{x}{100} = \log x - 2$

TRY IT : : 10.61

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_4 \frac{3}{4}$ ⓑ $\log \frac{x}{1000}$

TRY IT : : 10.62

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_2 \frac{5}{4}$ ⓑ $\log \frac{10}{y}$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_a M^p = p \log_a M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Power Property of Logarithms

If $M > 0$, $a > 0$, $a \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power as the product product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

EXAMPLE 10.32

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_5 4^3$ and ⓑ $\log x^{10}$

Solution

ⓐ

	$\log_5 4^3$
Use the Power Property, $\log_a M^p = p \log_a M$.	$3\log_5 4$
	$\log_5 4^3 = 3\log_5 4$

ⓑ

	$\log x^{10}$
Use the Power Property, $\log_a M^p = p \log_a M$.	$10 \log x$
	$\log x^{10} = 10\log x$

TRY IT : : 10.63

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_7 5^4$ ⓑ $\log x^{100}$

TRY IT : : 10.64

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_2 3^7$ ⓑ $\log x^{20}$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Properties of Logarithms

If $M > 0$, $a > 0$, $a \neq 1$ and p is any real number then,

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

Now that we have the properties we can use them to "expand" a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

EXAMPLE 10.33

Use the Properties of Logarithms to expand the logarithm $\log_4\left(2x^3 y^2\right)$. Simplify, if possible.

Solution

	$\log_4\left(2x^3y^2\right)$
Use the Product Property, $\log_a M\cdot N=\log_a M+\log_a N$.	$\log_4 2+\log_4 x^3+\log_4 y^2$
Use the Power Property, $\log_a M^p=p\log_a M$, on the last two terms.	$\log_4 2+3\log_4 x+2\log_4 y$
Simplify.	$\frac{1}{2}+3\log_4 x+2\log_4 y$

$$\log_4\left(2x^3y^2\right)=\frac{1}{2}+3\log_4 x+2\log_4 y$$

TRY IT : : 10.65

Use the Properties of Logarithms to expand the logarithm $\log_2\left(5x^4y^2\right)$. Simplify, if possible.

TRY IT : : 10.66

Use the Properties of Logarithms to expand the logarithm $\log_3\left(7x^5y^3\right)$. Simplify, if possible.

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

EXAMPLE 10.34

Use the Properties of Logarithms to expand the logarithm $\log_2\sqrt[4]{\frac{x^3}{3y^2z}}$. Simplify, if possible.

Solution

	$\log_2\sqrt[4]{\frac{x^3}{3y^2z}}$
Rewrite the radical with a rational exponent.	$\log_2\left(\frac{x^3}{3y^2z}\right)^{\frac{1}{4}}$
Use the Power Property, $\log_a M^p=p\log_a M$.	$\frac{1}{4}\log_2\left(\frac{x^3}{3y^2z}\right)$
Use the Quotient Property, $\log_a M\cdot N=\log_a M-\log_a N$.	$\frac{1}{4}\left(\log_2\left(x^3\right)-\log_2\left(3y^2z\right)\right)$
Use the Product Property, $\log_a M\cdot N=\log_a M+\log_a N$, in the second term.	$\frac{1}{4}\left(\log_2\left(x^3\right)-\left(\log_2 3+\log_2 y^2+\log_2 z\right)\right)$
Use the Power Property, $\log_a M^p=p\log_a M$, inside the parentheses.	$\frac{1}{4}\left(3\log_2 x-\left(\log_2 3+2\log_2 y+\log_2 z\right)\right)$
Simplify by distributing.	$\frac{1}{4}\left(3\log_2 x-\log_2 3-2\log_2 y-\log_2 z\right)$

$$\log_2\sqrt[4]{\frac{x^3}{3y^2z}}=\frac{1}{4}\left(3\log_2 x-\log_2 3-2\log_2 y-\log_2 z\right)$$

TRY IT : : 10.67

Use the Properties of Logarithms to expand the logarithm $\log_4 \sqrt[5]{\frac{x^4}{2y^3z^2}}$. Simplify, if possible.

TRY IT : : 10.68

Use the Properties of Logarithms to expand the logarithm $\log_3 \sqrt[3]{\frac{x^2}{5yz}}$. Simplify, if possible.

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

EXAMPLE 10.35

Use the Properties of Logarithms to condense the logarithm $\log_4 3 + \log_4 x - \log_4 y$. Simplify, if possible.

Solution

The log expressions all have the same base, 4.

The fir t two terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.

Since the logs are subtracted, we use the Quotient Property, $\log_a M - \log_a N = \log_a \frac{M}{N}$.

$$\log_4 3 + \log_4 x - \log_4 y$$

$$\log_4 3x - \log_4 y$$

$$\log_4 \frac{3x}{y}$$

$$\log_4 3 + \log_4 x - \log_4 y = \log_4 \frac{3x}{y}$$

TRY IT : : 10.69

Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

TRY IT : : 10.70

Use the Properties of Logarithms to condense the logarithm $\log_3 6 - \log_3 x - \log_3 y$. Simplify, if possible.

EXAMPLE 10.36

Use the Properties of Logarithms to condense the logarithm $2\log_3 x + 4\log_3 (x+1)$. Simplify, if possible.

Solution

The log expressions have the same base, 3.

Use the Power Property, $\log_a M + \log_a N = \log_a M \cdot N$.

The terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.

$$2\log_3 x + 4\log_3 (x+1)$$

$$\log_3 x^2 + \log_3 (x+1)^4$$

$$\log_3 x^2 (x+1)^4$$

$$2\log_3 x + 4\log_3 (x+1) = \log_3 x^2 (x+1)^4$$

TRY IT : : 10.71

Use the Properties of Logarithms to condense the logarithm $3\log_2 x + 2\log_2 (x-1)$. Simplify, if possible.

TRY IT : : 10.72

Use the Properties of Logarithms to condense the logarithm $2\log x + 2\log(x+1)$. Simplify, if possible.

Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the Change-of-Base Formula. We will show how this is derived.

Suppose we want to evaluate $\log_a M$.	$\log_a M$
Let $y = \log_a M$.	$y = \log_a M$
Rewrite the expression in exponential form.	$a^y = M$
Take the $\log_b$ of each side.	$\log_b a^y = \log_b M$
Use the Power Property.	$y\log_b a = \log_b M$
Solve for y.	$y = \frac{\log_b M}{\log_b a}$
Substitute $y = \log_a M$.	$\log_a M = \frac{\log_b M}{\log_b a}$

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b > 0$, $b \neq 1$. Because our calculators have keys for logarithms base 10 and base e, we will rewrite the Change-of-Base Formula with the new base as 10 or e.

Change-of-Base Formula

For any logarithmic bases a, b and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$

new base b | new base 10 | new base e

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol $(\approx)$.

EXAMPLE 10.37

Rounding to three decimal places, approximate $\log_4 35$.

Solution

	$\log_4 35$
Use the Change-of-Base Formula.	$\log_a M = \frac{\log_b M}{\log_b a}$
Identify a and M. Choose 10 for b.	$\log_4 35 = \frac{\log 35}{\log 4}$
Enter the expression $\frac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places.	$\log_4 35 \approx 2.565$

TRY IT : : 10.73 Rounding to three decimal places, approximate $\log_3 42$.

TRY IT : : 10.74 Rounding to three decimal places, approximate $\log_5 46$.

MEDIA : :

Access these online resources for additional instruction and practice with using the properties of logarithms.

- **Using Properties of Logarithms to Expand Logs (https://openstax.org/l/37Logproperties)**
- **Using Properties of Logarithms to Condense Logs (https://openstax.org/l/37Condenselogs)**
- **Change of Base (https://openstax.org/l/37Changeofbase)**

10.4 EXERCISES

Practice Makes Perfect

Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

218. ⓐ $\log_4 1$ ⓑ $\log_8 8$

219. ⓐ $\log_{12} 1$ ⓑ $\ln e$

220. ⓐ $3^{\log_3 6}$ ⓑ $\log_2 2^7$

221. ⓐ $5^{\log_5 10}$ ⓑ $\log_4 4^{10}$

222. ⓐ $8^{\log_8 7}$ ⓑ $\log_6 6^{-2}$

223. ⓐ $6^{\log_6 15}$ ⓑ $\log_8 8^{-4}$

224. ⓐ $10^{\log\sqrt{5}}$ ⓑ $\log 10^{-2}$

225. ⓐ $10^{\log\sqrt{3}}$ ⓑ $\log 10^{-1}$

226. ⓐ $e^{\ln 4}$ ⓑ $\ln e^2$

227. ⓐ $e^{\ln 3}$ ⓑ $\ln e^7$

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

228. $\log_4 6x$

229. $\log_5 8y$

230. $\log_2 32xy$

231. $\log_3 81xy$

232. $\log 100x$

233. $\log 1000y$

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

234. $\log_3 \frac{3}{8}$

235. $\log_6 \frac{5}{6}$

236. $\log_4 \frac{16}{y}$

237. $\log_5 \frac{125}{x}$

238. $\log \frac{x}{10}$

239. $\log \frac{10{,}000}{y}$

240. $\ln \frac{e^3}{3}$

241. $\ln \frac{e^4}{16}$

In the following exercises, use the Power Property of Logarithms to expand each. Simplify if possible.

242. $\log_3 x^2$

243. $\log_2 x^5$

244. $\log x^{-2}$

245. $\log x^{-3}$

246. $\log_4 \sqrt{x}$

247. $\log_5 \sqrt[3]{x}$

248. $\ln x^{\sqrt{3}}$

249. $\ln x^{\sqrt[3]{4}}$

In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.

250. $\log_5\left(4x^6y^4\right)$

251. $\log_2\left(3x^5y^3\right)$

252. $\log_3\left(\sqrt{2}x^2\right)$

253. $\log_5\left(\sqrt[4]{21}y^3\right)$

254. $\log_3\dfrac{xy^2}{z^2}$

255. $\log_5\dfrac{4ab^3c^4}{d^2}$

256. $\log_4\dfrac{\sqrt{x}}{16y^4}$

257. $\log_3\dfrac{\sqrt[3]{x^2}}{27y^4}$

258. $\log_2\dfrac{\sqrt{2x+y^2}}{z^2}$

259. $\log_3\dfrac{\sqrt{3x+2y^2}}{5z^2}$

260. $\log_2\sqrt[4]{\dfrac{5x^3}{2y^2z^4}}$

261. $\log_5\sqrt[3]{\dfrac{3x^2}{4y^3z}}$

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.

262. $\log_6 4+\log_6 9$

263. $\log 4+\log 25$

264. $\log_2 80-\log_2 5$

265. $\log_3 36-\log_3 4$

266. $\log_3 4+\log_3(x+1)$

267. $\log_2 5-\log_2(x-1)$

268. $\log_7 3+\log_7 x-\log_7 y$

269. $\log_5 2-\log_5 x-\log_5 y$

270. $4\log_2 x+6\log_2 y$

271. $6\log_3 x+9\log_3 y$

272. $\log_3\left(x^2-1\right)-2\log_3(x-1)$

273. $\log\left(x^2+2x+1\right)-2\log(x+1)$

274. $4\log x-2\log y-3\log z$

275. $3\ln x+4\ln y-2\ln z$

276. $\frac{1}{3}\log x-3\log(x+1)$

277. $2\log(2x+3)+\frac{1}{2}\log(x+1)$

Use the Change-of-Base Formula

In the following exercises, use the Change-of-Base Formula, rounding to three decimal places, to approximate each logarithm.

278. $\log_3 42$

279. $\log_5 46$

280. $\log_{12} 87$

281. $\log_{15} 93$

282. $\log_{\sqrt{2}} 17$

283. $\log_{\sqrt{3}} 21$

Writing Exercises

284. Write the Product Property in your own words. Does it apply to each of the following? $\log_a 5x$, $\log_a(5+x)$. Why or why not?

285. Write the Power Property in your own words. Does it apply to each of the following? $\log_a x^p$, $(\log_a x)^r$. Why or why not?

286. Use an example to show that $\log(a+b) \neq \log a + \log b$?

287. Explain how to find the value of $\log_7 15$ using your calculator.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No. I don't get it!
use the properties of logarithms.			
use the Change of Base Formula.			

ⓑ *On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

10.5 Solve Exponential and Logarithmic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve logarithmic equations using the properties of logarithms
- Solve exponential equations using logarithms
- Use exponential models in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 16$.
 If you missed this problem, review Example 6.46.
2. Solve: $x^2 - 5x + 6 = 0$.
 If you missed this problem, review Example 6.45.
3. Solve: $x(x + 6) = 2x + 5$.
 If you missed this problem, review Example 6.47.

Solve Logarithmic Equations Using the Properties of Logarithms

In the section on logarithmic functions, we solved some equations by rewriting the equation in exponential form. Now that we have the properties of logarithms, we have additional methods we can use to solve logarithmic equations.

If our equation has two logarithms we can use a property that says that if $\log_a M = \log_a N$ then it is true that $M = N$. This is the **One-to-One Property of Logarithmic Equations**.

One-to-One Property of Logarithmic Equations

For $M > 0$, $N > 0$, $a>0$, and $a \neq 1$ is any real number:

$$\text{If } \log_a M = \log_a N, \text{ then } M = N.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Remember that logarithms are defined only for positive real numbers. Check your results in the original equation. You may have obtained a result that gives a logarithm of zero or a negative number.

EXAMPLE 10.38

Solve: $2\log_5 x = \log_5 81$.

Solution

$$2\log_5 x = \log_5 81$$

Use the Power Property. $\log_5 x^2 = \log_5 81$

Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$. $x^2 = 81$

Solve using the Square Root Property. $x = \pm 9$

We eliminate $x = -9$ as we cannot take the logarithm of a negative number. $x = 9, \cancel{x = -9}$

Check.

$x = 9$

$$2\log_5 x = \log_5 81$$
$$2\log_5 9 \stackrel{?}{=} \log_5 81$$
$$\log_5 9^2 \stackrel{?}{=} \log_5 81$$
$$\log_5 81 = \log_5 81 \checkmark$$

TRY IT : : 10.75 Solve: $2\log_3 x = \log_3 36$

TRY IT : : 10.76 Solve: $3\log x = \log 64$

Another strategy to use to solve logarithmic equations is to condense sums or differences into a single logarithm.

EXAMPLE 10.39

Solve: $\log_3 x + \log_3 (x-8) = 2$.

Solution

$$\log_3 x + \log_3 (x-8) = 2$$

Use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$. $\log_3 x(x-8) = 2$

Rewrite in exponential form. $3^2 = x(x-8)$

Simplify. $9 = x^2 - 8x$

Subtract 9 from each side. $0 = x^2 - 8x - 9$

Factor. $0 = (x-9)(x+1)$

Use the Zero-Product Property. $x - 9 = 0, \quad x + 1 = 0$

Solve each equation. $x = 9, \quad \cancel{x = -1}$

Check.

$x = -1$

$$\log_3 x + \log_3 (x-8) = 2$$
$$\log_3 (-1) + \log_3 (-1-8) \stackrel{?}{=} 2$$

We cannot take the log of a negative number.

$x = 9$

$$\log_3 x + \log_3 (x-8) = 2$$
$$\log_3 9 + \log_3 (9-8) \stackrel{?}{=} 2$$
$$2 + 0 \stackrel{?}{=} 2$$
$$2 = 2 \checkmark$$

TRY IT : : 10.77 Solve: $\log_2 x + \log_2 (x-2) = 3$

TRY IT : : 10.78 Solve: $\log_2 x + \log_2 (x-6) = 4$

When there are logarithms on both sides, we condense each side into a single logarithm. Remember to use the Power Property as needed.

EXAMPLE 10.40

Solve: $\log_4 (x+6) - \log_4 (2x+5) = -\log_4 x$.

Solution

	$\log_4 (x+6) - \log_4 (2x+5) = -\log_4 x$
Use the Quotient Property on the left side and the Power Property on the right.	$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 x^{-1}$
Rewrite $x^{-1} = \frac{1}{x}$.	$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 \frac{1}{x}$
Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.	$\frac{x+6}{2x+5} = \frac{1}{x}$
Solve the rational equation.	$x(x+6) = 2x+5$
Distribute.	$x^2 + 6x = 2x+5$
Write in standard form.	$x^2 + 4x - 5 = 0$
Factor.	$(x+5)(x-1) = 0$
Use the Zero-Product Property.	$x+5=0, \quad x-1=0$
Solve each equation.	$\cancel{x=-5}, \quad x=1$
Check. We leave the check for you.	

TRY IT : : 10.79 Solve: $\log(x+2) - \log(4x+3) = -\log x$.

TRY IT : : 10.80 Solve: $\log(x-2) - \log(4x+16) = \log\frac{1}{x}$.

Solve Exponential Equations Using Logarithms

In the section on exponential functions, we solved some equations by writing both sides of the equation with the same base. Next we wrote a new equation by setting the exponents equal.

It is not always possible or convenient to write the expressions with the same base. In that case we often take the common logarithm or natural logarithm of both sides once the exponential is isolated.

EXAMPLE 10.41

Solve $5^x = 11$. Find the exact answer and then approximate it to three decimal places.

Solution

	$5^x = 11$
Since the exponential is isolated, take the logarithm of both sides.	$\log 5^x = \log 11$
Use the Power Property to get the x as a factor, not an exponent.	$x\log 5 = \log 11$
Solve for x. Find the exact answer.	$x = \frac{\log 11}{\log 5}$
Approximate the answer.	$x \approx 1.490$
Since $5^1 = 5$ and $5^2 = 25$, does it makes sense that $5^{1.490} \approx 11$?	

TRY IT : : 10.81 Solve $7^x = 43$. Find the exact answer and then approximate it to three decimal places.

TRY IT : : 10.82 Solve $8^x = 98$. Find the exact answer and then approximate it to three decimal places.

When we take the logarithm of both sides we will get the same result whether we use the common or the natural logarithm (try using the natural log in the last example. Did you get the same result?) When the exponential has base *e*, we use the natural logarithm.

EXAMPLE 10.42

Solve $3e^{x+2} = 24$. Find the exact answer and then approximate it to three decimal places.

Solution

	$3e^{x+2}$	$=$	24
Isolate the exponential by dividing both sides by 3.	e^{x+2}	$=$	8
Take the natural logarithm of both sides.	$\ln e^{x+2}$	$=$	$\ln 8$
Use the Power Property to get the x as a factor, not an exponent.	$(x+2)\ln e$	$=$	$\ln 8$
Use the property $\ln e = 1$ to simplify.	$x+2$	$=$	$\ln 8$
Solve the equation. Find the exact answer.	x	$=$	$\ln 8 - 2$
Approximate the answer.	x	$\approx$	0.079

TRY IT : : 10.83 Solve $2e^{x-2} = 18$. Find the exact answer and then approximate it to three decimal places.

TRY IT : : 10.84 Solve $5e^{2x} = 25$. Find the exact answer and then approximate it to three decimal places.

Use Exponential Models in Applications

In previous sections we were able to solve some applications that were modeled with exponential equations. Now that we have so many more options to solve these equations, we are able to solve more applications.

We will again use the Compound Interest Formulas and so we list them here for reference.

Compound Interest

For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A* is:

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ when compounded n times a year.

$A = Pe^{rt}$ when compounded continuously.

EXAMPLE 10.43

Jermael's parents put $10,000 in investments for his college expenses on his first birthday. They hope the investments will be worth $50,000 when he turns 18. If the interest compounds continuously, approximately what rate of growth will they need to achieve their goal?

Solution

	A	$=$	$\$50{,}000$
	P	$=$	$\$10{,}000$
Identify the variables in the formula.	r	$=$	$?$
	t	$=$	17 years
	A	$=$	Pe^{rt}
Substitute the values into the formula.	$50{,}000$	$=$	$10{,}000e^{r\cdot 17}$
Solve for r.Divide each side by 10,000.	5	$=$	e^{17r}
Take the natural log of each side.	$\ln 5$	$=$	$\ln e^{17r}$
Use the Power Property.	$\ln 5$	$=$	$17r\ln e$
Simplify.	$\ln 5$	$=$	$17r$
Divide each side by 17.	$\frac{\ln 5}{17}$	$=$	r
Approximate the answer.	r	$\approx$	0.095
Convert to a percentage.	r	$\approx$	9.5%

They need the rate of growth to be approximately 9.5%.

TRY IT : : 10.85

Hector invests $\$10{,}000$ at age 21. He hopes the investments will be worth $\$150{,}000$ when he turns 50. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal?

TRY IT : : 10.86

Rachel invests $\$15{,}000$ at age 25. She hopes the investments will be worth $\$90{,}000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

We have seen that growth and decay are modeled by exponential functions. For growth and decay we use the formula $A = A_0 e^{kt}$. Exponential growth has a positive rate of growth or growth constant, k, and exponential decay has a negative rate of growth or decay constant, *k*.

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, *k*, for a certain time, *t*, the final amount, *A*, is:

$$A = A_0 e^{kt}$$

We can now solve applications that give us enough information to determine the rate of growth. We can then use that rate of growth to predict other situations.

EXAMPLE 10.44

Researchers recorded that a certain bacteria population grew from 100 to 300 in 3 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

Solution

This problem requires two main steps. First we must find the unknown rate, *k*. Then we use that value of *k* to help us find the unknown number of bacteria.

Identify the variables in the formula.

$$\begin{aligned} A &= 300 \\ A_0 &= 100 \\ k &= ? \\ t &= 3 \text{ hours} \\ A &= A_0 e^{kt} \end{aligned}$$

Substitute the values in the formula. $300 = 100e^{k \cdot 3}$

Solve for k. Divide each side by 100. $3 = e^{3k}$

Take the natural log of each side. $\ln 3 = \ln e^{3k}$

Use the Power Property. $\ln 3 = 3k \ln e$

Simplify. $\ln 3 = 3k$

Divide each side by 3. $\frac{\ln 3}{3} = k$

Approximate the answer. $k \approx 0.366$

We use this rate of growth to predict the number of bacteria there will be in 24 hours.

$$\begin{aligned} A &= ? \\ A_0 &= 100 \\ k &= \frac{\ln 3}{3} \\ t &= 24 \text{ hours} \\ A &= A_0 e^{kt} \end{aligned}$$

Substitute in the values. $A = 100e^{\frac{\ln 3}{3} \cdot 24}$

Evaluate. $A \approx 656{,}100$

At this rate of growth, they can expect 656,100 bacteria.

TRY IT : : 10.87

Researchers recorded that a certain bacteria population grew from 100 to 500 in 6 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

TRY IT : : 10.88

Researchers recorded that a certain bacteria population declined from 700,000 to 400,000 in 5 hours after the administration of medication. At this rate of decay, how many bacteria will there be 24 hours from the start of the experiment?

Radioactive substances decay or decompose according to the exponential decay formula. The amount of time it takes for the substance to decay to half of its original amount is called the half-life of the substance.

Similar to the previous example, we can use the given information to determine the constant of decay, and then use that constant to answer other questions.

EXAMPLE 10.45

The half-life of radium-226 is 1,590 years. How much of a 100 mg sample will be left in 500 years?

Solution

This problem requires two main steps. First we must find the decay constant k. If we start with 100-mg, at the half-life there will be 50-mg remaining. We will use this information to find k. Then we use that value of k to help us find the amount of sample that will be left in 500 years.

Identify the variables in the formula.

$$\begin{aligned} A &= 50 \\ A_0 &= 100 \\ k &= ? \\ t &= 1590\text{years} \end{aligned}$$

$$A = A_0 e^{kt}$$

Substitute the values in the formula. $50 = 100e^{k \cdot 1590}$

Solve for k. Divide each side by 100. $0.5 = e^{1590k}$

Take the natural log of each side. $\ln 0.5 = \ln e^{1590k}$

Use the Power Property. $\ln 0.5 = 1590k \ln e$

Simplify. $\ln 0.5 = 1590k$

Divide each side by 1590. $\frac{\ln 0.5}{1590} = k$ exact answer

We use this rate of growth to predict the amount that will be left in 500 years.

$$\begin{aligned} A &= ? \\ A_0 &= 100 \\ k &= \frac{\ln 0.5}{1590} \\ t &= 500\text{years} \end{aligned}$$

$$A = A_0 e^{kt}$$

Substitute in the values. $A = 100e^{\frac{\ln 0.5}{1590} \cdot 500}$

Evaluate. $A \approx 80.4$ mg

In 500 years there would be approximately 80.4 mg remaining.

TRY IT : : 10.89

The half-life of magnesium-27 is 9.45 minutes. How much of a 10-mg sample will be left in 6 minutes?

TRY IT : : 10.90

The half-life of radioactive iodine is 60 days. How much of a 50-mg sample will be left in 40 days?

MEDIA : :

Access these online resources for additional instruction and practice with solving exponential and logarithmic equations.

- **Solving Logarithmic Equations (https://openstax.org/l/37Solvelog)**
- **Solving Logarithm Equations (https://openstax.org/l/37sSolvelogeqs2)**
- **Finding the rate or time in a word problem on exponential growth or decay (https://openstax.org/l/37Solveforrate)**
- **Finding the rate or time in a word problem on exponential growth or decay (https://openstax.org/l/37Solveforrate2)**

10.5 EXERCISES

Practice Makes Perfect

Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for x.

288. $\log_4 64 = 2\log_4 x$

289. $\log 49 = 2\log x$

290. $3\log_3 x = \log_3 27$

291. $3\log_6 x = \log_6 64$

292. $\log_5 (4x-2) = \log_5 10$

293. $\log_3 \left(x^2+3\right) = \log_3 4x$

294. $\log_3 x + \log_3 x = 2$

295. $\log_4 x + \log_4 x = 3$

296. $\log_2 x + \log_2 (x-3) = 2$

297. $\log_3 x + \log_3 (x+6) = 3$

298. $\log x + \log(x+3) = 1$

299. $\log x + \log(x-15) = 2$

300. $\log(x+4) - \log(5x+12) = -\log x$

301. $\log(x-1) - \log(x+3) = \log\frac{1}{x}$

302. $\log_5 (x+3) + \log_5 (x-6) = \log_5 10$

303. $\log_5 (x+1) + \log_5 (x-5) = \log_5 7$

304. $\log_3 (2x-1) = \log_3 (x+3) + \log_3 3$

305. $\log(5x+1) = \log(x+3) + \log 2$

Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

306. $3^x = 89$

307. $2^x = 74$

308. $5^x = 110$

309. $4^x = 112$

310. $e^x = 16$

311. $e^x = 8$

312. $\left(\frac{1}{2}\right)^x = 6$

313. $\left(\frac{1}{3}\right)^x = 8$

314. $4e^{x+1} = 16$

315. $3e^{x+2} = 9$

316. $6e^{2x} = 24$

317. $2e^{3x} = 32$

318. $\frac{1}{4}e^x = 3$

319. $\frac{1}{3}e^x = 2$

320. $e^{x+1} + 2 = 16$

321. $e^{x-1} + 4 = 12$

In the following exercises, solve each equation.

322. $3^{3x+1} = 81$

323. $6^{4x-17} = 216$

324. $\frac{e^{x^2}}{e^{14}} = e^{5x}$

325. $\frac{e^{x^2}}{e^{x}} = e^{20}$

326. $\log_a 64 = 2$

327. $\log_a 81 = 4$

328. $\ln x = -8$

329. $\ln x = 9$

330. $\log_5(3x - 8) = 2$

331. $\log_4(7x + 15) = 3$

332. $\ln e^{5x} = 30$

333. $\ln e^{6x} = 18$

334. $3\log x = \log 125$

335. $7\log_3 x = \log_3 128$

336. $\log_6 x + \log_6(x - 5) = 24$

337. $\log_9 x + \log_9(x - 4) = 12$

338. $\log_2(x + 2) - \log_2(2x + 9) = -\log_2 x$

339. $\log_6(x + 1) - \log_6(4x + 10) = \log_6 \frac{1}{x}$

In the following exercises, solve for x, giving an exact answer as well as an approximation to three decimal places.

340. $6^x = 91$

341. $\left(\frac{1}{2}\right)^x = 10$

342. $7e^{x-3} = 35$

343. $8e^{x+5} = 56$

Use Exponential Models in Applications

In the following exercises, solve.

344. Sung Lee invests $\$5{,}000$ at age 18. He hopes the investments will be worth $\$10{,}000$ when he turns 25. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?

345. Alice invests $\$15{,}000$ at age 30 from the signing bonus of her new job. She hopes the investments will be worth $\$30{,}000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

346. Coralee invests $\$5{,}000$ in an account that compounds interest monthly and earns 7%. How long will it take for her money to double?

347. Simone invests $\$8{,}000$ in an account that compounds interest quarterly and earns 5%. How long will it take for his money to double?

348. Researchers recorded that a certain bacteria population declined from 100,000 to 100 in 24 hours. At this rate of decay, how many bacteria will there be in 16 hours?

349. Researchers recorded that a certain bacteria population declined from 800,000 to 500,000 in 6 hours after the administration of medication. At this rate of decay, how many bacteria will there be in 24 hours?

350. A virus takes 6 days to double its original population $(A = 2A_0)$. How long will it take to triple its population?

351. A bacteria doubles its original population in 24 hours $(A = 2A_0)$. How big will its population be in 72 hours?

352. Carbon-14 is used for archeological carbon dating. Its half-life is 5,730 years. How much of a 100-gram sample of Carbon-14 will be left in 1000 years?

353. Radioactive technetium-99m is often used in diagnostic medicine as it has a relatively short half-life but lasts long enough to get the needed testing done on the patient. If its half-life is 6 hours, how much of the radioactive material form a 0.5 ml injection will be in the body in 24 hours?

Writing Exercises

354. Explain the method you would use to solve these equations: $3^{x+1} = 81$, $3^{x+1} = 75$. Does your method require logarithms for both equations? Why or why not?

355. What is the difference between the equation for exponential growth versus the equation for exponential decay?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve logarithmic equations using the properties of logarithms.			
solve exponential equations using logarithms.			
use exponential models in applications.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

CHAPTER 10 REVIEW

KEY TERMS

asymptote A line which a graph of a function approaches closely but never touches.

common logarithmic function The function $f(x) = \log x$ is the common logarithmic function with base 10, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

exponential function An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form $f(x) = a^x$.

logarithmic function The function $f(x) = \log_a x$ is the logarithmic function with base a, where $a > 0$, $x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

natural base The number *e* is defined as the value of $\left(1 + \frac{1}{n}\right)^n$, as *n* gets larger and larger. We say, as *n* increases without bound, $e \approx 2.718281827\ldots$

natural exponential function The natural exponential function is an exponential function whose base is *e*: $f(x) = e^x$. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

natural logarithmic function The function $f(x) = \ln x$ is the natural logarithmic function with base e, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

one-to-one function A function is one-to-one if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

KEY CONCEPTS

10.1 Finding Composite and Inverse Functions

- **Composition of Functions:** The composition of functions f and g, is written $f \circ g$ and is defined by

$$(f \circ g)(x) = f(g(x))$$

 We read $f(g(x))$ as f of g of x.

- **Horizontal Line Test:** If every horizontal line, intersects the graph of a function in at most one point, it is a one-to-one function.
- **Inverse of a Function Defined by Ordered Pairs:** If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).
- **Inverse Functions:** For every x in the domain of one-to-one function f and f^{-1},

$$\begin{aligned} f^{-1}(f(x)) &= x \\ f\left(f^{-1}(x)\right) &= x \end{aligned}$$

- **How to Find the Inverse of a One-to-One Function:**

 Step 1. Substitute *y* for $f(x)$.

 Step 2. Interchange the variables *x* and *y*.

 Step 3. Solve for *y*.

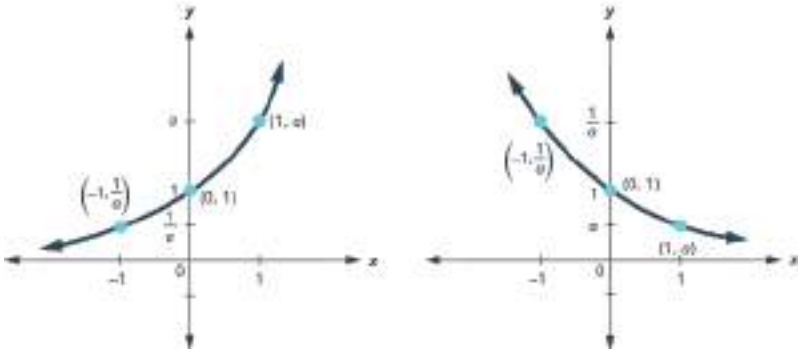
 Step 4. Substitute $f^{-1}(x)$ for y.

 Step 5. Verify that the functions are inverses.

10.2 Evaluate and Graph Exponential Functions

- **Properties of the Graph of** $f(x) = a^x$:

when $a > 1$		when $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x -intercept	none	x -intercept	none
y -intercept	$(0, 1)$	y -intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x -axis, the line $y = 0$	Asymptote	x -axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



- **One-to-One Property of Exponential Equations:**
 For $a > 0$ and $a \neq 1$,

$$A = A_0 e^{rt}$$

- **How to Solve an Exponential Equation**
 Step 1. Write both sides of the equation with the same base, if possible.
 Step 2. Write a new equation by setting the exponents equal.
 Step 3. Solve the equation.
 Step 4. Check the solution.
- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ when compounded n times a year.

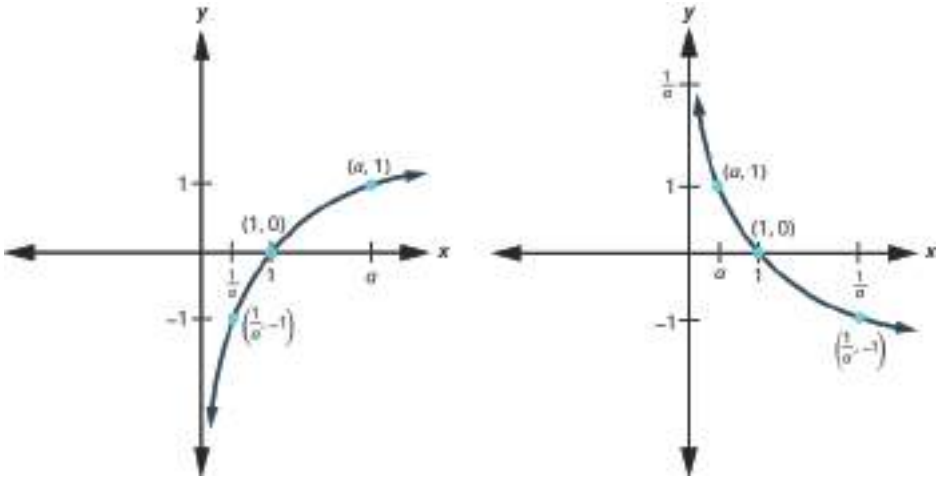
$A = Pe^{rt}$ when compounded continuously.

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount, A, is $A = A_0 e^{rt}$.

10.3 Evaluate and Graph Logarithmic Functions

- **Properties of the Graph of** $y = \log_a x$:

when $a > 1$		when $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	*x*-intercept	$(1, 0)$
y-intercept	none	*y*-intercept	none
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	*y*-axis	Asymptote	*y*-axis
Basic shape	increasing	Basic shape	decreasing



- **Decibel Level of Sound:** The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is $D = 10\log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

10.4 Use the Properties of Logarithms

- **Properties of Logarithms**

$$\log_a 1 = 0 \qquad \log_a a = 1$$

- **Inverse Properties of Logarithms**
 - For $a > 0$, $x > 0$ and $a \neq 1$

$$a^{\log_a x} = x \qquad \log_a a^x = x$$

- **Product Property of Logarithms**
 - If $M > 0$, $N > 0$, $a > 0$ and $a \neq 1$, then,

$$\log_a M \cdot N = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

- **Quotient Property of Logarithms**

- If $M > 0,\ N > 0, \text{a} > 0$ and $\text{a} \neq 1,$ then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**
 - If $M > 0,\ \text{a} > 0,\ \text{a} \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

- **Properties of Logarithms Summary**
If $M > 0,\ \text{a} > 0,\ \text{a} \neq 1$ and p is any real number then,

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

- **Change-of-Base Formula**
For any logarithmic bases a and b, and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$

new base b new base 10 new base e

10.5 Solve Exponential and Logarithmic Equations

- **One-to-One Property of Logarithmic Equations:** For $M > 0,\ N > 0,\ \text{a}>0,$ and $\text{a} \neq 1$ is any real number:

$$\text{If } \log_a M = \log_a N, \text{ then } M = N.$$

- **Compound Interest:**
For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A*, is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{when compounded } n \text{ times a year.}$$

$$A = Pe^{rt} \qquad \text{when compounded continuously.}$$

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, *r*, for a certain time *t*, the final amount, *A*, is $A = A_0 e^{rt}$.

REVIEW EXERCISES

10.1 Finding Composite and Inverse Functions

Find and Evaluate Composite Functions

In the following exercises, for each pair of functions, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

356. $f(x) = 7x - 2$ and $g(x) = 5x + 1$

357. $f(x) = 4x$ and $g(x) = x^2 + 3x$

In the following exercises, evaluate the composition.

358. For functions $f(x) = 3x^2 + 2$ and $g(x) = 4x - 3$, find

ⓐ $(f \circ g)(-3)$

ⓑ $(g \circ f)(-2)$

ⓒ $(f \circ f)(-1)$

359. For functions $f(x) = 2x^3 + 5$ and $g(x) = 3x^2 - 7$, find

ⓐ $(f \circ g)(-1)$

ⓑ $(g \circ f)(-2)$

ⓒ $(g \circ g)(1)$

Determine Whether a Function is One-to-One

In the following exercises, for each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

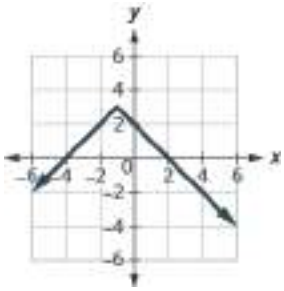
360. $\{(-3, -5), (-2, -4), (-1, -3), (0, -2), (-1, -1), (-2, 0), (-3, 1)\}$

361. $\{(-3, 0), (-2, -2), (-1, 0), (0, 1), (1, 2), (2, 1), (3, -1)\}$

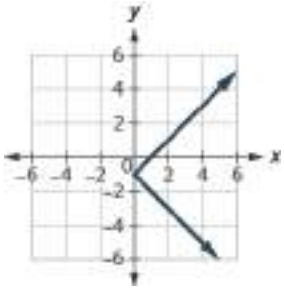
362. $\{(-3, 3), (-2, 1), (-1, -1), (0, -3), (1, -5), (2, -4), (3, -2)\}$

In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

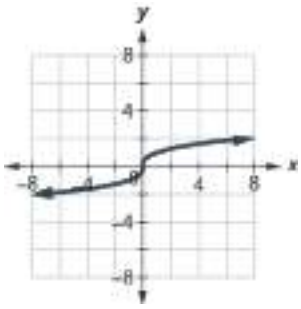
363. ⓐ



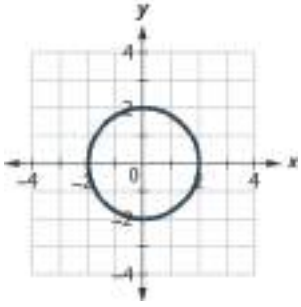
ⓑ



364. ⓐ



ⓑ



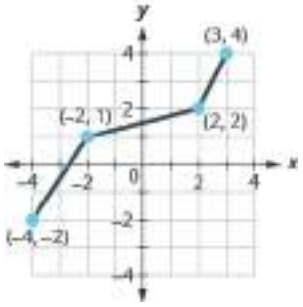
Find the Inverse of a Function

In the following exercise, find the inverse of the function. Determine the domain and range of the inverse function.

365. $\{(-3, 10), (-2, 5), (-1, 2), (0, 1)\}$

In the following exercise, graph the inverse of the one-to-one function shown.

366.



In the following exercises, verify that the functions are inverse functions.

367. $f(x) = 3x + 7$ and $g(x) = \frac{x-7}{3}$

368. $f(x) = 2x + 9$ and $g(x) = \frac{x+9}{2}$

In the following exercises, find the inverse of each function.

369. $f(x) = 6x - 11$

370. $f(x) = x^3 + 13$

371. $f(x) = \frac{1}{x+5}$

372. $f(x) = \sqrt[5]{x-1}$

10.2 Evaluate and Graph Exponential Functions

Graph Exponential Functions

In the following exercises, graph each of the following functions.

373. $f(x) = 4^x$

374. $f(x) = \left(\frac{1}{5}\right)^x$

375. $g(x) = (0.75)^x$

376. $g(x) = 3^{x+2}$

377. $f(x) = (2.3)^x - 3$

378. $f(x) = e^x + 5$

379. $f(x) = -e^x$

Solve Exponential Equations

In the following exercises, solve each equation.

380. $3^{5x-6} = 81$

381. $2^{x^2} = 16$

382. $9^x = 27$

383. $5^{x^2+2x} = \frac{1}{5}$

384. $e^{4x} \cdot e^7 = e^{19}$

385. $\frac{e^{x^2}}{e^{15}} = e^{2x}$

Use Exponential Models in Applications

In the following exercises, solve.

386. Felix invested $\$12{,}000$ in a savings account. If the interest rate is 4% how much will be in the account in 12 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously.

387. Sayed deposits $\$20{,}000$ in an investment account. What will be the value of his investment in 30 years if the investment is earning 7% per year and is compounded continuously?

388. A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. She starts her experiment with 150 of the bacteria that grows at a rate of 15% per hour. She will check on the bacteria every 24 hours. How many bacteria will he find in 24 hours?

389. In the last five years the population of the United States has grown at a rate of 0.7% per year to about 318,900,000. If this rate continues, what will be the population in 5 more years?

10.3 Evaluate and Graph Logarithmic Functions

Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

390. $5^4 = 625$

391. $10^{-3} = \frac{1}{1{,}000}$

392. $63^{\frac{1}{5}} = \sqrt[5]{63}$

393. $e^y = 16$

In the following exercises, convert each logarithmic equation to exponential form.

394. $7 = \log_2 128$

395. $5 = \log 100{,}000$

396. $4 = \ln x$

Evaluate Logarithmic Functions

In the following exercises, solve for x.

397. $\log_x 125 = 3$

398. $\log_7 x = -2$

399. $\log_{\frac{1}{2}} \frac{1}{16} = x$

In the following exercises, find the exact value of each logarithm without using a calculator.

400. $\log_2 32$

401. $\log_8 1$

402. $\log_3 \frac{1}{9}$

Graph Logarithmic Functions

In the following exercises, graph each logarithmic function.

403. $y = \log_5 x$

404. $y = \log_{\frac{1}{4}} x$

405. $y = \log_{0.8} x$

Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

406. $\log_a 36 = 5$

407. $\ln x = -3$

408. $\log_2 (5x - 7) = 3$

409. $\ln e^{3x} = 24$

410. $\log(x^2 - 21) = 2$

Use Logarithmic Models in Applications

411. What is the decibel level of a train whistle with intensity 10^{-3} watts per square inch?

10.4 Use the Properties of Logarithms

Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

412. ⓐ $\log_7 1$ ⓑ $\log_{12} 12$

413. ⓐ $5^{\log_5 13}$ ⓑ $\log_3 3^{-9}$

414. ⓐ $10^{\log\sqrt{5}}$ ⓑ $\log 10^{-3}$

415. ⓐ $e^{\ln 8}$ ⓑ $\ln e^5$

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

416. $\log_4(64xy)$

417. $\log 10{,}000m$

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

418. $\log_7 \frac{49}{y}$

419. $\ln \frac{e^5}{2}$

In the following exercises, use the Power Property of Logarithms to expand each logarithm. Simplify, if possible.

420. $\log x^{-9}$

421. $\log_4 \sqrt[7]{z}$

In the following exercises, use properties of logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

422. $\log_3\left(\sqrt{4}x^7 y^8\right)$

423. $\log_5 \frac{8a^2b^6c}{d^3}$

424. $\ln \frac{\sqrt{3x^2 - y^2}}{z^4}$

425. $\log_6 \sqrt[3]{\frac{7x^2}{6y^3z^5}}$

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.

426. $\log_2 56 - \log_2 7$

427. $3\log_3 x + 7\log_3 y$

428. $\log_5\left(x^2 - 16\right) - 2\log_5(x+4)$

429. $\frac{1}{4}\log y - 2\log(y-3)$

Use the Change-of-Base Formula

In the following exercises, rounding to three decimal places, approximate each logarithm.

430. $\log_5 97$

431. $\log_{\sqrt{3}} 16$

10.5 Solve Exponential and Logarithmic Equations

Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for x.

432. $3\log_5 x = \log_5 216$

433. $\log_2 x + \log_2(x-2) = 3$

434. $\log(x-1) - \log(3x+5) = -\log x$

435. $\log_4(x-2) + \log_4(x+5) = \log_4 8$

436. $\ln(3x-2) = \ln(x+4) + \ln 2$

Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

437. $2^x = 101$

438. $e^x = 23$

439. $\left(\frac{1}{3}\right)^x = 7$

440. $7e^{x+3} = 28$

441. $e^{x-4} + 8 = 23$

Use Exponential Models in Applications

442. Jerome invests $\$18{,}000$ at age 17. He hopes the investments will be worth $\$30{,}000$ when he turns 26. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?

443. Elise invests $\$4500$ in an account that compounds interest monthly and earns 6%. How long will it take for her money to double?

444. Researchers recorded that a certain bacteria population grew from 100 to 300 in 8 hours. At this rate of growth, how many bacteria will there be in 24 hours?

445. Mouse populations can double in 8 months $(A = 2A_0)$. How long will it take for a mouse population to triple?

446. The half-life of radioactive iodine is 60 days. How much of a 50 mg sample will be left in 40 days?

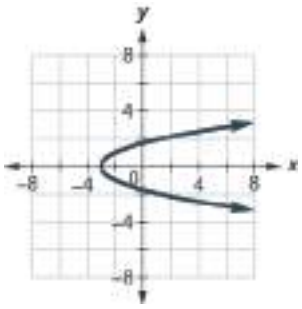
PRACTICE TEST

447. For the functions, $f(x) = 6x + 1$ and $g(x) = 8x - 3$, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

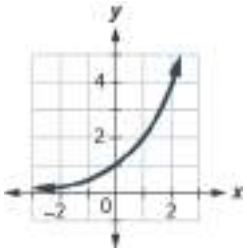
448. Determine if the following set of ordered pairs represents a function and if so, is the function one-to-one. $\{(-2, 2), (-1, -3), (0, 1), (1, -2), (2, -3)\}$

449. Determine whether each graph is the graph of a function and if so, is it one-to-one.

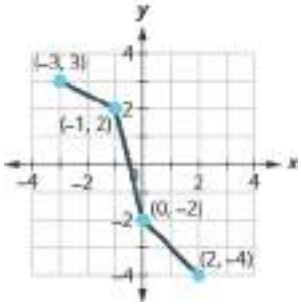
ⓐ



ⓑ



450. Graph, on the same coordinate system, the inverse of the one-to-one function shown.



451. Find the inverse of the function $f(x) = x^5 - 9$.

452. Graph the function $g(x) = 2^{x-3}$.

453. Solve the equation $2^{2x-4} = 64$.

454. Solve the equation $\dfrac{e^{x^2}}{e^4} = e^{3x}$.

455. Megan invested $\$21{,}000$ in a savings account. If the interest rate is 5%, how much will be in the account in 8 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously.

456. Convert the equation from exponential to logarithmic form: $10^{-2} = \dfrac{1}{100}$.

457. Convert the equation from logarithmic equation to exponential form: $3 = \log_7 343$

458. Solve for x: $\log_5 x = -3$

459. Evaluate $\log_{11} 1$.

460. Evaluate $\log_4 \dfrac{1}{64}$.

461. Graph the function $y = \log_3 x$.

462. Solve for x: $\log(x^2 - 39) = 1$

463. What is the decibel level of a small fan with intensity 10^{-8} watts per square inch?

464. Evaluate each. ⓐ $6^{\log_6 17}$ ⓑ $\log_9 9^{-3}$

In the following exercises, use properties of logarithms to write each expression as a sum of logarithms, simplifying if possible.

465. $\log_5 25ab$

466. $\ln\frac{e^{12}}{8}$

467. $\log_2 \sqrt[4]{\frac{5x^3}{16y^2z^7}}$

In the following exercises, use the Properties of Logarithms to condense the logarithm, simplifying if possible.

468. $5\log_4 x + 3\log_4 y$

469. $\frac{1}{6}\log x - 3\log(x+5)$

470. Rounding to three decimal places, approximate $\log_4 73$.

471. Solve for x: $\log_7(x+2) + \log_7(x-3) = \log_7 24$

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

472. $\left(\frac{1}{5}\right)^x = 9$

473. $5e^{x-4} = 40$

474. Jacob invests $14,000 in an account that compounds interest quarterly and earns 4%. How long will it take for his money to double?

475. Researchers recorded that a certain bacteria population grew from 500 to 700 in 5 hours. At this rate of growth, how many bacteria will there be in 20 hours?

476. A certain beetle population can double in 3 months $(A = 2A_0)$. How long will it take for that beetle population to triple?



11 CONICS

Figure 11.1 Aerospace engineers use rockets such as this one to launch people and objects into space. (credit: WikiImages/Pixabay)

Chapter Outline

11.1 Distance and Midpoint Formulas; Circles
11.2 Parabolas
11.3 Ellipses
11.4 Hyperbolas
11.5 Solve Systems of Nonlinear Equations

Introduction

Five, Four. Three. Two. One. Lift off. The rocket launches off the ground headed toward space. Unmanned spaceships, and spaceships in general, are designed by aerospace engineers. These engineers are investigating reusable rockets that return safely to Earth to be used again. Someday, rockets may carry passengers to the International Space Station and beyond. One essential math concept for aerospace engineers is that of conics. In this chapter, you will learn about conics, including circles, parabolas, ellipses, and hyperbolas. Then you will use what you learn to investigate systems of nonlinear equations.

11.1 Distance and Midpoint Formulas; Circles

Learning Objectives

By the end of this section, you will be able to:

- Use the Distance Formula
- Use the Midpoint Formula
- Write the equation of a circle in standard form
- Graph a circle

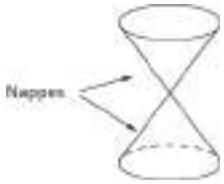
Be Prepared!

Before you get started, take this readiness quiz.

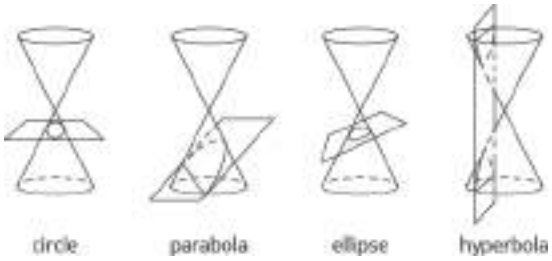
1. Find the length of the hypotenuse of a right triangle whose legs are 12 and 16 inches.
 If you missed this problem, review Example 2.34.
2. Factor: $x^2 - 18x + 81$.
 If you missed this problem, review Example 6.24.
3. Solve by completing the square: $x^2 - 12x - 12 = 0$.
 If you missed this problem, review Example 9.22.

In this chapter we will be looking at the conic sections, usually called the conics, and their properties. The conics are curves

that result from a plane intersecting a double cone—two cones placed point-to-point. Each half of a double cone is called a nappe.



There are four conics—the **circle**, **parabola**, **ellipse**, and **hyperbola**. The next figure shows how the plane intersecting the double cone results in each curve.



Each of the curves has many applications that affect your daily life, from your cell phone to acoustics and navigation systems. In this section we will look at the properties of a circle.

Use the Distance Formula

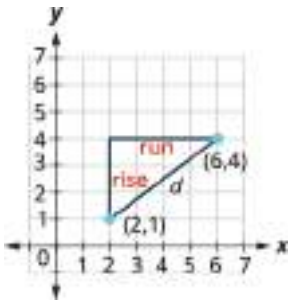
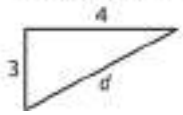
We have used the Pythagorean Theorem to find the lengths of the sides of a right triangle. Here we will use this theorem again to find distances on the rectangular coordinate system. By finding distance on the rectangular coordinate system, we can make a connection between the geometry of a conic and algebra—which opens up a world of opportunities for application.

Our first step is to develop a formula to find distances between points on the rectangular coordinate system. We will plot the points and create a right triangle much as we did when we found slope in **Graphs and Functions**. We then take it one step further and use the Pythagorean Theorem to find the length of the hypotenuse of the triangle—which is the distance between the points.

EXAMPLE 11.1

Use the rectangular coordinate system to find the distance between the points $(6, 4)$ and $(2, 1)$.

Solution

Plot the two points. Connect the two points with a line. Draw a right triangle as if you were going to find slope.	
Find the length of each leg.	The rise is 3. The run is 4. 
Use the Pythagorean Theorem to find d, the distance between the two points.	$a^2 + b^2 = c^2$
Substitute in the values.	$3^2 + 4^2 = d^2$

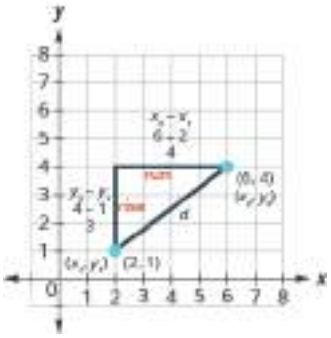
Simplify.	$9 + 16 = d^2$
	$25 = d^2$
Use the Square Root Property.	$d = 5$ ~~$d = -5$~~
Since distance, d is positive, we can eliminate $d = -5$.	The distance between the points $(6, 4)$ and $(2, 1)$ is 5.

TRY IT : : 11.1

Use the rectangular coordinate system to find the distance between the points $(6, 1)$ and $(2, -2)$.

TRY IT : : 11.2

Use the rectangular coordinate system to find the distance between the points $(5, 3)$ and $(-3, -3)$.



The method we used in the last example leads us to the formula to find the distance between the two points (x_1, y_1) and (x_2, y_2).

When we found the length of the horizontal leg we subtracted $6 - 2$ which is $x_2 - x_1$.

When we found the length of the vertical leg we subtracted $4 - 1$ which is $y_2 - y_1$.

If the triangle had been in a different position, we may have subtracted $x_1 - x_2$ or $y_1 - y_2$. The expressions $x_2 - x_1$ and $x_1 - x_2$ vary only in the sign of the resulting number. To get the positive value-since distance is positive- we can use absolute value. So to generalize we will say $|x_2 - x_1|$ and $|y_2 - y_1|$.

In the Pythagorean Theorem, we substitute the general expressions $|x_2 - x_1|$ and $|y_2 - y_1|$ rather than the numbers.

	$a^2 + b^2 = c^2$
Substitute in the values.	$(\lvert x_2 - x_1 \rvert)^2 + (\lvert y_2 - y_1 \rvert)^2 = d^2$
Squaring the expressions makes them positive, so we eliminate the absolute value bars.	$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$
Use the Square Root Property.	$d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Distance is positive, so eliminate the negative value.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

This is the Distance Formula we use to find the distance d between the two points (x_1, y_1) and (x_2, y_2).

Distance Formula

The distance d between the two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 11.2

Use the Distance Formula to find the distance between the points $(-5, -3)$ and $(7, 2)$.

Solution

Write the Distance Formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Label the points, $\begin{pmatrix} x_1, & y_1 \\ -5, & -3 \end{pmatrix}$, $\begin{pmatrix} x_2, & y_2 \\ 7, & 2 \end{pmatrix}$ and substitute.

$$d = \sqrt{(7 - (-5))^2 + (2 - (-3))^2}$$

Simplify.

$$\begin{aligned} d &= \sqrt{12^2 + 5^2} \\ d &= \sqrt{144 + 25} \\ d &= \sqrt{169} \\ d &= 13 \end{aligned}$$

TRY IT : : 11.3 Use the Distance Formula to find the distance between the points $(-4, -5)$ and $(5, 7)$.

TRY IT : : 11.4

Use the Distance Formula to find the distance between the points $(-2, -5)$ and $(-14, -10)$.

EXAMPLE 11.3

Use the Distance Formula to find the distance between the points $(10, -4)$ and $(-1, 5)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Solution

Write the Distance Formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Label the points, $\begin{pmatrix} x_1, & y_1 \\ 10, & -4 \end{pmatrix}$, $\begin{pmatrix} x_2, & y_2 \\ -1, & 5 \end{pmatrix}$ and substitute.

$$d = \sqrt{(-1 - 10)^2 + (5 - (-4))^2}$$

Simplify.

$$\begin{aligned} d &= \sqrt{(-11)^2 + 9^2} \\ d &= \sqrt{121 + 81} \\ d &= \sqrt{202} \end{aligned}$$

Since 202 is not a perfect square, we can leave the answer in exact form or find a decima approximation.

$$\begin{aligned} d &= \sqrt{202} \\ &\text{or} \\ d &\approx 14.2 \end{aligned}$$

TRY IT : : 11.5

Use the Distance Formula to find the distance between the points $(-4, -5)$ and $(3, 4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

TRY IT : : 11.6

Use the Distance Formula to find the distance between the points $(-2, -5)$ and $(-3, -4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Use the Midpoint Formula

It is often useful to be able to find the midpoint of a segment. For example, if you have the endpoints of the diameter of a circle, you may want to find the center of the circle which is the midpoint of the diameter. To find the midpoint of a line segment, we find the average of the *x*-coordinates and the average of the *y*-coordinates of the endpoints.

Midpoint Formula

The midpoint of the line segment whose endpoints are the two points (x_1, y_1) and (x_2, y_2) is

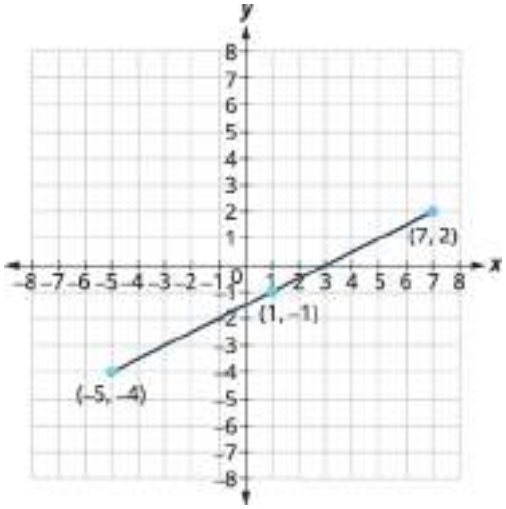
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

To find the midpoint of a line segment, we find the average of the *x*-coordinates and the average of the *y*-coordinates of the endpoints.

EXAMPLE 11.4

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-5, -4)$ and $(7, 2)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

Solution

Write the Midpoint Formula.	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Label the points, $\begin{pmatrix} x_1, y_1 \\ -5, -4 \end{pmatrix}, \begin{pmatrix} x_2, y_2 \\ 7, 2 \end{pmatrix}$ and substitute.	$\left(\frac{-5 + 7}{2}, \frac{-4 + 2}{2}\right)$
Simplify.	$\left(\frac{2}{2}, \frac{-2}{2}\right)$
	$(1, -1)$ The midpoint of the segment is the point $(1, -1)$.
Plot the endpoints and midpoint.	

TRY IT : : 11.7

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-3, -5)$ and $(5, 7)$.

Plot the endpoints and the midpoint on a rectangular coordinate system.

TRY IT : : 11.8

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-2, -5)$ and $(6, -1)$.

Plot the endpoints and the midpoint on a rectangular coordinate system.

Both the Distance Formula and the Midpoint Formula depend on two points, (x_1, y_1) and (x_2, y_2). It is easy to confuse which formula requires addition and which subtraction of the coordinates. If we remember where the formulas come from, is may be easier to remember the formulas.

Distance Formula	Midpoint Formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Subtract the coordinates.	Add the coordinates.

Write the Equation of a Circle in Standard Form

As we mentioned, our goal is to connect the geometry of a conic with algebra. By using the coordinate plane, we are able to do this easily.



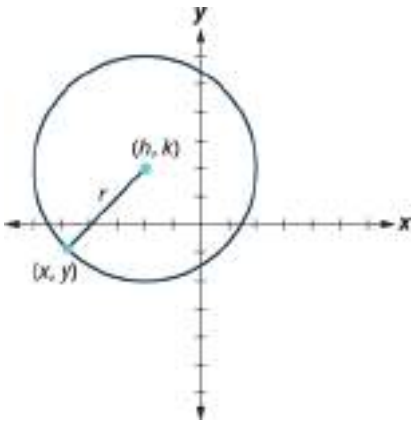
circle

We define a circle as all points in a plane that are a fixed distance from a given point in the plane. The given point is called the *center,* (h, k), and the fixed distance is called the *radius*, *r*, of the circle.

Circle

A circle is all points in a plane that are a fixed distance from a given point in the plane. The given point is called the **center**, (h, k), and the fixed distance is called the **radius**, *r*, of the circle.

We look at a circle in the rectangular coordinate system. The radius is the distance from the center, (h, k), to a point on the circle, (x, y).

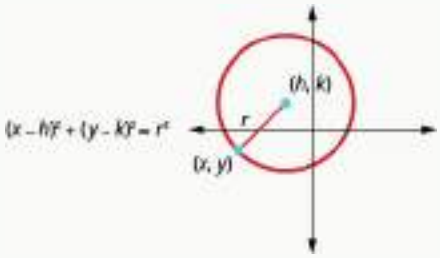


To derive the equation of a circle, we can use the distance formula with the points (h, k), (x, y) and the distance, r.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Substitute the values.	$r = \sqrt{(x - h)^2 + (y - k)^2}$
Square both sides.	$r^2 = (x - h)^2 + (y - k)^2$

This is the standard form of the equation of a circle with center, (h, k), and radius, r.

Standard Form of the Equation a Circle

The standard form of the equation of a circle with center, (h, k), and radius, r, is



EXAMPLE 11.5

Write the standard form of the equation of the circle with radius 3 and center $(0, 0)$.

Solution

Use the standard form of the equation of a circle	$(x - h)^2 + (y - k)^2 = r^2$
Substitute in the values $r = 3$, $h = 0$, and $k = 0$.	$(x - 0)^2 + (y - 0)^2 = 3^2$
Center: (h, k) $(0, 0)$	
Simplify.	$x^2 + y^2 = 9$

TRY IT : : 11.9 Write the standard form of the equation of the circle with a radius of 6 and center $(0, 0)$.

TRY IT : : 11.10 Write the standard form of the equation of the circle with a radius of 8 and center $(0, 0)$.

In the last example, the center was $(0, 0)$. Notice what happened to the equation. Whenever the center is $(0, 0)$, the standard form becomes $x^2 + y^2 = r^2$.

EXAMPLE 11.6

Write the standard form of the equation of the circle with radius 2 and center $(-1, 3)$.

Solution

Use the standard form of the equation of a circle.	$(x-h)^2+(y-k)^2=r^2$
Substitute in the values.	$(x-(-1))^2+(y-3)^2=2^2$
Center: (h, k) $(-1, 3)$	
Simplify.	$(x+1)^2+(y-3)^2=4$

TRY IT : : 11.11 Write the standard form of the equation of the circle with a radius of 7 and center $(2, -4)$.

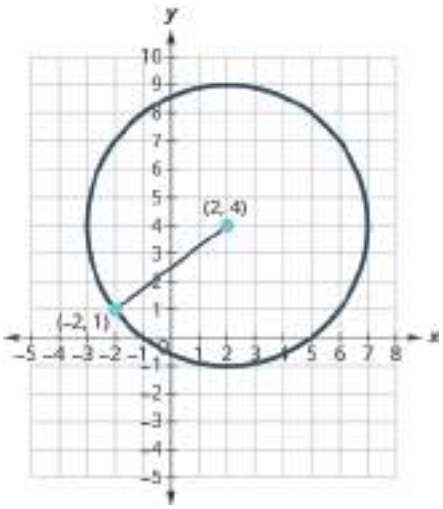
TRY IT : : 11.12

Write the standard form of the equation of the circle with a radius of 9 and center $(-3, -5)$.

In the next example, the radius is not given. To calculate the radius, we use the Distance Formula with the two given points.

EXAMPLE 11.7

Write the standard form of the equation of the circle with center $(2, 4)$ that also contains the point $(-2, 1)$.



Solution

The radius is the distance from the center to any point on the circle so we can use the distance formula to calculate it. We

will use the center $(2, 4)$ and point $(-2, 1)$

Use the Distance Formula to find he radius.	$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Substitute the values. $\begin{pmatrix} x_1, & y_1 \\ 2, & 4 \end{pmatrix}, \begin{pmatrix} x_2, & y_2 \\ -2, & 1 \end{pmatrix}$	$r = \sqrt{(-2 - 2)^2 + (1 - 4)^2}$
Simplify.	$r = \sqrt{(-4)^2 + (-3)^2}$
	$r = \sqrt{16 + 9}$
	$r = \sqrt{25}$
	$r = 5$

Now that we know the radius, $r = 5,$ and the center, $(2, 4),$ we can use the standard form of the equation of a circle to find the equation.

Use the standard form of the equation of a circle.	$(x - h)^2 + (y - k)^2 = r^2$
Substitute in the values.	$(x - 2)^2 + (y - 4)^2 = 5^2$
Simplify.	$(x - 2)^2 + (y - 4)^2 = 25$

TRY IT : : 11.13

Write the standard form of the equation of the circle with center $(2, 1)$ that also contains the point $(-2, -2).$

TRY IT : : 11.14

Write the standard form of the equation of the circle with center $(7, 1)$ that also contains the point $(-1, -5).$

Graph a Circle

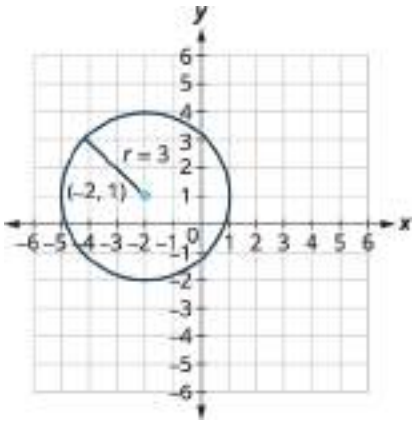
Any equation of the form $(x - h)^2 + (y - k)^2 = r^2$ is the standard form of the equation of a circle with center, (h, k), and radius, *r*. We can then graph the circle on a rectangular coordinate system.

Note that the standard form calls for subtraction from *x* and *y*. In the next example, the equation has $x + 2,$ so we need to rewrite the addition as subtraction of a negative.

EXAMPLE 11.8

Find the center and radius, then graph the circle: $(x + 2)^2 + (y - 1)^2 = 9.$

Solution

	$(x+2)^2+(y-1)^2=9$
Use the standard form of the equation of a circle. Identify the center, (h, k) and radius, r.	$(x-h)^2+(y-k)^2=r^2$ $(x-(-2))^2+(y-1)^2=3^2$
	Center: $(-2, 1)$ radius: 3
Graph the circle.	

TRY IT : : 11.15 ⓐ Find the center and radius, then ⓑ graph the circle: $(x-3)^2+(y+4)^2=4$.

TRY IT : : 11.16 ⓐ Find the center and radius, then ⓑ graph the circle: $(x-3)^2+(y-1)^2=16$.

To find the center and radius, we must write the equation in standard form. In the next example, we must first get the coefficient of x^2, y^2 to be one.

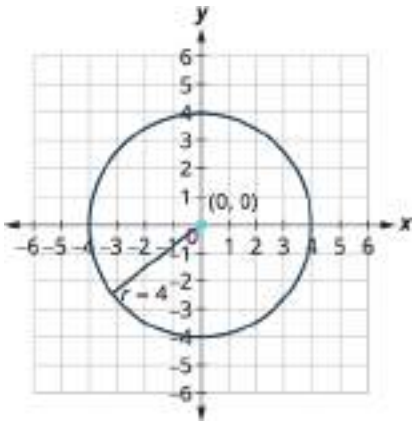
EXAMPLE 11.9

Find the center and radius and then graph the circle, $4x^2+4y^2=64$.

Solution

	$4x^2+4y^2=64$
Divide each side by 4.	$x^2+y^2=16$
Use the standard form of the equation of a circle. Identify the center, (h, k) and radius, r.	$(x-h)^2+(y-k)^2=r^2$ $(x-0)^2+(y-0)^2=4^2$
	Center: $(0, 0)$ radius: 4

Graph the circle.



TRY IT : : 11.17 ⓐ Find the center and radius, then ⓑ graph the circle: $3x^2 + 3y^2 = 27$

TRY IT : : 11.18 ⓐ Find the center and radius, then ⓑ graph the circle: $5x^2 + 5y^2 = 125$

If we expand the equation from Example 11.8, $(x+2)^2 + (y-1)^2 = 9$, the equation of the circle looks very different.

	$(x+2)^2 + (y-1)^2 = 9$
Square the binomials.	$x^2 + 4x + 4 + y^2 - 2y + 1 = 9$
Arrange the terms in descending degree order, and get zero on the right	$x^2 + y^2 + 4x - 2y - 4 = 0$

This form of the equation is called the general form of the equation of the circle.

General Form of the Equation of a Circle

The general form of the equation of a circle is

$$x^2 + y^2 + ax + by + c = 0$$

If we are given an equation in general form, we can change it to standard form by completing the squares in both *x* and *y*. Then we can graph the circle using its center and radius.

EXAMPLE 11.10

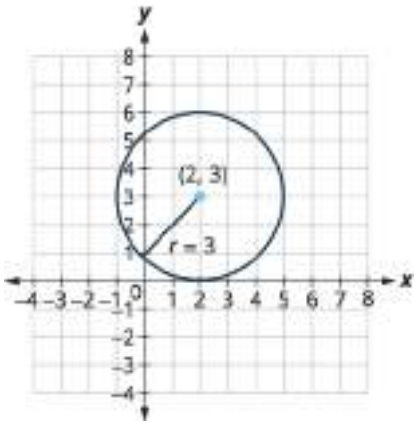
ⓐ Find the center and radius, then ⓑ graph the circle: $x^2 + y^2 - 4x - 6y + 4 = 0$.

Solution

We need to rewrite this general form into standard form in order to find the center and radius.

	$x^2 + y^2 - 4x - 6y + 4 = 0$
Group the *x*-terms and *y*-terms. Collect the constants on the right side.	$x^2 - 4x + y^2 - 6y = -4$
Complete the squares.	$x^2 - 4x + 4 + y^2 - 6y + 9 = -4 + 4 + 9$
Rewrite as binomial squares.	$(x-2)^2 + (y-3)^2 = 9$
Identify the center and radius.	Center: $(2, 3)$ radius: 3

Graph the circle.



TRY IT : : 11.19 ⓐ Find the center and radius, then ⓑ graph the circle: $x^2 + y^2 - 6x - 8y + 9 = 0$.

TRY IT : : 11.20 ⓐ Find the center and radius, then ⓑ graph the circle: $x^2 + y^2 + 6x - 2y + 1 = 0$.

In the next example, there is a *y*-term and a y^2-term. But notice that there is no *x*-term, only an x^2-term. We have seen this before and know that it means *h* is 0. We will need to complete the square for the *y* terms, but not for the *x* terms.

EXAMPLE 11.11

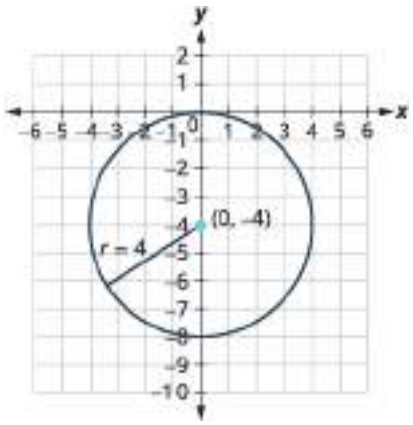
ⓐ Find the center and radius, then ⓑ graph the circle: $x^2 + y^2 + 8y = 0$.

Solution

We need to rewrite this general form into standard form in order to find the center and radius.

	$x^2 + y^2 + 8y = 0$
Group the *x*-terms and *y*-terms.	$x^2 + y^2 + 8y = 0$
There are no constants to collect on the right side.	
Complete the square for $y^2 + 8y$.	$x^2 + y^2 + 8y + 16 = 0 + 16$
Rewrite as binomial squares.	$(x - 0)^2 + (y + 4)^2 = 16$
Identify the center and radius.	Center: $(0, -4)$ radius: 4

Graph the circle.



TRY IT : : 11.21 ⓐ Find the center and radius, then ⓑ graph the circle: $x^2 + y^2 - 2x - 3 = 0$.

TRY IT : : 11.22 ⓐ Find the center and radius, then ⓑ graph the circle: $x^2 + y^2 - 12y + 11 = 0$.

MEDIA : :

Access these online resources for additional instructions and practice with using the distance and midpoint formulas, and graphing circles.

- **Distance-Midpoint Formulas and Circles (https://openstax.org/l/37distmidcircle)**
- **Finding the Distance and Midpoint Between Two Points (https://openstax.org/l/37distmid2pts)**
- **Completing the Square to Write Equation in Standard Form of a Circle (https://openstax.org/l/37stformcircle)**

11.1 EXERCISES

Practice Makes Perfect

Use the Distance Formula

In the following exercises, find the distance between the points. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

1. $(2, 0)$ and $(5, 4)$

2. $(-4, -3)$ and $(2, 5)$

3. $(-4, -3)$ and $(8, 2)$

4. $(-7, -3)$ and $(8, 5)$

5. $(-1, 4)$ and $(2, 0)$

6. $(-1, 3)$ and $(5, -5)$

7. $(1, -4)$ and $(6, 8)$

8. $(-8, -2)$ and $(7, 6)$

9. $(-3, -5)$ and $(0, 1)$

10. $(-1, -2)$ and $(-3, 4)$

11. $(3, -1)$ and $(1, 7)$

12. $(-4, -5)$ and $(7, 4)$

Use the Midpoint Formula

In the following exercises, ⓐ find the midpoint of the line segments whose endpoints are given and ⓑ plot the endpoints and the midpoint on a rectangular coordinate system.

13. $(0, -5)$ and $(4, -3)$

14. $(-2, -6)$ and $(6, -2)$

15. $(3, -1)$ and $(4, -2)$

16. $(-3, -3)$ and $(6, -1)$

Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given radius and center $(0, 0)$.

17. Radius: 7

18. Radius: 9

19. Radius: $\sqrt{2}$

20. Radius: $\sqrt{5}$

In the following exercises, write the standard form of the equation of the circle with the given radius and center

21. Radius: 1, center: $(3, 5)$

22. Radius: 10, center: $(-2, 6)$

23. Radius: 2.5, center: $(1.5, -3.5)$

24. Radius: 1.5, center: $(-5.5, -6.5)$

For the following exercises, write the standard form of the equation of the circle with the given center with point on the circle.

25. Center $(3, -2)$ with point $(3, 6)$

26. Center $(6, -6)$ with point $(2, -3)$

27. Center $(4, 4)$ with point $(2, 2)$

28. Center $(-5, 6)$ with point $(-2, 3)$

Graph a Circle

In the following exercises, ⓐ find the center and radius, then ⓑ graph each circle.

29. $(x+5)^2+(y+3)^2=1$

30. $(x-2)^2+(y-3)^2=9$

31. $(x-4)^2+(y+2)^2=16$

32. $(x+2)^2+(y-5)^2=4$

33. $x^2+(y+2)^2=25$

34. $(x-1)^2+y^2=36$

35. $(x-1.5)^2+(y+2.5)^2=0.25$

36. $(x-1)^2+(y-3)^2=\frac{9}{4}$

37. $x^2+y^2=64$

38. $x^2 + y^2 = 49$

39. $2x^2 + 2y^2 = 8$

40. $6x^2 + 6y^2 = 216$

In the following exercises, ⓐ identify the center and radius and ⓑ graph.

41. $x^2 + y^2 + 2x + 6y + 9 = 0$

42. $x^2 + y^2 - 6x - 8y = 0$

43. $x^2 + y^2 - 4x + 10y - 7 = 0$

44. $x^2 + y^2 + 12x - 14y + 21 = 0$

45. $x^2 + y^2 + 6y + 5 = 0$

46. $x^2 + y^2 - 10y = 0$

47. $x^2 + y^2 + 4x = 0$

48. $x^2 + y^2 - 14x + 13 = 0$

Writing Exercises

49. Explain the relationship between the distance formula and the equation of a circle.

50. Is a circle a function? Explain why or why not.

51. In your own words, state the definition of a circle.

52. In your own words, explain the steps you would take to change the general form of the equation of a circle to the standard form.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the distance formula.			
use the midpoint formula.			
write the equation of a circle in standard form.			
graph a circle.			

ⓑ If most of your checks were:

...confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

...with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

...no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

11.2 Parabolas

Learning Objectives

By the end of this section, you will be able to:

- Graph vertical parabolas
- Graph horizontal parabolas
- Solve applications with parabolas

Be Prepared!

Before you get started, take this readiness quiz.

1. Graph: $y = -3x^2 + 12x - 12$.
 If you missed this problem, review Example 9.47.
2. Solve by completing the square: $x^2 - 6x + 6 = 0$.
 If you missed this problem, review Example 9.12.
3. Write in standard form: $y = 3x^2 - 6x + 5$.
 If you missed this problem, review Example 9.59.

Graph Vertical Parabolas

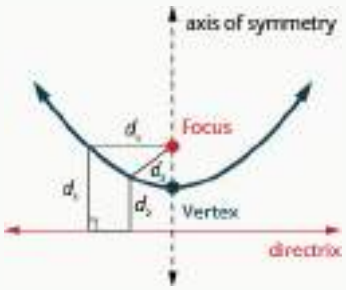
The next conic section we will look at is a **parabola**. We define a parabola as all points in a plane that are the same distance from a fixed point and a fixed line. The fixed point is called the **focus,** and the fixed line is called the **directrix** of the parabola.



parabola

Parabola

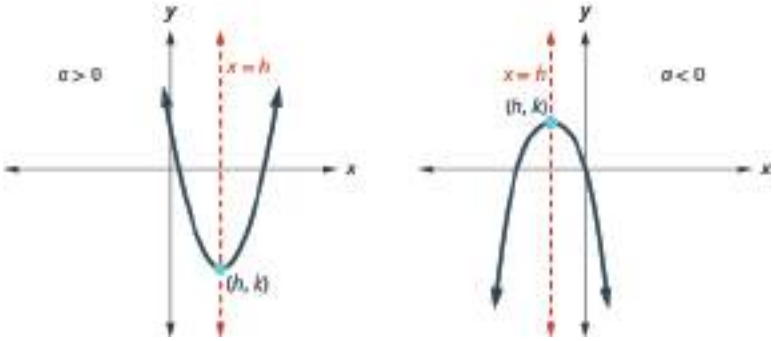
A **parabola** is all points in a plane that are the same distance from a fixed point and a fixed line. The fixed point is called the **focus,** and the fixed line is called the **directrix** of the parabola.



Previously, we learned to graph vertical parabolas from the general form or the standard form using properties. Those methods will also work here. We will summarize the properties here.

Vertical Parabolas		
	General form $y = ax^2 + bx + c$	**Standard form** $y = a(x - h)^2 + k$
Orientation	$a > 0$ up; $a < 0$ down	$a > 0$ up; $a < 0$ down
Axis of symmetry	$x = -\frac{b}{2a}$	$x = h$
Vertex	Substitute $x = -\frac{b}{2a}$ and solve for y.	(h, k)
***y*-intercept**	Let $x = 0$	Let $x = 0$
***x*-intercepts**	Let $y = 0$	Let $y = 0$

The graphs show what the parabolas look like when they open up or down. Their position in relation to the *x*- or *y*-axis is merely an example.



To graph a parabola from these forms, we used the following steps.

HOW TO : : GRAPH VERTICAL PARABOLAS $\left(y = ax^2 + bx + c \text{ or } f(x) = a(x - h)^2 + k\right)$ USING PROPERTIES.

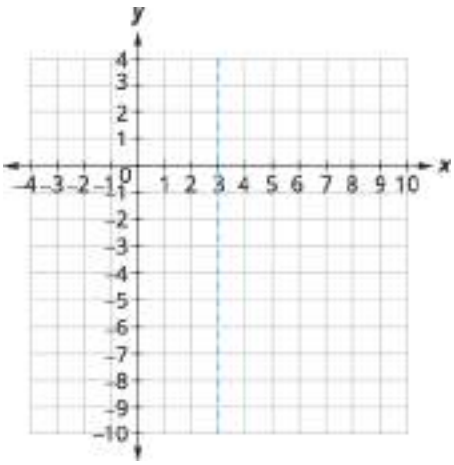
Step 1. Determine whether the parabola opens upward or downward.
Step 2. Find the axis of symmetry.
Step 3. Find the vertex.
Step 4. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
Step 5. Find the *x*-intercepts.
Step 6. Graph the parabola.

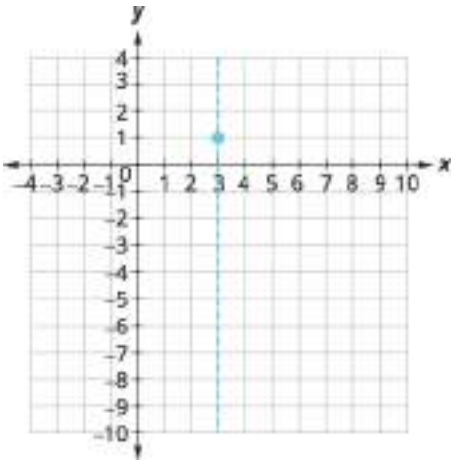
The next example reviews the method of graphing a parabola from the general form of its equation.

EXAMPLE 11.12

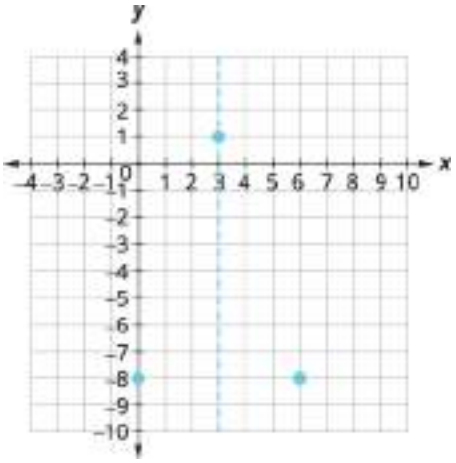
Graph $y = -x^2 + 6x - 8$ by using properties.

Solution

	$y = ax^2 + bx + c$ $y = -x^2 + 6x - 8$
Since a is -1, the parabola opens downward.	
	
To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{6}{2(-1)}$
	$x = 3$
	The axis of symmetry is $x = 3$.
	
The vertex is on the line $x = 3$.	$y = -x^2 + 6x - 8$
Let $x = 3$.	$y = -3^2 + 6 \cdot 3 - 8$
	$y = -9 + 18 - 8$
	$y = 1$
	The vertex is $(3, 1)$.

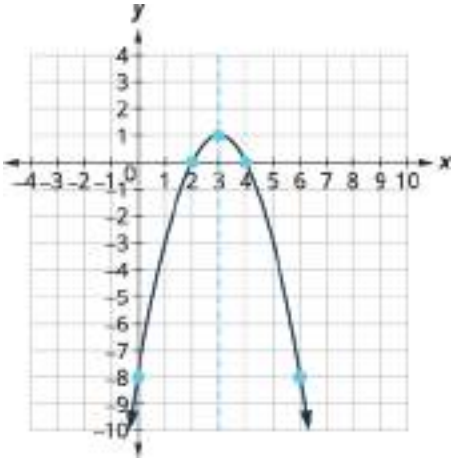


The y-intercept occurs when $x = 0$.	$y = -x^2 + 6x - 8$
Substitute $x = 0$.	$y = -0^2 + 6 \cdot 0 - 8$
Simplify.	$y = -8$
	The y-intercept is $(0, -8)$.
The point $(0, -8)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -8)$.	Point symmetric to the y-intercept is $(6, -8)$.



The x-intercept occurs when $y = 0$.	$y = -x^2 + 6x - 8$
Let $y = 0$.	$0 = -x^2 + 6x - 8$
Factor the GCF.	$0 = -(x^2 - 6x + 8)$
Factor the trinomial.	$0 = -(x - 4)(x - 2)$
Solve for x.	$x = 4, \quad x = 2$
	The x-intercepts are $(4, 0), (2, 0)$.

Graph the parabola.



TRY IT : : 11.23 Graph $y = -x^2 + 5x - 6$ by using properties.

TRY IT : : 11.24 Graph $y = -x^2 + 8x - 12$ by using properties.

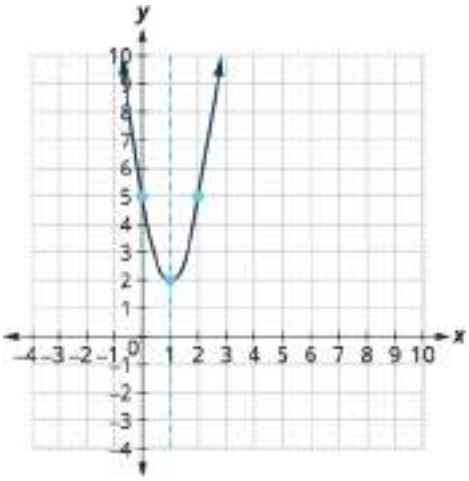
The next example reviews the method of graphing a parabola from the standard form of its equation, $y = a(x - h)^2 + k$.

EXAMPLE 11.13

Write $y = 3x^2 - 6x + 5$ in standard form and then use properties of standard form to graph the equation.

Solution

Rewrite the function in $y = a(x - h)^2 + k$ form by completing the square.	$y = 3x^2 - 6x + 5$
	$y = 3\left(x^2 - 2x \quad\right) + 5$
	$y = 3\left(x^2 - 2x + 1\right) + 5 - 3$
	$y = 3(x - 1)^2 + 2$
Identify the constants *a, h, k*.	$a = 3,\ h = 1,\ k = 2$
Since $a = 2,$ the parabola opens upward.	
	
The axis of symmetry is $x = h$.	The axis of symmetry is $x = 1$.
The vertex is (h, k).	The vertex is $(1, 2)$.

Find the *y*-intercept by substituting $x = 0$.	$y = 3(x-1)^2 + 2$ $y = 3 \cdot 0^2 - 6 \cdot 0 + 5$
	$y = 5$
	y-intercept $(0, 5)$
Find the point symmetric to $(0, 5)$ across the axis of symmetry.	$(2, 5)$
Find the *x*-intercepts.	$y = 3(x-1)^2 + 2$ $0 = 3(x-1)^2 + 2$ $-2 = 3(x-1)^2$ $-\frac{2}{3} = (x-1)^2$ $\pm\sqrt{-\frac{2}{3}} = x - 1$
	The square root of a negative number tells us the solutions are complex numbers. So there are no *x*-intercepts.
Graph the parabola.	

TRY IT : : 11.25

ⓐ Write $y = 2x^2 + 4x + 5$ in standard form and ⓑ use properties of standard form to graph the equation.

TRY IT : : 11.26

ⓐ Write $y = -2x^2 + 8x - 7$ in standard form and ⓑ use properties of standard form to graph the equation.

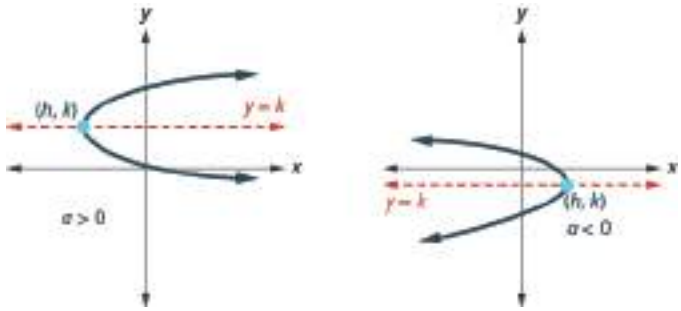
Graph Horizontal Parabolas

Our work so far has only dealt with parabolas that open up or down. We are now going to look at horizontal parabolas. These parabolas open either to the left or to the right. If we interchange the *x* and *y* in our previous equations for parabolas, we get the equations for the parabolas that open to the left or to the right.

Horizontal Parabolas		
	General form $x = ay^2 + by + c$	**Standard form** $x = a(y-k)^2 + h$
Orientation	$a > 0$ right; $a < 0$ left	$a > 0$ right; $a < 0$ left
Axis of symmetry	$y = -\frac{b}{2a}$	$y = k$
Vertex	Substitute $y = -\frac{b}{2a}$ and solve for x.	(h, k)
***y*-intercepts**	Let $x = 0$	Let $x = 0$
***x*-intercept**	Let $y = 0$	Let $y = 0$

Table 11.11

The graphs show what the parabolas look like when they to the left or to the right. Their position in relation to the *x*- or *y*-axis is merely an example.



Looking at these parabolas, do their graphs represent a function? Since both graphs would fail the vertical line test, they do not represent a function.

To graph a parabola that opens to the left or to the right is basically the same as what we did for parabolas that open up or down, with the reversal of the *x* and *y* variables.

HOW TO :: GRAPH HORIZONTAL PARABOLAS $\left(x = ay^2 + by + c \text{ or } x = a(y-k)^2 + h\right)$ USING PROPERTIES.

Step 1. Determine whether the parabola opens to the left or to the right.
Step 2. Find the axis of symmetry.
Step 3. Find the vertex.
Step 4. Find the *x*-intercept. Find the point symmetric to the *x*-intercept across the axis of symmetry.
Step 5. Find the *y*-intercepts.
Step 6. Graph the parabola.

EXAMPLE 11.14

Graph $x = 2y^2$ by using properties.

Solution

	$x = ay^2 + by + c$ $x = 2y^2$
Since $a = 2,$ the parabola opens to the right.	
	
To find the axis of symmetry, find $y = -\frac{b}{2a}$.	$y = -\frac{b}{2a}$
	$y = -\frac{0}{2(2)}$
	$y = 0$
	The axis of symmetry is $y = 0.$
The vertex is on the line $y = 0.$	$x = 2y^2$
Let $y = 0.$	$x = 2 \cdot 0^2$
	$x = 0$
	The vertex is $(0, 0).$

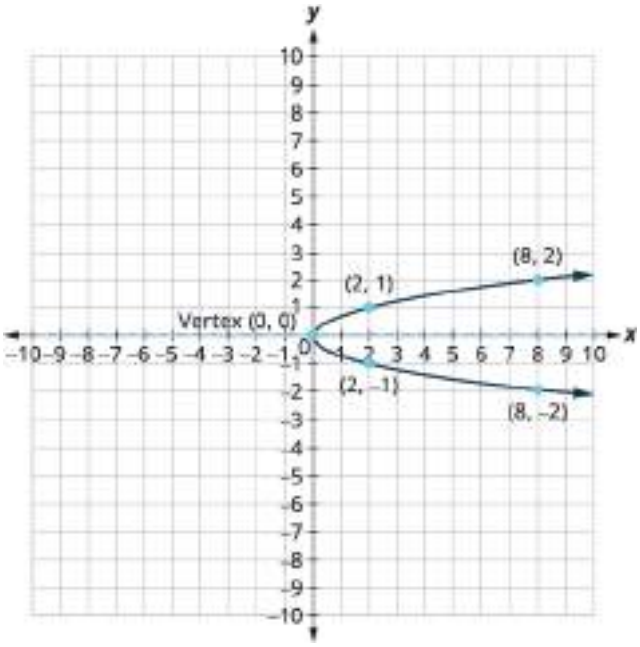
Since the vertex is $(0, 0),$ both the *x*- and *y*-intercepts are the point $(0, 0).$ To graph the parabola we need more points. In this case it is easiest to choose values of *y*.

$y = 1$: $x = 2y^2$, $x = 2 \cdot 1^2$, $x = 2$ $y = 2$: $x = 2y^2$, $x = 2 \cdot 2^2$, $x = 8$

x	y	
2	1	(2, 1)
8	2	(8, 2)

We also plot the points symmetric to $(2, 1)$ and $(8, 2)$ across the *y*-axis, the points $(2, -1),$ $(8, -2).$

Graph the parabola.



TRY IT : : 11.27 Graph $x = y^2$ by using properties.

TRY IT : : 11.28 Graph $x = -y^2$ by using properties.

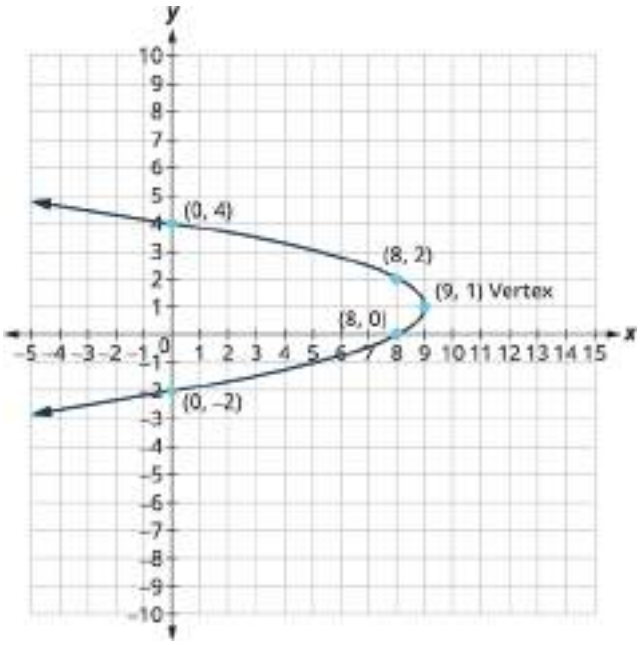
In the next example, the vertex is not the origin.

EXAMPLE 11.15

Graph $x = -y^2 + 2y + 8$ by using properties.

Solution

	$x = ay^2 + by + c$ $x = -y^2 + 2y + 8$
Since $a = -1,$ the parabola opens to the left.	
To find the axis of symmetry, find $y = -\frac{b}{2a}$.	$y = -\frac{b}{2a}$
	$y = -\frac{2}{2(-1)}$
	$y = 1$
	The axis of symmetry is $y = 1$.
The vertex is on the line $y = 1$.	$x = -y^2 + 2y + 8$
Let $y = 1$.	$x = -1^2 + 2 \cdot 1 + 8$

	$x = 9$
	The vertex is $(9, 1)$.
The x-intercept occurs when $y = 0$.	$x = -y^2 + 2y + 8$
	$x = -0^2 + 2 \cdot 0 + 8$
	$x = 8$
	The x-intercept is $(8, 0)$.
The point $(8, 0)$ is one unit below the line of symmetry. The symmetric point one unit above the line of symmetry is $(8, 2)$	Symmetric point is $(8, 2)$.
The y-intercept occurs when $x = 0$.	$x = -y^2 + 2y + 8$
Substitute $x = 0$.	$0 = -y^2 + 2y + 8$
Solve.	$y^2 - 2y - 8 = 0$
	$(y - 4)(y + 2) = 0$
	$y = 4, y = -2$
	The y-intercepts are $(0, 4)$ and $(0, -2)$.
Connect the points to graph the parabola.	

TRY IT : : 11.29 Graph $x = -y^2 - 4y + 12$ by using properties.

TRY IT : : 11.30 Graph $x = -y^2 + 2y - 3$ by using properties.

In Table 11.11, we see the relationship between the equation in standard form and the properties of the parabola. The

How To box lists the steps for graphing a parabola in the standard form $x = a(y-k)^2 + h$. We will use this procedure in the next example.

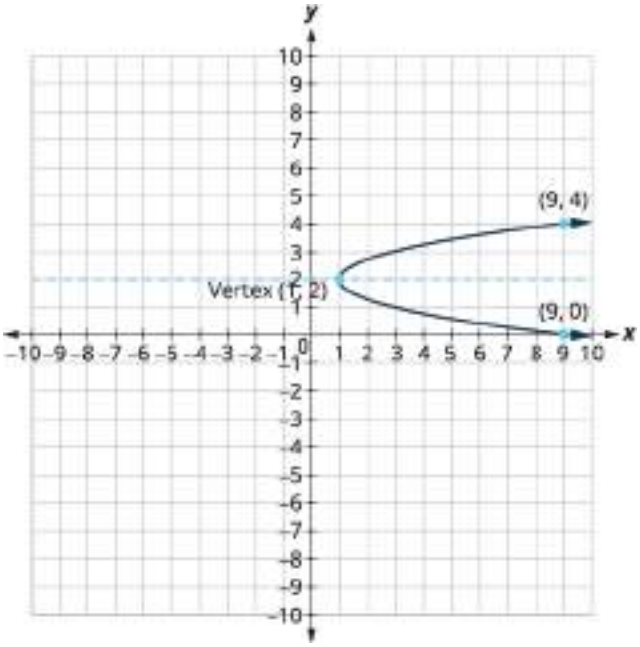
EXAMPLE 11.16

Graph $x = 2(y-2)^2 + 1$ using properties.

Solution

	$x = a(y-k)^2 + h$ $x = 2(y-2)^2 + 1$
Identify the constants *a, h, k*.	$a = 2, \quad h = 1, \quad k = 2$
Since $a = 2,$ the parabola opens to the right.	
	
The axis of symmetry is $y = k$.	The axis of symmetry is $y = 2$.
The vertex is (h, k).	The vertex is $(1, 2)$.
Find the *x*-intercept by substituting $y = 0$.	$x = 2(y-2)^2 + 1$ $x = 2(0-2)^2 + 1$ $x = 9$
	The *x*-intercept is $(9, 0)$.
Find the point symmetric to $(9, 0)$ across the axis of symmetry.	$(9, 4)$
Find the *y*-intercepts. Let $x = 0$.	$x = 2(y-2)^2 + 1$ $0 = 2(y-2)^2 + 1$ $-1 = 2(y-2)^2$
	A square cannot be negative, so there is no real solution. So there are no *y*-intercepts.

Graph the parabola.



TRY IT : : 11.31 Graph $x = 3(y-1)^2 + 2$ using properties.

TRY IT : : 11.32 Graph $x = 2(y-3)^2 + 2$ using properties.

In the next example, we notice the a is negative and so the parabola opens to the left.

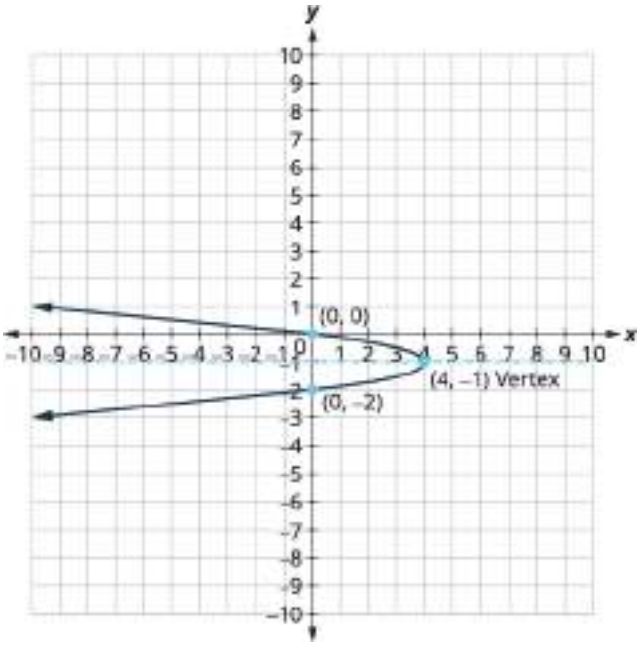
EXAMPLE 11.17

Graph $x = -4(y+1)^2 + 4$ using properties.

Solution

$x = a(y-k)^2 + h$
$x = -4(y+1)^2 + 4$

Identify the constants *a, h, k*.	$a = -4, \quad h = 4, \quad k = -1$
Since $a = -4,$ the parabola opens to the left.	
The axis of symmetry is $y = k$.	The axis of symmetry is $y = -1$.
The vertex is (h, k).	The vertex is $(4, -1)$.
Find the *x*-intercept by substituting $y = 0$.	$x = -4(y+1)^2 + 4$ $x = -4(0+1)^2 + 4$ $x = 0$

	The x-intercept is $(0,\ 0)$.
Find the point symmetric to $(0,\ 0)$ across the axis of symmetry.	$(0,\ -2)$
Find the y-intercepts.	$x = -4(y+1)^2 + 4$
Let $x = 0$.	$0 = -4(y+1)^2 + 4$ $-4 = -4(y+1)^2$ $1 = (y+1)^2$ $y+1 = \pm 1$
	$y = -1+1 \qquad y = -1-1$
	$y = 0 \qquad y = -2$
	The y-intercepts are $(0,\ 0)$ and $(0,\ -2)$.
Graph the parabola.	

TRY IT : : 11.33 Graph $x = -4(y+2)^2 + 4$ using properties.

TRY IT : : 11.34 Graph $x = -2(y+3)^2 + 2$ using properties.

The next example requires that we first put the equation in standard form and then use the properties.

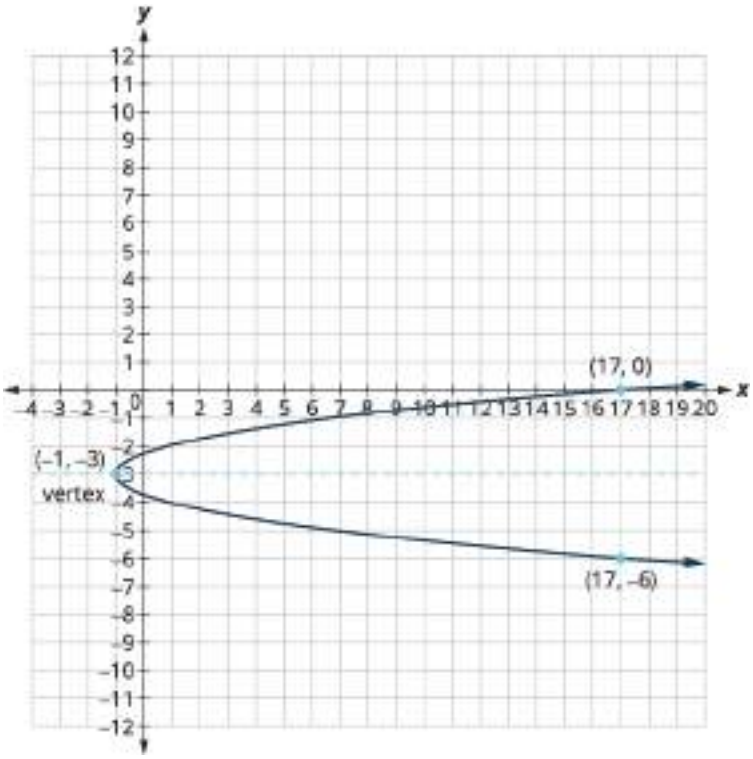
EXAMPLE 11.18

Write $x = 2y^2 + 12y + 17$ in standard form and then use the properties of the standard form to graph the equation.

Solution

	$x = 2y^2 + 12y + 17$
Rewrite the function in $x = a(y-k)^2 + h$ form by completing the square.	$x = 2(y^2 + 6y) + 17$
	$x = 2(y^2 + 6y + 9) + 17 - 18$
	$x = 2(y+3)^2 - 1$
	$x = a(y-k)^2 + h$ $x = 2(y+3)^2 - 1$
Identify the constants *a, h, k*.	$a = 2, \quad h = -1, \quad k = -3$
Since $a = 2,$ the parabola opens to the right.	
	
The axis of symmetry is $y = k$.	The axis of symmetry is $y = -3$.
The vertex is (h, k).	The vertex is $(-1, -3)$.
Find the *x*-intercept by substituting $y = 0$.	$x = 2(y+3)^2 - 1$ $x = 2(0+3)^2 - 1$ $x = 17$
	The *x*-intercept is $(17, 0)$.
Find the point symmetric to $(17, 0)$ across the axis of symmetry.	$(17, -6)$
Find the *y*-intercepts. Let $x = 0$.	$x = 2(y+3)^2 - 1$ $0 = 2(y+3)^2 - 1$ $1 = 2(y+3)^2$ $\frac{1}{2} = (y+3)^2$ $y + 3 = \pm\sqrt{\frac{1}{2}}$ $y = -3 \pm \frac{\sqrt{2}}{2}$
	$y = -3 + \frac{\sqrt{2}}{2} \quad y = -3 - \frac{\sqrt{2}}{2}$
	$y \approx -2.3 \quad y \approx -3.7$
	The *y*-intercepts are $\left(0, -3 + \frac{\sqrt{2}}{2}\right), \left(0, -3 - \frac{\sqrt{2}}{2}\right)$.

Graph the parabola.



TRY IT : : 11.35

ⓐ Write $x = 3y^2 + 6y + 7$ in standard form and ⓑ use properties of the standard form to graph the equation.

TRY IT : : 11.36

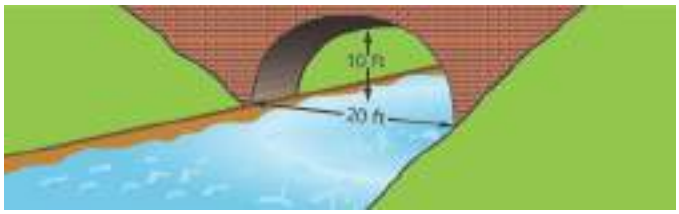
ⓐ Write $x = -4y^2 - 16y - 12$ in standard form and ⓑ use properties of the standard form to graph the equation.

Solve Applications with Parabolas

Many architectural designs incorporate parabolas. It is not uncommon for bridges to be constructed using parabolas as we will see in the next example.

EXAMPLE 11.19

Find the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.



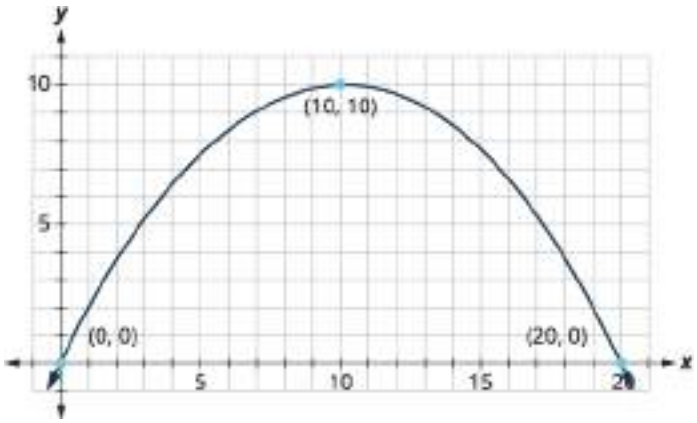
Solution

We will first set up a coordinate system and draw the parabola. The graph will give us the information we need to write the equation of the graph in the standard form $y = a(x - h)^2 + k$.

Let the lower left side of the bridge be the origin of the coordinate grid at the point $(0, 0)$.

Since the base is 20 feet wide the point $(20, 0)$ represents the lower right side.

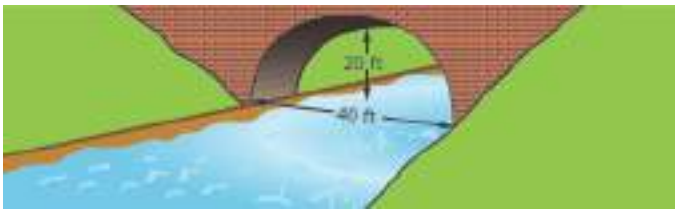
The bridge is 10 feet high at the highest point. The highest point is the vertex of the parabola so the y-coordinate of the vertex will be 10.
Since the bridge is symmetric, the vertex must fall halfway between the left most point, $(0, 0)$, and the rightmost point $(20, 0)$. From this we know that the x-coordinate of the vertex will also be 10.



Identify the vertex, (h, k).	$(h, k) = (10, 10)$
	$h = 10, \quad k = 10$
Substitute the values into the standard form. The value of a is still unknown. To find the value of a use one of the other points on the parabola.	$y = a(x - h)^2 + k$ $y = a(x - 10)^2 + 10$ $(x, y) = (0, 0)$
Substitute the values of the other point into the equation.	$y = a(x - 10)^2 + 10$ $0 = a(0 - 10)^2 + 10$
Solve for a.	$0 = a(0 - 10)^2 + 10$ $-10 = a(-10)^2$ $-10 = 100a$ $\frac{-10}{100} = a$ $a = -\frac{1}{10}$
	$y = a(x - 10)^2 + 10$
Substitute the value for a into the equation.	$y = -\frac{1}{10}(x - 10)^2 + 10$

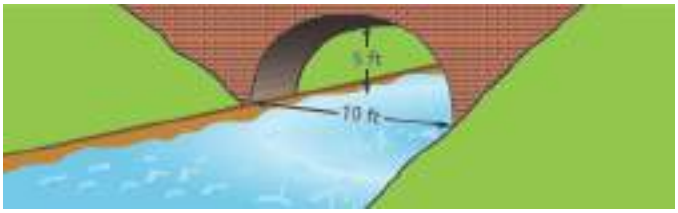
TRY IT : : 11.37

Find the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.



TRY IT : : 11.38

Find the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.



MEDIA : :

Access these online resources for additional instructions and practice with quadratic functions and parabolas.

- **Quadratic Functions (https://openstax.org/l/37quadfunc)**
- **Introduction to Conics and Graphing Horizontal Parabolas (https://openstax.org/l/37conhorizpbola)**

11.2 EXERCISES

Practice Makes Perfect

Graph Vertical Parabolas

In the following exercises, graph each equation by using properties.

53. $y = -x^2 + 4x - 3$

54. $y = -x^2 + 8x - 15$

55. $y = 6x^2 + 2x - 1$

56. $y = 8x^2 - 10x + 3$

In the following exercises, ⓐ write the equation in standard form and ⓑ use properties of the standard form to graph the equation.

57. $y = -x^2 + 2x - 4$

58. $y = 2x^2 + 4x + 6$

59. $y = -2x^2 - 4x - 5$

60. $y = 3x^2 - 12x + 7$

Graph Horizontal Parabolas

In the following exercises, graph each equation by using properties.

61. $x = -2y^2$

62. $x = 3y^2$

63. $x = 4y^2$

64. $x = -4y^2$

65. $x = -y^2 - 2y + 3$

66. $x = -y^2 - 4y + 5$

67. $x = y^2 + 6y + 8$

68. $x = y^2 - 4y - 12$

69. $x = (y - 2)^2 + 3$

70. $x = (y - 1)^2 + 4$

71. $x = -(y - 1)^2 + 2$

72. $x = -(y - 4)^2 + 3$

73. $x = (y + 2)^2 + 1$

74. $x = (y + 1)^2 + 2$

75. $x = -(y + 3)^2 + 2$

76. $x = -(y + 4)^2 + 3$

77. $x = -3(y - 2)^2 + 3$

78. $x = -2(y - 1)^2 + 2$

79. $x = 4(y + 1)^2 - 4$

80. $x = 2(y + 4)^2 - 2$

In the following exercises, ⓐ write the equation in standard form and ⓑ use properties of the standard form to graph the equation.

81. $x = y^2 + 4y - 5$

82. $x = y^2 + 2y - 3$

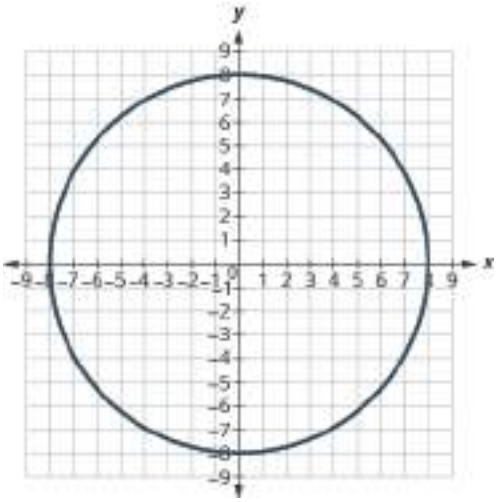
83. $x = -2y^2 - 12y - 16$

84. $x = -3y^2 - 6y - 5$

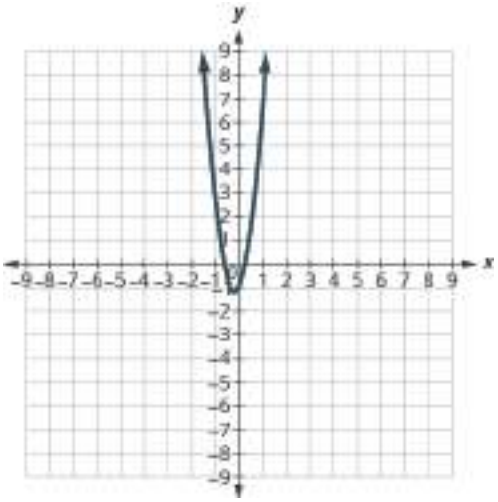
Mixed Practice

In the following exercises, match each graph to one of the following equations: ⓐ $x^2 + y^2 = 64$ ⓑ $x^2 + y^2 = 49$ ⓒ $(x + 5)^2 + (y + 2)^2 = 4$ ⓓ $(x - 2)^2 + (y - 3)^2 = 9$ ⓔ $y = -x^2 + 8x - 15$ ⓕ $y = 6x^2 + 2x - 1$

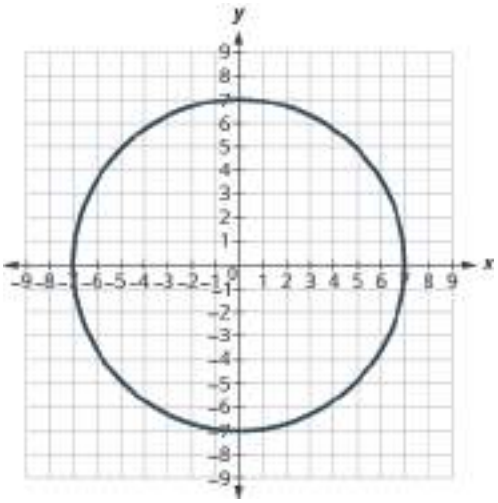
85.



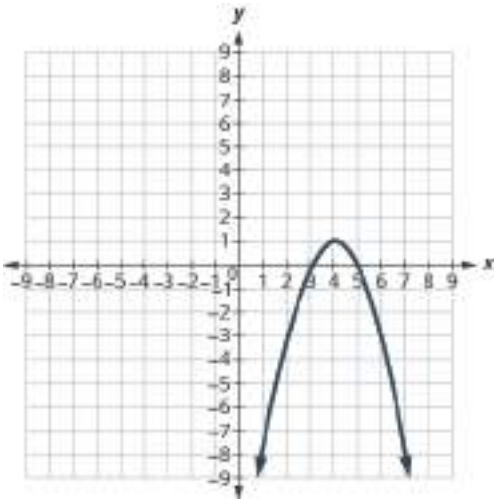
86.



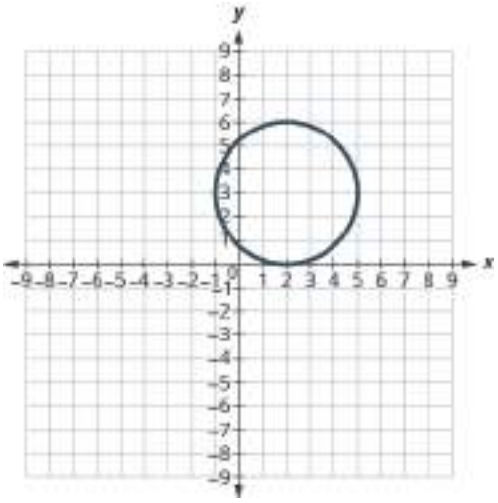
87.



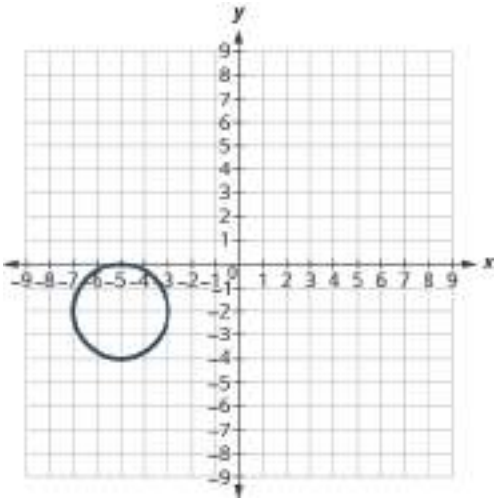
88.



89.



90.



Solve Applications with Parabolas

91. Write the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.



92. Write the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.



93. Write the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.



94. Write the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.



Writing Exercises

95. In your own words, define a parabola.

96. Is the parabola $y = x^2$ a function? Is the parabola $x = y^2$ a function? Explain why or why not.

97. Write the equation of a parabola that opens up or down in standard form and the equation of a parabola that opens left or right in standard form. Provide a sketch of the parabola for each one, label the vertex and axis of symmetry.

98. Explain in your own words, how you can tell from its equation whether a parabola opens up, down, left or right.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
graph vertical parabolas.			
graph horizontal parabolas.			
solve applications with parabolas.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

11.3 Ellipses

Learning Objectives

By the end of this section, you will be able to:

- Graph an ellipse with center at the origin
- Find the equation of an ellipse with center at the origin
- Graph an ellipse with center not at the origin
- Solve application with ellipses

Be Prepared!

Before you get started, take this readiness quiz.

1. Graph $y = (x - 1)^2 - 2$ using transformations.
 If you missed this problem, review **Example 9.57**.
2. Complete the square: $x^2 - 8x = 8$.
 If you missed this problem, review **Example 9.12**.
3. Write in standard form. $y = 2x^2 - 12x + 14$
 If you missed this problem, review **Example 9.59**.

Graph an Ellipse with Center at the Origin

The next conic section we will look at is an **ellipse**. We define an ellipse as all points in a plane where the sum of the distances from two fixed points is constant. Each of the given points is called a **focus** of the ellipse.

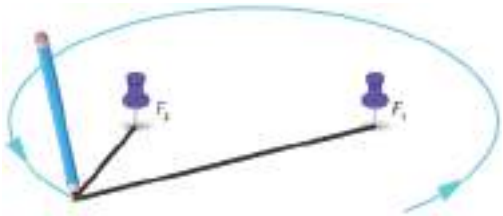
Ellipse

An **ellipse** is all points in a plane where the sum of the distances from two fixed points is constant. Each of the fixed points is called a **focus** of the ellipse.

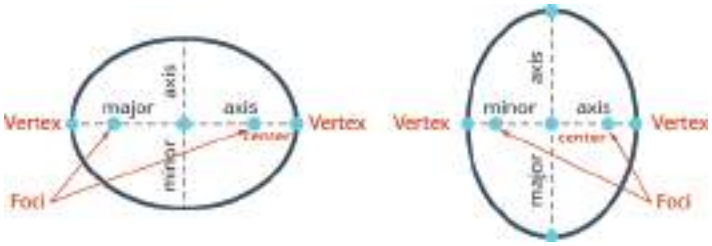


ellipse

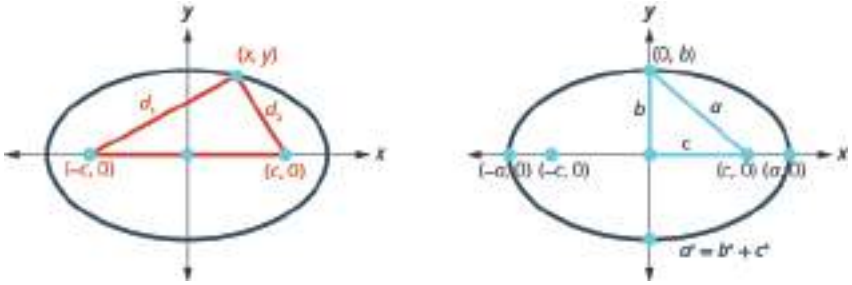
We can draw an ellipse by taking some fixed length of flexible string and attaching the ends to two thumbtacks. We use a pen to pull the string taut and rotate it around the two thumbtacks. The figure that results is an ellipse.



A line drawn through the foci intersect the ellipse in two points. Each point is called a **vertex** of the ellipse. The segment connecting the vertices is called the **major axis**. The midpoint of the segment is called the **center** of the ellipse. A segment perpendicular to the major axis that passes through the center and intersects the ellipse in two points is called the **minor axis**.



We mentioned earlier that our goal is to connect the geometry of a conic with algebra. Placing the ellipse on a rectangular coordinate system gives us that opportunity. In the figure, we placed the ellipse so the foci $((-c, 0), (c, 0))$ are on the x-axis and the center is the origin.



The definition states the sum of the distance from the foci to a point (x, y) is constant. So $d_1 + d_2$ is a constant that we will call $2a$ so, $d_1 + d_2 = 2a$. We will use the distance formula to lead us to an algebraic formula for an ellipse.

$$d_1 + d_2 = 2a$$

Use the distance formula to fin d_1, d_2.

$$\sqrt{(x-(-c))^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} = 2a$$

After eliminating radicals and simplifying, we get:

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} = 1$$

To simplify the equation of the ellipse, we let $a^2 - c^2 = b^2$.

So, the equation of an ellipse centered at the origin in standard form is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

To graph the ellipse, it will be helpful to know the intercepts. We will find the x-intercepts and y-intercepts using the formula.

y-intercepts

Let $x = 0$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$
$$\frac{0^2}{a^2}+\frac{y^2}{b^2} = 1$$
$$\frac{y^2}{b^2} = 1$$
$$y^2 = b^2$$
$$y = \pm b$$

The y-intercepts are $(0, b)$ and $(0, -b)$.

x-intercepts

Let $y = 0$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$
$$\frac{x^2}{a^2}+\frac{0^2}{b^2} = 1$$
$$\frac{x^2}{a^2} = 1$$
$$x^2 = a^2$$
$$x = \pm a$$

The x-intercepts are $(a, 0)$ and $(-a, 0)$.

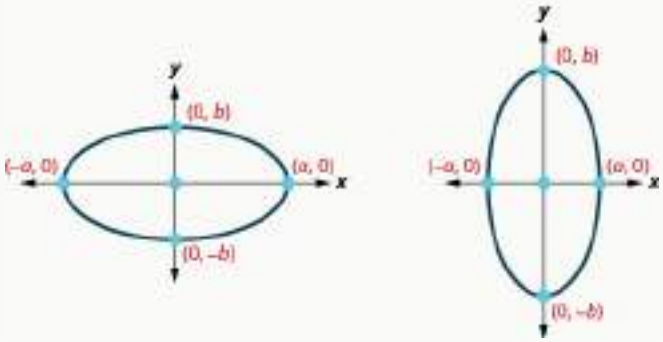
Standard Form of the Equation an Ellipse with Center $(0, 0)$

The standard form of the equation of an ellipse with center $(0, 0)$, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

The *x*-intercepts are $(a, 0)$ and $(-a, 0)$.

The *y*-intercepts are $(0, b)$ and $(0, -b)$.



Notice that when the major axis is horizontal, the value of *a* will be greater than the value of *b* and when the major axis is vertical, the value of *b* will be greater than the value of *a*. We will use this information to graph an ellipse that is centered at the origin.

Ellipse with Center $(0, 0)$		
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$a > b$	$b > a$
Major axis	on the *x*- axis.	on the *y*-axis.
***x*-intercepts**	$(-a, 0)$, $(a, 0)$	
***y*-intercepts**	$(0, -b)$, $(0, b)$	

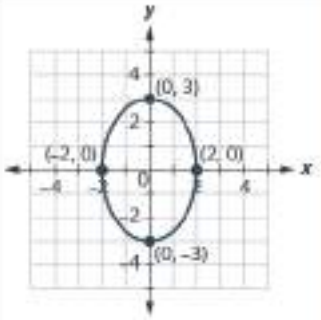
Table 11.18

EXAMPLE 11.20 HOW TO GRAPH AN ELLIPSE WITH CENTER (0, 0)

Graph: $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Solution

Step 1. Write the equation in standard form.	It is in standard form.	$\frac{x^2}{4}+\frac{y^2}{9}=1$
Step 2. Determine whether the major axis is horizontal or vertical.	Since $9 > 4$ and 9 is in the y^2 term, the major axis is vertical.	Major axis is vertical.
Step 3. Find the endpoints of the major axis.	The endpoints will be the *y*-intercepts. Since $b^2 = 9$, then $b = \pm 3$. The endpoints of the major axis are (0, 3), (0, −3).	The endpoints of the major axis are (0, 3), (0, −3).

Step 4. Find the endpoints of the minor axis.	The endpoints will be the x-intercepts. Since $a^2 = 4$, then $a = \pm 2$. The endpoints of the minor axis are $(2, 0), (-2, 0)$.	The endpoints of the major axis are $(2, 0), (-2, 0)$.
Step 5. Sketch the ellipse.		

TRY IT : : 11.39 Graph: $\frac{x^2}{4} + \frac{y^2}{16} = 1.$

TRY IT : : 11.40 Graph: $\frac{x^2}{9} + \frac{y^2}{16} = 1.$

We summarize the steps for reference.

HOW TO : : HOW TO GRAPH AN ELLIPSE WITH CENTER $(0, 0)$.

Step 1. Write the equation in standard form.
Step 2. Determine whether the major axis is horizontal or vertical.
Step 3. Find the endpoints of the major axis.
Step 4. Find the endpoints of the minor axis
Step 5. Sketch the ellipse.

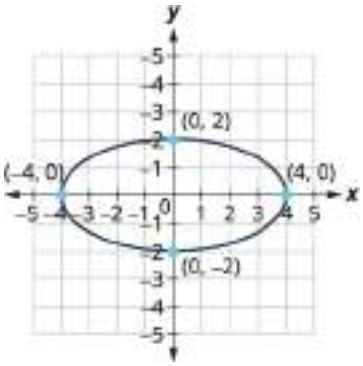
Sometimes our equation will first need to be put in standard form.

EXAMPLE 11.21

Graph $x^2 + 4y^2 = 16$.

Solution

We recognize this as the equation of an ellipse since both the x and y terms are squared and have different coefficients.	$x^2 + 4y^2 = 16$
To get the equation in standard form, divide both sides by 16 so that the equation is equal to 1.	$\frac{x^2}{16} + \frac{4y^2}{16} = \frac{16}{16}$

Simplify.	$\frac{x^2}{16}+\frac{y^2}{4}=1$
The equation is in standard form. The ellipse is centered at the origin.	The center is $(0, 0)$.
Since $16 > 4$ and 16 is in the x^2 term, the major axis is horizontal.	
$a^2 = 16,\ a = \pm 4$ $b^2 = 4,\quad b = \pm 2$	The vertices are $(4, 0), (-4, 0)$. The endpoints of the minor axis are $(0, 2), (0, -2)$.
Sketch the parabola.	

TRY IT : : 11.41 Graph $9x^2 + 16y^2 = 144$.

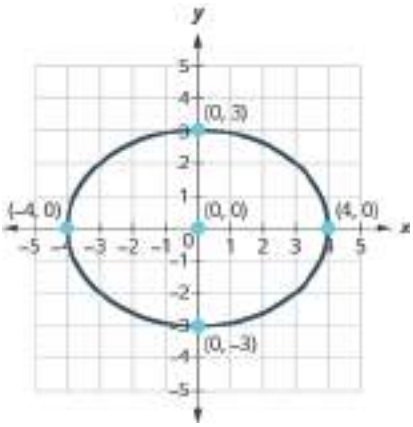
TRY IT : : 11.42 Graph $16x^2 + 25y^2 = 400$.

Find the Equation of an Ellipse with Center at the Origin

If we are given the graph of an ellipse, we can find the equation of the ellipse.

EXAMPLE 11.22

Find the equation of the ellipse shown.



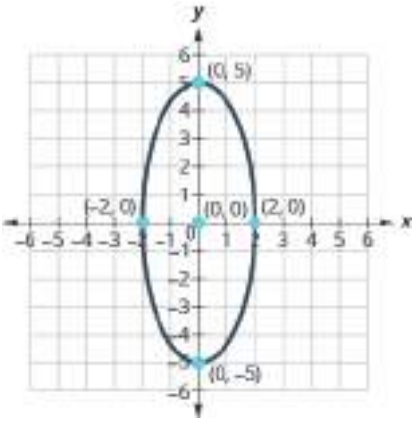
Solution

We recognize this as an ellipse that is centered at the origin. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

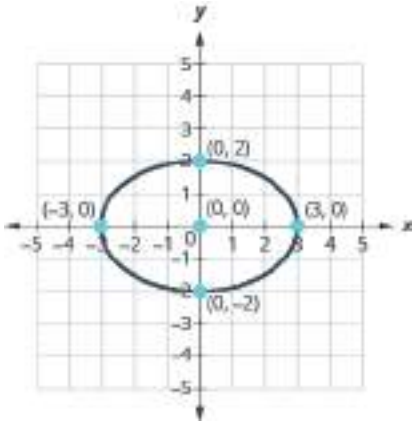
Since the major axis is horizontal and the distance from the center to the vertex is 4, we know $a=4$ and so $a^2=16$. $\frac{x^2}{16}+\frac{y^2}{b^2}=1$

The minor axis is vertical and the distance from the center to the ellipse is 3, we know $b=3$ and so $b^2=9$. $\frac{x^2}{16}+\frac{y^2}{9}=1$

TRY IT : : 11.43 Find the equation of the ellipse shown.



TRY IT : : 11.44 Find the equation of the ellipse shown.



Graph an Ellipse with Center Not at the Origin

The ellipses we have looked at so far have all been centered at the origin. We will now look at ellipses whose center is (h, k).

The equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ and when $a>b$, the major axis is horizontal so the distance from the center to the vertex is *a*. When $b>a$, the major axis is vertical so the distance from the center to the vertex is *b*.

Standard Form of the Equation an Ellipse with Center (h, k)

The standard form of the equation of an ellipse with center (h, k), is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

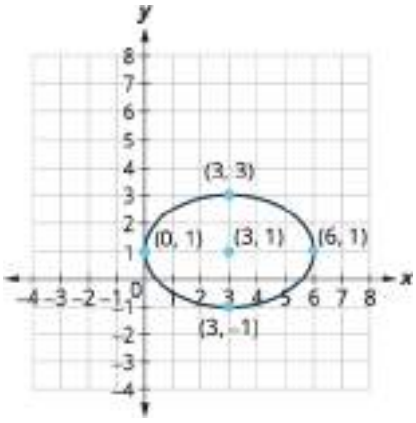
When $a>b$, the major axis is horizontal so the distance from the center to the vertex is *a*.

When $b>a$, the major axis is vertical so the distance from the center to the vertex is *b*.

EXAMPLE 11.23

Graph: $\frac{(x-3)^2}{9}+\frac{(y-1)^2}{4}=1.$

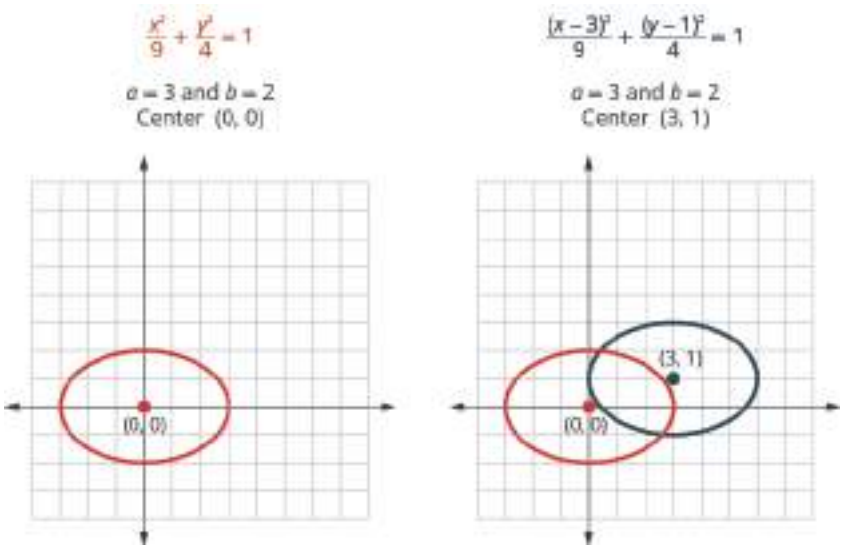
Solution

The equation is in standard form, $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$	$\frac{(x-3)^2}{9}+\frac{(y-1)^2}{4}=1$
The ellipse is centered at $(h,\ k)$.	The center is $(3,\ 1)$.
Since $9>4$ and 9 is in the x^2 term, the major axis is horizontal.	
$a^2=9,\ a=\pm 3$ $b^2=4,\ b=\pm 2$	The distance from the center to the vertices is 3. The distance from the center to the endpoints of the minor axis is 2.
Sketch the ellipse.	

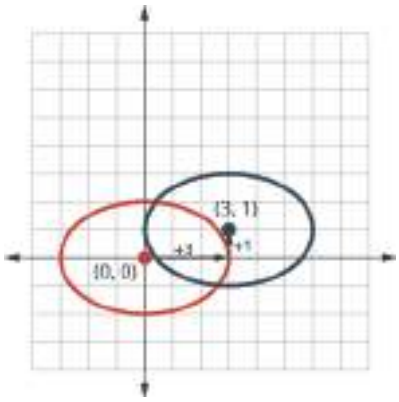
TRY IT : : 11.45 Graph: $\frac{(x+3)^2}{4}+\frac{(y-5)^2}{16}=1.$

TRY IT : : 11.46 Graph: $\frac{(x-1)^2}{25}+\frac{(y+3)^2}{16}=1.$

If we look at the equations of $\frac{x^2}{9}+\frac{y^2}{4}=1$ and $\frac{(x-3)^2}{9}+\frac{(y-1)^2}{4}=1,$ we see that they are both ellipses with $a=3$ and $b=2.$ So they will have the same size and shape. They are different in that they do not have the same center.



Notice in the graph above that we could have graphed $\frac{(x-3)^2}{9}+\frac{(y-1)^2}{4}=1$ by translations. We moved the original ellipse to the right 3 units and then up 1 unit.



In the next example we will use the translation method to graph the ellipse.

EXAMPLE 11.24

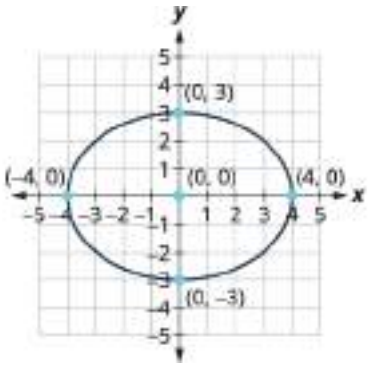
Graph $\frac{(x+4)^2}{16}+\frac{(y-6)^2}{9}=1$ by translation.

Solution

This ellipse will have the same size and shape as $\frac{x^2}{16}+\frac{y^2}{9}=1$ whose center is $(0, 0)$. We graph this ellipse first.

The center is $(0, 0)$.	Center $(0, 0)$
Since $16 > 9$, the major axis is horizontal.	
$a^2 = 16,\ a = \pm 4$ $b^2 = 9,\ \ b = \pm 3$	The vertices are $(4, 0), (-4, 0)$. The endpoints of the minor axis are $(0, 3), (0, -3)$.

Sketch the ellipse.



The original equation is in standard form, $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

$\frac{(x-(-4))^2}{16}+\frac{(y-6)^2}{9}=1$

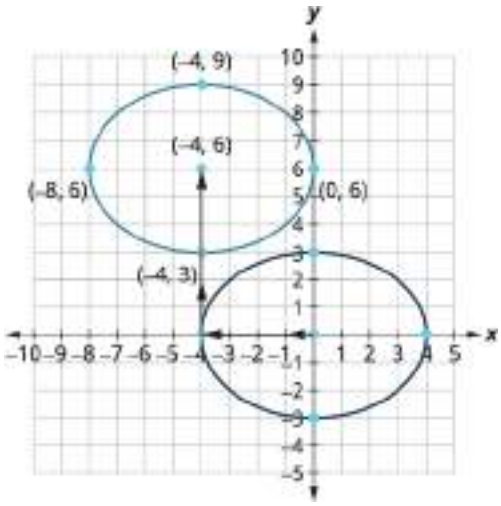
The ellipse is centered at (h, k).

The center is $(-4, 6)$.

We translate the graph of $\frac{x^2}{16}+\frac{y^2}{9}=1$ four units to the left and then up 6 units.
Verify that the center is $(-4, 6)$.

The new ellipse is the ellipse whose equation is

$\frac{(x+4)^2}{16}+\frac{(y-6)^2}{9}=1$.



TRY IT : : 11.47 Graph $\frac{(x-5)^2}{9}+\frac{(y+4)^2}{4}=1$ by translation.

TRY IT : : 11.48 Graph $\frac{(x+6)^2}{16}+\frac{(y+2)^2}{25}=1$ by translation.

When an equation has both an x^2 and a y^2 with different coefficients, we verify that it is an ellipsis by putting it in standard form. We will then be able to graph the equation.

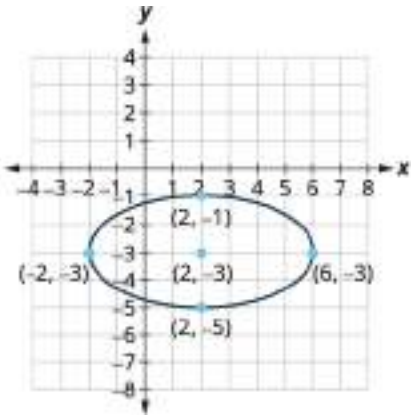
EXAMPLE 11.25

Write the equation $x^2+4y^2-4x+24y+24=0$ in standard form and graph.

Solution

We put the equation in standard form by completing the squares in both *x* and *y*.

$$x^2 + 4y^2 - 4x + 24y + 24 = 0$$

Rewrite grouping the x terms and y terms.	$(x^2 - 4x + _) + (4y^2 + 24y + _) = -24$
Make the coefficients of x^2 and y^2 equal 1.	$(x^2 - 4x + _) + 4(y^2 + 6y + _) = -24$
Complete the squares.	$(x^2 - 4x + 4) + 4(y^2 + 6y + 9) = -24 + 4 + 36$
Write as binomial squares.	$(x-2)^2 + 4(y+3)^2 = 16$
Divide both sides by 16 to get 1 on the right.	$\frac{(x-2)^2}{16} + \frac{4(y+3)^2}{16} = \frac{16}{16}$
Simplify.	$\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$
The equation is in standard form, $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$
The ellipse is centered at (h, k).	The center is $(2, -3)$.
Since $16 > 4$ and 16 is in the x^2 term, the major axis is horizontal. $a^2 = 16,\ a = \pm 4$ $b^2 = 4,\ \ b = \pm 2$	The distance from the center to the vertices is 4. The distance from the center to the endpoints of the minor axis is 2.
Sketch the ellipse.	

TRY IT : : 11.49 ⓐ Write the equation $6x^2 + 4y^2 + 12x - 32y + 34 = 0$ in standard form and ⓑ graph.

TRY IT : : 11.50 ⓐ Write the equation $4x^2 + y^2 - 16x - 6y + 9 = 0$ in standard form and ⓑ graph.

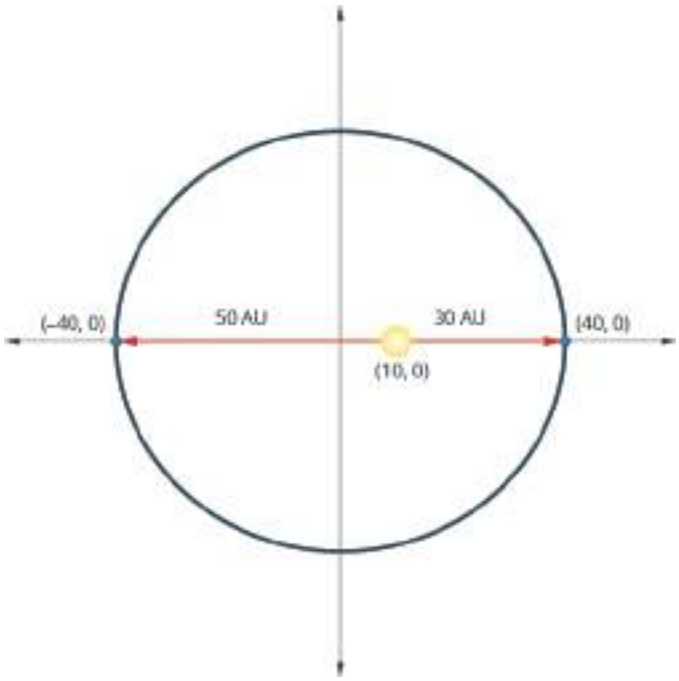
Solve Application with Ellipses

The orbits of the planets around the sun follow elliptical paths.

EXAMPLE 11.26

Pluto (a dwarf planet) moves in an elliptical orbit around the Sun. The closest Pluto gets to the Sun is approximately 30 astronomical units (AU) and the furthest is approximately 50 AU. The Sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write

an equation for the elliptical orbit of Pluto.



Solution

We recognize this as an ellipse that is centered at the origin.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since the major axis is horizontal and the distance from the center to the vertex is 40, we know $a = 40$ and so $a^2 = 1600$.

$$\frac{x^2}{1600}+\frac{y^2}{b^2}=1$$

The minor axis is vertical but the end points aren't given. To fin b we will use the location of the Sun. Since the Sun is a focus of the ellipse at the point $(10, 0)$, we know $c = 10$. Use this to solve for b^2.

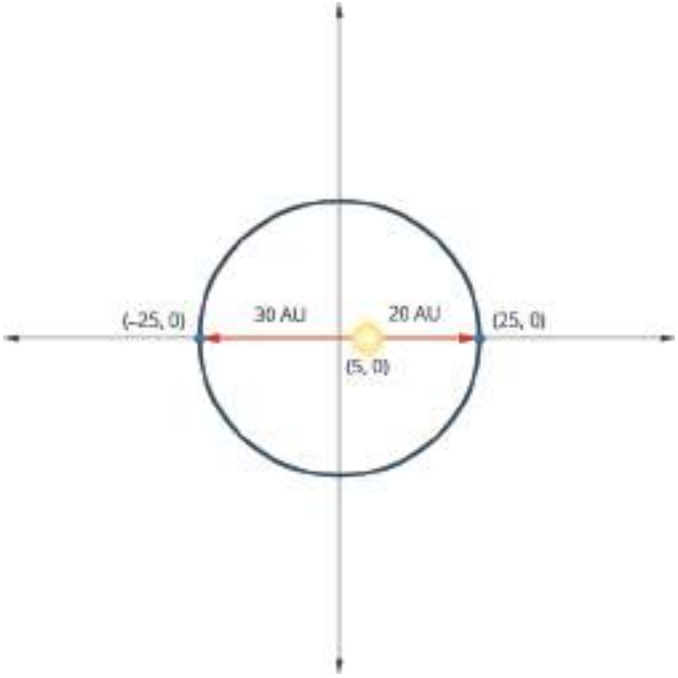
$$b^2 = a^2 - c^2$$
$$b^2 = 40^2 - 10^2$$
$$b^2 = 1600 - 100$$
$$b^2 = 1500$$

Substitute a^2 and b^2 into the standard form of the ellipse.

$$\frac{x^2}{1600}+\frac{y^2}{1500}=1$$

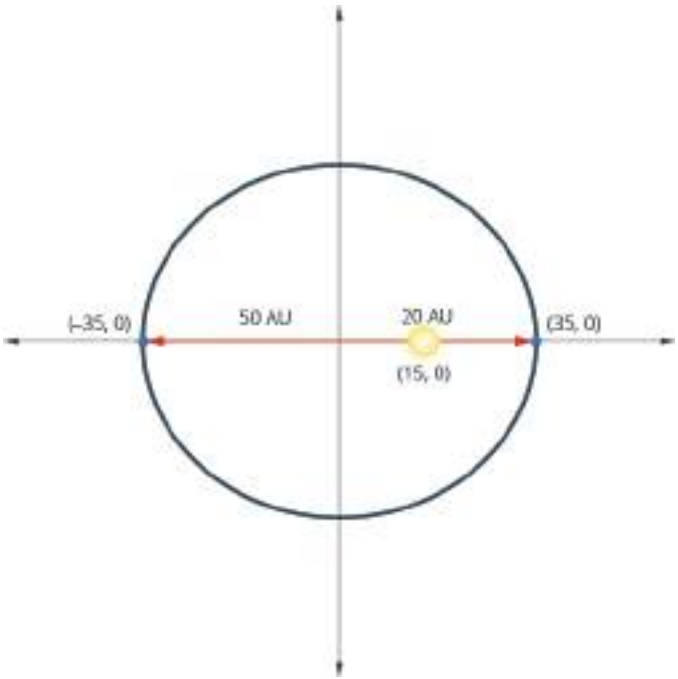
TRY IT : : 11.51

A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 20 AU and the furthest is approximately 30 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.



TRY IT : : 11.52

A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 20 AU and the furthest is approximately 50 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.



MEDIA : :

Access these online resources for additional instructions and practice with ellipses.

- **Conic Sections: Graphing Ellipses Part 1 (https://openstax.org/l/37graphellipse1)**
- **Conic Sections: Graphing Ellipses Part 2 (https://openstax.org/l/37graphellipse2)**
- **Equation for Ellipse From Graph (https://openstax.org/l/37eqellipse)**

11.3 EXERCISES

Practice Makes Perfect

Graph an Ellipse with Center at the Origin

In the following exercises, graph each ellipse.

99. $\frac{x^2}{4}+\frac{y^2}{25}=1$

100. $\frac{x^2}{9}+\frac{y^2}{25}=1$

101. $\frac{x^2}{25}+\frac{y^2}{36}=1$

102. $\frac{x^2}{16}+\frac{y^2}{36}=1$

103. $\frac{x^2}{36}+\frac{y^2}{16}=1$

104. $\frac{x^2}{25}+\frac{y^2}{9}=1$

105. $x^2+\frac{y^2}{4}=1$

106. $\frac{x^2}{9}+y^2=1$

107. $4x^2+25y^2=100$

108. $16x^2+9y^2=144$

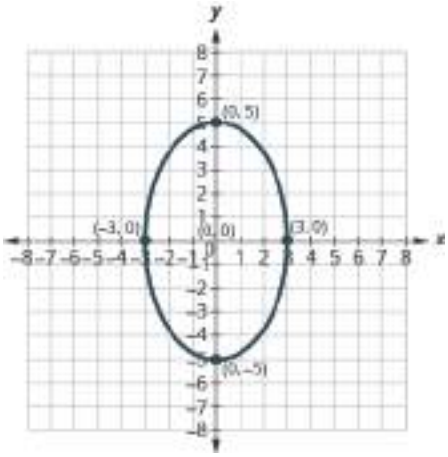
109. $16x^2+36y^2=576$

110. $9x^2+25y^2=225$

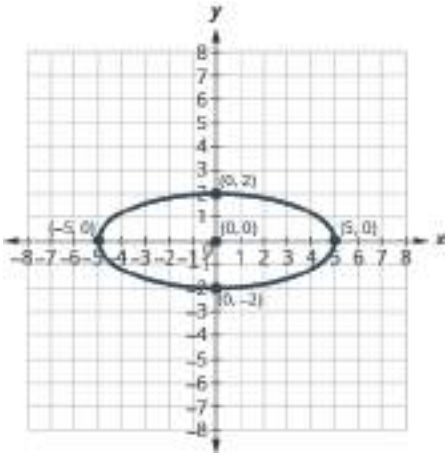
Find the Equation of an Ellipse with Center at the Origin

In the following exercises, find the equation of the ellipse shown in the graph.

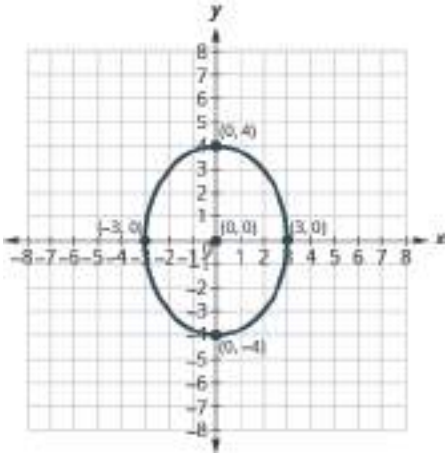
111.



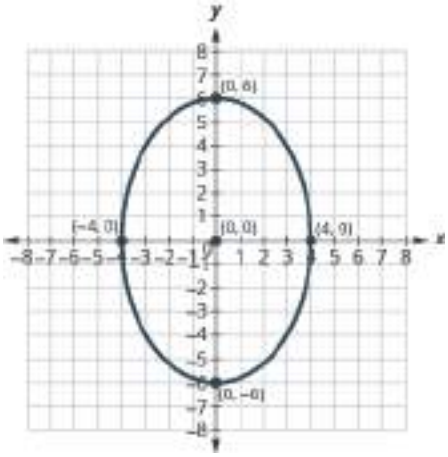
112.



113.



114.



Graph an Ellipse with Center Not at the Origin

In the following exercises, graph each ellipse.

115. $\frac{(x+1)^2}{4}+\frac{(y+6)^2}{25}=1$

116. $\frac{(x-3)^2}{25}+\frac{(y+2)^2}{9}=1$

117. $\frac{(x+4)^2}{4}+\frac{(y-2)^2}{9}=1$

118. $\frac{(x-4)^2}{9}+\frac{(y-1)^2}{16}=1$

In the following exercises, graph each equation by translation.

119. $\frac{(x-3)^2}{4}+\frac{(y-7)^2}{25}=1$

120. $\frac{(x+6)^2}{16}+\frac{(y+5)^2}{4}=1$

121. $\frac{(x-5)^2}{9}+\frac{(y+4)^2}{25}=1$

122. $\frac{(x+5)^2}{36}+\frac{(y-3)^2}{16}=1$

In the following exercises, ⓐ write the equation in standard form and ⓑ graph.

123. $25x^2+9y^2-100x-54y-44=0$

124. $4x^2+25y^2+8x+100y+4=0$

125. $4x^2+25y^2-24x-64=0$

126. $9x^2+4y^2+56y+160=0$

In the following exercises, graph the equation.

127. $x=-2(y-1)^2+2$

128. $x^2+y^2=49$

129. $(x+5)^2+(y+2)^2=4$

130. $y=-x^2+8x-15$

131. $\frac{(x+3)^2}{16}+\frac{(y+1)^2}{4}=1$

132. $(x-2)^2+(y-3)^2=9$

133. $\frac{x^2}{25}+\frac{y^2}{36}=1$

134. $x=4(y+1)^2-4$

135. $x^2+y^2=64$

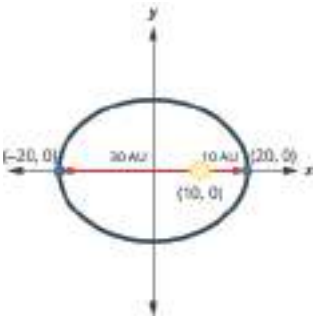
136. $\frac{x^2}{9}+\frac{y^2}{25}=1$

137. $y=6x^2+2x-1$

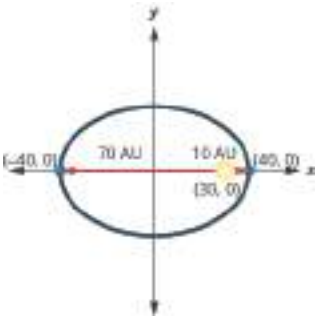
138. $\frac{(x-2)^2}{9}+\frac{(y+3)^2}{25}=1$

Solve Application with Ellipses

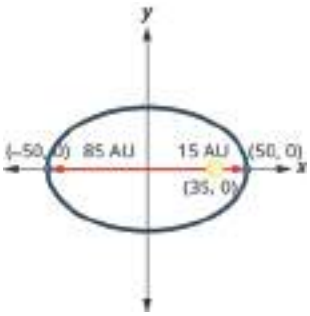
139. A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 10 AU and the furthest is approximately 30 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.



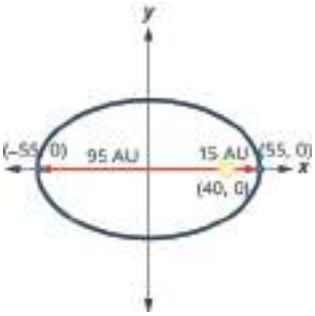
140. A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 10 AU and the furthest is approximately 70 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.



141. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 15 AU and the furthest is approximately 85 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.



142. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 15 AU and the furthest is approximately 95 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.



Writing Exercises

143. In your own words, define an ellipse and write the equation of an ellipse centered at the origin in standard form. Draw a sketch of the ellipse labeling the center, vertices and major and minor axes.

144. Explain in your own words how to get the axes from the equation in standard form.

145. Compare and contrast the graphs of the equations $\frac{x^2}{4}+\frac{y^2}{9}=1$ and $\frac{x^2}{9}+\frac{y^2}{4}=1$.

146. Explain in your own words, the difference between a vertex and a focus of the ellipse.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
graph an ellipse with center at the origin.			
find the equation of an ellipse with center at the origin.			
graph an ellipse with center not at the origin.			
solve applications with ellipses.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

11.4 Hyperbolas

Learning Objectives

By the end of this section, you will be able to:

- Graph a hyperbola with center at $(0, 0)$
- Graph a hyperbola with center at (h, k)
- Identify conic sections by their equations

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 12$.
 If you missed this problem, review **Example 9.1**.
2. Expand: $(x - 4)^2$.
 If you missed this problem, review **Example 5.32**.
3. Graph $y = -\frac{2}{3}x$.
 If you missed this problem, review **Example 3.4**.

Graph a Hyperbola with Center at (0, 0)

The last conic section we will look at is called a **hyperbola**. We will see that the equation of a hyperbola looks the same as the equation of an ellipse, except it is a difference rather than a sum. While the equations of an ellipse and a hyperbola are very similar, their graphs are very different.

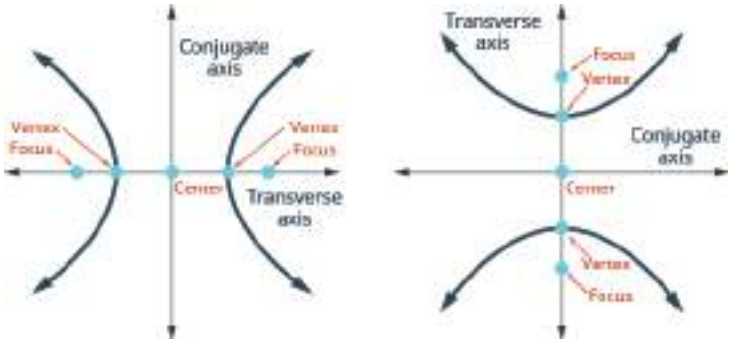
We define a **hyperbola** as all points in a plane where the difference of their distances from two fixed points is constant. Each of the fixed points is called a **focus** of the hyperbola.

Hyperbola

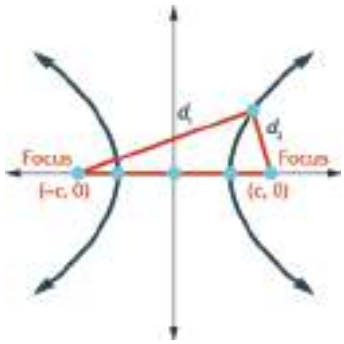
A **hyperbola** is all points in a plane where the difference of their distances from two fixed points is constant. Each of the fixed points is called a **focus** of the hyperbola.



The line through the foci, is called the **transverse axis**. The two points where the transverse axis intersects the hyperbola are each a **vertex** of the hyperbola. The midpoint of the segment joining the foci is called the **center** of the hyperbola. The line perpendicular to the transverse axis that passes through the center is called the **conjugate axis**. Each piece of the graph is called a **branch** of the hyperbola.



Again our goal is to connect the geometry of a conic with algebra. Placing the hyperbola on a rectangular coordinate system gives us that opportunity. In the figure, we placed the hyperbola so the foci $((-c, 0), (c, 0))$ are on the x-axis and the center is the origin.



The definition states the difference of the distance from the foci to a point (x, y) is constant. So $|d_1 - d_2|$ is a constant that we will call $2a$ so $|d_1 - d_2| = 2a$. We will use the distance formula to lead us to an algebraic formula for an ellipse.

	$\|d_1 - d_2\| = 2a$
Use the distance formula to fin d_1, d_2	$\left\|\sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2}\right\| = 2a$
Eliminate the radicals.	
To simplify the equation of the ellipse, we let $c^2 - a^2 = b^2$.	$\frac{x^2}{a^2} + \frac{y^2}{c^2-a^2} = 1$
So, the equation of a hyperbola centered at the origin in standard form is:	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

To graph the hyperbola, it will be helpful to know about the intercepts. We will find the x-intercepts and y-intercepts using the formula.

x-intercepts

Let $y = 0$.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{a^2} - \frac{0^2}{b^2} = 1$$
$$\frac{x^2}{a^2} = 1$$
$$x^2 = a^2$$
$$x = \pm a$$

The x-intercepts are $(a, 0)$ and $(-a, 0)$.

y-intercepts

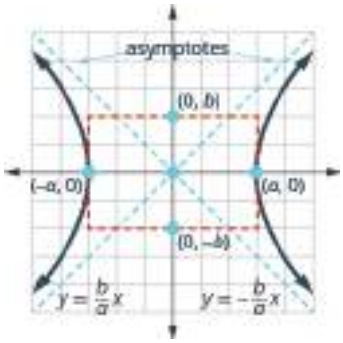
Let $x = 0$.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{0^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$-\frac{y^2}{b^2} = 1$$
$$y^2 = -b^2$$
$$y = \pm\sqrt{-b^2}$$

There are no y-intercepts.

The a, b values in the equation also help us find the asymptotes of the hyperbola. The asymptotes are intersecting straight lines that the branches of the graph approach but never intersect as the x, y values get larger and larger.

To find the asymptotes, we sketch a rectangle whose sides intersect the *x*-axis at the vertices $(-a, 0), \quad (a, 0)$ and intersect the *y*-axis at $(0, -b), \quad (0, b)$. The lines containing the diagonals of this rectangle are the asymptotes of the hyperbola. The rectangle and asymptotes are not part of the hyperbola, but they help us graph the hyperbola.



The asymptotes pass through the origin and we can evaluate their slope using the rectangle we sketched. They have equations $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

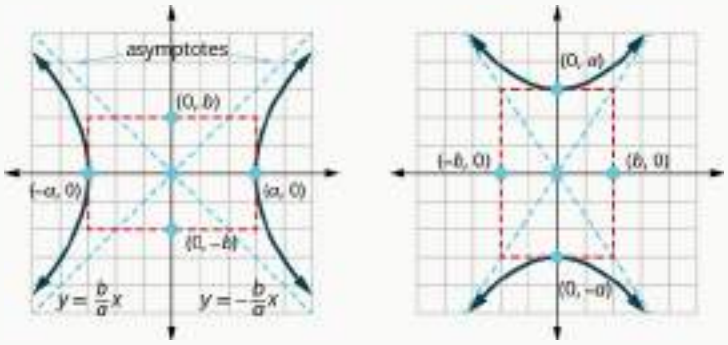
There are two equations for hyperbolas, depending whether the transverse axis is vertical or horizontal. We can tell whether the transverse axis is horizontal by looking at the equation. When the equation is in standard form, if the x^2-term is positive, the transverse axis is horizontal. When the equation is in standard form, if the y^2-term is positive, the transverse axis is vertical.

The second equations could be derived similarly to what we have done. We will summarize the results here.

Standard Form of the Equation a Hyperbola with Center $(0, 0)$

The standard form of the equation of a hyperbola with center $(0, 0)$, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$



Notice that, unlike the equation of an ellipse, the denominator of x^2 is not always a^2 and the denominator of y^2 is not always b^2.

Notice that when the x^2-term is positive, the transverse axis is on the *x*-axis. When the y^2-term is positive, the transverse axis is on the *y*-axis.

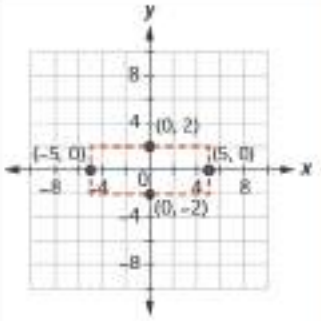
Standard Forms of the Equation a Hyperbola with Center $(0, 0)$		
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Orientation	Transverse axis on the *x*-axis. Opens left and right	Transverse axis on the *y*-axis. Opens up and down
Vertices	$(-a, 0), \quad (a, 0)$	$(0, -a), \quad (0, a)$
***x*-intercepts**	$(-a, 0), \quad (a, 0)$	none
***y*-intercepts**	none	$(0, -a), \quad (0, a)$
Rectangle	Use $(\pm a, 0) \quad (0, \pm b)$	Use $(0, \pm a) \quad (\pm b, 0)$
asymptotes	$y=\frac{b}{a}x, \quad y=-\frac{b}{a}x$	$y=\frac{a}{b}x, \quad y=-\frac{a}{b}x$

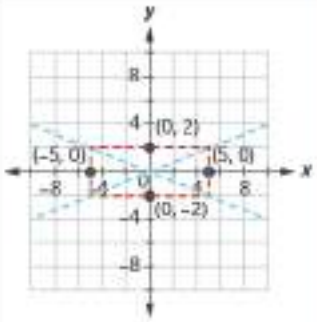
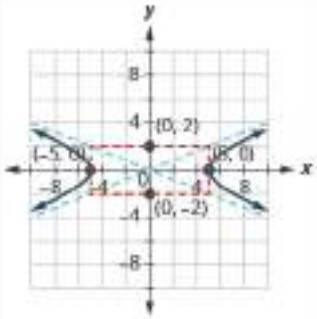
We will use these properties to graph hyperbolas.

EXAMPLE 11.27 HOW TO GRAPH A HYPERBOLA WITH CENTER $(0, 0)$

Graph $\frac{x^2}{25}-\frac{y^2}{4}=1$.

Solution

Step 1. Write the equation in standard form.	The equation is in standard form.	$\frac{x^2}{25}-\frac{y^2}{4}=1$
Step 2. Determine whether the transverse axis is horizontal or vertical.	Since the x^2-term is positive, the transverse axis is horizontal.	The transverse axis is horizontal.
Step 3. Find the vertices.	Since $a^2 = 25$ then $a = \pm 5$. The vertices are on the x-axis.	(–5, 0), (5, 0)
Step 4. Sketch the rectangle centered at the origin intersecting one axis at $\pm a$ and the other at $\pm b$.	Since $a = \pm 5$, the rectangle will intersect the x-axis at the vertices. Since $b = \pm 2$, the rectangle will intersect the y-axis at (0, –2) and (0, 2).	

Step 5. Sketch the asymptotes—the lines through the diagonals of the rectangle.	The asymptotes have the equations $y = \frac{5}{2}x, y = -\frac{5}{2}x$	
Step 6. Draw the two branches of the hyperbola.	Start at each vertex and use the asymptotes as a guide.	

TRY IT : : 11.53 Graph $\frac{x^2}{16} - \frac{y^2}{4} = 1.$

TRY IT : : 11.54 Graph $\frac{x^2}{9} - \frac{y^2}{16} = 1.$

We summarize the steps for reference.

HOW TO : : GRAPH A HYPERBOLA CENTERED AT $(0, 0)$.

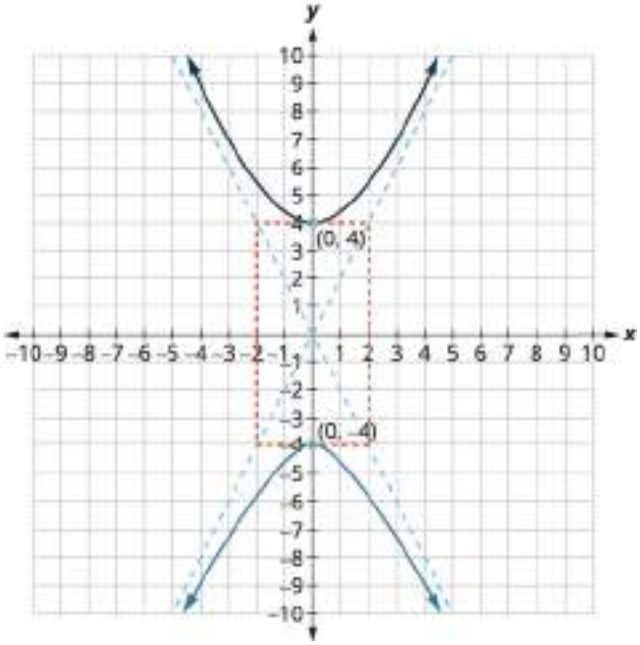
Step 1. Write the equation in standard form.
Step 2. Determine whether the transverse axis is horizontal or vertical.
Step 3. Find the vertices.
Step 4. Sketch the rectangle centered at the origin intersecting one axis at $\pm a$ and the other at $\pm b$.
Step 5. Sketch the asymptotes—the lines through the diagonals of the rectangle.
Step 6. Draw the two branches of the hyperbola.

Sometimes the equation for a hyperbola needs to be first placed in standard form before we graph it.

EXAMPLE 11.28

Graph $4y^2 - 16x^2 = 64.$

Solution

	$4y^2 - 16x^2 = 64$
To write the equation in standard form, divide each term by 64 to make the equation equal to 1.	$\frac{4y^2}{64} - \frac{16x^2}{64} = \frac{64}{64}$
Simplify.	$\frac{y^2}{16} - \frac{x^2}{4} = 1$
Since the y^2-term is positive, the transverse axis is vertical. Since $a^2 = 16$ then $a = \pm 4$.	
The vertices are on the y-axis, $(0, -a), \quad (0, a)$. Since $b^2 = 4$ then $b = \pm 2$.	$(0, -4), \quad (0, 4)$
Sketch the rectangle intersecting the x-axis at $(-2, 0), \quad (2, 0)$ and the y-axis at the vertices. Sketch the asymptotes through the diagonals of the rectangle. Draw the two branches of the hyperbola.	

TRY IT : : 11.55 Graph $4y^2 - 25x^2 = 100$.

TRY IT : : 11.56 Graph $25y^2 - 9x^2 = 225$.

Graph a Hyperbola with Center at (h, k)

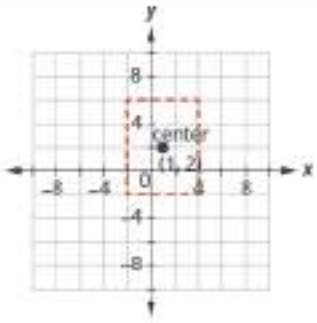
Hyperbolas are not always centered at the origin. When a hyperbola is centered at (h, k) the equations changes a bit as reflected in the table.

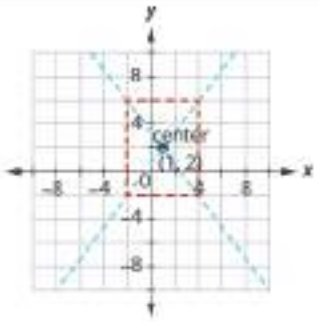
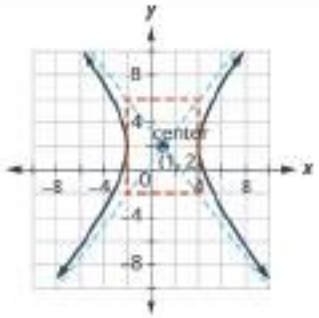
Standard Forms of the Equation a Hyperbola with Center (h, k)		
	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Orientation	Transverse axis is horizontal. Opens left and right	Transverse axis is vertical. Opens up and down
Center	(h, k)	(h, k)
Vertices	a units to the left and right of the center	a units above and below the center
Rectangle	Use a units left/right of center b units above/ below the center	Use a units above/below the center b units left/right of center

EXAMPLE 11.29 HOW TO GRAPH A HYPERBOLA WITH CENTER (h, k)

Graph $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$

Solution

Step 1. Write the equation in standard form.	The equation is in standard form.	$\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$
Step 2. Determine whether the transverse axis is horizontal or vertical.	Since the x^2-term is positive, the hyperbola opens left and right.	The transverse axis is horizontal. The hyperbola opens left and right.
Step 3. Find the center and a, b.	$h = 1$ and $k = 2$ $a^2 = 9$ $b^2 = 16$	$\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$ Center: $(1, 2)$ $a = 3$ $b = 4$
Step 4. Sketch the rectangle centered at (h, k) using a, b.	Mark the center, $(1, 2)$. Sketch the rectangle that goes through the points 3 units to the left/ right of the center and 4 units above and below the center.	

Step 5. Sketch the asymptotes—the lines through the diagonals of the rectangle. Mark the vertices.	Sketch the diagonals. Mark the vertices, which are on the rectangle 3 units to the left and right of the center.	
Step 6. Draw the two branches of the hyperbola.	Start at each vertex and use the asymptotes as a guide.	

TRY IT : : 11.57 Graph $\frac{(x-3)^2}{25}-\frac{(y-1)^2}{9}=1$.

TRY IT : : 11.58 Graph $\frac{(x-2)^2}{4}-\frac{(y-2)^2}{9}=1$.

We summarize the steps for easy reference.

HOW TO : : GRAPH A HYPERBOLA CENTERED AT (h, k).

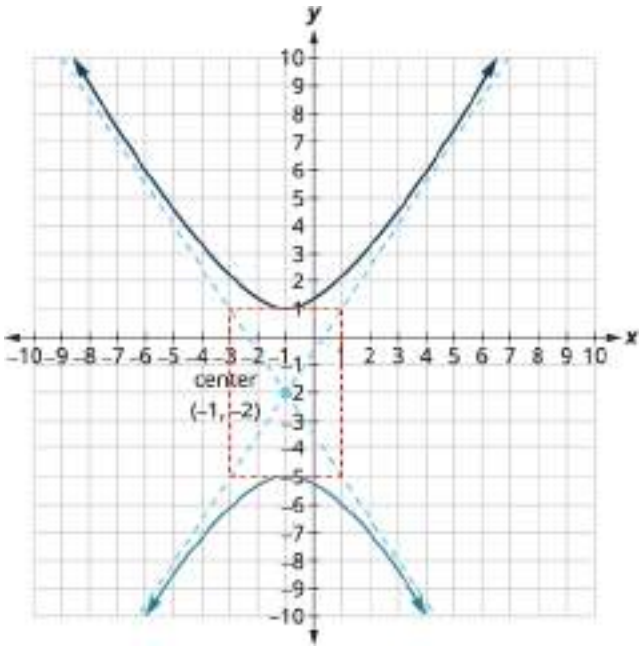
Step 1. Write the equation in standard form.
Step 2. Determine whether the transverse axis is horizontal or vertical.
Step 3. Find the center and *a, b*.
Step 4. Sketch the rectangle centered at (h, k) using *a, b*.
Step 5. Sketch the asymptotes—the lines through the diagonals of the rectangle. Mark the vertices.
Step 6. Draw the two branches of the hyperbola.

Be careful as you identify the center. The standard equation has $x-h$ and $y-k$ with the center as (h, k).

EXAMPLE 11.30

Graph $\frac{(y+2)^2}{9}-\frac{(x+1)^2}{4}=1$.

Solution

	$\frac{(y+2)^2}{9}-\frac{(x+1)^2}{4}=1$
Since the y^2 - term is positive, the hyperbola opens up and down.	$\frac{(y-k)^2}{9}-\frac{(x-h)^2}{4}=1$
Find the center, (h, k).	Center: $(-1, -2)$
Find a, b.	$a=3 \quad b=2$
Sketch the rectangle that goes through the points 3 units above and below the center and 2 units to the left/right of the center. Sketch the asymptotes—the lines through the diagonals of the rectangle. Mark the vertices. Graph the branches.	

TRY IT : : 11.59 Graph $\frac{(y+3)^2}{16}-\frac{(x+2)^2}{9}=1.$

TRY IT : : 11.60 Graph $\frac{(y+2)^2}{9}-\frac{(x+2)^2}{9}=1.$

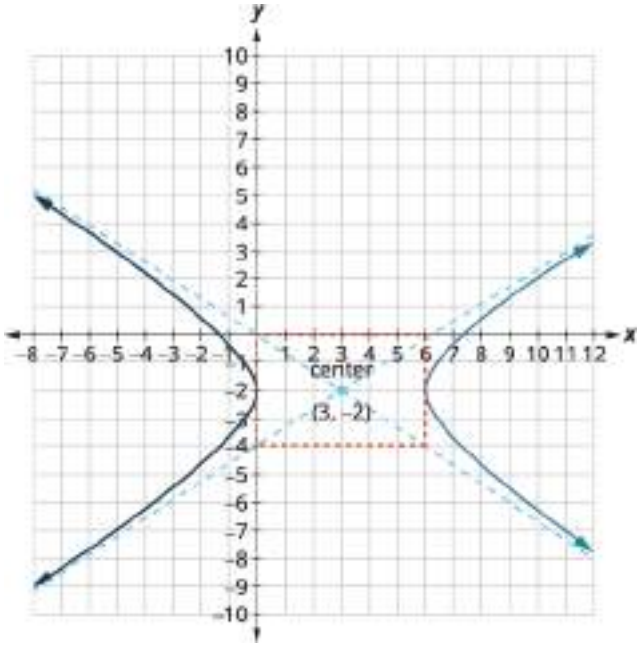
Again, sometimes we have to put the equation in standard form as our first step.

EXAMPLE 11.31

Write the equation in standard form and graph $4x^2-9y^2-24x-36y-36=0.$

Solution

	$4x^2-9y^2-24x-36y-36=0$
To get to standard form, complete the squares.	$4(x^2-6x)-9(y^2+4y)=36$
	$4(x^2-6x+9)-9(y^2+4y+4)=36+36-36$

	$4(x-3)^2-9(y+2)^2=36$
Divide each term by 36 to get the constant to be 1.	$\frac{4(x-3)^2}{36}-\frac{9(y+2)^2}{36}=\frac{36}{36}$
	$\frac{(x-3)^2}{9}-\frac{(y+2)^2}{4}=1$
Since the x^2- term is positive, the hyperbola opens left and right.	
Find the center, (h, k).	Center: $(3, -2)$
Find a, b.	$a=3$ $b=4$
Sketch the rectangle that goes through the points 3 units to the left/right of the center and 2 units above and below the center. Sketch the asymptotes—the lines through the diagonals of the rectangle. Mark the vertices. Graph the branches.	

TRY IT : : 11.61 ⓐ Write the equation in standard form and ⓑ graph $9x^2-16y^2+18x+64y-199=0$.

TRY IT : : 11.62 ⓐ Write the equation in standard form and ⓑ graph $16x^2-25y^2+96x-50y-281=0$.

Identify Conic Sections by their Equations

Now that we have completed our study of the conic sections, we will take a look at the different equations and recognize some ways to identify a conic by its equation. When we are given an equation to graph, it is helpful to identify the conic so we know what next steps to take.

To identify a conic from its equation, it is easier if we put the variable terms on one side of the equation and the constants on the other.

Conic	Characteristics of x^2- and y^2- terms	Example
Parabola	Either x^2 OR y^2. Only one variable is squared.	$x = 3y^2 - 2y + 1$
Circle	x^2- and y^2- terms have the same coefficients	$x^2 + y^2 = 49$
Ellipse	x^2- and y^2- terms have the **same** sign, different coefficients	$4x^2 + 25y^2 = 100$
Hyperbola	x^2- and y^2- terms have **different** signs, different coefficients	$25y^2 - 4x^2 = 100$

EXAMPLE 11.32

Identify the graph of each equation as a circle, parabola, ellipse, or hyperbola.

ⓐ $9x^2 + 4y^2 + 56y + 160 = 0$

ⓑ $9x^2 - 16y^2 + 18x + 64y - 199 = 0$

ⓒ $x^2 + y^2 - 6x - 8y = 0$

ⓓ $y = -2x^2 - 4x - 5$

Solution

ⓐ

The x^2- and y^2-terms have the same sign and diffe ent coefficient

$$9x^2 + 4y^2 + 56y + 160 = 0$$

Ellipse

ⓑ

The x^2- and y^2-terms have diffe ent signs and diffe ent coefficient

$$9x^2 - 16y^2 + 18x + 64y - 199 = 0$$

Hyperbola

ⓒ

The x^2- and y^2-terms have the same coefficient

$$x^2 + y^2 - 6x - 8y = 0$$

Circle

ⓓ

Only one variable, x, is squared.

$$y = -2x^2 - 4x - 5$$

Parabola

TRY IT : : 11.63

Identify the graph of each equation as a circle, parabola, ellipse, or hyperbola.

ⓐ $x^2 + y^2 - 8x - 6y = 0$

ⓑ $4x^2 + 25y^2 = 100$

ⓒ $y = 6x^2 + 2x - 1$

ⓓ $16y^2 - 9x^2 = 144$

TRY IT : : 11.64 Identify the graph of each equation as a circle, parabola, ellipse, or hyperbola.

ⓐ $16x^2 + 9y^2 = 144$

ⓑ $y = 2x^2 + 4x + 6$

ⓒ $x^2 + y^2 + 2x + 6y + 9 = 0$

ⓓ $4x^2 - 16y^2 = 64$

MEDIA : :

Access these online resources for additional instructions and practice with hyperbolas.

- **Graph a Hyperbola with Center at the Origin (https://openstax.org/l/37ghyperborig)**
- **Graph a Hyperbola with Center not at the Origin (https://openstax.org/l/37ghyperbnorig)**
- **Graph a Hyperbola in General Form (https://openstax.org/l/37ghyperbgen)**
- **Identifying Conic Sections in General Form (https://openstax.org/l/37conicsgen)**

11.4 EXERCISES

Practice Makes Perfect

Graph a Hyperbola with Center at $(0, 0)$

In the following exercises, graph.

147. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

148. $\frac{x^2}{25} - \frac{y^2}{9} = 1$

149. $\frac{x^2}{16} - \frac{y^2}{25} = 1$

150. $\frac{x^2}{9} - \frac{y^2}{36} = 1$

151. $\frac{y^2}{25} - \frac{x^2}{4} = 1$

152. $\frac{y^2}{36} - \frac{x^2}{16} = 1$

153. $16y^2 - 9x^2 = 144$

154. $25y^2 - 9x^2 = 225$

155. $4y^2 - 9x^2 = 36$

156. $16y^2 - 25x^2 = 400$

157. $4x^2 - 16y^2 = 64$

158. $9x^2 - 4y^2 = 36$

Graph a Hyperbola with Center at (h, k)

In the following exercises, graph.

159. $\frac{(x-1)^2}{16} - \frac{(y-3)^2}{4} = 1$

160. $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{16} = 1$

161. $\frac{(y-4)^2}{9} - \frac{(x-2)^2}{25} = 1$

162. $\frac{(y-1)^2}{25} - \frac{(x-4)^2}{16} = 1$

163. $\frac{(y+4)^2}{25} - \frac{(x+1)^2}{36} = 1$

164. $\frac{(y+1)^2}{16} - \frac{(x+1)^2}{4} = 1$

165. $\frac{(y-4)^2}{16} - \frac{(x+1)^2}{25} = 1$

166. $\frac{(y+3)^2}{16} - \frac{(x-3)^2}{36} = 1$

167. $\frac{(x-3)^2}{25} - \frac{(y+2)^2}{9} = 1$

168. $\frac{(x+2)^2}{4} - \frac{(y-1)^2}{9} = 1$

In the following exercises, ⓐ write the equation in standard form and ⓑ graph.

169. $9x^2 - 4y^2 - 18x + 8y - 31 = 0$

170. $16x^2 - 4y^2 + 64x - 24y - 36 = 0$

171. $y^2 - x^2 - 4y + 2x - 6 = 0$

172. $4y^2 - 16x^2 - 24y + 96x - 172 = 0$

173.
$9y^2 - x^2 + 18y - 4x - 4 = 0$

Identify the Graph of each Equation as a Circle, Parabola, Ellipse, or Hyperbola

In the following exercises, identify the type of graph.

174.

ⓐ $x = -y^2 - 2y + 3$

ⓑ $9y^2 - x^2 + 18y - 4x - 4 = 0$

ⓒ $9x^2 + 25y^2 = 225$

ⓓ $x^2 + y^2 - 4x + 10y - 7 = 0$

175.

ⓐ $x = -2y^2 - 12y - 16$

ⓑ $x^2 + y^2 = 9$

ⓒ $16x^2 - 4y^2 + 64x - 24y - 36 = 0$

ⓓ $16x^2 + 36y^2 = 576$

Mixed Practice

In the following exercises, graph each equation.

176. $\frac{(y-3)^2}{9} - \frac{(x+2)^2}{16} = 1$

177. $x^2 + y^2 - 4x + 10y - 7 = 0$

178. $y = (x-1)^2 + 2$

179. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

180. $(x+2)^2 + (y-5)^2 = 4$

181. $y^2 - x^2 - 4y + 2x - 6 = 0$

182. $x = -y^2 - 2y + 3$

183. $16x^2 + 9y^2 = 144$

Writing Exercises

184. In your own words, define a hyperbola and write the equation of a hyperbola centered at the origin in standard form. Draw a sketch of the hyperbola labeling the center, vertices, and asymptotes.

185. Explain in your own words how to create and use the rectangle that helps graph a hyperbola.

186. Compare and contrast the graphs of the equations $\frac{x^2}{4} - \frac{y^2}{9} = 1$ and $\frac{y^2}{9} - \frac{x^2}{4} = 1$.

187. Explain in your own words, how to distinguish the equation of an ellipse with the equation of a hyperbola.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
graph a hyperbola with center at (0, 0).			
graph a hyperbola with center at (*h*, *k*).			
identify conic sections by their equations.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

11.5 Solve Systems of Nonlinear Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of nonlinear equations using graphing
- Solve a system of nonlinear equations using substitution
- Solve a system of nonlinear equations using elimination
- Use a system of nonlinear equations to solve applications

Be Prepared!

1. Solve the system by graphing: $\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$.
 If you missed this problem, review Example 4.2.
2. Solve the system by substitution: $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$.
 If you missed this problem, review Example 4.7.
3. Solve the system by elimination: $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$.
 If you missed this problem, review Example 4.9.

Solve a System of Nonlinear Equations Using Graphing

We learned how to solve systems of linear equations with two variables by graphing, substitution and elimination. We will be using these same methods as we look at nonlinear systems of equations with two equations and two variables. A **system of nonlinear equations** is a system where at least one of the equations is not linear.

For example each of the following systems is a **system of nonlinear equations**.

$$\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases} \qquad \begin{cases} 9x^2+y^2=9 \\ y=3x-3 \end{cases} \qquad \begin{cases} x+y=4 \\ y=x^2+2 \end{cases}$$

System of Nonlinear Equations

A **system of nonlinear equations** is a system where at least one of the equations is not linear.

Just as with systems of linear equations, a solution of a nonlinear system is an ordered pair that makes both equations true. In a nonlinear system, there may be more than one solution. We will see this as we solve a system of nonlinear equations by graphing.

When we solved systems of linear equations, the solution of the system was the point of intersection of the two lines. With systems of nonlinear equations, the graphs may be circles, parabolas or hyperbolas and there may be several points of intersection, and so several solutions. Once you identify the graphs, visualize the different ways the graphs could intersect and so how many solutions there might be.

To solve systems of nonlinear equations by graphing, we use basically the same steps as with systems of linear equations modified slightly for nonlinear equations. The steps are listed below for reference.



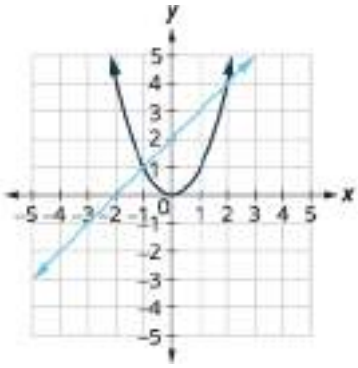
HOW TO : : SOLVE A SYSTEM OF NONLINEAR EQUATIONS BY GRAPHING.

Step 1. Identify the graph of each equation. Sketch the possible options for intersection.
Step 2. Graph the first equation.
Step 3. Graph the second equation on the same rectangular coordinate system.
Step 4. Determine whether the graphs intersect.
Step 5. Identify the points of intersection.
Step 6. Check that each ordered pair is a solution to both original equations.

EXAMPLE 11.33

Solve the system by graphing: $\begin{cases} x - y = -2 \\ y = x^2 \end{cases}$.

Solution

Identify each graph.	$\begin{cases} x - y = -2 & \text{line} \\ y = x^2 & \text{parabola} \end{cases}$
Sketch the possible options for intersection of a parabola and a line.	
Graph the line, $x - y = -2$. Slope-intercept form $y = x + 2$. Graph the parabola, $y = x^2$.	
Identify the points of intersection.	The points of intersection appear to be $(2, 4)$ and $(-1, 1)$.
Check to make sure each solution makes both equations true. $(2, 4)$ $x - y = -2$ $2 - 4 \stackrel{?}{=} -2$ $-2 = -2$ ✓ $y = x^2$ $4 \stackrel{?}{=} 2^2$ $4 = 4$ ✓ $(-1, 1)$ $x - y = -2$ $-1 - 1 \stackrel{?}{=} -2$ $-2 = -2$ ✓ $y = x^2$ $1 \stackrel{?}{=} (-1)^2$ $1 = 1$ ✓	
	The solutions are $(2, 4)$ and $(-1, 1)$.

TRY IT : : 11.65 Solve the system by graphing: $\begin{cases} x + y = 4 \\ y = x^2 + 2 \end{cases}$.

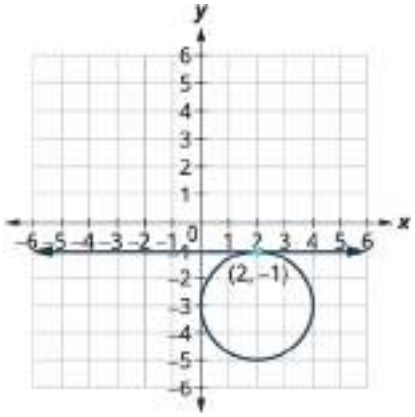
TRY IT : : 11.66 Solve the system by graphing: $\begin{cases} x - y = -1 \\ y = -x^2 + 3 \end{cases}$.

To identify the graph of each equation, keep in mind the characteristics of the x^2 and y^2 terms of each conic.

EXAMPLE 11.34

Solve the system by graphing: $\begin{cases} y = -1 \\ (x-2)^2 + (y+3)^2 = 4 \end{cases}$.

Solution

Identify each graph.	$\begin{cases} y = -1 & \text{line} \\ (x-2)^2 + (y+3)^2 = 4 & \text{circle} \end{cases}$
Sketch the possible options for the intersection of a circle and a line.	
Graph the circle, $(x-2)^2 + (y+3)^2 = 4$ Center: $(2, -3)$ radius: 2 Graph the line, $y = -1$. It is a horizontal line.	
Identify the points of intersection.	The point of intersection appears to be $(2, -1)$.
Check to make sure the solution makes both equations true. $(2, -1)$ $(x-2)^2 + (y+3)^2 = 4$ $(2-2)^2 + (-1+3)^2 \stackrel{?}{=} 4$ $(0)^2 + (2)^2 \stackrel{?}{=} 4$ $4 = 4$ ✓ $y = -1$ $-1 = -1$ ✓	
	The solution is $(2, -1)$.

TRY IT : : 11.67 Solve the system by graphing: $\begin{cases} x = -6 \\ (x+3)^2 + (y-1)^2 = 9 \end{cases}$.

TRY IT : : 11.68 Solve the system by graphing: $\begin{cases} y = 4 \\ (x-2)^2 + (y+3)^2 = 4 \end{cases}$.

Solve a System of Nonlinear Equations Using Substitution

The graphing method works well when the points of intersection are integers and so easy to read off the graph. But more

often it is difficult to read the coordinates of the points of intersection. The substitution method is an algebraic method that will work well in many situations. It works especially well when it is easy to solve one of the equations for one of the variables.

The substitution method is very similar to the substitution method that we used for systems of linear equations. The steps are listed below for reference.



HOW TO : : SOLVE A SYSTEM OF NONLINEAR EQUATIONS BY SUBSTITUTION.

Step 1. Identify the graph of each equation. Sketch the possible options for intersection.
Step 2. Solve one of the equations for either variable.
Step 3. Substitute the expression from Step 2 into the other equation.
Step 4. Solve the resulting equation.
Step 5. Substitute each solution in Step 4 into one of the original equations to find the other variable.
Step 6. Write each solution as an ordered pair.
Step 7. Check that each ordered pair is a solution to **both** original equations.

EXAMPLE 11.35

Solve the system by using substitution: $\begin{cases} 9x^2 + y^2 = 9 \\ y = 3x - 3 \end{cases}$.

Solution

Identify each graph.	$\begin{cases} 9x^2 + y^2 = 9 & \text{ellipse} \\ y = 3x - 3 & \text{line} \end{cases}$
Sketch the possible options for intersection of an ellipse and a line.	
The equation $y = 3x - 3$ is solved for y.	$y = 3x - 3$
	$9x^2 + y^2 = 9$
Substitute $3x - 3$ for y in the first equation.	$9x^2 + (3x - 3)^2 = 9$
Solve the equation for x.	$9x^2 + 9x^2 - 18x + 9 = 9$
	$18x^2 - 18x = 0$ $18x(x - 1) = 0$ $x = 0 \quad x = 1$
Substitute $x = 0$ and $x = 1$ into $y = 3x - 3$ to find y.	$y = 3x - 3 \qquad y = 3x - 3$
	$y = 3 \cdot 0 - 3 \qquad y = 3 \cdot 1 - 3$ $y = -3 \qquad y = 0$
	The ordered pairs are $(0, -3), \quad (1, 0)$.

Check **both** ordered pairs in **both** equations.

$(0, -3)$

$$\begin{aligned} 9x^2+y^2 &= 9 & y &= 3x-3 \\ 9\cdot 0^2+(-3)^2 &\stackrel{?}{=} 9 & -3 &\stackrel{?}{=} 3\cdot 0-3 \\ 0+9 &\stackrel{?}{=} 9 & -3 &\stackrel{?}{=} 0-3 \\ 9 &= 9\ \checkmark & -3 &= -3\ \checkmark \end{aligned}$$

$(1, 0)$

$$\begin{aligned} 9x^2+y^2 &= 9 & y &= 3x-3 \\ 9\cdot 1^2+0^2 &\stackrel{?}{=} 9 & 0 &\stackrel{?}{=} 3\cdot 1-3 \\ 9+0 &\stackrel{?}{=} 9 & 0 &\stackrel{?}{=} 3-3 \\ 9 &= 9\ \checkmark & 0 &= 0\ \checkmark \end{aligned}$$

The solutions are $(0, -3), (1, 0)$.

TRY IT : : 11.69 Solve the system by using substitution: $\begin{cases} x^2+9y^2=9 \\ y=\frac{1}{3}x-3 \end{cases}$.

TRY IT : : 11.70 Solve the system by using substitution: $\begin{cases} 4x^2+y^2=4 \\ y=x+2 \end{cases}$.

So far, each system of nonlinear equations has had at least one solution. The next example will show another option.

EXAMPLE 11.36

Solve the system by using substitution: $\begin{cases} x^2-y=0 \\ y=x-2 \end{cases}$.

Solution

Identify each graph.

$$\begin{cases} x^2-y=0 & \text{parabola} \\ y=x-2 & \text{line} \end{cases}$$

Sketch the possible options for intersection of a parabola and a line



The equation $y=x-2$ is solved for y.

$$y=x-2$$

$$x^2-y=0$$

Substitute $x-2$ for y in the first equation.

$$x^2-(x-2)=0$$

Solve the equation for x.

$$x^2-x+2=0$$

This doesn't factor easily, so we can check the discriminant.

$b^2 - 4ac$
$(-1)^2 - 4 \cdot 1 \cdot 2$
-7

The discriminant is negative, so there is no real solution. The system has no solution.

TRY IT : : 11.71 Solve the system by using substitution: $\begin{cases} x^2 - y = 0 \\ y = 2x - 3 \end{cases}$.

TRY IT : : 11.72 Solve the system by using substitution: $\begin{cases} y^2 - x = 0 \\ y = 3x - 2 \end{cases}$.

Solve a System of Nonlinear Equations Using Elimination

When we studied systems of linear equations, we used the method of elimination to solve the system. We can also use elimination to solve systems of nonlinear equations. It works well when the equations have both variables squared. When using elimination, we try to make the coefficients of one variable to be opposites, so when we add the equations together, that variable is eliminated.

The elimination method is very similar to the elimination method that we used for systems of linear equations. The steps are listed for reference.

HOW TO : : SOLVE A SYSTEM OF EQUATIONS BY ELIMINATION.

Step 1. Identify the graph of each equation. Sketch the possible options for intersection.
Step 2. Write both equations in standard form.
Step 3. Make the coefficients of one variable opposites.
Decide which variable you will eliminate.
Multiply one or both equations so that the coefficients of that variable are opposites.
Step 4. Add the equations resulting from Step 3 to eliminate one variable.
Step 5. Solve for the remaining variable.
Step 6. Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
Step 7. Write each solution as an ordered pair.
Step 8. Check that each ordered pair is a solution to **both** original equations.

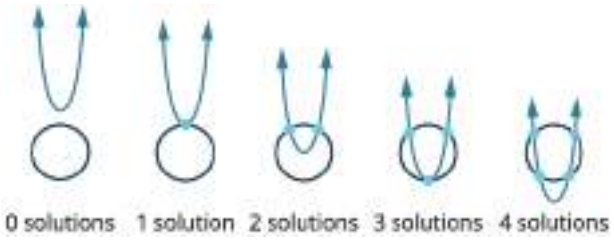
EXAMPLE 11.37

Solve the system by elimination: $\begin{cases} x^2 + y^2 = 4 \\ x^2 - y = 4 \end{cases}$.

Solution

Identify each graph.

$x^2 + y^2 = 4$ circle
$x^2 - y = 4$ parabola

Sketch the possible options for intersection of a circle and a parabola.	
Both equations are in standard form.	$\begin{aligned} x^2+y^2&=4\\ x^2-y&=4 \end{aligned}$
To get opposite coefficients of x^2, we will multiply the second equation by -1.	$\begin{aligned} x^2+y^2&=4\\ -1(x^2-y)&=-1(4) \end{aligned}$
Simplify.	$\begin{aligned} x^2+y^2&=4\\ -x^2+y&=4 \end{aligned}$
Add the two equations to eliminate x^2.	$\begin{aligned} x^2+y^2&=4\\ -x^2+y&=4\\ \hline y^2+y&=0 \end{aligned}$
Solve for y.	$y(y+1)=0$
	$y=0 \quad y+1=0$ $y=-1$
Substitute $y=0$ and $y=-1$ into one of the original equations. Then solve for x.	$y=0 \quad y=-1$
	$\begin{aligned} x^2-y&=4\\ x^2-0&=4\\ x^2&=4\\ x&=\pm 2 \end{aligned}$ $\qquad$ $\begin{aligned} x^2-y&=4\\ x^2-(-1)&=4\\ x^2&=3\\ x&=\pm\sqrt{3} \end{aligned}$
Write each solution as an ordered pair.	The ordered pairs are $(-2,\ 0)\ \ (2,\ 0).$ $(\sqrt{3},\ -1)(-\sqrt{3},\ -1)$
Check that each ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2,\ 0),\ \ (2,\ 0),\ \ (\sqrt{3},\ -1),$ and $(-\sqrt{3},\ -1).$

TRY IT : : 11.73 Solve the system by elimination: $\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases}$.

TRY IT : : 11.74 Solve the system by elimination: $\begin{cases} x^2+y^2=1 \\ -x+y^2=1 \end{cases}$.

There are also four options when we consider a circle and a hyperbola.

EXAMPLE 11.38

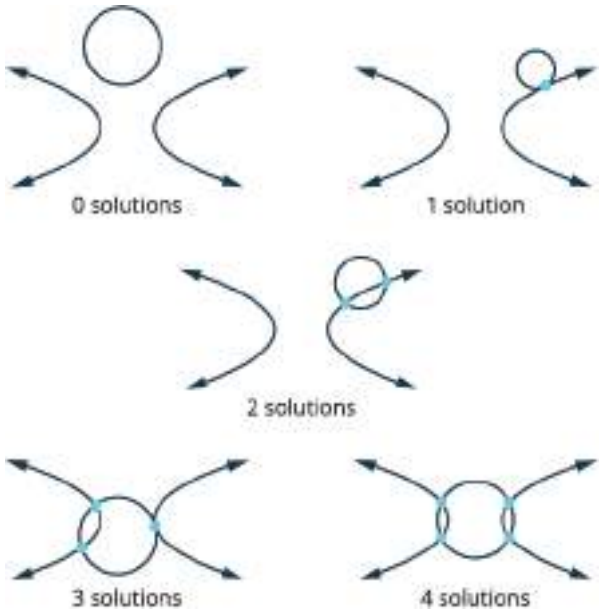
Solve the system by elimination: $\begin{cases} x^2 + y^2 = 7 \\ x^2 - y^2 = 1 \end{cases}$.

Solution

Identify each graph.

$$\begin{cases} x^2 + y^2 = 7 & \text{circle} \\ x^2 - y^2 = 1 & \text{hyperbola} \end{cases}$$

Sketch the possible options for intersection of a circle and hyperbola.



Both equations are in standard form.

$$\begin{cases} x^2 + y^2 = 7 \\ x^2 - y^2 = 1 \end{cases}$$

The coefficients of y^2 are opposite, so we will add the equations.

$$\begin{cases} x^2 + y^2 = 7 \\ \underline{x^2 - y^2 = 1} \end{cases}$$
$$2x^2 \quad = 8$$

Simplify.

$$\begin{aligned} x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$
$$x = 2 \quad x = -2$$

Substitute $x = 2$ and $x = -2$ into one of the original equations. Then solve for *y*.

$$\begin{aligned} x^2 + y^2 &= 7 \\ 2^2 + y^2 &= 7 \\ 4 + y^2 &= 7 \\ y^2 &= 3 \\ y &= \pm\sqrt{3} \end{aligned} \qquad \begin{aligned} x^2 + y^2 &= 7 \\ (-2)^2 + y^2 &= 7 \\ 4 + y^2 &= 7 \\ y^2 &= 3 \\ y &= \pm\sqrt{3} \end{aligned}$$

Write each solution as an ordered pair.

The ordered pairs are $(-2, \sqrt{3})$, $(-2, -\sqrt{3})$, $(2, \sqrt{3})$, and $(2, -\sqrt{3})$.

Check that the ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2, \sqrt{3})$, $(-2, -\sqrt{3})$, $(2, \sqrt{3})$, and $(2, -\sqrt{3})$.

TRY IT : : 11.75 Solve the system by elimination: $\begin{cases} x^2 + y^2 = 25 \\ y^2 - x^2 = 7 \end{cases}$.

TRY IT : : 11.76 Solve the system by elimination: $\begin{cases} x^2 + y^2 = 4 \\ x^2 - y^2 = 4 \end{cases}$.

Use a System of Nonlinear Equations to Solve Applications

Systems of nonlinear equations can be used to model and solve many applications. We will look at an everyday geometric situation as our example.

EXAMPLE 11.39

The difference of the squares of two numbers is 15. The sum of the numbers is 5. Find the numbers.

Solution

Identify what we are looking for.	Two different numbers.
Define the variables.	$x =$ first number $y =$ second number
Translate the information into a system of equations.	
First sentence.	The difference of the squares of two numbers is 15.
	$x^2 - y^2 = 15$
Second sentence.	The sum of the numbers is 5.
	$x + y = 5$
Solve the system by substitution	$\begin{cases} x^2 - y^2 = 15 \\ x + y = 5 \end{cases}$
Solve the second equation for x.	$x = 5 - y$
Substitute x into the first equation.	$x^2 - y^2 = 15$
	$(5 - y)^2 - y^2 = 15$
Expand and simplify.	$(25 - 10y + y^2) - y^2 = 15$
	$25 - 10y + y^2 - y^2 = 15$ $25 - 10y = 15$
Solve for y.	$-10y = -10$
	$y = 1$

Substitute back into the second equation.	$x+y=5$
	$x+(1)=5$ $x=4$
	The numbers are 1 and 4.

TRY IT : : 11.77

The difference of the squares of two numbers is -20. The sum of the numbers is 10. Find the numbers.

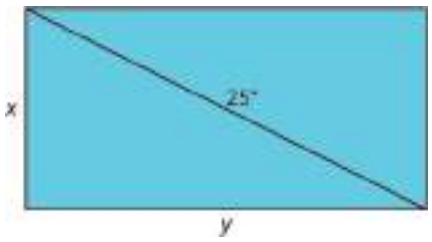
TRY IT : : 11.78

The difference of the squares of two numbers is 35. The sum of the numbers is -1. Find the numbers.

EXAMPLE 11.40

Myra purchased a small 25" TV for her kitchen. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 300 square inches. What are the length and width of the TV screen?

Solution

Identify what we are looking for.	The length and width of the rectangle
Define the variables.	Let $x=$ width of the rectangle $y=$ length of the rectangle
Draw a diagram to help visualize the situation.	
	Area is 300 square inches.
Translate the information into a system of equations.	The diagonal of the right triangle is 25 inches.
	$x^2+y^2=25^2$ $x^2+y^2=625$
	The area of the rectangle is 300 square inches.
	$x\cdot y=300$ $x^2+y^2=625$ $x\cdot y=300$
Solve the system using substitution.	$x\cdot y=300$
Solve the second equation for x.	$x=\frac{300}{y}$
Substitute x into the first equation.	$x^2+y^2=625$
	$\left(\frac{300}{y}\right)^2+y^2=625$

Simplify.	$\frac{90000}{y^2} + y^2 = 625$
Multiply by y^2 to clear the fractions.	$90000 + y^4 = 625y^2$
Put in standard form.	$y^4 - 625y^2 + 90000 = 0$
Solve by factoring.	$(y^2 - 225)(y^2 - 400) = 0$
	$y^2 - 225 = 0 \quad y^2 - 400 = 0$
	$y^2 = 225 \quad y^2 = 400$ $y = \pm 15 \quad y = \pm 20$
Since y is a side of the rectangle, we discard the negative values.	$y = 15 \quad y = 20$
Substitute back into the second equation.	$x \cdot y = 300 \quad x \cdot y = 300$
	$x \cdot 15 = 300 \quad x \cdot 20 = 300$ $x = 20 \quad x = 15$
	If the length is 15 inches, the width is 20 inches.
	If the length is 20 inches, the width is 15 inches.

TRY IT : : 11.79

Edgar purchased a small 20" TV for his garage. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 192 square inches. What are the length and width of the TV screen?

TRY IT : : 11.80

The Harper family purchased a small microwave for their family room. The diagonal of the door measures 15 inches. The door also has an area of 108 square inches. What are the length and width of the microwave door?

MEDIA : :

Access these online resources for additional instructions and practice with solving nonlinear equations.

- **Nonlinear Systems of Equations (https://openstax.org/l/37nonsyseq)**
- **Solve a System of Nonlinear Equations (https://openstax.org/l/37nonsyseq2)**
- **Solve a System of Nonlinear Equations by Elimination (https://openstax.org/l/37nonsyselim)**
- **System of Nonlinear Equations – Area and Perimeter Application (https://openstax.org/l/37nonsysapps)**

11.5 EXERCISES

Practice Makes Perfect

Solve a System of Nonlinear Equations Using Graphing

In the following exercises, solve the system of equations by using graphing.

188. $\begin{cases} y = 2x + 2 \\ y = -x^2 + 2 \end{cases}$

189. $\begin{cases} y = 6x - 4 \\ y = 2x^2 \end{cases}$

190. $\begin{cases} x + y = 2 \\ x = y^2 \end{cases}$

191. $\begin{cases} x - y = -2 \\ x = y^2 \end{cases}$

192. $\begin{cases} y = \frac{3}{2}x + 3 \\ y = -x^2 + 2 \end{cases}$

193. $\begin{cases} y = x - 1 \\ y = x^2 + 1 \end{cases}$

194. $\begin{cases} x = -2 \\ x^2 + y^2 = 4 \end{cases}$

195. $\begin{cases} y = -4 \\ x^2 + y^2 = 16 \end{cases}$

196. $\begin{cases} x = 2 \\ (x+2)^2 + (y+3)^2 = 16 \end{cases}$

197. $\begin{cases} y = -1 \\ (x-2)^2 + (y-4)^2 = 25 \end{cases}$

198. $\begin{cases} y = -2x + 4 \\ y = \sqrt{x} + 1 \end{cases}$

199. $\begin{cases} y = -\frac{1}{2}x + 2 \\ y = \sqrt{x} - 2 \end{cases}$

Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

200. $\begin{cases} x^2 + 4y^2 = 4 \\ y = \frac{1}{2}x - 1 \end{cases}$

201. $\begin{cases} 9x^2 + y^2 = 9 \\ y = 3x + 3 \end{cases}$

202. $\begin{cases} 9x^2 + y^2 = 9 \\ y = x + 3 \end{cases}$

203. $\begin{cases} 9x^2 + 4y^2 = 36 \\ x = 2 \end{cases}$

204. $\begin{cases} 4x^2 + y^2 = 4 \\ y = 4 \end{cases}$

205. $\begin{cases} x^2 + y^2 = 169 \\ x = 12 \end{cases}$

206. $\begin{cases} 3x^2 - y = 0 \\ y = 2x - 1 \end{cases}$

207. $\begin{cases} 2y^2 - x = 0 \\ y = x + 1 \end{cases}$

208. $\begin{cases} y = x^2 + 3 \\ y = x + 3 \end{cases}$

209. $\begin{cases} y = x^2 - 4 \\ y = x - 4 \end{cases}$

210. $\begin{cases} x^2 + y^2 = 25 \\ x - y = 1 \end{cases}$

211. $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$

Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

212. $\begin{cases} x^2 + y^2 = 16 \\ x^2 - 2y = 8 \end{cases}$

213. $\begin{cases} x^2 + y^2 = 16 \\ x^2 - y = 4 \end{cases}$

214. $\begin{cases} x^2 + y^2 = 4 \\ x^2 + 2y = 1 \end{cases}$

215. $\begin{cases} x^2 + y^2 = 4 \\ x^2 - y = 2 \end{cases}$

216. $\begin{cases} x^2 + y^2 = 9 \\ x^2 - y = 3 \end{cases}$

217. $\begin{cases} x^2 + y^2 = 4 \\ y^2 - x = 2 \end{cases}$

218. $\begin{cases} x^2 + y^2 = 25 \\ 2x^2 - 3y^2 = 5 \end{cases}$

219. $\begin{cases} x^2 + y^2 = 20 \\ x^2 - y^2 = -12 \end{cases}$

220. $\begin{cases} x^2 + y^2 = 13 \\ x^2 - y^2 = 5 \end{cases}$

221. $\begin{cases} x^2+y^2=16 \\ x^2-y^2=16 \end{cases}$

222. $\begin{cases} 4x^2+9y^2=36 \\ 2x^2-9y^2=18 \end{cases}$

223. $\begin{cases} x^2-y^2=3 \\ 2x^2+y^2=6 \end{cases}$

224. $\begin{cases} 4x^2-y^2=4 \\ 4x^2+y^2=4 \end{cases}$

225. $\begin{cases} x^2-y^2=-5 \\ 3x^2+2y^2=30 \end{cases}$

226. $\begin{cases} x^2-y^2=1 \\ x^2-2y=4 \end{cases}$

227. $\begin{cases} 2x^2+y^2=11 \\ x^2+3y^2=28 \end{cases}$

Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.

228. The sum of two numbers is -6 and the product is 8. Find the numbers.

229. The sum of two numbers is 11 and the product is -42. Find the numbers.

230. The sum of the squares of two numbers is 65. The difference of the number is 3. Find the numbers.

231. The sum of the squares of two numbers is 113. The difference of the number is 1. Find the numbers.

232. The difference of the squares of two numbers is 15. The difference of twice the square of the first number and the square of the second number is 30. Find the numbers.

233. The difference of the squares of two numbers is 20. The difference of the square of the first number and twice the square of the second number is 4. Find the numbers.

234. The perimeter of a rectangle is 32 inches and its area is 63 square inches. Find the length and width of the rectangle.

235. The perimeter of a rectangle is 52 cm and its area is 165 cm^2. Find the length and width of the rectangle.

236. Dion purchased a new microwave. The diagonal of the door measures 17 inches. The door also has an area of 120 square inches. What are the length and width of the microwave door?

237. Jules purchased a microwave for his kitchen. The diagonal of the front of the microwave measures 26 inches. The front also has an area of 240 square inches. What are the length and width of the microwave?

238. Roman found a widescreen TV on sale, but isn't sure if it will fit his entertainment center. The TV is 60". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1728 square inches. His entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Roman's entertainment center?

239. Donnette found a widescreen TV at a garage sale, but isn't sure if it will fit her entertainment center. The TV is 50". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1200 square inches. Her entertainment center has an insert for the TV with a length of 38 inches and width of 27 inches. What are the length and width of the TV screen and will it fit Donnette's entertainment center?

Writing Exercises

240. In your own words, explain the advantages and disadvantages of solving a system of equations by graphing.

241. Explain in your own words how to solve a system of equations using substitution.

242. Explain in your own words how to solve a system of equations using elimination.

243. A circle and a parabola can intersect in ways that would result in 0, 1, 2, 3, or 4 solutions. Draw a sketch of each of the possibilities.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve a system of nonlinear equations using graphing.			
solve a system of nonlinear equations using substitution.			
solve a system of nonlinear equations using elimination.			
use a system of nonlinear equations to solve applications.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

CHAPTER 11 REVIEW

KEY TERMS

circle A circle is all points in a plane that are a fixed distance from a fixed point in the plane.

ellipse An ellipse is all points in a plane where the sum of the distances from two fixed points is constant.

hyperbola A hyperbola is defined as all points in a plane where the difference of their distances from two fixed points is constant.

parabola A parabola is all points in a plane that are the same distance from a fixed point and a fixed line.

system of nonlinear equations A system of nonlinear equations is a system where at least one of the equations is not linear.

KEY CONCEPTS

11.1 Distance and Midpoint Formulas; Circles

- **Distance Formula:** The distance *d* between the two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

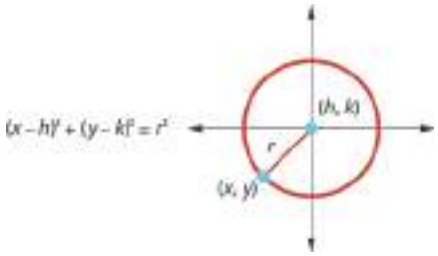
- **Midpoint Formula:** The midpoint of the line segment whose endpoints are the two points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

 To find the midpoint of a line segment, we find the average of the *x*-coordinates and the average of the *y*-coordinates of the endpoints.

- **Circle:** A circle is all points in a plane that are a fixed distance from a fixed point in the plane. The given point is called the *center,* (h, k), and the fixed distance is called the *radius, r,* of the circle.

- **Standard Form of the Equation a Circle:** The standard form of the equation of a circle with center, (h, k), and radius, *r,* is



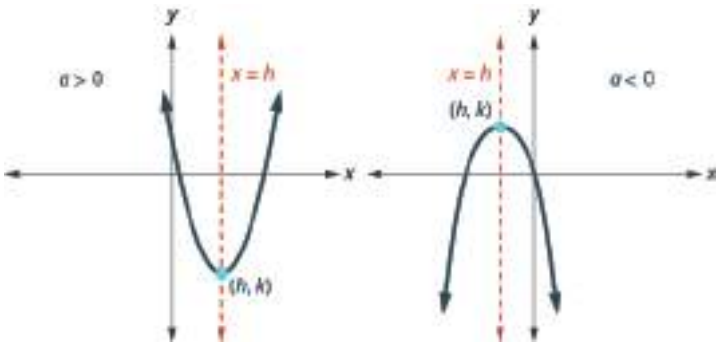
- **General Form of the Equation of a Circle:** The general form of the equation of a circle is

$$x^2 + y^2 + ax + by + c = 0$$

11.2 Parabolas

- **Parabola:** A **parabola** is all points in a plane that are the same distance from a fixed point and a fixed line. The fixed point is called the **focus,** and the fixed line is called the **directrix** of the parabola.

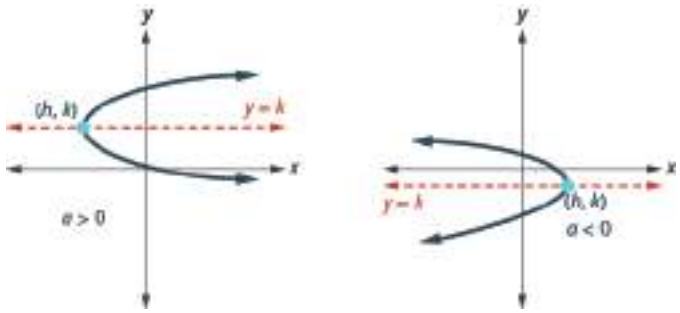
Vertical Parabolas		
	General form $y = ax^2 + bx + c$	**Standard form** $y = a(x - h)^2 + k$
Orientation	$a > 0$ up; $a < 0$ down	$a > 0$ up; $a < 0$ down
Axis of symmetry	$x = -\frac{b}{2a}$	$x = h$
Vertex	Substitute $x = -\frac{b}{2a}$ and solve for y.	(h, k)
***y*- intercept**	Let $x = 0$	Let $x = 0$
***x*-intercepts**	Let $y = 0$	Let $y = 0$



- **How to graph vertical parabolas $(y = ax^2 + bx + c$ or $f(x) = a(x - h)^2 + k)$ using properties.**

 Step 1. Determine whether the parabola opens upward or downward.
 Step 2. Find the axis of symmetry.
 Step 3. Find the vertex.
 Step 4. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
 Step 5. Find the *x*-intercepts.
 Step 6. Graph the parabola.

Horizontal Parabolas		
	General form $x = ay^2 + by + c$	**Standard form** $x = a(y-k)^2 + h$
Orientation	$a > 0$ right; $a < 0$ left	$a > 0$ right; $a < 0$ left
Axis of symmetry	$y = -\frac{b}{2a}$	$y = k$
Vertex	Substitute $y = -\frac{b}{2a}$ and solve for x.	(h, k)
***y*-intercepts**	Let $x = 0$	Let $x = 0$
***x*-intercept**	Let $y = 0$	Let $y = 0$



- **How to graph horizontal parabolas $(x = ay^2 + by + c$ or $x = a(y-k)^2 + h)$ using properties.**

 Step 1. Determine whether the parabola opens to the left or to the right.

 Step 2. Find the axis of symmetry.

 Step 3. Find the vertex.

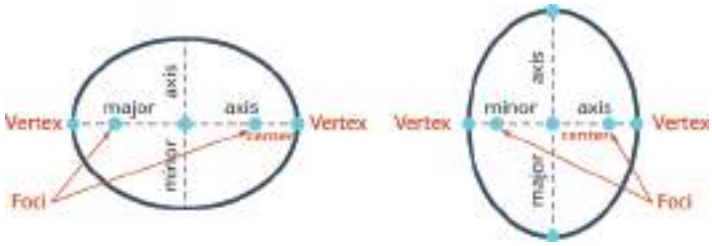
 Step 4. Find the *x*-intercept. Find the point symmetric to the *x*-intercept across the axis of symmetry.

 Step 5. Find the *y*-intercepts.

 Step 6. Graph the parabola.

11.3 Ellipses

- **Ellipse:** An **ellipse** is all points in a plane where the sum of the distances from two fixed points is constant. Each of the fixed points is called a **focus** of the ellipse.



If we draw a line through the foci intersects the ellipse in two points—each is called a **vertex** of the ellipse.
The segment connecting the vertices is called the **major axis**.
The midpoint of the segment is called the **center** of the ellipse.
A segment perpendicular to the major axis that passes through the center and intersects the ellipse in two points is called the **minor axis**.

- **Standard Form of the Equation an Ellipse with Center** $(0, 0)$: The standard form of the equation of an ellipse with center $(0, 0)$, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The *x*-intercepts are $(a, 0)$ and $(-a, 0)$.
The *y*-intercepts are $(0, b)$ and $(0, -b)$.

- **How to an Ellipse with Center** $(0, 0)$

Step 1. Write the equation in standard form.
Step 2. Determine whether the major axis is horizontal or vertical.
Step 3. Find the endpoints of the major axis.
Step 4. Find the endpoints of the minor axis
Step 5. Sketch the ellipse.

- **Standard Form of the Equation an Ellipse with Center** (h, k) : The standard form of the equation of an ellipse with center (h, k), is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

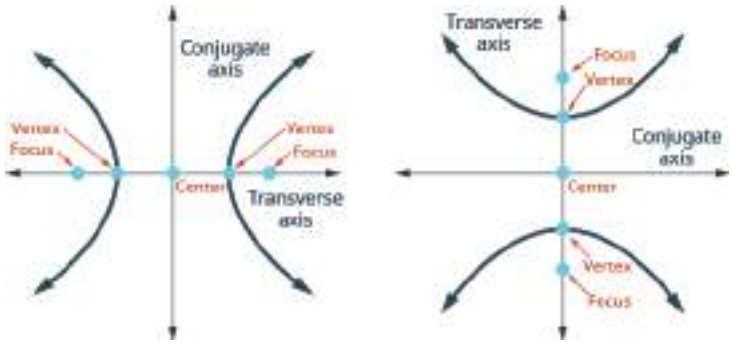
When $a > b$, the major axis is horizontal so the distance from the center to the vertex is *a*.
When $b > a$, the major axis is vertical so the distance from the center to the vertex is *b*.

11.4 Hyperbolas

- **Hyperbola:** A **hyperbola** is all points in a plane where the difference of their distances from two fixed points is constant.


hyperbola

Each of the fixed points is called a **focus** of the hyperbola.
The line through the foci, is called the **transverse axis**.
The two points where the transverse axis intersects the hyperbola are each a **vertex** of the hyperbola.
The midpoint of the segment joining the foci is called the **center** of the hyperbola.
The line perpendicular to the transverse axis that passes through the center is called the **conjugate axis**.
Each piece of the graph is called a **branch** of the hyperbola.



Standard Forms of the Equation a Hyperbola with Center $(0, 0)$		
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Orientation	Transverse axis on the *x*-axis. Opens left and right	Transverse axis on the *y*-axis. Opens up and down
Vertices	$(-a, 0), \quad (a, 0)$	$(0, -a), \quad (0, a)$
***x*-intercepts**	$(-a, 0), \quad (a, 0)$	none
***y*-intercepts**	none	$(0, -a), (0, a)$
Rectangle	Use $(\pm a, 0) \quad (0, \pm b)$	Use $(0, \pm a) \quad (\pm b, 0)$
asymptotes	$y=\frac{b}{a}x, \quad y=-\frac{b}{a}x$	$y=\frac{a}{b}x, \quad y=-\frac{a}{b}x$

- **How to graph a hyperbola centered at $(0, 0)$.**

 Step 1. Write the equation in standard form.
 Step 2. Determine whether the transverse axis is horizontal or vertical.
 Step 3. Find the vertices.
 Step 4. Sketch the rectangle centered at the origin intersecting one axis at $\pm a$ and the other at $\pm b$.
 Step 5. Sketch the asymptotes—the lines through the diagonals of the rectangle.
 Step 6. Draw the two branches of the hyperbola.

Standard Forms of the Equation a Hyperbola with Center (h, k)		
	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
Orientation	Transverse axis is horizontal. Opens left and right	Transverse axis is vertical. Opens up and down
Center	(h, k)	(h, k)
Vertices	*a* units to the left and right of the center	*a* units above and below the center
Rectangle	Use *a* units left/right of center *b* units above/below the center	Use *a* units above/below the center *b* units left/right of center

- **How to graph a hyperbola centered at (h, k).**

 Step 1. Write the equation in standard form.
 Step 2. Determine whether the transverse axis is horizontal or vertical.
 Step 3. Find the center and a, b.
 Step 4. Sketch the rectangle centered at (h, k) using a, b.
 Step 5. Sketch the asymptotes—the lines through the diagonals of the rectangle. Mark the vertices.
 Step 6. Draw the two branches of the hyperbola.

Conic	Characteristics of x^2- and y^2- terms	Example
Parabola	Either x^2 OR y^2. Only one variable is squared.	$x = 3y^2 - 2y + 1$
Circle	x^2- and y^2- terms have the same coefficients	$x^2 + y^2 = 49$
Ellipse	x^2- and y^2- terms have the **same** sign, different coefficients	$4x^2 + 25y^2 = 100$
Hyperbola	x^2- and y^2- terms have **different** signs, different coefficients	$25y^2 - 4x^2 = 100$

11.5 Solve Systems of Nonlinear Equations

- **How to solve a system of nonlinear equations by graphing.**
 Step 1. Identify the graph of each equation. Sketch the possible options for intersection.
 Step 2. Graph the first equation.
 Step 3. Graph the second equation on the same rectangular coordinate system.
 Step 4. Determine whether the graphs intersect.
 Step 5. Identify the points of intersection.
 Step 6. Check that each ordered pair is a solution to both original equations.
- **How to solve a system of nonlinear equations by substitution.**
 Step 1. Identify the graph of each equation. Sketch the possible options for intersection.
 Step 2. Solve one of the equations for either variable.
 Step 3. Substitute the expression from Step 2 into the other equation.
 Step 4. Solve the resulting equation.
 Step 5. Substitute each solution in Step 4 into one of the original equations to find the other variable.
 Step 6. Write each solution as an ordered pair.
 Step 7. Check that each ordered pair is a solution to **both** original equations.
- **How to solve a system of equations by elimination.**
 Step 1. Identify the graph of each equation. Sketch the possible options for intersection.
 Step 2. Write both equations in standard form.
 Step 3. Make the coefficients of one variable opposites.
 Decide which variable you will eliminate.
 Multiply one or both equations so that the coefficients of that variable are opposites.
 Step 4. Add the equations resulting from Step 3 to eliminate one variable.
 Step 5. Solve for the remaining variable.
 Step 6. Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
 Step 7. Write each solution as an ordered pair.
 Step 8. Check that each ordered pair is a solution to **both** original equations.

REVIEW EXERCISES

11.1 Distance and Midpoint Formulas; Circles

Use the Distance Formula

In the following exercises, find the distance between the points. Round to the nearest tenth if needed.

244. $(-5, 1)$ and $(-1, 4)$ **245.** $(-2, 5)$ and $(1, 5)$ **246.** $(8, 2)$ and $(-7, -3)$

247. $(1, -4)$ and $(5, -5)$

Use the Midpoint Formula

In the following exercises, find the midpoint of the line segments whose endpoints are given.

248. $(-2, -6)$ and $(-4, -2)$

249. $(3, 7)$ and $(5, 1)$

250. $(-8, -10)$ and $(9, 5)$

251. $(-3, 2)$ and $(6, -9)$

Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given information.

252. radius is 15 and center is $(0, 0)$

253. radius is $\sqrt{7}$ and center is $(0, 0)$

254. radius is 9 and center is $(-3, 5)$

255. radius is 7 and center is $(-2, -5)$

256. center is $(3, 6)$ and a point on the circle is $(3, -2)$

257. center is $(2, 2)$ and a point on the circle is $(4, 4)$

Graph a Circle

In the following exercises, ⓐ find the center and radius, then ⓑ graph each circle.

258. $2x^2 + 2y^2 = 450$

259. $3x^2 + 3y^2 = 432$

260. $(x + 3)^2 + (y - 5)^2 = 81$

261. $(x + 2)^2 + (y + 5)^2 = 49$

262. $x^2 + y^2 - 6x - 12y - 19 = 0$

263. $x^2 + y^2 - 4y - 60 = 0$

11.2 Parabolas

Graph Vertical Parabolas

In the following exercises, graph each equation by using its properties.

264. $y = x^2 + 4x - 3$

265. $y = 2x^2 + 10x + 7$

266. $y = -6x^2 + 12x - 1$

267. $y = -x^2 + 10x$

In the following exercises, ⓐ write the equation in standard form, then ⓑ use properties of the standard form to graph the equation.

268. $y = x^2 + 4x + 7$

269. $y = 2x^2 - 4x - 2$

270. $y = -3x^2 - 18x - 29$

271. $y = -x^2 + 12x - 35$

Graph Horizontal Parabolas

In the following exercises, graph each equation by using its properties.

272. $x = 2y^2$

273. $x = 2y^2 + 4y + 6$

274. $x = -y^2 + 2y - 4$

275. $x = -3y^2$

In the following exercises, ⓐ write the equation in standard form, then ⓑ use properties of the standard form to graph the equation.

276. $x = 4y^2 + 8y$

277. $x = y^2 + 4y + 5$

278. $x = -y^2 - 6y - 7$

279. $x = -2y^2 + 4y$

Solve Applications with Parabolas

In the following exercises, create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in standard form.

280.



281.



11.3 Ellipses

Graph an Ellipse with Center at the Origin

In the following exercises, graph each ellipse.

282. $\frac{x^2}{36} + \frac{y^2}{25} = 1$

283. $\frac{x^2}{4} + \frac{y^2}{81} = 1$

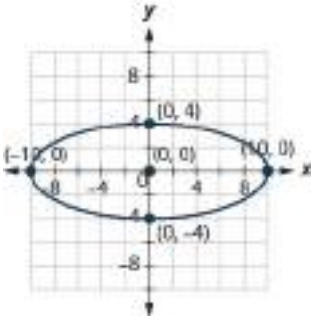
284. $49x^2 + 64y^2 = 3136$

285. $9x^2 + y^2 = 9$

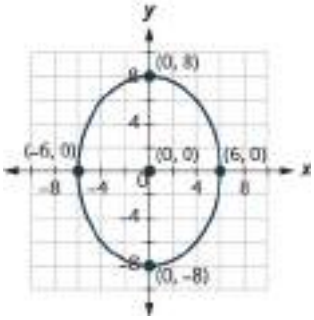
Find the Equation of an Ellipse with Center at the Origin

In the following exercises, find the equation of the ellipse shown in the graph.

286.



287.



Graph an Ellipse with Center Not at the Origin

In the following exercises, graph each ellipse.

288. $\frac{(x-1)^2}{25} + \frac{(y-6)^2}{4} = 1$

289. $\frac{(x+4)^2}{16} + \frac{(y+1)^2}{9} = 1$

290. $\frac{(x-5)^2}{16} + \frac{(y+3)^2}{36} = 1$

291. $\frac{(x+3)^2}{9} + \frac{(y-2)^2}{25} = 1$

In the following exercises, ⓐ write the equation in standard form and ⓑ graph.

292. $x^2 + y^2 + 12x + 40y + 120 = 0$

293. $25x^2 + 4y^2 - 150x - 56y + 321 = 0$

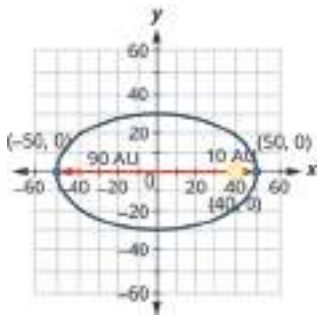
294. $25x^2 + 4y^2 + 150x + 125 = 0$

295. $4x^2 + 9y^2 - 126x + 405 = 0$

Solve Applications with Ellipses

In the following exercises, write the equation of the ellipse described.

296. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 10 AU and the furthest is approximately 90 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.



11.4 Hyperbolas

Graph a Hyperbola with Center at $(0, 0)$

In the following exercises, graph.

297. $\frac{x^2}{25} - \frac{y^2}{9} = 1$

298. $\frac{y^2}{49} - \frac{x^2}{16} = 1$

299. $9y^2 - 16x^2 = 144$

300. $16x^2 - 4y^2 = 64$

Graph a Hyperbola with Center at (h, k)

In the following exercises, graph.

301. $\frac{(x+1)^2}{4} - \frac{(y+1)^2}{9} = 1$

302. $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{16} = 1$

303. $\frac{(y+2)^2}{9} - \frac{(x+1)^2}{9} = 1$

304. $\frac{(y-1)^2}{25} - \frac{(x-2)^2}{9} = 1$

In the following exercises, ⓐ write the equation in standard form and ⓑ graph.

305. $4x^2 - 16y^2 + 8x + 96y - 204 = 0$

306. $16x^2 - 4y^2 - 64x - 24y - 36 = 0$

307. $4y^2 - 16x^2 + 32x - 8y - 76 = 0$

308. $36y^2 - 16x^2 - 96x + 216y - 396 = 0$

Identify the Graph of each Equation as a Circle, Parabola, Ellipse, or Hyperbola

In the following exercises, identify the type of graph.

309.

ⓐ $16y^2 - 9x^2 - 36x - 96y - 36 = 0$

ⓑ $x^2 + y^2 - 4x + 10y - 7 = 0$

ⓒ $y = x^2 - 2x + 3$

ⓓ $25x^2 + 9y^2 = 225$

310.

ⓐ $x^2 + y^2 + 4x - 10y + 25 = 0$

ⓑ $y^2 - x^2 - 4y + 2x - 6 = 0$

ⓒ $x = -y^2 - 2y + 3$

ⓓ $16x^2 + 9y^2 = 144$

11.5 Solve Systems of Nonlinear Equations

Solve a System of Nonlinear Equations Using Graphing

In the following exercises, solve the system of equations by using graphing.

311. $\begin{cases} 3x^2 - y = 0 \\ y = 2x - 1 \end{cases}$

312. $\begin{cases} y = x^2 - 4 \\ y = x - 4 \end{cases}$

313. $\begin{cases} x^2 + y^2 = 169 \\ x = 12 \end{cases}$

314. $\begin{cases} x^2 + y^2 = 25 \\ y = -5 \end{cases}$

Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

315. $\begin{cases} y = x^2 + 3 \\ y = -2x + 2 \end{cases}$

316. $\begin{cases} x^2 + y^2 = 4 \\ x - y = 4 \end{cases}$

317. $\begin{cases} 9x^2 + 4y^2 = 36 \\ y - x = 5 \end{cases}$

318. $\begin{cases} x^2 + 4y^2 = 4 \\ 2x - y = 1 \end{cases}$

Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

319. $\begin{cases} x^2 + y^2 = 16 \\ x^2 - 2y - 1 = 0 \end{cases}$

320. $\begin{cases} x^2 - y^2 = 5 \\ -2x^2 - 3y^2 = -30 \end{cases}$

321. $\begin{cases} 4x^2 + 9y^2 = 36 \\ 3y^2 - 4x = 12 \end{cases}$

322. $\begin{cases} x^2 + y^2 = 14 \\ x^2 - y^2 = 16 \end{cases}$

Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.

323. The sum of the squares of two numbers is 25. The difference of the numbers is 1. Find the numbers.

324. The difference of the squares of two numbers is 45. The difference of the square of the first number and twice the square of the second number is 9. Find the numbers.

325. The perimeter of a rectangle is 58 meters and its area is 210 square meters. Find the length and width of the rectangle.

326. Colton purchased a larger microwave for his kitchen. The diagonal of the front of the microwave measures 34 inches. The front also has an area of 480 square inches. What are the length and width of the microwave?

PRACTICE TEST

In the following exercises, find the distance between the points and the midpoint of the line segment with the given endpoints. Round to the nearest tenth as needed.

327. $(-4, -3)$ and $(-10, -11)$

328. $(6, 8)$ and $(-5, -3)$

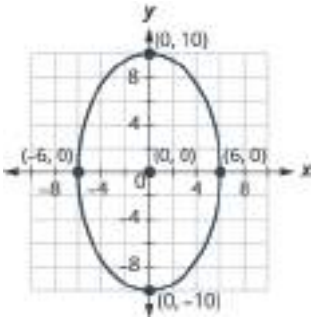
In the following exercises, write the standard form of the equation of the circle with the given information.

329. radius is 11 and center is $(0, 0)$

330. radius is 12 and center is $(10, -2)$

331. center is $(-2, 3)$ and a point on the circle is $(2, -3)$

332. Find the equation of the ellipse shown in the graph.



In the following exercises, ⓐ identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola, and ⓑ graph the equation.

333. $4x^2 + 49y^2 = 196$

334. $y = 3(x - 2)^2 - 2$

335. $3x^2 + 3y^2 = 27$

336. $\frac{y^2}{100} - \frac{x^2}{36} = 1$

337. $\frac{x^2}{16} + \frac{y^2}{81} = 1$

338. $x = 2y^2 + 10y + 7$

339. $64x^2 - 9y^2 = 576$

In the following exercises, ⓐ identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola, ⓑ write the equation in standard form, and ⓒ graph the equation.

340. $25x^2 + 64y^2 + 200x - 256y - 944 = 0$

341. $x^2 + y^2 + 10x + 6y + 30 = 0$

342. $x = -y^2 + 2y - 4$

343. $9x^2 - 25y^2 - 36x - 50y - 214 = 0$

344. $y = x^2 + 6x + 8$

345. Solve the nonlinear system of equations by graphing:
$\begin{cases} 3y^2 - x = 0 \\ y = -2x - 1 \end{cases}$.

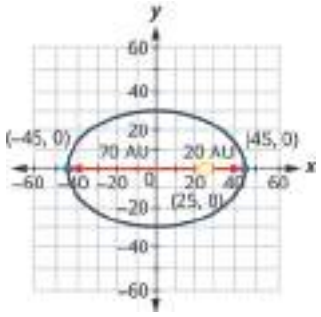
346. Solve the nonlinear system of equations using substitution:
$\begin{cases} x^2 + y^2 = 8 \\ y = -x - 4 \end{cases}$.

347. Solve the nonlinear system of equations using elimination:
$\begin{cases} x^2 + 9y^2 = 9 \\ 2x^2 - 9y^2 = 18 \end{cases}$.

348. Create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in $y = ax^2 + bx + c$ form.

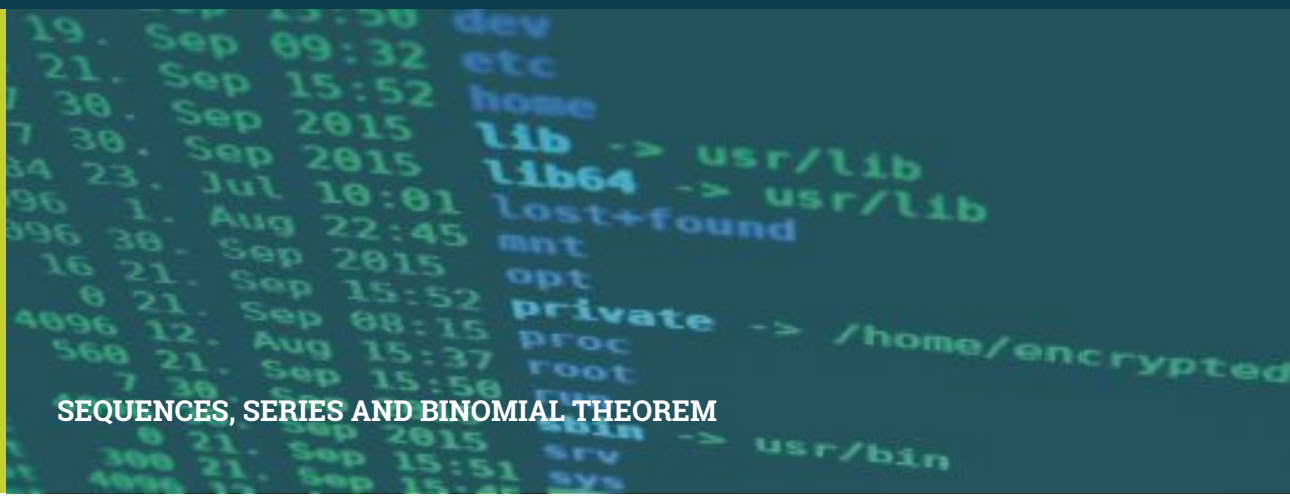


349. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 20 AU and the furthest is approximately 70 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.



350. The sum of two numbers is 22 and the product is -240. Find the numbers.

351. For her birthday, Olive's grandparents bought her a new widescreen TV. Before opening it she wants to make sure it will fit her entertainment center. The TV is 55". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1452 square inches. Her entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Olive's entertainment center?



12 SEQUENCES, SERIES AND BINOMIAL THEOREM

Figure 12.1 Cryptographers protect private data by encrypting it; this means they convert the data into a code that hackers and thieves cannot easily break. (credit: "joffi"/pixabay)

Chapter Outline

12.1 Sequences
12.2 Arithmetic Sequences
12.3 Geometric Sequences and Series
12.4 Binomial Theorem

Introduction

A strange charge suddenly appears on your credit card. But your card is in your wallet—it's not even lost or stolen. Sadly, you may have been a victim of cyber crime. In this day and age, most transactions take advantage of the benefit of computers in some way. Cyber crime is any type of crime that uses a computer or computer network. Thankfully, many people are working to prevent cyber crime. Sometimes known as cryptographers, these people develop complex patterns in computer codes that block access to would-be thieves as well as write codes to intercept and decode information from them so that they may be identified. In this chapter, you will explore basic sequences and series related to those used by computer programmers to prevent cyber crime.

12.1 Sequences

Learning Objectives

By the end of this section, you will be able to:

- Write the first few terms of a sequence
- Find a formula for the general term (nth term) of a sequence
- Use factorial notation
- Find the partial sum
- Use summation notation to write a sum

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $2n+3$ for the integers 1, 2, 3, and 4.
 If you missed this problem, review Example 1.6.
2. Evaluate $(-1)^n$ for the integers 1, 2, 3, and 4.
 If you missed this problem, review Example 1.19.
3. If $f(n)=n^2+2$, find $f(1)+f(2)+f(3)$.

If you missed this problem, review **Example 3.49**.

Write the First Few Terms of a Sequence

Let's look at the function $f(x) = 2x$ and evaluate it for just the counting numbers.

$f(x) = 2x$	
x	$2x$
1	2
2	4
3	6
4	8
5	10
...	...

If we list the function values in order as 2, 4, 6, 8, and 10, ... we have a sequence. A **sequence** is a function whose domain is the counting numbers.

Sequences

A **sequence** is a function whose domain is the counting numbers.

A sequence can also be seen as an ordered list of numbers and each number in the list is a *term*. A sequence may have an infinite number of terms or a finite number of terms. Our sequence has three dots (ellipsis) at the end which indicates the list never ends. If the domain is the set of all counting numbers, then the sequence is an **infinite sequence**. Its domain is all counting numbers and there is an infinite number of counting numbers.

$$2,\ 4,\ 6,\ 8,\ 10,\ \ldots,$$

If we limit the domain to a finite number of counting numbers, then the sequence is a **finite sequence**. If we use only the first four counting numbers, 1, 2, 3, 4 our sequence would be the finite sequence,

$$2,\ 4,\ 6,\ 8$$

Often when working with sequences we do not want to write out all the terms. We want more compact way to show how each term is defined. When we worked with functions, we wrote $f(x) = 2x$ and we said the expression $2x$ was the rule that defined values in the range. While a sequence is a function, we do not use the usual function notation. Instead of writing the function as $f(x) = 2x$, we would write it as $a_n = 2n$. The a_n is the nth term of the sequence, the term in the nth position where n is a value in the domain. The formula for writing the nth term of the sequence is called the **general term** or formula of the sequence.

General Term of a Sequence

The **general term** of the sequence is found from the formula for writing the nth term of the sequence. The nth term of the sequence, a_n, is the term in the nth position where n is a value in the domain.

When we are given the general term of the sequence, we can find the terms by replacing n with the counting numbers in order. For $a_n = 2n$,

n	1	2	3	4	5	a_n
a_n	$2 \cdot 1$ 2	$2 \cdot 2$ 4	$2 \cdot 3$ 6	$2 \cdot 4$ 8	$2 \cdot 5$ 10	$2n$

$$\begin{array}{cccccccc} a_1, & a_2, & a_3, & a_4, & a_5, \ldots, & a_n, \ldots \\ 2, & 4, & 6, & 8, & 10, \ldots \end{array}$$

To find the values of a sequence, we substitute in the counting numbers in order into the general term of the sequence.

EXAMPLE 12.1

Write the first five terms of the sequence whose general term is $a_n = 4n - 3$.

Solution

We substitute the values 1, 2, 3, 4, and 5 into the formula, $a_n = 4n - 3$, in order.

$a_n = 4n - 3$ $\quad a_n = 4n - 3$ $\quad a_n = 4n - 3$ $\quad a_n = 4n - 3$ $\quad a_n = 4n - 3$

$a_1 = 4 \cdot 1 - 3$ $\quad a_2 = 4 \cdot 2 - 3$ $\quad a_3 = 4 \cdot 3 - 3$ $\quad a_4 = 4 \cdot 4 - 3$ $\quad a_5 = 4 \cdot 5 - 3$

$a_1 = 1$ $\quad a_2 = 5$ $\quad a_3 = 9$ $\quad a_4 = 13$ $\quad a_5 = 17$

The first five terms of the sequence are 1, 5, 9, 13, and 17.

TRY IT : : 12.1 Write the first five terms of the sequence whose general term is $a_n = 3n - 4$.

TRY IT : : 12.2 Write the first five terms of the sequence whose general term is $a_n = 2n - 5$.

For some sequences, the variable is an exponent.

EXAMPLE 12.2

Write the first five terms of the sequence whose general term is $a_n = 2^n + 1$.

Solution

We substitute the values 1, 2, 3, 4, and 5 into the formula, $a_n = 2^n + 1$, in order.

$a_n = 2^n + 1$ $\quad a_n = 2^n + 1$ $\quad a_n = 2^n + 1$ $\quad a_n = 2^n + 1$ $\quad a_n = 2^n + 1$

$a_1 = 2^1 + 1$ $\quad a_2 = 2^2 + 1$ $\quad a_3 = 2^3 + 1$ $\quad a_4 = 2^4 + 1$ $\quad a_5 = 2^5 + 1$

$a_1 = 3$ $\quad a_2 = 5$ $\quad a_3 = 9$ $\quad a_4 = 17$ $\quad a_5 = 33$

The first five terms of the sequence are 3, 5, 9, 17, and 33.

TRY IT : : 12.3 Write the first five terms of the sequence whose general term is $a_n = 3^n + 4$.

TRY IT : : 12.4 Write the first five terms of the sequence whose general term is $a_n = 2^n - 5$.

It is not uncommon to see the expressions $(-1)^n$ or $(-1)^{n+1}$ in the general term for a sequence. If we evaluate each of these expressions for a few values, we see that this expression alternates the sign for the terms.

n	1	2	3	4	5
$(-1)^n$	$(-1)^1$ -1	$(-1)^2$ 1	$(-1)^3$ -1	$(-1)^4$ 1	$(-1)^5$ -1
$(-1)^{n+1}$	$(-1)^{1+1}$ 1	$(-1)^{2+1}$ -1	$(-1)^{3+1}$ 1	$(-1)^{4+1}$ -1	$(-1)^{5+1}$ 1

$$\begin{array}{cccccc} a_1, & a_2, & a_3, & a_4, & a_5, \ldots, & a_n, \ldots \\ -1, & 1, & -1, & 1, & -1 \ldots & \\ 1, & -1, & 1, & -1, & 1 \ldots & \end{array}$$

The terms in the next example will alternate signs as a result of the powers of -1.

EXAMPLE 12.3

Write the first five terms of the sequence whose general term is $a_n = (-1)^n n^3$.

Solution

We substitute the values 1, 2, 3, 4, and 5 into the formula, $a_n = (-1)^n n^3$, in order.

$a_1 = (-1)^1 n^3$ $a_2 = (-1)^2 n^3$ $a_3 = (-1)^3 n^3$ $a_4 = (-1)^4 n^3$ $a_5 = (-1)^5 n^3$

$a_1 = (-1)^1 \cdot 1^3$ $a_2 = (-1)^2 \cdot 2^3$ $a_3 = (-1)^3 \cdot 3^3$ $a_4 = (-1)^4 \cdot 4^3$ $a_5 = (-1)^5 \cdot 5^3$

$a_1 = -1$ $a_2 = 8$ $a_3 = -27$ $a_4 = 64$ $a_5 = -125$

The first five terms of the sequence are $-1,\ 8,\ -27,\ 64,$ and -125.

TRY IT : : 12.5 Write the first five terms of the sequence whose general term is $a_n = (-1)^n n^2$.

TRY IT : : 12.6 Write the first five terms of the sequence whose general term is $a_n = (-1)^{n+1} n^3$.

Find a Formula for the General Term (*n*th Term) of a Sequence

Sometimes we have a few terms of a sequence and it would be helpful to know the general term or *n*th term. To find the general term, we look for patterns in the terms. Often the patterns involve multiples or powers. We also look for a pattern in the signs of the terms.

EXAMPLE 12.4

Find a general term for the sequence whose first five terms are shown.

$$4,\ 8,\ 12,\ 16,\ 20,\ \ldots$$

Solution

		4,	8,	12,	16,	20, ...
	n:	1,	2,	3,	4,	5, ...n
We look for a pattern in the terms.	Terms:	4,	8,	12,	16,	20, ...
The numbers are all multiples of 4.	Pattern: $4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, 4 \cdot 5, \ldots, 4 \cdot n$					
	The general term of the sequence is $a_n = 4n$.					

TRY IT : : 12.7 Find a general term for the sequence whose first five terms are shown.

$3,\ 6,\ 9,\ 12,\ 15,\ \ldots$

TRY IT : : 12.8 Find a general term for the sequence whose first five terms are shown.

$5,\ 10,\ 15,\ 20,\ 25,\ \ldots$

EXAMPLE 12.5

Find a general term for the sequence whose first five terms are shown.

$$2,\ -4,\ 8,\ -16,\ 32,\ \ldots$$

Solution

		2, −4, 8, −16, 32, …
	n:	1, 2, 3, 4, 5, …n
We look for a pattern in the terms.	Terms:	2, −4, 8, −16, 32, …
The numbers are powers of 2. The signs are alternating, with even n negative.	Pattern:	$(-1)^{1+1}2^1, (-1)^{2+1}2^2, (-1)^{3+1}2^3, (-1)^{4+1}2^4, (-1)^{5+1}2^5, \ldots, (-1)^{n+1}2^n$
		The general term of the sequence is $a_n = (-1)^{n+1}2^n$.

TRY IT : : 12.9 Find a general term for the sequence whose first five terms are shown.

$-3, 9, -27, 81, -243, \ldots$

TRY IT : : 12.10 Find a general term for the sequence whose first five terms are shown

$1, -4, 9, -16, 25, \ldots$

EXAMPLE 12.6

Find a general term for the sequence whose first five terms are shown.

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \cdots$$

Solution

		$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \cdots$
	n:	1, 2, 3, 4, 5, …n
We look for a pattern in the terms.	Terms:	$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \cdots$
The numerators are all 1.	Pattern:	$\frac{1}{3^1}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \frac{1}{3^5}, \cdots \frac{1}{3^n}$
The denominators are powers of 3.		The general term of the sequence is $a_n = \frac{1}{3^n}$.

TRY IT : : 12.11 Find a general term for the sequence whose first five terms are shown.

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots$

TRY IT : : 12.12 Find a general term for the sequence whose first five terms are shown.

$\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \cdots$

Use Factorial Notation

Sequences often have terms that are products of consecutive integers. We indicate these products with a special notation called *factorial notation*. For example, $5!$, read 5 factorial, means $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. The exclamation point is not punctuation here; it indicates the **factorial notation**.

Factorial Notation

If n is a positive integer, then $n!$ is

$$n! = n(n-1)(n-2)\ldots$$

We define $0!$ as 1, so $0! = 1$.

The values of $n!$ for the first 5 positive integers are shown.

$1!$	$2!$	$3!$	$4!$	$5!$
1	$2 \cdot 1$	$3 \cdot 2 \cdot 1$	$4 \cdot 3 \cdot 2 \cdot 1$	$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
1	2	6	24	120

EXAMPLE 12.7

Write the first five terms of the sequence whose general term is $a_n = \frac{1}{n!}$.

Solution

We substitute the values 1, 2, 3, 4, 5 into the formula, $a_n = \frac{1}{n!}$, in order.

$a_1 = \frac{1}{n!}$ $a_2 = \frac{1}{n!}$ $a_3 = \frac{1}{n!}$ $a_4 = \frac{1}{n!}$ $a_5 = \frac{1}{n!}$

$a_1 = \frac{1}{1!}$ $a_2 = \frac{1}{2!}$ $a_3 = \frac{1}{3!}$ $a_4 = \frac{1}{4!}$ $a_5 = \frac{1}{5!}$

$a_1 = \frac{1}{1}$ $a_2 = \frac{1}{2 \cdot 1}$ $a_3 = \frac{1}{3 \cdot 2 \cdot 1}$ $a_4 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1}$ $a_5 = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$a_1 = 1$ $a_2 = \frac{1}{2}$ $a_3 = \frac{1}{6}$ $a_4 = \frac{1}{24}$ $a_5 = \frac{1}{120}$

The first five terms of the sequence are $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}$.

TRY IT : : 12.13 Write the first five terms of the sequence whose general term is $a_n = \frac{2}{n!}$.

TRY IT : : 12.14 Write the first five terms of the sequence whose general term is $a_n = \frac{3}{n!}$.

When there is a fraction with factorials in the numerator and denominator, we line up the factors vertically to make our calculations easier.

EXAMPLE 12.8

Write the first five terms of the sequence whose general term is $a_n = \frac{(n+1)!}{(n-1)!}$.

Solution

We substitute the values 1, 2, 3, 4, 5 into the formula, $a_n = \frac{(n+1)!}{(n-1)!}$, in order.

$$a_n=\frac{(n+1)!}{(n-1)!} \quad a_n=\frac{(n+1)!}{(n-1)!} \quad a_n=\frac{(n+1)!}{(n-1)!} \quad a_n=\frac{(n+1)!}{(n-1)!} \quad a_n=\frac{(n+1)!}{(n-1)!}$$

$$a_1=\frac{(1+1)!}{(1-1)!} \quad a_2=\frac{(2+1)!}{(2-1)!} \quad a_3=\frac{(3+1)!}{(3-1)!} \quad a_4=\frac{(4+1)!}{(4-1)!} \quad a_5=\frac{(5+1)!}{(5-1)!}$$

$$a_1=\frac{(2)!}{(0)!} \quad a_2=\frac{(3)!}{(1)!} \quad a_3=\frac{(4)!}{(2)!} \quad a_4=\frac{(5)!}{(3)!} \quad a_5=\frac{(6)!}{(4)!}$$

$$a_1=\frac{2\cdot 1}{1} \quad a_2=\frac{3\cdot 2\cdot 1}{1} \quad a_3=\frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 1} \quad a_4=\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1} \quad a_5=\frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1}$$

$$a_1=2 \quad a_2=6 \quad a_3=12 \quad a_4=20 \quad a_5=30$$

The first five terms of the sequence are 2, 6, 12, 20, and 30.

TRY IT : : 12.15 Write the first five terms of the sequence whose general term is $a_n=\frac{(n-1)!}{(n+1)!}$.

TRY IT : : 12.16 Write the first five terms of the sequence whose general term is $a_n=\frac{n!}{(n+1)!}$.

Find the Partial Sum

Sometimes in applications, rather than just list the terms, it is important for us to add the terms of a sequence. Rather than just connect the terms with plus signs, we can use **summation notation**.

For example, $a_1+a_2+a_3+a_4+a_5$ can be written as $\sum_{i=1}^{5} a_i$. We read this as "the sum of *a* sub *i* from *i* equals one to five." The symbol $\sum$ means to add and the *i* is the index of summation. The 1 tells us where to start (initial value) and the 5 tells us where to end (terminal value).

Summation Notation

The sum of the first *n* terms of a sequence whose *n*th term is a_n is written in summation notation as:

$$\sum_{i=1}^{n} a_i=a_1+a_2+a_3+a_4+a_5+\ \ldots\ +a_n$$

The *i* is the index of summation and the 1 tells us where to start and the *n* tells us where to end.

When we add a finite number of terms, we call the sum a **partial sum**.

EXAMPLE 12.9

Expand the partial sum and find its value: $\sum_{i=1}^{5} 2i$.

Solution

$$\sum_{i=1}^{5} 2i$$

We substitute the values 1, 2, 3, 4, 5 in order.	$2\cdot 1+2\cdot 2+2\cdot 3+2\cdot 4+2\cdot 5$
Simplify.	$2+4+6+8+10$
Add.	30

$$\sum_{i=1}^{5} 2i = 30$$

TRY IT : : 12.17 Expand the partial sum and find its value: $\sum_{i=1}^{5} 3i$.

TRY IT : : 12.18 Expand the partial sum and find its value: $\sum_{i=1}^{5} 4i$.

The index does not always have to be *i* we can use any letter, but *i* and *k* are commonly used. The index does not have to start with 1 either—it can start and end with any positive integer.

EXAMPLE 12.10

Expand the partial sum and find its value: $\sum_{k=0}^{3} \frac{1}{k!}$.

Solution

$$\sum_{k=0}^{3} \frac{1}{k!}$$

We substitute the values 0, 1, 2, 3, in order.	$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}$
Evaluate the factorials.	$\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{6}$
Simplify.	$1+1+\frac{3}{6}+\frac{1}{6}$
Simplify.	$\frac{16}{6}$
Simplify.	$\frac{8}{3}$

$$\sum_{k=0}^{3} \frac{1}{k!} = \frac{8}{3}$$

TRY IT : : 12.19 Expand the partial sum and find its value: $\sum_{k=0}^{3} \frac{2}{k!}$.

TRY IT : : 12.20 Expand the partial sum and find its value: $\sum_{k=0}^{3} \frac{3}{k!}$.

Use Summation Notation to Write a Sum

In the last two examples, we went from summation notation to writing out the sum. Now we will start with a sum and change it to summation notation. This is very similar to finding the general term of a sequence. We will need to look at the terms and find a pattern. Often the patterns involve multiples or powers.

EXAMPLE 12.11

Write the sum using summation notation: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$.

Solution

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$$

n: 1, 2, 3, 4, 5

We look for a pattern in the terms.

Terms: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

The numerators are all one.

Pattern: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \frac{1}{n}$

The denominators are the counting numbers from one to fi e.

The sum written in summation notation is

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\sum_{n=1}^{5}\frac{1}{n}.$$

TRY IT : : 12.21 Write the sum using summation notation: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$.

TRY IT : : 12.22 Write the sum using summation notation: $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$.

When the terms of a sum have negative coefficients, we must carefully analyze the pattern of the signs.

EXAMPLE 12.12

Write the sum using summation notation: $-1+8-27+64-125$.

Solution

$$-1+8-27+64-125$$

	n:	1,	2,	3,	4,	5
We look for a pattern in the terms.	Terms:	−1,	8,	−27,	64,	−125
The signs of the terms alternate, and the odd terms are negative.	Pattern: $(-1)^1 \cdot 1$,	$(-1)^2 \cdot 8$,	$(-1)^3 \cdot 27$,	$(-1)^4 \cdot 64$,	$(-1)^5 \cdot 125$	
The numbers are the cubes of the counting numbers from one to five.	Pattern: $(-1)^1 \cdot 1^3$,	$(-1)^2 \cdot 2^3$,	$(-1)^3 \cdot 3^3$,	$(-1)^4 \cdot 4^3$,	$(-1)^5 \cdot 5^3$	
	Pattern: $(-1)^n \cdot n^3$					
	The sum written in summation notation is					
	$-1+8-27+64-125=\sum_{n=1}^{5}(-1)^n \cdot n^3$					

TRY IT : : 12.23 Write each sum using summation notation: $1-4+9-16+25$.

TRY IT : : 12.24 Write each sum using summation notation: $-2+4-6+8-10$.

MEDIA : :

Access this online resource for additional instruction and practice with sequences.

- **Series and Sequences-Finding Patterns (https://openstax.org/l/37serseqfindpat)**

12.1 EXERCISES

Practice Makes Perfect

Write the First Few Terms of a Sequence

In the following exercises, write the first five terms of the sequence whose general term is given.

1. $a_n = 2n - 7$

2. $a_n = 5n - 1$

3. $a_n = 3n + 1$

4. $a_n = 4n + 2$

5. $a_n = 2^n + 3$

6. $a_n = 3^n - 1$

7. $a_n = 3^n - 2n$

8. $a_n = 2^n - 3n$

9. $a_n = \frac{2^n}{n^2}$

10. $a_n = \frac{3^n}{n^3}$

11. $a_n = \frac{4n-2}{2^n}$

12. $a_n = \frac{3n+3}{3^n}$

13. $a_n = (-1)^n \cdot 2n$

14. $a_n = (-1)^n \cdot 3n$

15. $a_n = (-1)^{n+1} n^2$

16. $a_n = (-1)^{n+1} n^4$

17. $a_n = \frac{(-1)^{n+1}}{n^2}$

18. $a_n = \frac{(-1)^{n+1}}{2n}$

Find a Formula for the General Term (*n*th Term) of a Sequence

In the following exercises, find a general term for the sequence whose first five terms are shown.

19. 8, 16, 24, 32, 40, …

20. 7, 14, 21, 28, 35, …

21. 6, 7, 8, 9, 10, …

22. −3, −2, −1, 0, 1, …

23. $e^3, e^4, e^5, e^6, e^7, \ldots$

24. $\frac{1}{e^2}, \frac{1}{e}, 1, e, e^2, \ldots$

25. −5, 10, −15, 20, −25, …

26. −6, 11, −16, 21, −26, …

27. −1, 8, −27, 64, −125, …

28. 2, −5, 10, −17, 26, …

29. −2, 4, −6, 8, −10, …

30. 1, −3, 5, −7, 9, …

31. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1{,}024}, \ldots$

32. $\frac{1}{1}, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, \frac{1}{125}, \ldots$

33. $-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, -\frac{4}{5}, -\frac{5}{6}, \ldots$

34. $-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, -\frac{6}{5}, \ldots$

35. $-\frac{5}{2}, -\frac{5}{4}, -\frac{5}{8}, -\frac{5}{16}, -\frac{5}{32}, \ldots$

36. $4, \frac{1}{2}, \frac{4}{27}, \frac{4}{64}, \frac{4}{125}, \ldots$

Use Factorial Notation

In the following exercises, using factorial notation, write the first five terms of the sequence whose general term is given.

37. $a_n = \frac{4}{n!}$

38. $a_n = \frac{5}{n!}$

39. $a_n = 3n!$

40. $a_n = 2n!$

41. $a_n = (2n)!$

42. $a_n = (3n)!$

43. $a_n = \frac{(n-1)!}{(n)!}$

44. $a_n = \frac{n!}{(n+1)!}$

45. $a_n = \frac{n!}{n^2}$

46. $a_n = \frac{n^2}{n!}$

47. $a_n = \frac{(n+1)!}{n^2}$

48. $a_n = \frac{(n+1)!}{2n}$

Find the Partial Sum

In the following exercises, expand the partial sum and find its value.

49. $\sum_{i=1}^{5} i^2$

50. $\sum_{i=1}^{5} i^3$

51. $\sum_{i=1}^{6} (2i+3)$

52. $\sum_{i=1}^{6} (3i-2)$

53. $\sum_{i=1}^{4} 2^i$

54. $\sum_{i=1}^{4} 3^i$

55. $\sum_{k=0}^{3} \frac{4}{k!}$

56. $\sum_{k=0}^{4} -\frac{1}{k!}$

57. $\sum_{k=1}^{5} k(k+1)$

58. $\sum_{k=1}^{5} k(2k-3)$

59. $\sum_{n=1}^{5} \frac{n}{n+1}$

60. $\sum_{n=1}^{4} \frac{n}{n+2}$

Use Summation Notation to write a Sum

In the following exercises, write each sum using summation notation.

61. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$

62. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$

63. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125}$

64. $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625}$

65. $2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5}$

66. $3 + \frac{3}{2} + 1 + \frac{3}{4} + \frac{3}{5} + \frac{1}{2}$

67. $3-6+9-12+15$

68. $-5+10-15+20-25$

69. $-2+4-6+8-10+\ \ldots\ +20$

70. $1-3+5-7+9+\ \ldots\ +21$

71. $14+16+18+20+22+24+26$

72. $9+11+13+15+17+19+21$

Writing Exercises

73. In your own words, explain how to write the terms of a sequence when you know the formula. Show an example to illustrate your explanation.

74. Which terms of the sequence are negative when the n^{th} term of the sequence is $a_n=(-1)^n(n+2)$?

75. In your own words, explain what is meant by $n!$ Show some examples to illustrate your explanation.

76. Explain what each part of the notation $\sum_{k=1}^{12} 2k$ means.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
write the first few terms of a sequence.			
find a formula for the *n*th term of a sequence.			
use factorial notation.			
find the partial sum.			
use summation notation to write a sum.			

ⓑ *If most of your checks were:*

...confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

...with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

...no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

12.2 Arithmetic Sequences

Learning Objectives

By the end of this section, you will be able to:

- Determine if a sequence is arithmetic
- Find the general term (n th term) of an arithmetic sequence
- Find the sum of the first n terms of an arithmetic sequence

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $4n-1$ for the integers 1, 2, 3, and 4.
 If you missed this problem, review **Example 1.6**.
2. Solve the system of equations: $\begin{cases} x+y=7 \\ 3x+4y=23 \end{cases}$.
 If you missed this problem, review **Example 4.9**.
3. If $f(n)=\frac{n}{2}(3n+5)$, find $f(1)+f(20)$.
 If you missed this problem, review **Example 3.49**.

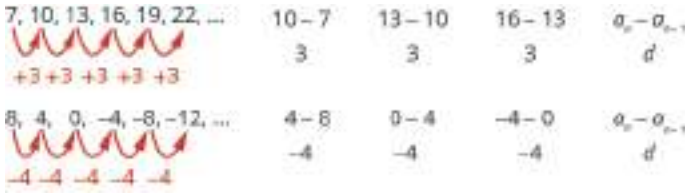
Determine if a Sequence is Arithmetic

The last section introduced sequences and now we will look at two specific types of sequences that each have special properties. In this section we will look at arithmetic sequences and in the next section, geometric sequences.

An **arithmetic sequence** is a sequence where the difference between consecutive terms is constant. The difference between consecutive terms in an arithmetic sequence, a_n-a_{n-1}, is d, the **common difference**, for n greater than or equal to two.

Arithmetic Sequence

An **arithmetic sequence** is a sequence where the difference between consecutive terms is always the same.
The difference between consecutive terms, a_n-a_{n-1}, is d, the **common difference**, for n greater than or equal to two.



In each of these sequences, the difference between consecutive terms is constant, and so the sequence is arithmetic.

EXAMPLE 12.13

Determine if each sequence is arithmetic. If so, indicate the common difference.

ⓐ $5, 9, 13, 17, 21, 25, \ldots$

ⓑ $4, 9, 12, 17, 20, 25, \ldots$

ⓒ $10, 3, -4, -11, -18, -25, \ldots$

Solution

To determine if the sequence is arithmetic, we find the difference of the consecutive terms shown.

ⓐ

	5,	9,	13,	17	21,	25, …
Find the diffe ence of the consecutive terms.	$9-5$	$13-9$	$17-13$	$21-17$	$25-21$	
	4	4	4	4	4	

The sequence is arithmetic. The common diffe ence is $d = 4$.

ⓑ

	4,	9,	12,	17	20,	25, …
Find the diffe ence of the consecutive terms.	$9-4$	$12-9$	$17-12$	$20-17$	$25-20$	
	2	3	5	3	5	

The sequence is not arithmetic as all the diffe ences between the consecutive terms are not the same.
There is no common diffe ence.

ⓒ

	10,	3,	-4,	-11	-18,	-25, …
Find the diffe ence of the consecutive terms.	$3-10$	$-4-3$	$-11-(-4)$	$-18-(-11)$	$-25-(-18)$	
	-7	-7	-7	-7	-7	

The sequence is arithmetic. The common diffe ence is $d = -7$.

TRY IT : : 12.25 Determine if each sequence is arithmetic. If so, indicate the common difference.

ⓐ 9, 20, 31, 42, 53, 64, … ⓑ 12, 6, 0, −6, −12, −18, … ⓒ 7, 1, 10, 4, 13, 7, …

TRY IT : : 12.26 Determine if each sequence is arithmetic. If so, indicate the common difference.

ⓐ −4, 4, 2, 10, 8, 16, … ⓑ −3, −1, 1, 3, 5, 7, … ⓒ 7, 2, −3, −8, −13, −18, …

If we know the first term, a_1, and the common difference, *d*, we can list a finite number of terms of the sequence.

EXAMPLE 12.14

Write the first five terms of the sequence where the first term is 5 and the common difference is $d = -6$.

Solution

We start with the first term and add the common difference. Then we add the common difference to that result to get the next term, and so on.

a_1	a_2	a_3	a_4	a_5
5	$5+(-6)$	$-1+(-6)$	$-7+(-6)$	$-13+(-6)$
	-1	-7	-13	-19

The sequence is $5, -1, -7, -13, -19, \ldots$

TRY IT : : 12.27

Write the first five terms of the sequence where the first term is 7 and the common difference is $d = -4$.

TRY IT : : 12.28

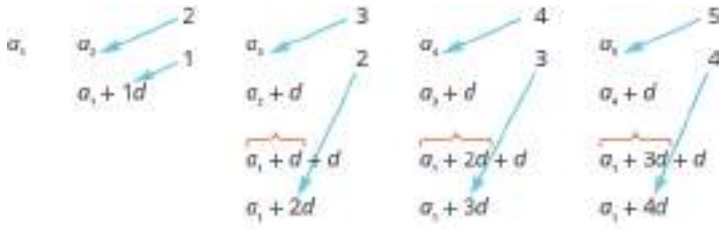
Write the first five terms of the sequence where the first term is 11 and the common difference is $d = -8$.

Find the General Term (*n*th Term) of an Arithmetic Sequence

Just as we found a formula for the general term of a sequence, we can also find a formula for the general term of an arithmetic sequence.

Let's write the first few terms of a sequence where the first term is a_1 and the common difference is *d*. We will then look for a pattern.

As we look for a pattern we see that each term starts with a_1.



The first term adds 0*d* to the a_1, the second term adds 1*d*, the third term adds 2*d*, the fourth term adds 3*d*, and the fifth term adds 4*d*. The number of *d*s that were added to a_1 is one less than the number of the term. This leads us to the following

$$a_n = a_1 + (n-1)d$$

General Term (*n*th term) of an Arithmetic Sequence

The general term of an arithmetic sequence with first term a_1 and the common difference *d* is

$$a_n = a_1 + (n-1)d$$

We will use this formula in the next example to find the 15th term of a sequence.

EXAMPLE 12.15

Find the fifteenth term of a sequence where the first term is 3 and the common difference is 6.

Solution

To find he fi teenth term, a_{15}, use the formula with $a_1 = 3$ and $d = 6$. $a_n = a_1 + (n-1)d$

Substitute in the values. $a_{15} = 3 + (15-1)6$

Simplify. $a_{15} = 3 + (14)6$

$a_{15} = 87$

TRY IT : : 12.29

Find the twenty-seventh term of a sequence where the first term is 7 and the common difference is 9.

TRY IT : : 12.30

Find the eighteenth term of a sequence where the first term is 13 and the common difference is -7.

Sometimes we do not know the first term and we must use other given information to find it before we find the requested term.

EXAMPLE 12.16

Find the twelfth term of a sequence where the seventh term is 10 and the common difference is -2. Give the formula for the general term.

Solution

To find the first term, a_1, use the formula with $a_7 = 10$, $n = 7$, and $d = -2$.	$a_n = a_1 + (n-1)d$
Substitute in the values.	$10 = a_1 + (7-1)(-2)$
Simplify.	$10 = a_1 + (6)(-2)$
	$10 = a_1 - 12$
	$a_1 = 22$
Find the twelfth term, a_{12}, using the formula with $a_1 = 22$, $n = 12$, and $d = -2$.	$a_n = a_1 + (n-1)d$
Substitute in the values.	$a_{12} = 22 + (12-1)(-2)$
Simplify.	$a_{12} = 22 + (11)(-2)$
	$a_{12} = 0$
	The twelfth term of the sequence is 0, $a_{12} = 0$.
To find the general term, substitute the values into the formula.	$a_n = a_1 + (n-1)d$
	$a_n = 22 + (n-1)(-2)$
	$a_n = 22 - 2n + 2$
	The general term is $a_n = -2n + 24$.

TRY IT : : 12.31

Find the eleventh term of a sequence where the ninth term is 8 and the common difference is -3. Give the formula for the general term.

TRY IT : : 12.32

Find the nineteenth term of a sequence where the fifth term is 1 and the common difference is -4. Give the formula for the general term.

Sometimes the information given leads us to two equations in two unknowns. We then use our methods for solving systems of equations to find the values needed.

EXAMPLE 12.17

Find the first term and common difference of a sequence where the fifth term is 19 and the eleventh term is 37. Give the formula for the general term.

Solution

Since we know two terms, we can make a system of equations using the formula for the general term.

	$a_n = a_1 + (n-1)d$
We know the value of a_5 and a_{11}, so we will use $n = 5$ and $n = 11$.	$a_5 = a_1 + (5-1)d$ $a_{11} = a_1 + (11-1)d$
Substitute in the values, $a_5 = 19$ and $a_{11} = 37$.	$19 = a_1 + (5-1)d$ $37 = a_1 + (11-1)d$
Simplify.	$19 = a_1 + 4d$ $37 = a_1 + 10d$
Prepare to eliminate the a_1 term by multiplying the top equation by -1. Add the equations.	$-19 = -a_1 - 4d$ $37 = a_1 + 10d$ $18 = 6d$ $3 = d$
Substituting $d = 3$ back into the first equation.	$19 = a_1 + 4 \cdot 3$
Solve for a_1.	$19 = a_1 + 12$ $7 = a_1$
Use the formula with $a_1 = 7$ and $d = 3$.	$a_n = a_1 + (n-1)d$
Substitute in the values.	$a_n = 7 + (n-1)3$
Simplify.	$a_n = 7 + 3n - 3$ $a_n = 3n + 4$
	The first term is $a_1 = 7$. The common difference is $d = 3$.
	The general term of the sequence is $a_n = 3n + 4$.

TRY IT : : 12.33

Find the first term and common difference of a sequence where the fourth term is 17 and the thirteenth term is 53. Give the formula for the general term.

TRY IT : : 12.34

Find the first term and common difference of a sequence where the third term is 2 and the twelfth term is -25. Give the formula for the general term.

Find the Sum of the First *n* Terms of an Arithmetic Sequence

As with the general sequences, it is often useful to find the sum of an arithmetic sequence. The sum, S_n, of the first n terms of any arithmetic sequence is written as $S_n = a_1 + a_2 + a_3 + \ldots + a_n$. To find the sum by merely adding all the terms can be tedious. So we can also develop a formula to find the sum of a sequence using the first and last term of the sequence.

We can develop this new formula by first writing the sum by starting with the first term, a_1, and keep adding a *d* to get the next term as:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ \ldots \ + a_n.$$

We can also reverse the order of the terms and write the sum by starting with a_n and keep subtracting *d* to get the next term as

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \ \ldots \ + a_1.$$

If we add these two expressions for the sum of the first *n* terms of an arithmetic sequence, we can derive a formula for the sum of the first *n* terms of any arithmetic series.

$$\begin{array}{rcccccccc} S_n &=& a_1 & + & (a_1+d) & + & (a_1+2d) & + \ \ldots \ + & a_n \\ +S_n &=& a_n & + & (a_n-d) & + & (a_n-2d) & + \ \ldots \ + & a_1 \\ \hline 2S_n &=& (a_1+a_n) & + & (a_1+a_n) & + & (a_1+a_n) & + \ \ldots \ + & (a_1+a_n) \end{array}$$

Because there are *n* sums of $(a_1 + a_n)$ on the right side of the equation, we rewrite the right side as $n(a_1 + a_n)$.

$$2S_n = n(a_1 + a_n)$$

We divide by two to solve for S_n.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

This give us a general formula for the sum of the first *n* terms of an arithmetic sequence.

Sum of the First *n* Terms of an Arithmetic Sequence

The sum, S_n, of the first *n* terms of an arithmetic sequence is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

where a_1 is the first term and a_n is the *n*th term.

We apply this formula in the next example where the first few terms of the sequence are given.

EXAMPLE 12.18

Find the sum of the first 30 terms of the arithmetic sequence: 8, 13, 18, 23, 28, ...

Solution

To find the sum, we will use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. We know $a_1 = 8$, $d = 5$ and $n = 30$, but we need to find a_n in order to use the sum formula.

	$a_n = a_1 + (n-1)d$
Find a_n where $a_1 = 8$, $d = 5$ and $n = 30$.	$a_{30} = 8 + (30-1)5$
Simplify.	$a_{30} = 8 + (29)5$
	$a_{30} = 153$
Knowing $a_1 = 8$, $n = 30$, and $a_{30} = 153$, use the sum formula.	$S_n = \frac{n}{2}(a_1 + a_n)$
Substitute in the values.	$S_{30} = \frac{30}{2}(8 + 153)$
Simplify.	$S_{30} = 15(161)$
Simplify.	$S_{30} = 2{,}415$

TRY IT : : 12.35 Find the sum of the first 30 terms of the arithmetic sequence: 5, 9, 13, 17, 21, ...

TRY IT : : 12.36 Find the sum of the first 30 terms of the arithmetic sequence: 7, 10, 13, 16, 19, ...

In the next example, we are given the general term for the sequence and are asked to find the sum of the first 50 terms.

EXAMPLE 12.19

Find the sum of the first 50 terms of the arithmetic sequence whose general term is $a_n = 3n - 4$.

Solution

To find the sum, we will use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. We know $n = 50,$ but we need to find a_1 and a_n in order to use the sum formula.

	$a_n = 3n - 4$
Find $a_1,$ by substituting $n = 1.$	$a_1 = 3 \cdot 1 - 4$ $a_1 = -1$
Find a_n by substituting $n = 50.$	$a_n = 3n - 4$ $a_{50} = 3 \cdot 50 - 4$
Simplify.	$a_{50} = 146$
Knowing $n = 50,$ $a_1 = -1,$ and $a_{50} = 146$ use the sum formula.	$S_n = \frac{n}{2}(a_1 + a_n)$
Substitute in the values.	$S_{50} = \frac{50}{2}(-1 + 146)$
Simplify.	$S_{50} = 25(145)$
Simplify.	$S_{50} = 3{,}625$

TRY IT : : 12.37

Find the sum of the first 50 terms of the arithmetic sequence whose general term is $a_n = 2n - 5.$

TRY IT : : 12.38

Find the sum of the first 50 terms of the arithmetic sequence whose general term is $a_n = 4n + 3.$

In the next example we are given the sum in summation notation. To add all the terms would be tedious, so we extract the information needed to use the formula to find the sum of the first *n* terms.

EXAMPLE 12.20

Find the sum: $\sum_{i=1}^{25} (4i + 7).$

Solution

To find the sum, we will use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. We know $n = 25,$ but we need to find a_1 and a_n in order to use the sum formula.

Expand the summation notation.	$\sum_{i=1}^{25} 4i+7 = (4\cdot 1+7)+(4\cdot 2+7)+(4\cdot 3+7)+\ldots+(4\cdot 25+7)$
Simplify.	$\sum_{i=1}^{25} 4i+7 = 11 + 15 + 19 + \ldots + 107$
Identify a_1.	$a_1 = 11$
Identify a_{25}.	$a_{25} = 107$
Knowing $n = 25,\ a_1 = 11,$ and $a_{25} = 107$ use the sum formula.	$S_n = \frac{n}{2}(a_1 + a_n)$
Substitute in the values.	$S_{25} = \frac{25}{2}(11 + 107)$
Simplify.	$S_{25} = \frac{25}{2}(118)$
Simplify.	$S_{25} = 1{,}475$

TRY IT : : 12.39 Find the sum: $\sum_{i=1}^{30} (6i - 4)$.

TRY IT : : 12.40 Find the sum: $\sum_{i=1}^{35} (5i - 3)$.

MEDIA : :

Access these online resources for additional instruction and practice with arithmetic sequences

- **Arithmetic Sequences (https://openstax.org/l/37ArithSequenc)**
- **Arithmetic Sequences: A Formula for the 'n-th' Term (https://openstax.org/l/37AritSeqnthter)**
- **Arithmetic Series (https://openstax.org/l/37ArithSeries)**

12.2 EXERCISES

Practice Makes Perfect

Determine if a Sequence is Arithmetic

In the following exercises, determine if each sequence is arithmetic, and if so, indicate the common difference.

77. $4, 12, 20, 28, 36, 44, \ldots$

78. $-7, -2, 3, 8, 13, 18, \ldots$

79. $-15, -16, 3, 12, 21, 30, \ldots$

80. $11, 5, -1, -7 - 13, -19, \ldots$

81. $8, 5, 2, -1, -4, -7, \ldots$

82. $15, 5, -5, -15, -25, -35, \ldots$

In the following exercises, write the first five terms of each sequence with the given first term and common difference.

83. $a_1 = 11$ and $d = 7$

84. $a_1 = 18$ and $d = 9$

85. $a_1 = -7$ and $d = 4$

86. $a_1 = -8$ and $d = 5$

87. $a_1 = 14$ and $d = -9$

88. $a_1 = -3$ and $d = -3$

Find the General Term (*n*th Term) of an Arithmetic Sequence

In the following exercises, find the term described using the information provided.

89. Find the twenty-first term of a sequence where the first term is three and the common difference is eight.

90. Find the twenty-third term of a sequence where the first term is six and the common difference is four.

91. Find the thirtieth term of a sequence where the first term is -14 and the common difference is five.

92. Find the fortieth term of a sequence where the first term is -19 and the common difference is seven.

93. Find the sixteenth term of a sequence where the first term is 11 and the common difference is -6.

94. Find the fourteenth term of a sequence where the first term is eight and the common difference is -3.

95. Find the twentieth term of a sequence where the fifth term is -4 and the common difference is -2. Give the formula for the general term.

96. Find the thirteenth term of a sequence where the sixth term is -1 and the common difference is -4. Give the formula for the general term.

97. Find the eleventh term of a sequence where the third term is 19 and the common difference is five. Give the formula for the general term.

98. Find the fifteenth term of a sequence where the tenth term is 17 and the common difference is seven. Give the formula for the general term.

99. Find the eighth term of a sequence where the seventh term is -8 and the common difference is -5. Give the formula for the general term.

100. Find the fifteenth term of a sequence where the tenth term is -11 and the common difference is -3. Give the formula for the general term.

In the following exercises, find the first term and common difference of the sequence with the given terms. Give the formula for the general term.

101. The second term is 14 and the thirteenth term is 47.

102. The third term is 18 and the fourteenth term is 73.

103. The second term is 13 and the tenth term is -51.

104. The third term is four and the tenth term is -38.

105. The fourth term is -6 and the fifteenth term is 27.

106. The third term is -13 and the seventeenth term is 15.

Find the Sum of the First *n* Terms of an Arithmetic Sequence

In the following exercises, find the sum of the first 30 terms of each arithmetic sequence.

107. 11, 14, 17, 20, 23, …

108. 12, 18, 24, 30, 36, …

109. 8, 5, 2, −1, −4, …

110. 16, 10, 4, −2, −8, …

111. −17, −15, −13, −11, −9, …

112. −15, −12, −9, −6, −3, …

In the following exercises, find the sum of the first 50 terms of the arithmetic sequence whose general term is given.

113. $a_n = 5n - 1$

114. $a_n = 2n + 7$

115. $a_n = -3n + 5$

116. $a_n = -4n + 3$

In the following exercises, find each sum.

117. $\sum_{i=1}^{40} (8i - 7)$

118. $\sum_{i=1}^{45} (7i - 5)$

119. $\sum_{i=1}^{50} (3i + 6)$

120. $\sum_{i=1}^{25} (4i + 3)$

121. $\sum_{i=1}^{35} (-6i - 2)$

122. $\sum_{i=1}^{30} (-5i + 1)$

Writing Exercises

123. In your own words, explain how to determine whether a sequence is arithmetic.

124. In your own words, explain how the first two terms are used to find the tenth term. Show an example to illustrate your explanation.

125. In your own words, explain how to find the general term of an arithmetic sequence.

126. In your own words, explain how to find the sum of the first n terms of an arithmetic sequence without adding all the terms.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
determine if a sequence is arithmetic.			
find the general term (nth term) of an arithmetic sequence.			
find the sum of the first *n* terms of an arithmetic sequence			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

12.3 Geometric Sequences and Series

Learning Objectives

By the end of this section, you will be able to:

- Determine if a sequence is geometric
- Find the general term (nth term) of a geometric sequence
- Find the sum of the first n terms of a geometric sequence
- Find the sum of an infinite geometric series
- Apply geometric sequences and series in the real world

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{24}{32}$.
 If you missed this problem, review **Example 1.24**.
2. Evaluate: ⓐ 3^4 ⓑ $\left(\frac{1}{2}\right)^4$.
 If you missed this problem, review **Example 1.19**.
3. If $f(x) = 4 \cdot 3^x$, find ⓐ $f(1)$ ⓑ $f(2)$ ⓒ $f(3)$.
 If you missed this problem, review **Example 3.49**.

Determine if a Sequence is Geometric

We are now ready to look at the second special type of sequence, the geometric sequence.

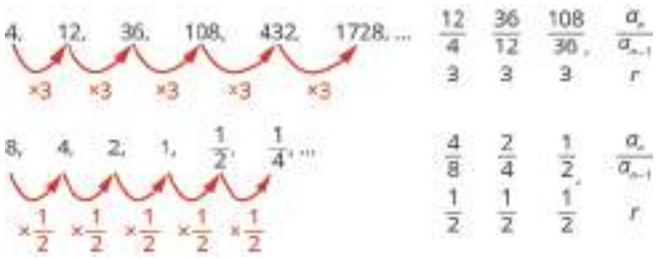
A sequence is called a **geometric sequence** if the ratio between consecutive terms is always the same. The ratio between consecutive terms in a geometric sequence is *r*, the **common ratio**, where *r* is greater than or equal to two.

Geometric Sequence

A **geometric sequence** is a sequence where the ratio between consecutive terms is always the same.

The ratio between consecutive terms, $\frac{a_n}{a_{n-1}}$, is *r*, the **common ratio**. *r* is greater than or equal to two.

Consider these sequences.



EXAMPLE 12.21

Determine if each sequence is geometric. If so, indicate the common ratio.

ⓐ 4, 8, 16, 32, 64, 128, …

ⓑ −2, 6, −12, 36, −72, 216, …

ⓒ 27, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, …

Solution

To determine if the sequence is geometric, we find the ratio of the consecutive terms shown.

ⓐ

$$4,\quad 8,\quad 16,\quad 32,\quad 64,\quad 128,\ \ldots$$

Find the ratio of the consecutive terms.

$$\begin{array}{ccccc} \frac{8}{4} & \frac{16}{8} & \frac{32}{16} & \frac{64}{32} & \frac{128}{64} \\ 2 & 2 & 2 & 2 & 2 \end{array}$$

The sequence is geometric. The common ratio is $r = 2$.

ⓑ

$$-2,\quad 6,\quad -12,\quad 36,\quad -72,\quad 216,\ \ldots$$

Find the ratio of the consecutive terms.

$$\begin{array}{ccccc} \frac{6}{-2} & \frac{-12}{6} & \frac{36}{-12} & \frac{-72}{36} & \frac{216}{-72} \\ -3 & -2 & -3 & -2 & -3 \end{array}$$

The sequence is not geometric. There is no common ratio.

ⓒ

$$27,\ 9,\ 3,\ 1,\ \frac{1}{3},\ \frac{1}{9},\ \ldots$$

Find the ratio of the consecutive terms.

$$\begin{array}{ccccc} \frac{9}{27} & \frac{3}{9} & \frac{1}{3} & \frac{\frac{1}{3}}{1} & \frac{\frac{1}{9}}{\frac{1}{3}} \end{array}$$

$$\begin{array}{ccccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}$$

The sequence is geometric. The common ratio is $r = \frac{1}{3}$.

TRY IT : : 12.41 Determine if each sequence is geometric. If so indicate the common ratio.

ⓐ $7,\ 21,\ 63,\ 189,\ 567,\ 1{,}701,\ \ldots$

ⓑ $64,\ 16,\ 4,\ 1,\ \frac{1}{4},\ \frac{1}{16},\ \ldots$

ⓒ $2,\ 4,\ 12,\ 48,\ 240,\ 1{,}440,\ \ldots$

TRY IT : : 12.42 Determine if each sequence is geometric. If so indicate the common ratio.

ⓐ $-150,\ -30,\ -15,\ -5,\ -\frac{5}{2},\ 0,\ \ldots$

ⓑ $5,\ 10,\ 20,\ 40,\ 80,\ 160,\ \ldots$

ⓒ $8,\ 4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \ldots$

If we know the first term, a_1, and the common ratio, r, we can list a finite number of terms of the sequence.

EXAMPLE 12.22

Write the first five terms of the sequence where the first term is 3 and the common ratio is $r = -2$.

Solution

We start with the first term and multiply it by the common ratio. Then we multiply that result by the common ratio to get the next term, and so on.

a_1	a_2	a_3	a_4	a_5
3	$3 \cdot (-2)$	$-6 \cdot (-2)$	$12 \cdot (-2)$	$-24 \cdot (-2)$
	-6	12	-24	48

The sequence is $3, -6, 12, -24, 48, \ldots$

TRY IT : : 12.43

Write the first five terms of the sequence where the first term is 7 and the common ratio is $r = -3$.

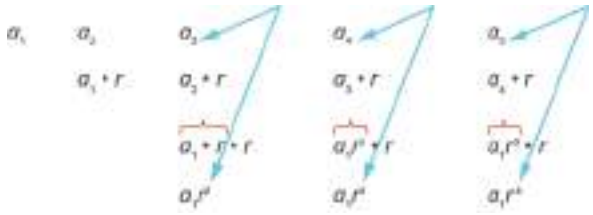
TRY IT : : 12.44

Write the first five terms of the sequence where the first term is 6 and the common ratio is $r = -4$.

Find the General Term (*n*th Term) of a Geometric Sequence

Just as we found a formula for the general term of a sequence and an arithmetic sequence, we can also find a formula for the general term of a geometric sequence.

Let's write the first few terms of the sequence where the first term is a_1 and the common ratio is *r*. We will then look for a pattern.



As we look for a pattern in the five terms above, we see that each of the terms starts with a_1.

The first term, a_1, is not multiplied by any *r*. In the second term, the a_1 is multiplied by *r*. In the third term, the a_1 is multiplied by *r* two times ($r \cdot r$ or r^2). In the fourth term, the a_1 is multiplied by *r* three times ($r \cdot r \cdot r$ or r^3) and in the fifth term, the a_1 is multiplied by *r* four times. In each term, the number of times a_1 is multiplied by *r* is one less than the number of the term. This leads us to the following

$$a_n = a_1 r^{n-1}$$

General Term (*n*th term) of a Geometric Sequence

The general term of a geometric sequence with first term a_1 and the common ratio *r* is

$$a_n = a_1 r^{n-1}$$

We will use this formula in the next example to find the fourteenth term of a sequence.

EXAMPLE 12.23

Find the fourteenth term of a sequence where the first term is 64 and the common ratio is $r = \frac{1}{2}$.

Solution

To find he fourteenth term, a_{14}, use the formula with $a_1 = 64$ and $r = \frac{1}{2}$.

$$a_n = a_1 r^{n-1}$$

Substitute in the values.

$$a_{14} = 64\left(\frac{1}{2}\right)^{14-1}$$

Simplify.

$$a_{14} = 64\left(\frac{1}{2}\right)^{13}$$

$$a_{14} = \frac{1}{128}$$

TRY IT : : 12.45

Find the thirteenth term of a sequence where the first term is 81 and the common ratio is $r = \frac{1}{3}$.

TRY IT : : 12.46

Find the twelfth term of a sequence where the first term is 256 and the common ratio is $r = \frac{1}{4}$.

Sometimes we do not know the common ratio and we must use the given information to find it before we find the requested term.

EXAMPLE 12.24

Find the twelfth term of the sequence 3, 6, 12, 24, 48, 96, ... Find the general term for the sequence.

Solution

To find the twelfth term, we use the formula, $a_n = a_1 r^{n-1}$, and so we need to first determine a_1 and the common ratio *r*.

$3, 6, 12, 24, 48, 96, \ldots$

The fir t term is three. $a_1 = 3$

Find the common ratio.

$$\frac{6}{3}\ \frac{12}{6}\ \frac{24}{12}\ \frac{48}{24}\ \frac{96}{48}$$

$2\ \ 2\ \ \ 2\ \ \ 2\ \ \ 2$

The common ratio is $r = 2$.

To find he twelfth term, a_{12}, use the formula with $a_1 = 3$ and $r = 2$.

$$a_n = a_1 r^{n-1}$$

Substitute in the values.

$$a_{12} = 3 \cdot 2^{12-1}$$

Simplify.

$$a_{12} = 3 \cdot 2^{11}$$

$$a_{12} = 6{,}144$$

Find the general term.

$$a_n = a_1 r^{n-1}$$

We use the formula with $a_1 = 3$ and $r = 2$.

$$a_n = 3(2)^{n-1}$$

TRY IT : : 12.47

Find the ninth term of the sequence 6, 18, 54, 162, 486, 1,458, ... Then find the general term for the sequence.

TRY IT : : 12.48

Find the eleventh term of the sequence 7, 14, 28, 56, 112, 224, ... Then find the general term for the sequence.

Find the Sum of the First *n* Terms of a Geometric Sequence

We found the sum of both general sequences and arithmetic sequence. We will now do the same for geometric sequences. The sum, S_n, of the first *n* terms of a geometric sequence is written as $S_n = a_1 + a_2 + a_3 + ... + a_n$. We can write this sum by starting with the first term, a_1, and keep multiplying by *r* to get the next term as:

$$S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1}$$

Let's also multiply both sides of the equation by *r*.

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^n$$

Next, we subtract these equations. We will see that when we subtract, all but the first term of the top equation and the last term of the bottom equation subtract to zero.

$$\begin{array}{rcl} S_n & = & a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ \ldots \ + a_1 r^{n-1} \\ rS_n & = & \qquad a_1 r + a_1 r^2 + a_1 r^3 + \ \ldots \ + a_1 r^{n-1} + a_1 r^n \\ \hline S_n - rS_n & = & a_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -a_1 r^n \end{array}$$

We factor both sides. $S_n(1-r) = a_1(1-r^n)$

To obtain the formula for S_n, divide both sides by $(1-r)$. $S_n = \dfrac{a_1(1-r^n)}{1-r}$

Sum of the First *n* Terms of a Geometric Series

The sum, S_n, of the first *n* terms of a geometric sequence is

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

where a_1 is the first term and *r* is the common ratio, and *r* is not equal to one.

We apply this formula in the next example where the first few terms of the sequence are given. Notice the sum of a geometric sequence typically gets very large when the common ratio is greater than one.

EXAMPLE 12.25

Find the sum of the first 20 terms of the geometric sequence 7, 14, 28, 56, 112, 224, ...

Solution

To find the sum, we will use the formula $S_n = \dfrac{a_1(1-r^n)}{1-r}$. We know $a_1 = 7$, $r = 2$, and $n = 20$.

Knowing $a_1 = 7$, $r = 2$, and $n = 20$, use the sum formula. $S_n = \dfrac{a_1(1-r^n)}{1-r}$

Substitute in the values. $S_{20} = \dfrac{7\left(1-2^{20}\right)}{1-2}$

Simplify. $S_{20} = 7{,}340{,}025$

TRY IT : : 12.49 Find the sum of the first 20 terms of the geometric sequence 3, 6, 12, 24, 48, 96, ...

TRY IT : : 12.50 Find the sum of the first 20 terms of the geometric sequence 6, 18, 54, 162, 486, 1,458, ...

In the next example, we are given the sum in summation notation. While adding all the terms might be possible, most

often it is easiest to use the formula to find the sum of the first *n* terms.

To use the formula, we need *r*. We can find it by writing out the first few terms of the sequence and find their ratio. Another option is to realize that in summation notation, a sequence is written in the form $\sum_{i=1}^{k} a(r)^i$, where *r* is the common ratio.

EXAMPLE 12.26

Find the sum: $\sum_{i=1}^{15} 2(3)^i$.

Solution

To find the sum, we will use the formula $S_n = \frac{a_1(1-r^n)}{1-r}$, which requires a_1 and *r*. We will write out a few of the terms, so we can get the needed information.

	$\sum_{i=1}^{15} 2(3)^i$
Write out the first few terms.	$2 \cdot 3^1, 2 \cdot 3^2, 2 \cdot 3^3, \ldots$ $6, \quad 18, \quad 54, \ldots$
Identify a_1.	$a_1 = 6$
Find the common ratio.	$\frac{18}{6} \quad \frac{54}{18}$ or $\sum_{i=1}^{15} 2(3)^{\text{ratio}}$ $3 \quad 3$ The common ratio is $r = 3$.
Knowing $a_1 = 6, \quad r = 3, \quad$ and $\quad n = 15,$ use the sum formula.	$S_n = \frac{a_1(1-r^n)}{1-r}$
Substitute in the values.	$S_{15} = \frac{6(1-3^{15})}{1-3}$
Simplify.	$S_{15} = 43{,}046{,}718$

TRY IT : : 12.51 Find the sum: $\sum_{i=1}^{15} 6(2)^i$.

TRY IT : : 12.52 Find the sum: $\sum_{i=1}^{10} 5(2)^i$.

Find the Sum of an Infinite Geometric Series

If we take a geometric sequence and add the terms, we have a sum that is called a geometric series. An **infinite geometric series** is an infinite sum whose first term is a_1 and common ratio is *r* and is written

$$a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} + \ldots$$

Infinite Geometric Series

An **infinite geometric series** is an infinite sum whose first term is a_1 and common ratio is *r* and is written

$$a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} + \ldots$$

We know how to find the sum of the first n terms of a geometric series using the formula, $S_n = \frac{a_1(1-r^n)}{1-r}$. But how do we find the sum of an infinite sum?

Let's look at the infinite geometric series $3 + 6 + 12 + 24 + 48 + 96 + \ldots$. Each term gets larger and larger so it makes sense that the sum of the infinite number of terms gets larger. Let's look at a few partial sums for this series. We see $a_1 = 3$ and $r = 2$

$$\begin{aligned} S_n &= \frac{a_1(1-r^n)}{1-r} & S_n &= \frac{a_1(1-r^n)}{1-r} & S_n &= \frac{a_1(1-r^n)}{1-r} \\ S_{10} &= \frac{3\left(1-2^{10}\right)}{1-2} & S_{30} &= \frac{3\left(1-2^{30}\right)}{1-2} & S_{50} &= \frac{3\left(1-2^{50}\right)}{1-2} \\ S_{10} &= 3{,}069 & S_{30} &= 3{,}221{,}225{,}469 & S_{50} &\approx 3.38 \times 10^{15} \end{aligned}$$

As n gets larger and larger, the sum gets larger and larger. This is true when $|r| \geq 1$ and we call the series divergent. We cannot find a sum of an infinite geometric series when $|r| \geq 1$.

Let's look at an infinite geometric series whose common ratio is a fraction less than one, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots$. Here the terms get smaller and smaller as n gets larger. Let's look at a few finite sums for this series. We see $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$.

$$\begin{aligned} S_n &= \frac{a_1(1-r^n)}{1-r} & S_n &= \frac{a_1(1-r^n)}{1-r} & S_n &= \frac{a_1(1-r^n)}{1-r} \\ S_{10} &= \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}} & S_{20} &= \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{20}\right)}{1-\frac{1}{2}} & S_{30} &= \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{30}\right)}{1-\frac{1}{2}} \\ S_{10} &\approx .9990234375 & S_{20} &\approx 0.9999990463 & S_{30} &\approx 0.9999999991 \end{aligned}$$

Notice the sum gets larger and larger but also gets closer and closer to one. When $|r| < 1$, the expression r^n gets smaller and smaller. In this case, we call the series convergent. As n approaches infinity, (gets infinitely large), r^n gets closer and closer to zero. In our sum formula, we can replace the r^n with zero and then we get a formula for the sum, S, for an infinite geometric series when $|r| < 1$.

$$\begin{aligned} S_n &= \frac{a_1(1-r^n)}{1-r} \\ S &= \frac{a_1(1-0)}{1-r} \\ S &= \frac{a_1}{1-r} \end{aligned}$$

This formula gives us the sum of the infinite geometric sequence. Notice the S does not have the subscript n as in S_n as we are not adding a finite number of terms.

Sum of an Infinite Geometric Series

For an infinite geometric series whose first term is a_1 and common ratio r,

If $|r| < 1$, the sum is

$$S = \frac{a_1}{1-r}$$

If $|r| \geq 1$, the infinite eometric series does not have a sum. We say the series diverges.

EXAMPLE 12.27

Find the sum of the infinite geometric series $54 + 18 + 6 + 2 + \frac{2}{3} + \frac{2}{9} + \ldots$

Solution

To find the sum, we first have to verify that the common ratio $|r| < 1$ and then we can use the sum formula $S = \frac{a_1}{1-r}$.

Find the common ratio. $r = \frac{18}{54} \quad r = \frac{6}{18} \ldots$

$r = \frac{1}{3} \quad r = \frac{1}{3} \quad |r| < 1$

Identify a_1. $a_1 = 54$

Knowing $a_1 = 54$, $r = \frac{1}{3}$, use the sum formula. $S = \frac{a_1}{1-r}$

Substitute in the values. $S = \frac{54}{1-\frac{1}{3}}$

Simplify. $S = 81$

TRY IT : : 12.53 Find the sum of the infinite geometric series $48 + 24 + 12 + 6 + 3 + \frac{3}{2} + \ldots$

TRY IT : : 12.54 Find the sum of the infinite geometric series $64 + 16 + 4 + 1 + \frac{1}{4} + \frac{1}{16} + \ldots$

An interesting use of infinite geometric series is to write a repeating decimal as a fraction.

EXAMPLE 12.28

Write the repeating decimal $0.\bar{5}$ as a fraction.

Solution

Rewrite the $0.\overline{5}$ showing the repeating fi e.	$0.5555555555555\ldots$
Use place value to rewrite this as a sum.	$0.5 + 0.05 + 0.005 + 0.0005 + ...$
This is an infinite eometric series.	
Find the common ratio.	$r = \frac{0.05}{0.5} \quad r = \frac{0.005}{0.05} \ldots$
	$r = 0.1 \quad r = 0.1 \quad \lvert r\rvert < 1$
Identify a_1.	$a_1 = 0.5$
Knowing $a_1 = 0.5,\ r = 0.1,$ use the sum formula.	$S = \frac{a_1}{1-r}$
Substitute in the values.	$S = \frac{0.5}{1-0.1}$
Simplify.	$S = \frac{0.5}{0.9}$
Multiply numerator and denominator by 10.	$S = \frac{5}{9}$
We are asked to find he fraction form.	$0.\overline{5} = \frac{5}{9}$

TRY IT : : 12.55 Write the repeating decimal $0.\overline{4}$ as a fraction.

TRY IT : : 12.56 Write the repeating decimal $0.\overline{8}$ as a fraction.

Apply Geometric Sequences and Series in the Real World

One application of geometric sequences has to do with consumer spending. If a tax rebate is given to each household, the effect on the economy is many times the amount of the individual rebate.

EXAMPLE 12.29

The government has decided to give a $1,000 tax rebate to each household in order to stimulate the economy. The government statistics say that each household will spend 80% of the rebate in goods and services. The businesses and individuals who benefitted from that 80% will then spend 80% of what they received and so on. The result is called the multiplier effect. What is the total effect of the rebate on the economy?

Solution

Every time money goes into the economy, 80% of it is spent and is then in the economy to be spent. Again, 80% of this money is spent in the economy again. This situation continues and so leads us to an infinite geometric series.

$$1000 + 1000(0.8) + 1000(0.8)^2 + \ldots$$

Here the first term is 1,000, $a_1 = 1000$. The common ratio is 0.8, $r = 0.8$. We can evaluate this sum since $0.8 < 1$. We use the formula for the sum on an infinite geometric series.

	$S = \frac{a_1}{1-r}$
Substitute in the values, $a_1 = 1{,}000$ and $r = 0.8$.	$S = \frac{1{,}000}{1-0.8}$
Evaluate.	$S = 5{,}000$

The total effect of the $1,000 received by each household will be a $5,000 growth in the economy.

TRY IT : : 12.57

What is the total effect on the economy of a government tax rebate of $1,000 to each household in order to stimulate the economy if each household will spend 90% of the rebate in goods and services?

TRY IT : : 12.58

What is the total effect on the economy of a government tax rebate of $500 to each household in order to stimulate the economy if each household will spend 85% of the rebate in goods and services?

We have looked at a compound interest formula where a principal, *P*, is invested at an interest rate, *r*, for *t* years. The new balance, *A*, is $A = P\left(1 + \frac{r}{n}\right)^{nt}$ when interest is compounded *n* times a year. This formula applies when a lump sum was invested upfront and tells us the value after a certain time period.

An **annuity** is an investment that is a sequence of equal periodic deposits. We will be looking at annuities that pay the interest at the time of the deposits. As we develop the formula for the value of an annuity, we are going to let $n = 1$. That means there is one deposit per year.

$$\begin{aligned} & & A &= P\left(1 + \frac{r}{n}\right)^{nt} \\ \text{Let } n = 1. & & A &= P\left(1 + \frac{r}{1}\right)^{1t} \\ \text{Simplify.} & & A &= P(1 + r)^t \end{aligned}$$

Suppose *P* dollars is invested at the end of each year. One year later that deposit is worth $P(1 + r)^1$ dollars, and another year later it is worth $P(1 + r)^2$ dollars. After *t* years, it will be worth $A = P(1 + r)^t$ dollars.

	End of year 1	End of year 2	End of year 3
First Deposit *P* @ end of year 1	*P*	Amount 1 year later $P(1 + r)^1$	Amount 2 years later $P(1 + r)^2$
2nd Deposit *P* @ end of year 2		*P*	Amount 1 year later $P(1 + r)^1$
3rd Deposit *P* @ end of year 3			*P*

After three years, the value of the annuity is



This a sum of the terms of a geometric sequence where the first term is *P* and the common ratio is $1 + r$. We substitute these values into the sum formula. Be careful, we have two different uses of *r*. The *r* in the sum formula is the common ratio of the sequence. In this case, that is $1 + r$ where *r* is the interest rate.

$$\begin{aligned} & & S_t &= \frac{a_1(1 - r^t)}{1 - r} \\ \text{Substitute in the values.} & & S_t &= \frac{P\left(1 - (1 + r)^t\right)}{1 - (1 + r)} \\ \text{Simplify.} & & S_t &= \frac{P\left(1 - (1 + r)^t\right)}{-r} \\ & & S_t &= \frac{P\left((1 + r)^t - 1\right)}{r} \end{aligned}$$

Remember our premise was that one deposit was made at the end of each year.

We can adapt this formula for *n* deposits made per year and the interest is compounded *n* times a year.

Value of an Annuity with Interest Compounded n Times a Year

For a principal, P, invested at the end of a compounding period, with an interest rate, r, which is compounded n times a year, the new balance, A, after t years, is

$$A_t = \frac{P\left(\left(1+\frac{r}{n}\right)^{nt}-1\right)}{\frac{r}{n}}$$

EXAMPLE 12.30

New parents decide to invest $100 per month in an annuity for their baby daughter. The account will pay 5% interest per year which is compounded monthly. How much will be in the child's account at her eighteenth birthday?

Solution

To find the Annuity formula, $A_t = \frac{P\left(\left(1+\frac{r}{n}\right)^{nt}-1\right)}{\frac{r}{n}}$, we need to identify P, r, n, and t.

Identify P, the amount invested each month.	$P = 100$
Identify r, the annual interest rate, in decimal form.	$r = 0.05$
Identify n, the number of times the deposit will be made and the interest compounded each year.	$n = 12$
Identify t, the number of years.	$t = 18$
Knowing $P = 100$, $r = 0.05$, $n = 12$ and $t = 18$, use the sum formula.	$A_t = \frac{P\left(\left(1+\frac{r}{n}\right)^{nt}-1\right)}{\frac{r}{n}}$
Substitute in the values.	$A_t = \frac{100\left(\left(1+\frac{0.05}{12}\right)^{12\cdot 18}-1\right)}{\frac{0.05}{12}}$
Use the calculator to evaluate. Be sure to use parentheses as needed.	$A_t = 34{,}920.20$ The child will have $34,920.20 when she turns 18.

TRY IT : : 12.59

New grandparents decide to invest $200 per month in an annuity for their grandson. The account will pay 5% interest per year which is compounded monthly. How much will be in the child's account at his twenty-first birthday?

TRY IT : : 12.60

Arturo just got his first full-time job after graduating from college at age 27. He decided to invest $200 per month in an IRA (an annuity). The interest on the annuity is 8%, which is compounded monthly. How much will be in the Arturo's account when he retires at his sixty-seventh birthday?

MEDIA : :

Access these online resources for additional instruction and practice with sequences.

- **Geometric Sequences (https://openstax.org/l/37GeomSequence)**
- **Geometric Series (https://openstax.org/l/37GeometricSer)**
- **Future Value Annuities and Geometric Series (https://openstax.org/l/37FutValAnnGeoS)**
- **Application of a Geometric Series: Tax Rebate (https://openstax.org/l/37AppGeomSerTax)**

12.3 EXERCISES

Practice Makes Perfect

Determine if a Sequence is Geometric

In the following exercises, determine if the sequence is geometric, and if so, indicate the common ratio.

127. $3, 12, 48, 192, 768, 3072, \ldots$

128. $2, 10, 50, 250, 1250, 6250, \ldots$

129. $48, 24, 12, 6, 3, \frac{3}{2}, \ldots$

130. $54, 18, 6, 2, \frac{2}{3}, \frac{2}{9}, \ldots$

131. $-3, 6, -12, 24, -48, 96, \ldots$

132. $2, -6, 18, -54, 162, -486, \ldots$

In the following exercises, determine if each sequence is arithmetic, geometric or neither. If arithmetic, indicate the common difference. If geometric, indicate the common ratio.

133. $48, 24, 12, 6, 3, \frac{3}{2}, \ldots$

134. $12, 6, 0, -6, -12, -18, \ldots$

135. $-7, -2, 3, 8, 13, 18, \ldots$

136. $5, 9, 13, 17, 21, \ldots$

137. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$

138. $4, 8, 12, 24, 48, 96, \ldots$

In the following exercises, write the first five terms of each geometric sequence with the given first term and common ratio.

139. $a_1 = 4$ and $r = 3$

140. $a_1 = 9$ and $r = 2$

141. $a_1 = -4$ and $r = -2$

142. $a_1 = -5$ and $r = -3$

143. $a_1 = 27$ and $r = \frac{1}{3}$

144. $a_1 = 64$ and $r = \frac{1}{4}$

Find the General Term (*n*th Term) of a Geometric Sequence

In the following exercises, find the indicated term of a sequence where the first term and the common ratio is given.

145. Find a_{11} given $a_1 = 8$ and $r = 3$.

146. Find a_{13} given $a_1 = 7$ and $r = 2$.

147. Find a_{10} given $a_1 = -6$ and $r = -2$.

148. Find a_{15} given $a_1 = -4$ and $r = -3$.

149. Find a_{10} given $a_1 = 100{,}000$ and $r = 0.1$.

150. Find a_8 given $a_1 = 1{,}000{,}000$ and $r = 0.01$.

In the following exercises, find the indicated term of the given sequence. Find the general term for the sequence.

151. Find a_9 of the sequence, $9, 18, 36, 72, 144, 288, \ldots$

152. Find a_{12} of the sequence, $5, 15, 45, 135, 405, 1215, \ldots$

153. Find a_{15} of the sequence, $-486, 162, -54, 18, -6, 2, \ldots$

154. Find a_{16} of the sequence, $224, -112, 56, -28, 14, -7, \ldots$

155. Find a_{10} of the sequence, $1, 0.1, 0.01, 0.001, 0.0001, 0.00001, \ldots$

156. Find a_9 of the sequence, $1000, 100, 10, 1, 0.1, 0.01, \ldots$

Find the Sum of the First *n* terms of a Geometric Sequence

In the following exercises, find the sum of the first fifteen terms of each geometric sequence.

157. 8, 24, 72, 216, 648, 1944, …

158. 7, 14, 28, 56, 112, 224, …

159. −6, 12, −24, 48, −96, 192, …

160. −4, 12, −36, 108, −324, 972, …

161. 81, 27, 9, 3, 1, $\frac{1}{3}$, …

162. 256, 64, 16, 4, 1, $\frac{1}{4}$, $\frac{1}{16}$, …

In the following exercises, find the sum of the geometric sequence.

163. $\sum_{i=1}^{15} (2)^i$

164. $\sum_{i=1}^{10} (3)^i$

165. $\sum_{i=1}^{9} 4(2)^i$

166. $\sum_{i=1}^{8} 5(3)^i$

167. $\sum_{i=1}^{10} 9\left(\frac{1}{3}\right)^i$

168. $\sum_{i=1}^{15} 4\left(\frac{1}{2}\right)^i$

Find the Sum of an Infinite Geometric Series

In the following exercises, find the sum of each infinite geometric series.

169. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}+\ \ldots$

170. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ \ldots$

171. $6-2+\frac{2}{3}-\frac{2}{9}+\frac{2}{27}-\frac{2}{81}+\ \ldots$

172. $-4+2-1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\ \ldots$

173. $6+12+24+48+96+192+\ \ldots$

174. $5+15+45+135+405+1215+\ \ldots$

175. $1{,}024+512+256+128+64+32+\ \ldots$

176. $6{,}561+2187+729+243+81+27+\ \ldots$

In the following exercises, write each repeating decimal as a fraction.

177. $0.\overline{3}$

178. $0.\overline{6}$

179. $0.\overline{7}$

180. $0.\overline{2}$

181. $0.\overline{45}$

182. $0.\overline{27}$

Apply Geometric Sequences and Series in the Real World

In the following exercises, solve the problem.

183. Find the total effect on the economy of each government tax rebate to each household in order to stimulate the economy if each household will spend the indicated percent of the rebate in goods and services.

	Tax rebate to each household	Percent spent on goods and services	Total Effect on the economy
ⓐ	$1,000	85%	
ⓑ	$1,000	85%	
ⓒ	$1,500	90%	
ⓓ	$1,500	80%	

184. New grandparents decide to invest $100 per month in an annuity for their grandchild. The account will pay 6% interest per year which is compounded monthly (12 times a year). How much will be in the child's account at their twenty-first birthday?

185. Berenice just got her first full-time job after graduating from college at age 30. She decided to invest $500 per quarter in an IRA (an annuity). The interest on the annuity is 7% which is compounded quarterly (4 times a year). How much will be in the Berenice's account when she retires at age 65?

186. Alice wants to purchase a home in about five years. She is depositing $500 a month into an annuity that earns 5% per year that is compounded monthly (12 times a year). How much will Alice have for her down payment in five years?

187. Myra just got her first full-time job after graduating from college. She plans to get a master's degree, and so is depositing $2,500 a year from her year-end bonus into an annuity. The annuity pays 6.5% per year and is compounded yearly. How much will she have saved in five years to pursue her master's degree?

Writing Exercises

188. In your own words, explain how to determine whether a sequence is geometric.

189. In your own words, explain how to find the general term of a geometric sequence.

190. In your own words, explain the difference between a geometric sequence and a geometric series.

191. In your own words, explain how to determine if an infinite geometric series has a sum and how to find it.

Self Check

I can...	Confidently	With some help	No-I don't get it!
determine if a sequence is geometric.			
find the general term (nth term) of a geometric sequence.			
find the sum of the first n terms of a geometric sequence.			
find the sum of an infinite geometric series.			
use geometric sequences to solve applications.			

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

12.4 Binomial Theorem

Learning Objectives

By the end of this section, you will be able to:

- Use Pascal's Triangle to expand a binomial
- Evaluate a binomial coefficient
- Use the Binomial Theorem to expand a binomial

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1}$.
 If you missed this problem, review **Example 1.25**.
2. Expand: $(3x+5)^2$.
 If you missed this problem, review **Example 5.32**.
3. Expand: $(x-y)^2$.
 If you missed this problem, review **Example 5.32**.

Use Pascal's Triangle to Expand a Binomial

In our previous work, we have squared binomials either by using FOIL or by using the Binomial Squares Pattern. We can also say that we expanded $(a+b)^2$.

$$(a+b)^2 = a^2 + 2ab + b^2$$

To expand $(a+b)^3$, we recognize that this is $(a+b)^2(a+b)$ and multiply.

$$\begin{gathered}(a+b)^3\\(a+b)^2(a+b)\\\left(a^2+2ab+b^2\right)(a+b)\\a^3+2a^2b+ab^2+a^2b+2ab^2+b^3\\a^3+3a^2b+3ab^2+b^3\\(a+b)^3 = a^3+3a^2b+3ab^2+b^3\end{gathered}$$

To find a method that is less tedious that will work for higher expansions like $(a+b)^7$, we again look for patterns in some expansions.

	Number of terms	First term	Last term
$(a+b)^1 = a+b$	2	a^1	b^1
$(a+b)^2 = a^2+2ab+b^2$	3	a^2	b^2
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$	4	a^3	b^3
$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$	5	a^4	b^4
$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$	6	a^5	b^5
$(a+b)^n$	n	a^n	b^n

Notice the first and last terms show only one variable. Recall that $a^0 = 1,$ so we could rewrite the first and last terms to include both variables. For example, we could expand $(a+b)^3$ to show each term with both variables.

$$(a+b)^3 = a^3b^0 + 3a^2b + 3ab^2 + a^0b^3$$

Generally, we don't show the zero exponents, just as we usually write *x* rather than 1*x*.

Patterns in the expansion of $(a+b)^n$

- The number of terms is $n+1$.
- The first term is a^n and the last term is b^n.
- The exponents on *a* decrease by one on each term going left to right.
- The exponents on *b* increase by one on each term going left to right.
- The sum of the exponents on any term is *n*.

Let's look at an example to highlight the last three patterns.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The exponents of *a* go from 5 → 4 → 3 → 2 → 1

The exponents of *b* go from 1 → 2 → 3 → 4 → 5

In each term, the sum of the exponents is 5.

From the patterns we identified, we see the variables in the expansion of $(a+b)^n,$ would be

$$(a+b)^n = a^n + ___a^{n-1}b^1 + ___a^{n-2}b^2 + \ldots + ___a^1 b^{n-1} + b^n.$$

To find the coefficients of the terms, we write our expansions again focusing on the coefficients. We rewrite the coefficients to the right forming an array of coefficients.

$(a+b)^0 = 1$ 1

$(a+b)^1 = 1a + 1b$ 1 1

$(a+b)^2 = 1a^2 + 2ab + 1b^2$ 1 2 1

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$ 1 3 3 1

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$ 1 4 6 4 1

$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$ 1 5 10 10 5 1

The array to the right is called **Pascal's Triangle**. Notice each number in the array is the sum of the two closest numbers in the row above. We can find the next row by starting and ending with one and then adding two adjacent numbers.

1

1 1

1 2 1

1 3 3 1

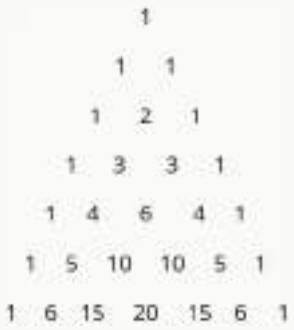
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

This triangle gives the coefficients of the terms when we expand binomials.

Pascal's Triangle



In the next example, we will use this triangle and the patterns we recognized to expand the binomial.

EXAMPLE 12.31

Use Pascal's Triangle to expand $(x+y)^6$.

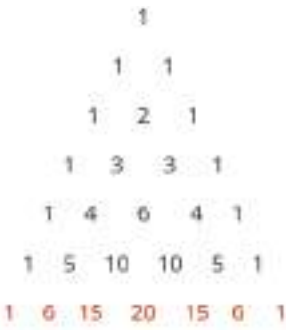
Solution

We know the variables for this expansion will follow the pattern we identified. The nonzero exponents of *x* will start at six and decrease to one. The nonzero exponents of *y* will start at one and increase to six. The sum of the exponents in each term will be six. In our pattern, $a = x$ and $b = y$.

$$(a+b)^n = a^n + ___a^{n-1}b^1 + ___a^{n-2}b^2 + \ldots + ___a^1 b^{n-1} + b^n$$

$$(x+y)^6 = x^6 + ___x^5 y^1 + ___x^4 y^2 + ___x^3 y^3 + ___x^2 y^4 + ___x^1 y^5 + y^6$$

To find the coefficients, we go to Pascal's Triangle and read off the coefficients from the row whose second entry is *n*, in this case, 6.



$(x+y)^6 = x^6 + __x^5y^1 + __x^4y^2 + __x^3y^3 + __x^2y^4 + __x^1y^5 + y^6$

$(x+y)^6 = 1x^6 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + 1y^6$

$(x+y)^6 = x^6 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + y^6$

TRY IT : : 12.61 Use Pascal's Triangle to expand $(x+y)^5$.

TRY IT : : 12.62 Use Pascal's Triangle to expand $(p+q)^7$.

In the next example we want to expand a binomial with one variable and one constant. We need to identify the *a* and *b* to carefully apply the pattern.

EXAMPLE 12.32

Use Pascal's Triangle to expand $(x+3)^5$.

Solution

We identify the *a* and *b* of the pattern.

$(a+b)^n$
$(x+3)^5$

In our pattern, $a = x$ and $b = 3$.

We know the variables for this expansion will follow the pattern we identified. The sum of the exponents in each term will be five.

$$(a+b)^n = a^n + ___a^{n-1}b^1 + ___a^{n-2}b^2 + \ldots + ___a^1 b^{n-1} + b^n$$

$$(x+3)^5 = x^5 + ___x^4 \cdot 3^1 + ___x^3 \cdot 3^2 + ___x^2 \cdot 3^3 + ___x^1 \cdot 3^4 + 3^5$$

To find the coefficients, we go to Pascal's Triangle and read off the coefficients from the row whose second entry is *n*, in this case, 5.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$(x+3)^5 = 1x^5 + 5 \cdot 3x^4 + 10 \cdot 9x^3 + 10 \cdot 27x^2 + 5 \cdot 81x^1 + 1 \cdot 243$

$(x+3)^5 = x^5 + 15x^4 + 90x^3 + 270x^2 + 405x^1 + 243$

TRY IT : : 12.63 Use Pascal's Triangle to expand $(x+2)^4$.

TRY IT : : 12.64 Use Pascal's Triangle to expand $(x+1)^6$.

In the next example, the binomial is a difference and the first term has a constant times the variable. Once we identify the *a* and *b* of the pattern, we must once again carefully apply the pattern.

EXAMPLE 12.33

Use Pascal's Triangle to expand $(3x-2)^4$.

Solution

We identify the *a* and *b* of the pattern.

$(a+b)^n$
$(3x-2)^4$

In our pattern, $a = 3x$ and $b = -2$.

To find the coefficients, we go to Pascal's Triangle and read off the coefficients from the row whose second entry is *n*, in this case, 4.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$(a+b)^n = a^n + ___a^{n-1}b^1 + ___a^{n-2}b^2 + \ldots + ___a^1 b^{n-1} + b^n$$
$$(3x-2)^4 = 1\cdot(^3 + 4(3x)^3(-2)^1 + 6(3x)^2(-2)^2 + 4(3x)^1(-2)^3 + 1\cdot(-2)^4$$
$$(3x-2)^4 = 81x^4 + 4\left(27x^3\right)(-2) + 6\left(9x^2\right)(4) + 4(3x)(-8) + 1\cdot 16$$
$$(3x-2)^4 = 81x^4 - 216x^3 + 216x^2 - 96x + 16$$

TRY IT : : 12.65 Use Pascal's Triangle to expand $(2x-3)^4$.

TRY IT : : 12.66 Use Pascal's Triangle to expand $(2x-1)^6$.

Evaluate a Binomial Coefficient

While Pascal's Triangle is one method to expand a binomial, we will also look at another method. Before we get to that, we need to introduce some more factorial notation. This notation is not only used to expand binomials, but also in the study and use of probability.

To find the coefficients of the terms of expanded binomials, we will need to be able to evaluate the notation $\binom{n}{r}$ which is called a binomial coefficient. We read $\binom{n}{r}$ as "*n* choose *r*" or "*n* taken *r* at a time".

Binomial Coefficient $\binom{n}{r}$

A binomial coefficient $\binom{n}{r}$, where *r* and *n* are integers with $0 \le r \le n$, is defined as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

We read $\binom{n}{r}$ as "*n* choose *r*" or "*n* taken *r* at a time".

EXAMPLE 12.34

Evaluate: ⓐ $\binom{5}{1}$ ⓑ $\binom{7}{7}$ ⓒ $\binom{4}{0}$ ⓓ $\binom{8}{5}$.

Solution

ⓐ We will use the definition of a binomial coefficient, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

$$\binom{5}{1}$$

Use the definition $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $n = 5,\ r = 1$.

$$\frac{5!}{1!(5-1)!}$$

Simplify.

$$\frac{5!}{1!(4)!}$$

Rewrite $5!$ as $5 \cdot 4!$

$$\frac{5 \cdot 4!}{1! \cdot 4!}$$

Simplify, by removing common factors.

$$\frac{5 \cdot \cancel{4!}}{1 \cdot \cancel{4!}}$$

Simplify.

$$5$$

$$\binom{5}{1} = 5$$

ⓑ

$$\binom{7}{7}$$

Use the definition $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $n = 7,\ r = 7$.

$$\frac{7!}{7!(7-7)!}$$

Simplify.

$$\frac{7!}{7!(0)!}$$

Simplify. Remember $0! = 1$.

$$1$$

$$\binom{7}{7} = 1$$

ⓒ

$$\binom{4}{0}$$

Use the definition $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $n = 4,\ r = 0$.

$$\frac{4!}{0!(4-0)!}$$

Simplify.

$$\frac{4!}{0!(4)!}$$

Simplify.

$$1$$

$$\binom{4}{0} = 1$$

ⓓ

$$\binom{8}{5}$$

Use the definition $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $n = 8,\ r = 5$.

$$\frac{8!}{5!(8-5)!}$$

Simplify.

$$\frac{8!}{5!(3)!}$$

Rewrite $8!$ as $8 \cdot 7 \cdot 6 \cdot 5!$ and remove common factors.

$$\frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5!}}{\cancel{5!} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}$$

Simplify.

$$56$$

$$\binom{8}{5} = 56$$

TRY IT : : 12.67 Evaluate each binomial coefficient:

ⓐ $\binom{6}{1}$ ⓑ $\binom{8}{8}$ ⓒ $\binom{5}{0}$ ⓓ $\binom{7}{3}$.

TRY IT : : 12.68 Evaluate each binomial coefficient:

ⓐ $\binom{2}{1}$ ⓑ $\binom{11}{11}$ ⓒ $\binom{9}{0}$ ⓓ $\binom{6}{5}$.

In the previous example, parts (a), (b), (c) demonstrate some special properties of binomial coefficients.

Properties of Binomial Coefficients

$$\binom{n}{1} = n \qquad \binom{n}{n} = 1 \qquad \binom{n}{0} = 1$$

Use the Binomial Theorem to Expand a Binomial

We are now ready to use the alternate method of expanding binomials. The **Binomial Theorem** uses the same pattern for the variables, but uses the binomial coefficient for the coefficient of each term.

Binomial Theorem

For any real numbers a and b, and positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

EXAMPLE 12.35

Use the Binomial Theorem to expand $(p+q)^4$.

Solution

We identify the a and b of the pattern.

$$(a+b)^n$$
$$(p+q)^4$$

In our pattern, $a = p$ and $b = q$.

We use the Binomial Theorem.

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

Substitute in the values $a = p,\ \ b = q$ and $n = 4$.

$$(p+q)^4 = \binom{4}{0}p^4 + \binom{4}{1}p^{4-1}q^1 + \binom{4}{2}p^{4-2}q^2 + \binom{4}{3}p^{4-3}q^3 + \binom{4}{4}q^4$$

Simplify the exponents.

$$(p+q)^4 = \binom{4}{0}p^4 + \binom{4}{1}p^3q + \binom{4}{2}p^2q^2 + \binom{4}{3}pq^3 + \binom{4}{4}q^4$$

Evaluate the coefficients. Remember, $\binom{n}{1} = n,\ \ \binom{n}{n} = 1,\ \ \binom{n}{0} = 1.$

$$(p+q)^4 = 1p^4 + 4p^3q^1 + \frac{4!}{2!(2)!}p^2q^2 + \frac{4!}{3!(4-3)!}p^1q^3 + 1q^4$$

$$(p+q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$$

TRY IT : : 12.69 Use the Binomial Theorem to expand $(x+y)^5$.

TRY IT : : 12.70 Use the Binomial Theorem to expand $(m+n)^6$.

Notice that when we expanded $(p+q)^4$ in the last example, using the Binomial Theorem, we got the same coefficients we would get from using Pascal's Triangle.

$$(p+q)^4 = \binom{4}{0}p^4 + \binom{4}{1}p^3q + \binom{4}{2}p^2q^2 + \binom{4}{3}pq^3 + \binom{4}{4}q^4$$

$$(p+q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The next example, the binomial is a difference. When the binomial is a difference, we must be careful in identifying the values we will use in the pattern.

EXAMPLE 12.36

Use the Binomial Theorem to expand $(x-2)^5$.

Solution

We identify the *a* and *b* of the pattern.

$$(a+b)^n$$
$$(x-2)^5$$

In our pattern, $a = x$ and $b = -2$.

We use the Binomial Theorem.

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

Substitute in the values $a = x,\ \ b = -2,\ \ $ and $n = 5$.

$$(x-2)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^{5-1}(-2)^1 + \binom{5}{2}x^{5-2}(-2)^2 + \binom{5}{3}x^{5-3}(-2)^3 + \binom{5}{4}x^{5-4}(-2)^4 + \binom{5}{5}(-2)^5$$

Simplify the exponents and evaluate the coefficients. Remember, $\binom{n}{1} = n,\ \ \binom{n}{n} = 1,\ \ \binom{n}{0} = 1.$

$$(x-2)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4(-2) + \binom{5}{2}x^3(-2)^2 + \binom{5}{3}x^2(-2)^3 + \binom{5}{4}x(-2)^4 + \binom{5}{5}(-2)^5$$

$$(x-2)^5 = 1x^5 + 5(-2)x^4 + \frac{5!}{2!\cdot 3!}(-2)^2x^3 + \frac{5!}{3!\cdot 2!}(-2)^3x^2 + \frac{5!}{4!\cdot 1!}(-2)^4x + 1(-2)^5$$

$$(x-2)^5 = x^5 + 5(-2)x^4 + 10\cdot 4\cdot x^3 + 10(-8)x^2 + 5\cdot 16\cdot x + 1(-32)$$

$$(x-2)^5 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$$

TRY IT : : 12.71 Use the Binomial Theorem to expand $(x-3)^5$.

TRY IT : : 12.72 Use the Binomial Theorem to expand $(y-1)^6$.

Things can get messy when both terms have a coefficient and a variable.

EXAMPLE 12.37

Use the Binomial Theorem to expand $(2x-3y)^4$.

Solution

We identify the *a* and *b* of the pattern.

$$(a+b)^n$$
$$(2x-3y)^4$$

In our pattern, $a = 2x$ and $b = -3y$.

We use the Binomial Theorem.

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

Substitute in the values $a = 2x$, $b = -3y$ and $n = 4$.

$$(2x-3y)^4 = \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^{4-1}(-3y)^1 + \binom{4}{2}(2x)^{4-2}(-3y)^2 + \binom{4}{3}(2x)^{4-3}(-3y)^3 + \binom{4}{4}(-3y)^4$$

Simplify the exponents.

$$(2x-3y)^4 = \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(-3y)^1 + \binom{4}{2}(2x)^2(-3y)^2 + \binom{4}{3}(2x)^1(-3y)^3 + \binom{4}{4}(-3y)^4$$

Evaluate the coefficients. Remember, $\binom{n}{1} = n$, $\binom{n}{n} = 1$, $\binom{n}{0} = 1$.

$$(2x-3y)^4 = 1(2x)^4 + 4(2x)^3(-3y)^1 + \frac{4!}{2!(2)!}(2x)^2(-3y)^2 + \frac{4!}{3!(4-3)!}(2x)^1(-3y)^3 + 1(-3y)^4$$

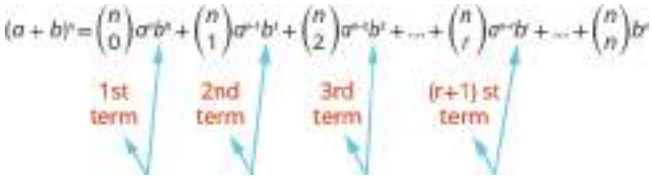
$$(2x-3y)^4 = 16x^4 + 4\cdot 8x^3(-3y) + 6\left(4x^2\right)\left(9y^2\right) + 4(2x)\left(-27y^3\right) + 81y^4$$

$$(2x-3y)^4 = 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

TRY IT : : 12.73 Use the Binomial Theorem to expand $(3x-2y)^5$.

TRY IT : : 12.74 Use the Binomial Theorem to expand $(4x-3y)^4$.

The real beauty of the Binomial Theorem is that it gives a formula for any particular term of the expansion without having to compute the whole sum. Let's look for a pattern in the Binomial Theorem.



Notice, that in each case the exponent on the *b* is one less than the number of the term. The $(r+1)st$ term is the term where the exponent of *b* is *r*. So we can use the format of the $(r+1)st$ term to find the value of a specific term.

Find a Specific Term in a Binomial Expansion

The $(r+1)^{st}$ term in the expansion of $(a+b)^n$ is

$$\binom{n}{r}a^{n-r}b^r$$

EXAMPLE 12.38

Find the fourth term of $(x+y)^7$.

Solution

In our pattern, $n=7$, $a=x$ and $b=y$.	$(a+b)^n$ $(x+y)^7$
We are looking for the fourth term. Since $r+1 = 4$, then $r = 3$.	
Write the formula.	$\binom{n}{r}a^{n-r}b^r$
Substitute in the values, $n=7$, $r=3$, $a=x$, and $b=y$.	$\binom{7}{3}a^{7-3}b^3$
Use $\binom{n}{r}=\frac{n!}{r!(n-r)!}$	$\frac{7!}{3!4!}a^{7-3}b^3$
Simplify.	$\frac{7\cdot6\cdot5\cdot\cancel{4!}}{\cancel{4!}\cdot3\cdot2\cdot1}a^4b^3$
Simplify.	$35a^4b^3$

TRY IT : : 12.75 Find the third term of $(x+y)^6$.

TRY IT : : 12.76 Find the fifth term of $(a+b)^8$.

EXAMPLE 12.39

Find the coefficient of the x^6 term of $(x+3)^9$.

Solution

In our pattern, then $n = 9,\ a = x,$ and $b = 3.$	$(a+b)^n$ $(x+3)^9$
We are looking for the coefficient of the x^6 term. Since $a = x,$ and $x^{9-r} = x^6,$ we know $r = 3.$	
Write the formula.	$\binom{n}{r} a^{n-r} b^r$
Substitute in the values, $n = 9,\ r = 3,\ a = x,$ and $b = 3.$	$\binom{9}{3} x^{9-3} \cdot 3^3$
Use $\binom{n}{r} = \frac{n!}{r!(n-r)!}$	$\frac{9!}{3!6!} x^{9-3} \cdot 3^3$
Simplify.	$\frac{9 \cdot 8 \cdot 7 \cdot 6!}{3! \cdot 2 \cdot 1 \cdot 6!} x^6 \cdot 27$
Simplify.	$9 \cdot 27 \cdot x^6$
Simplify.	$243x^6$
	The coefficient of the x^6 term is 243.

TRY IT : : 12.77 Find the coefficient of the x^5 term of $(x+4)^8.$

TRY IT : : 12.78 Find the coefficient of the x^4 term of $(x+2)^7.$

MEDIA : :

Access these online resources for additional instruction and practice with sequences.

- **Binomial Expansion Using Pascal's Triangle (https://openstax.org/l/37binexpanpastr)**
- **Binomial Coefficients (https://openstax.org/l/37binocoeffic)**

12.4 EXERCISES

Practice Makes Perfect

Use Pascal's Triangle to Expand a Binomial

In the following exercises, expand each binomial using Pascal's Triangle.

192. $(x+y)^4$

193. $(a+b)^8$

194. $(m+n)^{10}$

195. $(p+q)^9$

196. $(x-y)^5$

197. $(a-b)^6$

198. $(x+4)^4$

199. $(x+5)^3$

200. $(y+2)^5$

201. $(y+1)^7$

202. $(z-3)^5$

203. $(z-2)^6$

204. $(4x-1)^3$

205. $(3x-1)^5$

206. $(3x-4)^4$

207. $(3x-5)^3$

208. $(2x+3y)^3$

209. $(3x+5y)^3$

Evaluate a Binomial Coefficient

In the following exercises, evaluate.

210. ⓐ $\binom{8}{1}$ ⓑ $\binom{10}{10}$ ⓒ $\binom{6}{0}$ ⓓ $\binom{9}{3}$

211. ⓐ $\binom{7}{1}$ ⓑ $\binom{4}{4}$ ⓒ $\binom{3}{0}$ ⓓ $\binom{10}{8}$

212. ⓐ $\binom{3}{1}$ ⓑ $\binom{9}{9}$ ⓒ $\binom{7}{0}$ ⓓ $\binom{5}{3}$

213. ⓐ $\binom{4}{1}$ ⓑ $\binom{5}{5}$ ⓒ $\binom{8}{0}$ ⓓ $\binom{11}{9}$

Use the Binomial Theorem to Expand a Binomial

In the following exercises, expand each binomial.

214. $(x+y)^3$

215. $(m+n)^5$

216. $(a+b)^6$

217. $(s+t)^7$

218. $(x-2)^4$

219. $(y-3)^4$

220. $(p-1)^5$

221. $(q-4)^3$

222. $(3x-y)^5$

223. $(5x-2y)^4$

224. $(2x+5y)^4$

225. $(3x+4y)^5$

In the following exercises, find the indicated term in the expansion of the binomial.

226. Sixth term of $(x+y)^{10}$

227. Fifth term of $(a+b)^9$

228. Fourth term of $(x-y)^8$

229. Seventh term of $(x-y)^{11}$

In the following exercises, find the coefficient of the indicated term in the expansion of the binomial.

230. y^3 term of $(y+5)^4$

231. x^6 term of $(x+2)^8$

232. x^5 term of $(x-4)^6$

233. x^7 term of $(x-3)^9$

234. a^4b^2 term of $(2a+b)^6$

235. p^5q^4 term of $(3p+q)^9$

Writing Exercises

236. In your own words explain how to find the rows of the Pascal's Triangle. Write the first five rows of Pascal's Triangle.

237. In your own words, explain the pattern of exponents for each variable in the expansion of.

238. In your own words, explain the difference between $(a+b)^n$ and $(a-b)^n$.

239. In your own words, explain how to find a specific term in the expansion of a binomial without expanding the whole thing. Use an example to help explain.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
use Pascal's Triangle to expand a binomial.			
evaluate a binomial coefficient.			
use the binomial theorem to expand a binomial.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

CHAPTER 12 REVIEW

KEY TERMS

annuity An annuity is an investment that is a sequence of equal periodic deposits.

arithmetic sequence An arithmetic sequence is a sequence where the difference between consecutive terms is constant.

common difference The difference between consecutive terms in an arithmetic sequence, $a_n - a_{n-1}$, is d, the common difference, for n greater than or equal to two.

common ratio The ratio between consecutive terms in a geometric sequence, $\frac{a_n}{a_{n-1}}$, is r, the common ratio, where r greater than or equal to two.

finite sequence A sequence with a domain that is limited to a finite number of counting numbers.

general term of a sequence The general term of the sequence is the formula for writing the nth term of the sequence. The nth term of the sequence, a_n, is the term in the nth position where n is a value in the domain.

geometric sequence A geometric sequence is a sequence where the ratio between consecutive terms is always the same

infinite geometric series An infinite geometric series is an infinite sum infinite geometric sequence.

infinite sequence A sequence whose domain is all counting numbers and there is an infinite number of counting numbers.

partial sum When we add a finite number of terms of a sequence, we call the sum a partial sum.

sequence A sequence is a function whose domain is the counting numbers.

KEY CONCEPTS

12.1 Sequences

- **Factorial Notation**
 If n is a positive integer, then $n!$ is

 $$n! = n(n-1)(n-2)\ \ldots\ (3)(2)(1).$$

 We define $0!$ as 1, so $0! = 1$

- **Summation Notation**
 The sum of the first n terms of a sequence whose nth term a_n is written in summation notation as:

 $$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + \ \ldots\ + a_n$$

 The i is the index of summation and the 1 tells us where to start and the n tells us where to end.

12.2 Arithmetic Sequences

- **General Term (nth term) of an Arithmetic Sequence**
 The general term of an arithmetic sequence with first term a_1 and the common difference d is

 $$a_n = a_1 + (n-1)d$$

- **Sum of the First n Terms of an Arithmetic Sequence**
 The sum, S_n, of the first n terms of an arithmetic sequence, where a_1 is the first term and a_n is the nth term is

 $$S_n = \frac{n}{2}(a_1 + a_n)$$

12.3 Geometric Sequences and Series

- **General Term (nth term) of a Geometric Sequence:** The general term of a geometric sequence with first term a_1 and the common ratio r is

$$a_n = a_1 r^{n-1}$$

- **Sum of the First *n* Terms of a Geometric Series:** The sum, S_n, of the *n* terms of a geometric sequence is

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

where a_1 is the first term and *r* is the common ratio.

- **Infinite Geometric Series:** An infinite geometric series is an infinite sum whose first term is a_1 and common ratio is *r* and is written

$$a_1 + a_1 r + a_1 r^2 + \ \ldots\ + a_1 r^{n-1} + \ \ldots$$

- **Sum of an Infinite Geometric Series:** For an infinite geometric series whose first term is a_1 and common ratio *r*,

 If $|r| < 1$, the sum is

$$S = \frac{a_1}{1-r}$$

 We say the series converges.

 If $|r| \geq 1$, the infinite eometric series does not have a sum. We say the series diverges.

- **Value of an Annuity with Interest Compounded n Times a Year:** For a principal, *P*, invested at the end of a compounding period, with an interest rate, *r*, which is compounded *n* times a year, the new balance, *A*, after *t* years, is

$$A_t = \frac{P\left(\left(1+\frac{r}{n}\right)^{nt}-1\right)}{\frac{r}{n}}$$

12.4 Binomial Theorem

- **Patterns in the expansion of $(\mathbf{a+b})^n$**
 - The number of terms is $n+1$.
 - The first term is a^n and the last term is b^n.
 - The exponents on *a* decrease by one on each term going left to right.
 - The exponents on *b* increase by one on each term going left to right.
 - The sum of the exponents on any term is *n*.
- **Pascal's Triangle**

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

- **Binomial Coefficient $\binom{\mathbf{n}}{\mathbf{r}}$:** A binomial coefficient $\binom{n}{r}$, where *r* and *n* are integers with $0 \leq r \leq n$, is defined as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

We read $\binom{n}{r}$ as "*n* choose *r*" or "*n* taken *r* at a time".

- **Properties of Binomial Coefficients**
 $\binom{n}{1} = n \qquad \binom{n}{n} = 1 \qquad \binom{n}{0} = 1$
- **Binomial Theorem:** For any real numbers a, b, and positive integer n,
 $$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

REVIEW EXERCISES

12.1 Sequences

Write the First Few Terms of a Sequence

In the following exercises, write the first five terms of the sequence whose general term is given.

240. $a_n = 7n - 5$

241. $a_n = 3^n + 4$

242. $a_n = 2^n + n$

243. $a_n = \frac{2n+1}{4^n}$

244. $a_n = \frac{(-1)^n}{n^2}$

Find a Formula for the General Term (*n*th Term) of a Sequence

In the following exercises, find a general term for the sequence whose first five terms are shown.

245. $9, 18, 27, 36, 45, \ldots$

246. $-5, -4, -3, -2, -1, \ldots$

247. $\frac{1}{e^3}, \frac{1}{e^2}, \frac{1}{e}, 1, e, \ldots$

248. $1, -8, 27, -64, 125, \ldots$

249. $-\frac{1}{3}, -\frac{1}{2}, -\frac{3}{5}, -\frac{2}{3}, -\frac{5}{7}, \ldots$

Use Factorial Notation

In the following exercises, using factorial notation, write the first five terms of the sequence whose general term is given.

250. $a_n = 4n!$

251. $a_n = \frac{n!}{(n+2)!}$

252. $a_n = \frac{(n-1)!}{(n+1)^2}$

Find the Partial Sum

In the following exercises, expand the partial sum and find its value.

253. $\sum_{i=1}^{7}(2i-5)$

254. $\sum_{i=1}^{3}5^i$

255. $\sum_{k=0}^{4}\frac{4}{k!}$

256. $\sum_{k=1}^{4}(k+1)(2k+1)$

Use Summation Notation to write a Sum

In the following exercises, write each sum using summation notation.

257. $-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243}$

258. $4-8+12-16+20-24$

259. $4+2+\frac{4}{3}+1+\frac{4}{5}$

12.2 Arithmetic Sequences

Determine if a Sequence is Arithmetic

In the following exercises, determine if each sequence is arithmetic, and if so, indicate the common difference.

260. $1, 2, 4, 8, 16, 32, \ldots$

261. $-7, -1, 5, 11, 17, 23, \ldots$

262. $13, 9, 5, 1, -3, -7, \ldots$

In the following exercises, write the first five terms of each arithmetic sequence with the given first term and common difference.

263. $a_1 = 5$ and $d = 3$

264. $a_1 = 8$ and $d = -2$

265. $a_1 = -13$ and $d = 6$

Find the General Term (*n*th Term) of an Arithmetic Sequence

In the following exercises, find the term described using the information provided.

266. Find the twenty-fifth term of a sequence where the first term is five and the common difference is three.

267. Find the thirtieth term of a sequence where the first term is 16 and the common difference is -5.

268. Find the seventeenth term of a sequence where the first term is -21 and the common difference is two.

In the following exercises, find the indicated term and give the formula for the general term.

269. Find the eighteenth term of a sequence where the fifth term is 12 and the common difference is seven.

270. Find the twenty-first term of a sequence where the seventh term is 14 and the common difference is -3.

In the following exercises, find the first term and common difference of the sequence with the given terms. Give the formula for the general term.

271. The fifth term is 17 and the fourteenth term is 53.

272. The third term is -26 and the sixteenth term is -91. .

Find the Sum of the First *n* Terms of an Arithmetic Sequence

In the following exercises, find the sum of the first 30 terms of each arithmetic sequence.

273. $7, 4, 1, -2, -5, \ldots$

274. $1, 6, 11, 16, 21, \ldots$

In the following exercises, find the sum of the first fifteen terms of the arithmetic sequence whose general term is given.

275. $a_n = 4n + 7$

276. $a_n = -2n + 19$

In the following exercises, find each sum.

277. $\sum_{i=1}^{50}(4i-5)$

278. $\sum_{i=1}^{30}(-3i-7)$

279. $\sum_{i=1}^{35}(i+10)$

12.3 Geometric Sequences and Series

Determine if a Sequence is Geometric

In the following exercises, determine if the sequence is geometric, and if so, indicate the common ratio.

280. $3, 12, 48, 192, 768, 3072, \ldots$

281. $5, 10, 15, 20, 25, 30, \ldots$

282. $112, 56, 28, 14, 7, \frac{7}{2}, \ldots$

283. $9, -18, 36, -72, 144, -288, \ldots$

In the following exercises, write the first five terms of each geometric sequence with the given first term and common ratio.

284. $a_1 = -3$ and $r = 5$

285. $a_1 = 128$ and $r = \frac{1}{4}$

286. $a_1 = 5$ and $r = -3$

Find the General Term (*n*th Term) of a Geometric Sequence

In the following exercises, find the indicated term of a sequence where the first term and the common ratio is given.

287. Find a_9 given $a_1 = 6$ and $r = 2$.

288. Find a_{11} given $a_1 = 10{,}000{,}000$ and $r = 0.1$.

In the following exercises, find the indicated term of the given sequence. Find the general term of the sequence.

289. Find a_{12} of the sequence, $6, -24, 96, -384, 1536, -6144, \ldots$

290. Find a_9 of the sequence, $4374, 1458, 486, 162, 54, 18, \ldots$

Find the Sum of the First *n* terms of a Geometric Sequence

In the following exercises, find the sum of the first fifteen terms of each geometric sequence.

291. $-4, 8, -16, 32, -64, 128\ldots$

292. $3, 12, 48, 192, 768, 3072\ldots$

293. $3125, 625, 125, 25, 5, 1\ldots$

In the following exercises, find the sum

294. $\sum_{i=1}^{8}7(3)^i$

295. $\sum_{i=1}^{6}24\left(\frac{1}{2}\right)^i$

Find the Sum of an Infinite Geometric Series

In the following exercises, find the sum of each infinite geometric series.

296. $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243}+\frac{1}{729}-\ldots$

297. $49+7+1+\frac{1}{7}+\frac{1}{49}+\frac{1}{343}+\ldots$

In the following exercises, write each repeating decimal as a fraction.

298. $0.\overline{8}$

299. $0.\overline{36}$

Apply Geometric Sequences and Series in the Real World

In the following exercises, solve the problem.

300. What is the total effect on the economy of a government tax rebate of $\$360$ to each household in order to stimulate the economy if each household will spend 60% of the rebate in goods and services?

301. Adam just got his first full-time job after graduating from high school at age 17. He decided to invest $\$300$ per month in an IRA (an annuity). The interest on the annuity is 7% which is compounded monthly. How much will be in Adam's account when he retires at his sixty-seventh birthday?

12.4 Binomial Theorem

Use Pascal's Triangle to Expand a Binomial

In the following exercises, expand each binomial using Pascal's Triangle.

302. $(a+b)^7$

303. $(x-y)^4$

304. $(x+6)^3$

305. $(2y-3)^5$

306. $(7x+2y)^3$

Evaluate a Binomial Coefficient

In the following exercises, evaluate.

307.

ⓐ $\binom{11}{1}$

ⓑ $\binom{12}{12}$

ⓒ $\binom{13}{0}$

ⓓ $\binom{8}{3}$

308.

ⓐ $\binom{7}{1}$

ⓑ $\binom{5}{5}$

ⓒ $\binom{9}{0}$

ⓓ $\binom{9}{5}$

309.

ⓐ $\binom{1}{1}$

ⓑ $\binom{15}{15}$

ⓒ $\binom{4}{0}$

ⓓ $\binom{11}{2}$

Use the Binomial Theorem to Expand a Binomial

In the following exercises, expand each binomial, using the Binomial Theorem.

310. $(p+q)^6$

311. $(t-1)^9$

312. $(2x+1)^4$

313. $(4x+3y)^4$

314. $(x-3y)^5$

In the following exercises, find the indicated term in the expansion of the binomial.

315. Seventh term of $(a+b)^9$

316. Third term of $(x-y)^7$

In the following exercises, find the coefficient of the indicated term in the expansion of the binomial.

317. y^4 term of $(y+3)^6$

318. x^5 term of $(x-2)^8$

319. a^3b^4 term of $(2a+b)^7$

PRACTICE TEST

In the following exercises, write the first five terms of the sequence whose general term is given.

320. $a_n = \frac{5n-3}{3^n}$

321. $a_n = \frac{(n+2)!}{(n+3)!}$

322. Find a general term for the sequence, $-\frac{2}{3}, -\frac{4}{5}, -\frac{6}{7}, -\frac{8}{9}, -\frac{10}{11}, \ldots$

323. Expand the partial sum and find its value. $\sum_{i=1}^{4} (-4)^i$

324. Write the following using summation notation. $-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25}$

325. Write the first five terms of the arithmetic sequence with the given first term and common difference. $a_1 = -13$ and $d = 3$

326. Find the twentieth term of an arithmetic sequence where the first term is two and the common difference is -7.

327. Find the twenty-third term of an arithmetic sequence whose seventh term is 11 and common difference is three. Then find a formula for the general term.

328. Find the first term and common difference of an arithmetic sequence whose ninth term is -1 and the sixteenth term is -15. Then find a formula for the general term.

329. Find the sum of the first 25 terms of the arithmetic sequence, $5, 9, 13, 17, 21, \ldots$

330. Find the sum of the first 50 terms of the arithmetic sequence whose general term is $a_n = -3n + 100$.

331. Find the sum. $\sum_{i=1}^{40} (5i - 21)$

In the following exercises, determine if the sequence is arithmetic, geometric, or neither. If arithmetic, then find the common difference. If geometric, then find the common ratio.

332. $14, 3, -8, -19, -30, -41, \ldots$

333. $324, 108, 36, 12, 4, \frac{4}{3}, \ldots$

334. Write the first five terms of the geometric sequence with the given first term and common ratio. $a_1 = 6$ and $r = -2$

335. In the geometric sequence whose first term and common ratio are $a_1 = 5$ and $r = 4$, find a_{11}.

336. Find a_{10} of the geometric sequence, $1250, 250, 50, 10, 2, \frac{2}{5}, \ldots$. Then find a formula for the general term.

337. Find the sum of the first thirteen terms of the geometric sequence, $2, -6, 18, -54, 162, -486\ldots$

In the following exercises, find the sum.

338. $\sum_{i=1}^{9} 5(2)^i$

339. $1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \frac{1}{625} - \frac{1}{3125} + \ldots$

340. Write the repeating decimal as a fraction. $0.\overline{81}$

341. Dave just got his first full-time job after graduating from high school at age 18. He decided to invest \$450 per month in an IRA (an annuity). The interest on the annuity is 6% which is compounded monthly. How much will be in Adam's account when he retires at his sixty-fifth birthday?

342. Expand the binomial using Pascal's Triangle. $(m-2n)^5$

343. Evaluate each binomial coefficient. ⓐ $\binom{8}{1}$ ⓑ $\binom{16}{16}$ ⓒ $\binom{12}{0}$ ⓓ $\binom{10}{6}$

344. Expand the binomial using the Binomial Theorem. $(4x+5y)^3$

ANSWER KEY

Chapter 1

Try It

1.1. ⓐ yes ⓑ yes ⓒ no ⓓ yes ⓔ no

1.2. ⓐ no ⓑ yes ⓒ yes ⓓ no ⓔ no

1.3. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

1.4. $2 \cdot 2 \cdot 3 \cdot 5$

1.5. 36

1.6. 72

1.7. 16

1.8. 23

1.9. 86

1.10. 1

1.11. ⓐ 9 ⓑ 64 ⓒ 40

1.12. ⓐ 216 ⓑ 64 ⓒ 185

1.13. $10x^2 + 16x + 17$

1.14. $12y^2 + 9y + 7$

1.15. ⓐ $14x^2 - 13$ ⓑ $12x \div 2$ ⓒ $z + 13$ ⓓ $8x - 18$

1.16. ⓐ $17y^2 + 19$ ⓑ $7y$ ⓒ $x + 11$ ⓓ $11a - 14$

1.17. ⓐ $4(p + q)$ ⓑ $4p + q$

1.18. ⓐ $2x - 8$ ⓑ $2(x - 8)$

1.19. $w - 7$

1.20. $l - 6$

1.21. $4q - 8$

1.22. $7n + 3$

1.23. ⓐ $>$ ⓑ $>$ ⓒ $<$ ⓓ $=$

1.24. ⓐ $>$ ⓑ $=$ ⓒ $>$ ⓓ $<$

1.25. 16

1.26. 9

1.27. ⓐ -6 ⓑ 2 ⓒ -2

1.28. ⓐ -7 ⓑ 3 ⓒ -3

1.29. ⓐ 2 ⓑ -2 ⓒ -10 ⓓ 10

1.30. ⓐ 3 ⓑ -3 ⓒ -11 ⓓ 11

1.31. ⓐ 8, 8 ⓑ -18, -18 ⓒ 19, 19 ⓓ -4, -4

1.32. ⓐ 8, 8 ⓑ -22, -22 ⓒ 23, 23 ⓓ 3, 3

1.33. 3

1.34. 13

1.35. ⓐ 23 ⓑ 60 ⓒ -63 ⓓ -9

1.36. ⓐ 39 ⓑ 39 ⓒ -28 ⓓ -7

1.37. ⓐ 81 ⓑ -81

1.38. ⓐ 49 ⓑ -49

1.39. ⓐ 4 ⓑ 21

1.40. ⓐ 9 ⓑ 6

1.41. 31

1.42. 67

1.43. $(9 + (-16)) + 4;\ -3$

1.44. $(-8 + (-12)) + 7;\ -13$

1.45. The difference in temperatures was 45 degrees Fahrenheit.

1.46. The difference in temperatures was 9 degrees.

1.47. $-\frac{23}{40}$

1.48. $-\frac{5}{8}$

1.49. $-33a$

1.50. $-26b$

1.51. $\frac{4}{15}$

1.52. $\frac{2}{3}$

1.53. $\frac{3}{4b}$

1.54. $\frac{4}{q}$

1.55. $\frac{79}{60}$

1.56. $\frac{103}{60}$

1.57. ⓐ $\frac{27a - 32}{36}$ ⓑ $\frac{2a}{3}$

1.58. ⓐ $\frac{24k - 5}{30}$ ⓑ $\frac{2k}{15}$

1.59. 4

1.60. 2

1.61. $\frac{1}{90}$

1.62. 272

1.63. 2

1.64. $\frac{2}{7}$

1.65. $-\frac{1}{2}$

1.66. $\frac{2}{3}$

1.67. ⓐ 6.58 ⓑ 6.6 ⓒ 7

1.68. ⓐ 15.218 ⓑ 15.22 ⓒ 15.2

1.69. ⓐ -16.49 ⓑ -0.42

1.70. ⓐ -23.593 ⓑ -12.58

1.71. -27.4815

1.72. -87.6148

1.73. ⓐ 25.8 ⓑ 258 ⓒ 2,580

1.74. ⓐ 142 ⓑ 1,420 ⓒ 14,200

1.75. 587.3

1.76. 34.25

1.77. ⓐ $\frac{117}{500}$ ⓑ -0.875

1.78. ⓐ $\frac{3}{125}$ ⓑ -0.375

1.79. ⓐ 0.09, 0.87, 0.039 ⓑ 17%, 175%, 8.25%

1.80. ⓐ 0.03, 0.91, 0.083 ⓑ 41%, 225%, 9.25%

1.81. ⓐ 6 ⓑ 13 ⓒ -15

1.82. ⓐ 4 ⓑ 14 ⓒ -10

1.83. ⓐ $4, \sqrt{49}$ ⓑ $-3, 4, \sqrt{49}$ ⓒ $-3, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$ ⓓ $-\sqrt{2}$ ⓔ $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$

1.84. ⓐ $6, \sqrt{121}$ ⓑ $-\sqrt{25}, -1, 6, \sqrt{121}$ ⓒ $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}$ ⓓ 2.041975... ⓔ $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}$, 2.041975...

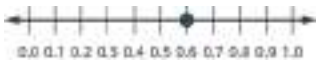
1.85.



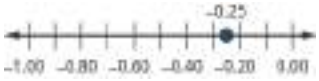
1.86.



1.87. ⓐ



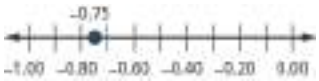
ⓑ



1.88. ⓐ



ⓑ



1.89. $32r + 29s$

1.90. $41m + 6n$

1.91. $1\frac{7}{15}$

1.92. $1\frac{2}{9}$

1.93. $-48a$

1.94. $-92x$

1.95. $\frac{5}{49}$

1.96. $\frac{11}{25}$

1.97. ⓐ 0 ⓑ undefined

1.98. ⓐ 0 ⓑ undefined

1.99. $4x + 8$

1.100. $6x + 42$

1.101. $5y + 3$

1.102. $4n + 9$

1.103. $70 + 15p$

1.104. $4 + 35d$

1.105. $-10 + 15a$

1.106. $-56 + 105y$

1.107. $-z + 11$

1.108. $-x + 4$

1.109. $3 - 3x$

1.110. $2x - 20$

1.111. $5x - 66$

1.112. $7x - 13$

Section Exercises

1. Divisible by 2, 3, 6

3. Divisible by 2

5. Divisible by 3, 5

7. $2 \cdot 43$

9. $5 \cdot 7 \cdot 13$

11. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

13. 24

15. 420

17. 440

19. 5

21. 58

23. 29

25. 149

27. 50

29. 5

31. ⓐ 64 ⓑ 16 ⓒ 7

33. 21

35. 9

37. $10x + 6$

39. $22a + 1$

41. $17x^2 + 20x + 16$

43. ⓐ $5x^2 - 6xy$ ⓑ $\frac{6y^2}{5x}$ ⓒ $y^2 + 21$ ⓓ $81x^2 - 6x$

45. ⓐ $4ab^2 + 3a^2b$ ⓑ $20xy^2$ ⓒ $m + 15$ ⓓ $121x^2 - 9x$ $9x < 121x^2$

47. ⓐ $8(y - 9)$ ⓑ $8y - 9$

49. ⓐ $5(3x + y)$ ⓑ $15x + y$

51. $14 > 2c$

53. $3n - 7$

55. Answers will vary.

57. Answers will vary.

59. ⓐ $>$ ⓑ $>$ ⓒ $>$ ⓓ $>$

61. ⓐ $=$ ⓑ $=$ ⓒ $>$ ⓓ $=$

63. 0

65. 8

67. 15

69. 1

71. ⓐ -11 ⓑ -3 ⓒ 3

73. 32

75. -22

77. 29

79. ⓐ 6 ⓑ -6 ⓒ -20 ⓓ 20

81. -59

83. 22

85. ⓐ 16 ⓑ16

87. ⓐ 45 ⓑ45

89. 0

91. 4

93. 6

95. ⓐ -32 ⓑ -65 ⓒ -4 ⓓ 13

97. ⓐ -4 ⓑ -12 ⓒ -39 ⓓ 14

99. ⓐ 64 ⓑ -64

101. -47

103. 90

105. 9

107. 41

109. -9

111. -29

113. 1

115. -3

117. 5

119. ⓐ -47 ⓑ 16

121. 121

123. -56

125. 6

127. $(3 + (-15)) + 7;\ -5$

129. ⓐ $10 - (-18);\ 28$ ⓑ $-25 - 11;\ -36$

131. $\frac{-6}{a + b}$

133. $96°$

135. 21

137. $-\$28$

139. Answers will vary.

141. Answers will vary.

143. $-\frac{12}{7}$

145. $\frac{10}{21}$

147. $\frac{2x^2}{3y}$

149. $-\frac{21a^2}{11b^2}$

151. $\frac{1}{3}$

153. $-\frac{21}{50}$

155. $\frac{11}{30}$

157. $9n$

159. $\frac{33}{4x}$

161. $-\frac{4}{9}$

163. $\frac{10u}{9v}$

165. $-\frac{1}{16}$

167. $-\frac{10}{9}$

169. $-\frac{2}{5}$

171. $\frac{2m}{3n}$

173. $\frac{29}{24}$

175. $\frac{1}{48}$

177. $\frac{17}{105}$

179. $-\frac{53}{40}$

181. $\frac{1}{12}$

183. $\frac{4x + 3}{12}$

185. ⓐ $\frac{5}{6}$ ⓑ 4

187. ⓐ $\frac{25n}{16}$ ⓑ $\frac{25n - 16}{30}$

189. ⓐ $\frac{-8x - 15}{18}$ ⓑ $-\frac{10k}{27}$

191. ⓐ $\frac{-5(a + 1)}{3}$ ⓑ a

193. $\frac{9}{7}$

195. -8

197. $\frac{11}{6}$

199. $\frac{5}{2}$

201. 54

203. $\frac{49}{25}$

205. $\frac{15}{4}$

207. $\frac{5}{21}$

209. $\frac{5}{4}$

211. $\frac{1}{24}$

213. $\frac{-28 - 15y}{60}$

215. $\frac{33}{64}$

217. $\frac{7}{9}$

219. -5

221. $\frac{23}{24}$

223. $\frac{11}{5}$

225. 1

227. $\frac{13}{3}$

229. ⓐ $\frac{1}{5}$ ⓑ $\frac{6}{5}$

231. $-\frac{1}{9}$

233. $-\frac{5}{11}$

235. Answers will vary.

237. Answers will vary.

239. ⓐ 5.78 ⓑ 5.8 ⓒ 6

241. ⓐ 0.30 ⓑ 0.3 ⓒ 0

243. ⓐ 63.48 ⓑ 63.5 ⓒ 63

245. -40.91

247. -7.22

249. -27.5

251. 02.212

253. 51.31

255. -4.89

257. -11.653

259. 337.8914

261. 1.305

263. 92.4

265. 2.5

267. 55200

269. \$2.44

271. -4.8

273. 2.08

275. $\frac{1}{25}$

277. $\frac{19}{200}$

279. 0.85

281. -12.4

283. 0.71

285. 0.393

287. 156%

289. 6.25%

291. 8

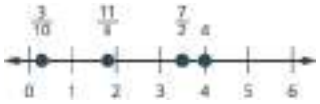
293. 12

295. -10

297. ⓐ $0, \sqrt{36}, 9$ ⓑ $-8, 0, \sqrt{36}, 9$ ⓒ $-8, 0, \sqrt{36}, 9$ ⓓ 1.95286..., ⓔ $-8, 0, 1.95286..., \frac{12}{5}, \sqrt{36}, 9$

299. ⓐ none ⓑ $-\sqrt{100}, -7, -1$ ⓒ $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$ ⓓ none ⓔ $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$

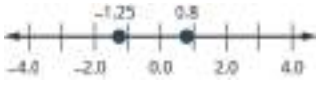
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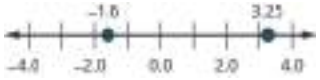
303.



305.



307.



309. Answers will vary.

311. Answers will vary.

313. $27m + (-21n)$

315. $\frac{5}{4}g + \frac{1}{2}h$

317. $2.43p + 8.26q$

319. -63

321. $1\frac{5}{6}$

323. 17

325. 14.88

327. $10p$

329. 44

331. $\frac{7}{8}$

333. d

335. $\frac{49}{11}$

337. 0

339. undefined

341. undefined

343. $32y + 72$

345. $6c - 78$

347. $\frac{3}{4}q + 3$

349. $5y - 3$

351. $3 + 8r$

353. $36d + 90$

355. $rs - 18r$

357. $yp + 4p$

359. $-28p - 7$

361. $-3x + 18$

363. $-3x + 7$

365. $-3y - 8$

367. $-33c + 26$

369. $-a + 19$

371. $4m - 10$

373. $72x - 25$

375. $22n + 9$

377. $6c + 34$

379. $12y + 63$

381. Answers will vary.

383. Answers will vary.

Review Exercises

385. Divisible by 2, 3, 5, 6

387. 120

389. 4

391. 3

393. $6x^2 - x + 5$

395. ⓐ $11(y-2)$ ⓑ $11y - 2$

397. ⓐ = ⓑ = ⓒ > ⓓ =

399. -9

401. ⓐ 8 ⓑ -8 ⓒ -22 ⓓ 22

403. ⓐ 40 ⓑ 40

405. ⓐ -3 ⓑ -15 ⓒ -56 ⓓ 17

407. 16

409. -12

411. 9

413. $(-4 + (-9)) + 23;\ 10$

415. 46°

417. $-\frac{15x^3}{11y^2}$

419. $\frac{8x}{15y}$

421. $\frac{31}{36}$

423. ⓐ $\frac{11}{8}$ ⓑ $\frac{5}{6}$

425. $-\frac{1}{6}$

427. 75

429. $-\frac{1}{5}$

431. 6.17

433. 96.978

435. 488.1813

437. 56.3

439. -23

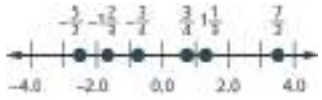
441. -9.6

443. $1.\overline{27}$

445. 4.75%

447. no real number

449.



451. $\frac{3}{4}x + y$

453. $1\frac{11}{15}$

455. $\frac{9}{17}$

457. undefined

459. $8b + 10$

461. $xp - 5p$

463. $-6x - 6$

465. $y + 16$

Practice Test

467. $7n + 7$

469. $-8 - 11;\ -19$
$(-8 - (-3)) + 5;\ 0$

471. ⓐ 28.15 ⓑ 28.146

473.



475. 1

477. -8

479. $\frac{15}{17}$

481. $-\frac{5}{3}$

483. 3

485. $-\frac{7}{6}$

487. -65.4

489. 160

491. $1\frac{8}{13}$

493. ⓐ 0 ⓑ undefined

495. $13y - 3$

Chapter 2

Try It

2.1. ⓐ no ⓑ yes

2.2. ⓐ yes ⓑ no

2.3. $m = 2$

2.4. $a = 0$

2.5. $u = 2$

2.6. $x = 4$

2.7. $p = -2$

2.8. $q = -8$

2.9. $y = -\frac{17}{5}$

2.10. $z = 0$

2.11. identity; all real numbers

2.12. identity; all real numbers

2.13. conditional equation; $q = -\frac{9}{11}$

2.14. conditional equation; $k = \frac{201}{14}$

2.15. contradiction; no solution

2.16. contradiction; no solution

2.17. $x = \frac{1}{2}$

2.18. $x = -2$

2.19. $x = 12$

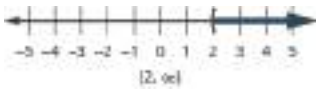
2.20. $u = -12$

2.21. $n = 2$

2.22. $m = -1$
2.23. $r = 1$
2.24. $s = -8$
2.25. $n = 9$
2.26. $d = 16$
2.27. He bought two notebooks.
2.28. He did seven crosswords puzzles.
2.29. 3
2.30. 6
2.31. $-15, -8$
2.32. $-29, 11$
2.33. $-33, -32, -31$
2.34. $-13, -12, -11$
2.35. 32, 34, 36
2.36. $-10, -8, -6$
2.37. The average cost was $5,000.
2.38. The median price was $19,300.
2.39. ⓐ 36 ⓑ $26 ⓒ 125%
2.40. ⓐ 33 ⓑ $36 ⓐ 175%
2.41. 25 grams
2.42. 2,375 mg
2.43. 50%
2.44. 40%
2.45. 8.8%
2.46. 50%
2.47. ⓐ $600 ⓑ $1,800
2.48. ⓐ $2,975 ⓑ $11,475
2.49. He will earn $2,500.
2.50. She earned $7,020.
2.51. The rate of simple interest was 6%.
2.52. The rate of simple interest was 5.5%.
2.53. He paid $17,590.
2.54. She deposited $9,600.
2.55. $b = \frac{2A}{h}$
2.56. $h = \frac{2A}{b}$
2.57. $C = \frac{5}{9}(F - 32)$
2.58. $b = \frac{2A - Bh}{h}$
2.59. $t = \frac{A - P}{Pr}$
2.60. $r = \frac{A - P}{Pt}$
2.61. $y = \frac{9 - 4x}{7}$
2.62. $y = \frac{1 - 5x}{8}$
2.63. The window's height is 12 meters.
2.64. The length of the base is 6 feet.
2.65. The measures of the angles are 20°, 70°, and 90°.
2.66. The measures of the angles are 30°, 60°, and 90°.
2.67. The length of the leg is 8.
2.68. The length of the leg is 12.
2.69. The length is 16 inches and the width is 39 inches.
2.70. The length is 17 yards and the width is 26 yards.
2.71. The lengths of the sides of the triangle are 5, 11 and 12 inches.
2.72. The lengths of the sides of the triangle are 4, 7 and 9 feet.
2.73. The length of the swimming pool is 70 feet and the width is 30 feet.
2.74. The length of the garden is 90 yards and the width is 60 yards.
2.75. The ladder reaches 12 feet.
2.76. He should attach the lights 8 feet from the base of the mast.
2.77. Jess has 41 nickels and 18 quarters.
2.78. Elane has 22 nickels and 59 dimes.
2.79. Eric bought thirty-two 49-cent stamps and twelve 35-cent stamps.
2.80. Kailee bought twenty-six 49-cent stamps and ten 20-cent stamps.
2.81. 84 adult tickets, 31 student tickets
2.82. 615 children's tickets and 195 adult tickets
2.83. Orlando mixed five pounds of cereal squares and 25 pounds of nuts.
2.84. Becca mixed 21 gallons of fruit punch and seven gallons of soda.
2.85. The speed of the local train is 48 mph and the speed of the express train is 60 mph.
2.86. Jeromy drove at a speed of 80 mph and his mother drove 60 mph.
2.87. Christopher's speed was 50 mph and his parents' speed was 40 mph.
2.88. Ashley's parents drove 55 mph and Ashley drove 62 mph.
2.89. Pierre and Monique will be 429 miles apart in 3 hours.
2.90. Thanh and Nhat will be 330 miles apart in 2.2 hours.
2.91. Suzy's speed uphill is 1.8 mph and downhill is three mph.
2.92. The boat's speed upstream is eight mph and downstream is12 mph.
2.93. Hamilton drove 40 mph in the city and 70 mph in the desert.

2.94. Phuong rode uphill at a speed of 12 mph and on the flat street at 20 mph.

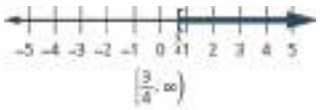
2.95. ⓐ



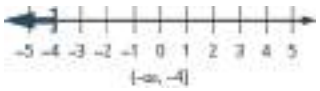
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ⓒ



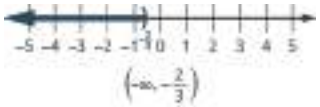
2.96. ⓐ



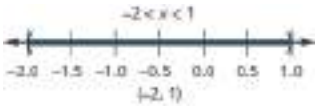
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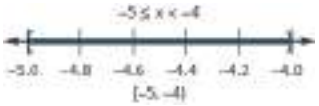
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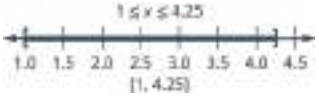
2.97. ⓐ



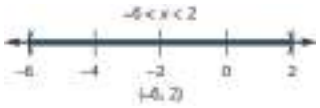
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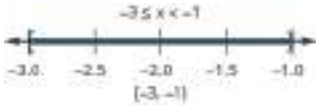
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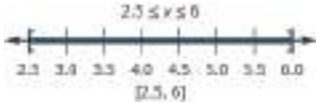
2.98. ⓐ



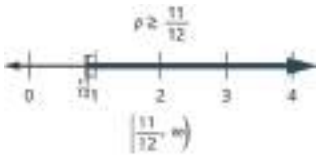
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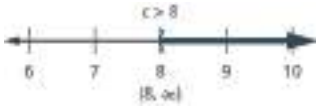
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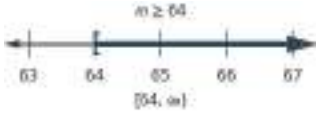
2.99. ⓐ



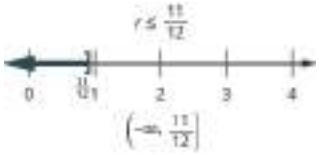
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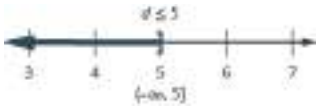
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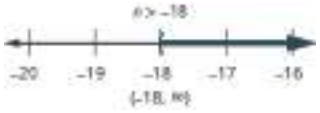
2.100. ⓐ



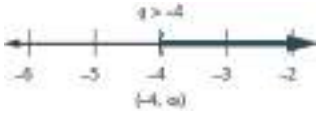
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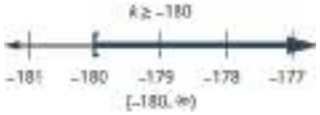
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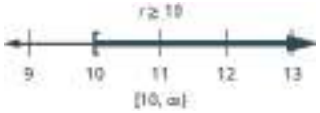
2.101. ⓐ



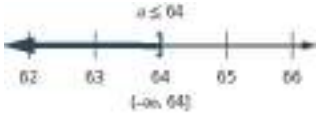
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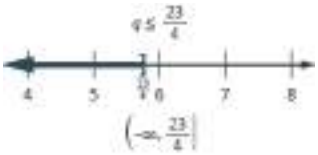
2.102. ⓐ



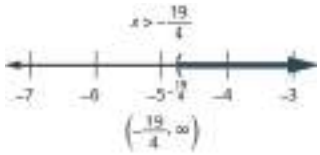
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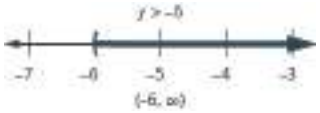
2.103.



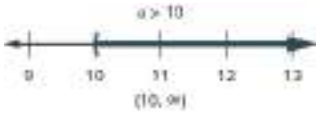
2.104.



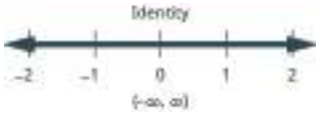
2.105.



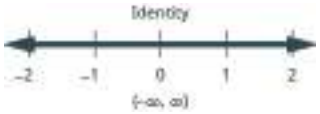
2.106.



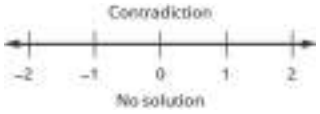
2.107.



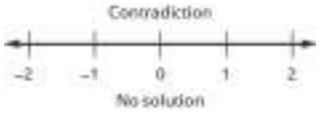
2.108.



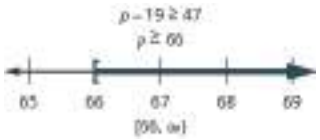
2.109.



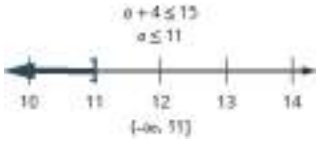
2.110.



2.111.



2.112.



2.113. Angie can buy 7 packs of juice.

2.114. Daniel can have 11 people at the party.

2.115. Sergio and Lizeth can travel no more than 500 miles.

2.116. Rameen can use no more than 76 therms.

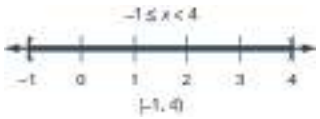
2.117. Caleb must work at least 96 hours.

2.118. Elliot must work at least 85 jobs.

2.119. Brenda must babysit at least 27 hours.

2.120. Josue must shovel at least 20 driveways.

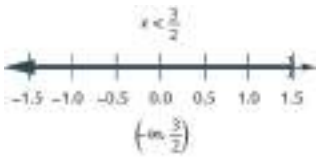
2.121.



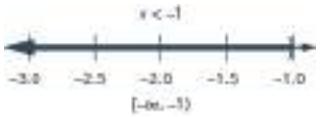
2.122.



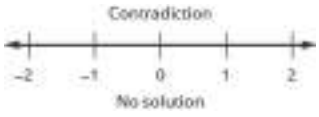
2.123.



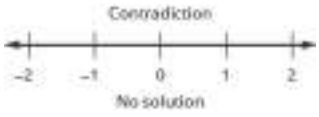
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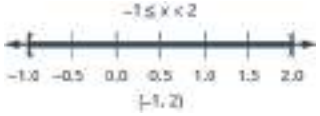
2.125.



2.126.



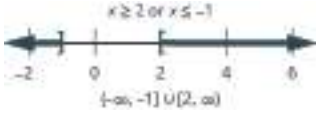
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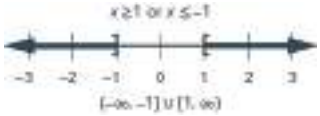
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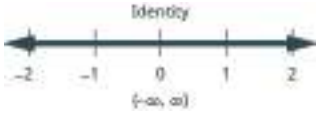
2.129.



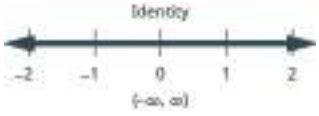
2.130.



2.131.



2.132.



2.133. The homeowner can use 5–20 hcf and still fall within the "conservation usage" billing range.

2.134. The homeowner can use 16–40 hcf and still fall within the "normal usage" billing range.

2.135. ⓐ ± 2 ⓑ no solution ⓒ 0

2.136. ⓐ ± 11 ⓑ no solution ⓒ 0

2.137. $x = 4,\ x = -\frac{2}{3}$

2.138. $x = -1,\ x = \frac{5}{2}$

2.139. $x = 8,\ x = 0$

2.140. $x = 8,\ x = 2$

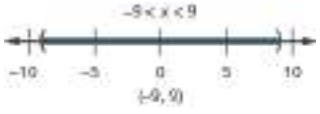
2.141. No solution

2.142. No solution

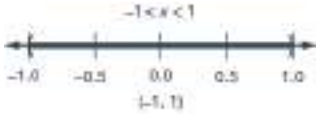
2.143. $x = -\frac{2}{5},\quad x = \frac{5}{2}$

2.144. $x = 3,\quad x = \frac{1}{9}$

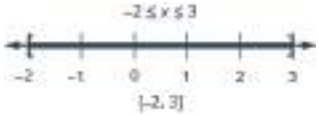
2.145.



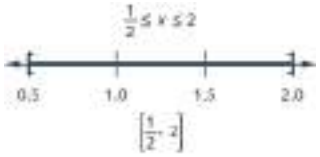
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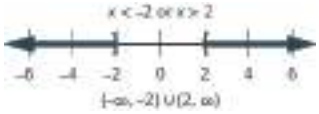
2.147.



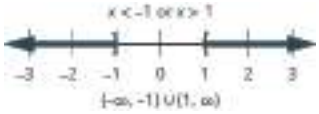
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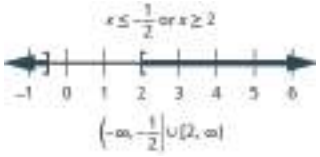
2.149.



2.150.



2.151.



2.152.



2.153. The diameter of the rod can be between 79.991 and 80.009 mm.

2.154. The diameter of the rod can be between 74.95 and 75.05 mm.

Section Exercises

1. ⓐ yes ⓑ no
3. ⓐ no ⓑ yes
5. $y = 5$
7. $w = -18$
9. $q = 3$
11. $x = 14$
13. $c = 4$
15. $c = \frac{5}{2}$
17. $n = 2$
19. $y = -5$
21. $s = 10$
23. $p = -4$
25. $c = -4$
27. $h = \frac{3}{4}$
29. $m = 6$
31. identity; all real numbers
33. conditional equation; $j = \frac{2}{5}$
35. conditional equation; $m = \frac{16}{5}$
37. contradiction; no solution
39. contradiction; no solution
41. identity; all real numbers
43. $x = -1$
45. $y = -1$
47. $a = \frac{3}{4}$
49. $x = 4$
51. $w = \frac{9}{4}$
53. $b = 12$
55. $p = -41$
57. $x = -\frac{5}{2}$
59. $n = -3$
61. $x = -2$
63. $u = 3$
65. $x = 18$
67. $x = 20$
69. $n = 9$
71. $d = 8$
73. $L = 19.75$ feet
75. Answers will vary.
77. Answers will vary.
79. Answers will vary.
81. Answers will vary.
83. six boys
85. 58 hardback books
87. 15
89. 5
91. 12
93. -5
95. 18, 24
97. 8, 12
99. 32, 46
101. 4, 10
103. 38, 39
105. 25, 26, 27
107. $-11, -12, -13$
109. 84, 86, 88
111. $-69, -71, -73$
113. $750
115. $13,500
117. ⓐ 54 ⓑ 108 ⓐ 30%
119. ⓐ 162.5 ⓑ $35 ⓐ 150%
121. $11.88
123. 24.2 g
125. 2407 mg
127. 45%
129. 25%
131. 12%
133. 200%
135. -2.5%
137. -11%
139. ⓐ $26.97 ⓑ $17.98
141. ⓐ $576 ⓑ 30%
143. ⓐ $7.20 ⓑ $23.20
145. ⓐ $0.20 ⓑ $0.80
147. $116
149. $4836
151. 3%
153. 3.75%
155. $35,000
157. $3345
159. 17.5%
161. Answers will vary.
163. Answers will vary.
165. $d = \frac{C}{\pi}$
167. $L = \frac{V}{WH}$
169. $b = \frac{2A}{h}$
171. $d_1 = \frac{2A}{d_2}$
173. $b_1 = \frac{2A}{h} - b_2$
175. $a = \frac{2h - 108t}{t^2}$
177. $a = 180 - b - c$
179. $p = \frac{2A - 2B}{l}$

181. $L = \frac{P - 2W}{2}$

183. $y = 15 - 8x$

185. $y = -6 + 4x$

187. $y = 4 + x$

189. $y = \frac{7 - 4x}{3}$

191. $y = \frac{12 - 2x}{3}$

193. $y = \frac{18 - 3x}{-2}$

195. 1 foot

197. 23 inches

199. $45°, 45°, 90°$

201. $30°, 60°, 90°$

203. 15

205. 25

207. 8

209. 12

211. 10.2

213. 9.8

215. 18 meters, 11 meters

217. 13.5 m, 12.8 m

219. 25 ft, 50 ft

221. 7 m, 11 m

223. 12 ft, 13 ft, 14 ft

225. 3 ft, 6 ft, 8 ft

227. 120 yd, 160 yd

229. 40 ft, 85 ft

231. 5 feet

233. 14.1 feet

235. $104°$ F

237. 24 ft

239. Answers will vary.

241. ⓐ Answers will vary. ⓑ The areas are the same. The 2×8 rectangle has a larger perimeter than the 4×4 square.

ⓒ Answers will vary.

243. nine nickels, 16 dimes

245. ten $10 bills, five $5 bills

247. 63 dimes, 20 quarters

249. 16 nickels, 12 dimes, seven quarters

251. 330 day passes, 367 tournament passes

253. 40 postcards, 100 stamps

255. 15 $10 shares, five $12 shares

257. 31 general, 61 youth

259. 114 general, 246 student

261. Four pounds of macadamia nuts, eight pounds almonds

263. 3.6 lbs Bermuda seed, 5.4 lbs Fescue seed

265. $33,000 in Fund A, $22,000 in Fund B

267. 5.9%

269. Kathy 5 mph, Cheryl 3 mph

271. commercial 540 mph, private plane 330 mph

273. Violet 65 mph, Charlie 55 mph

275. Ethan 22 mph, Leo 16 mph

277. DaMarcus 16 mph, Fabian 22 mph

279. four hours

281. 4.5 hours

283. uphill 1.6 mph, downhill 4.8 mph

285. light traffic 54 mph, heavy traffic 30 mph

287. freeway 72 mph, mountain road 24 mph

289. running eight mph, walking three mph

291. ⓐ 15 minutes ⓑ 20 minutes

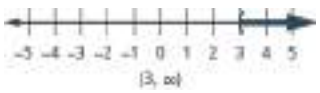
ⓒ one hour (d) 1:25

293. Answers will vary.

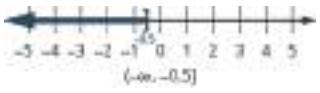
295. Answers will vary.

297.

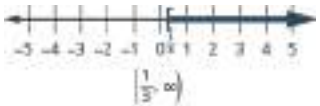
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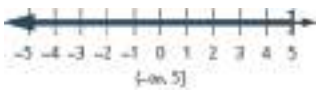


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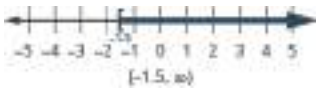


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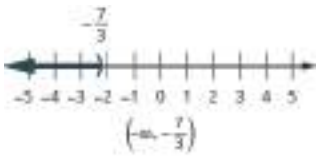
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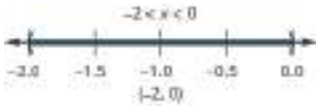


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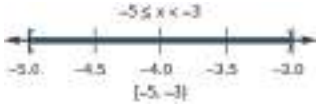


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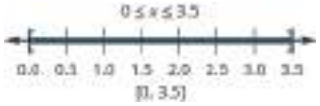
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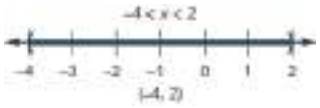


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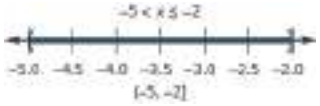


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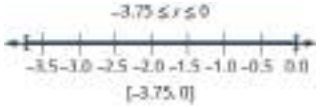
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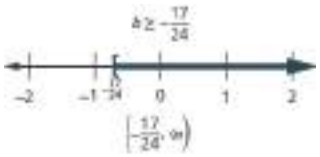


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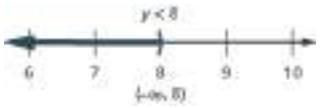


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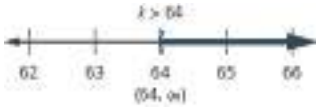
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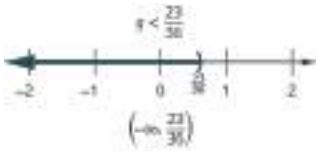


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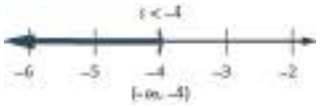


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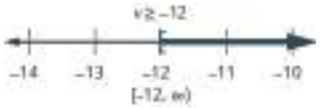


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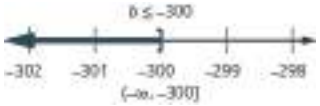


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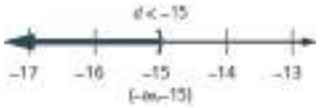


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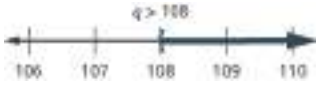


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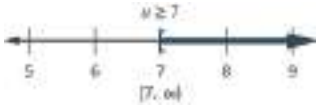
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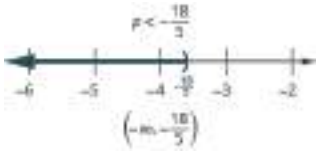
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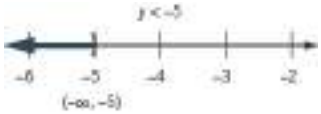
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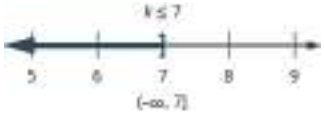
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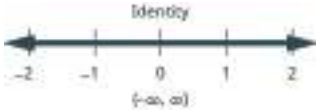
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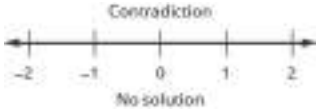
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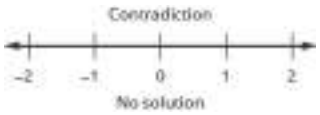
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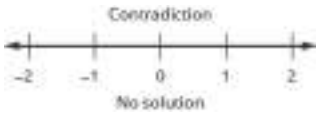
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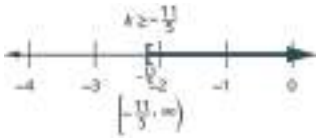
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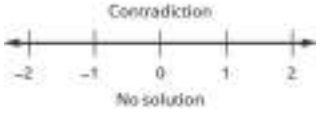
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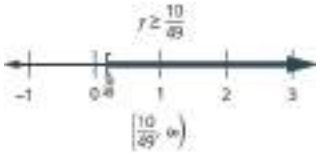
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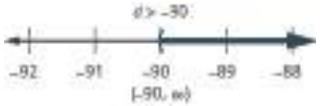
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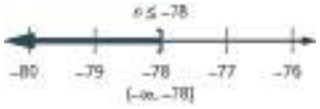
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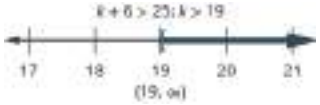
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337.



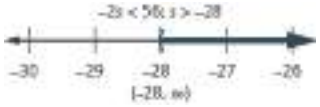
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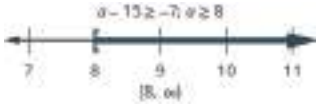
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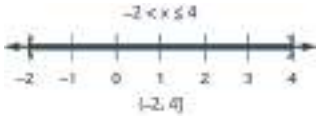


345.

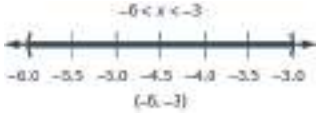


347. A maximum of 14 people can safely ride in the elevator.

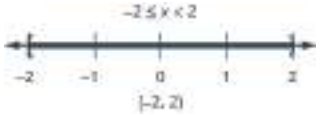
349. five drinks
351. 86 cars
353. $16,875
355. $110,000
357. 260 messages
359. 35 people
361. 32 jobs
363. 62 necklaces
365. seven lawns
367. 20 hours
369. 20 people
371. 42 guests
373. Answers will vary.
375. Answers will vary.
377.



379.



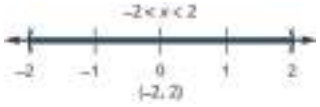
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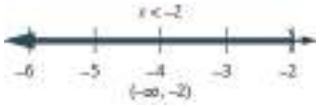
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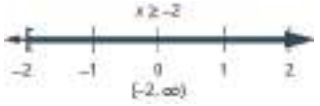
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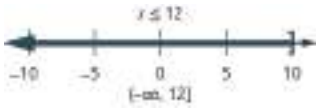
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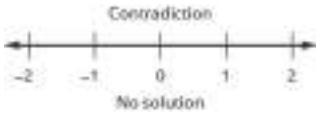
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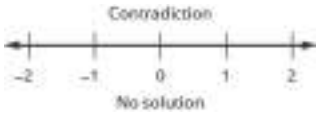
391.



393.



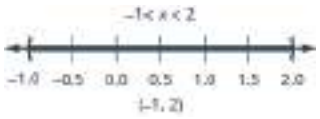
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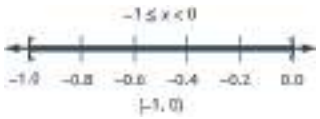
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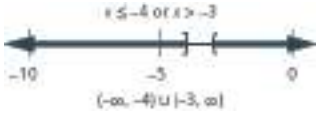
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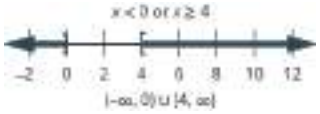
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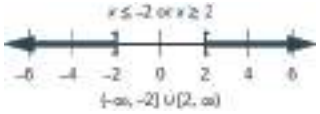
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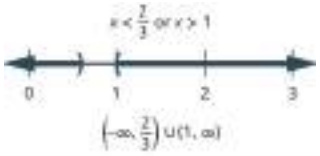
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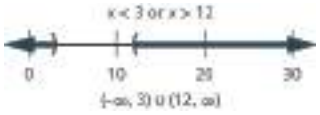
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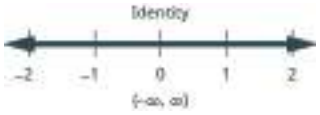
409.



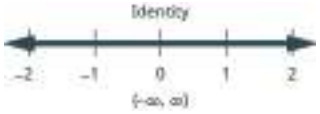
411.



413.



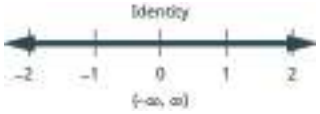
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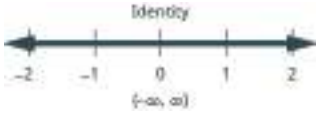
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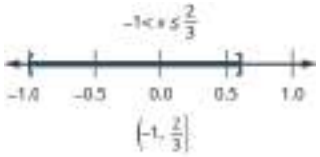
419.



421.



423.



425.



427. $5 \le n \le 24$

429. $6 \le w \le 12$

431. ⓐ answers vary ⓑ answers vary

433. Answers will vary.

435. ⓐ $x = 4,\ x = -4$ ⓑ no solution ⓒ $z = 0$

437. ⓐ $x = 3,\ x = -3$ ⓑ no solution ⓒ $z = 0$

439. $x = 1,\ x = -\frac{1}{2}$

441. $x = -1,\ x = -\frac{5}{2}$

443. $x = 7,\ x = 1$

445. $x = 4,\ x = -1$

447. $x = 7,\ x = 1$

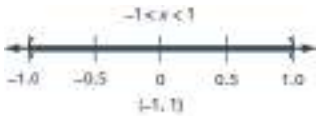
449. no solution

451. no solution

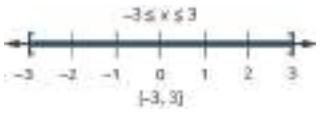
453. $x = -1,\ x = \frac{2}{3}$

455. $x = -3,\ x = 3$

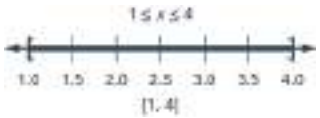
457.



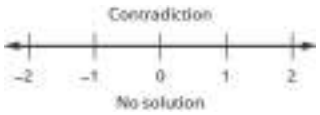
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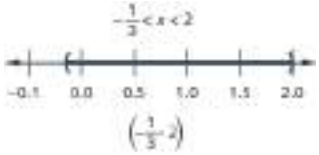
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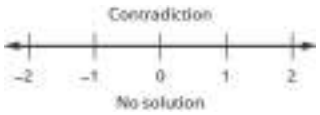
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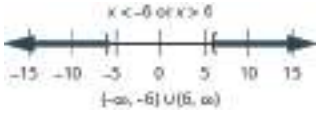
465.



467.



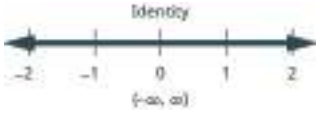
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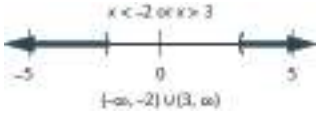
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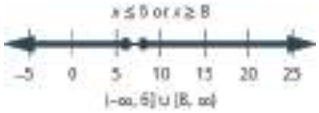
473.



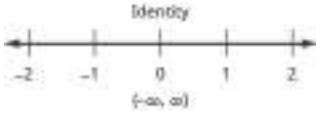
475.



477.



479.



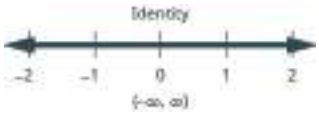
481. $x = 4,\ x = \frac{2}{7}$

483. $x = 3,\ x = 2$

485. $x = 3,\ x = -\frac{11}{3}$

487. $x = \frac{3}{2},\ x = -\frac{1}{2}$

489.



491. The minimum to maximum expected production is 207,500 to 2,225,000 bottles

493. The acceptable weight is 22.5 to 25.5 ounces.

495. Answers will vary.

Review Exercises

497. no

499. $s = -22$

501. $m = -14$

503. $q = 18$

505. $x = -1$

507. $k = \frac{3}{4}$

509. contradiction; no solution

511. $n = 2$

513. $k = 23$

515. $x = 5$

517. There are 116 people.

519. 38

521. $-3,\ -10$

523. 76, 78, 80

525. $922

527. 160

529. $\$3.89$

531. 35%

533. 32%

535. ⓐ $105 ⓑ 52.5%

537. $428.22

539. $1,900

541. $d_2 = \frac{2A}{d_1}$

543. $y = \frac{4x}{3} - 4$

545. $22.5°,\ 67.5°,\ 90°$

547. 26

549. 6 feet

551. 24.5 cm, 12.5 cm

553. 9 ft, 14 ft, 12 ft

555. nine pennies, six dimes, 12 quarters

557. 57 students, 68 adults

559. 2.2 lbs of raisins, 7.8 lbs of nuts

561. 9.7%

563. Louellen 65 mph, Tracy 66 mph

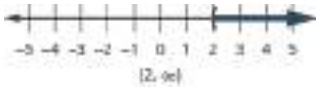
565. upstream 3 mph, downstream 5 mph

567. heavy traffic 32 mph, light traffic 66 mph

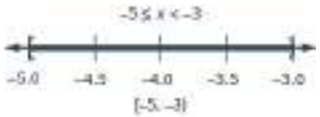
569.



571.



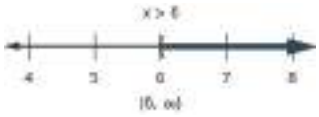
573.



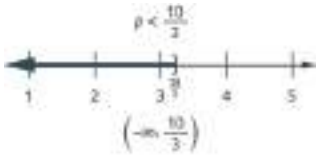
575.



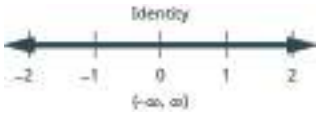
577.



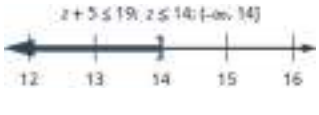
579.



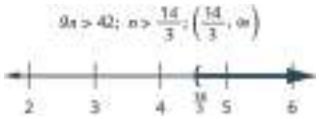
581.



583.



585.



587. $33 per day

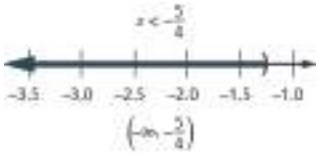
589. at least $300,000

591. at least 112 jobs

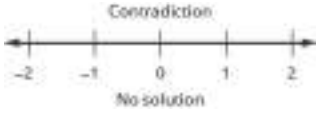
593.



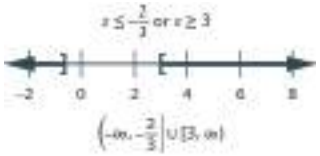
595.



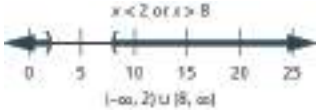
597.



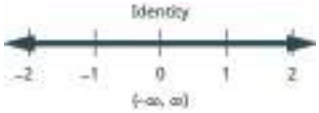
599.



601.



603.



605. $6 \le w \le 12$

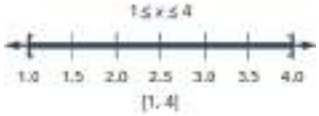
607. no solution

609. $x = 2,\ x = \frac{2}{3}$

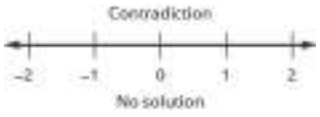
611. $x = 9,\ x = -3$

613. $x = 2,\ x = \frac{1}{4}$

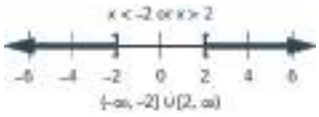
615.



617.



619.



621.



623. The minimum to maximum expected usage is 210,000 to 220,000 bottles

Practice Test

625. $x = -5$

627. $a = 41$

629. contradiction; no solution

631. $x = 6$

633. $x = -2,\ x = -\frac{1}{3}$

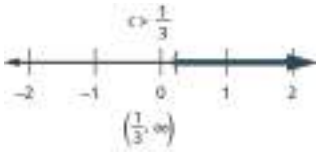
635.



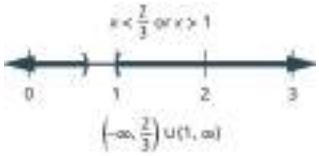
637.



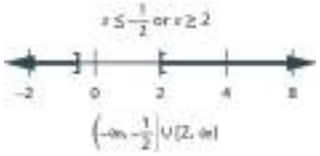
639.



641.



643.



645. 10.8

647. $-57, -55$

649. 12 dimes, seven quarters

651. $30°, 60°, 90°$

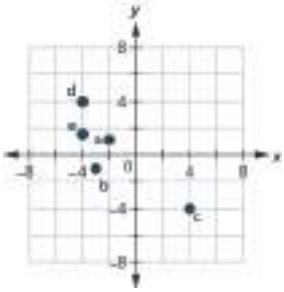
653. 2.5 hours

655. At most $55.56 per costume.

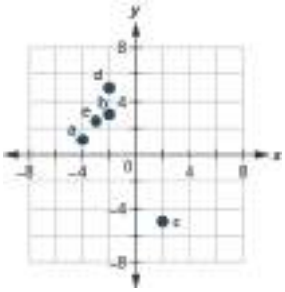
Chapter 3

Try It

3.1.



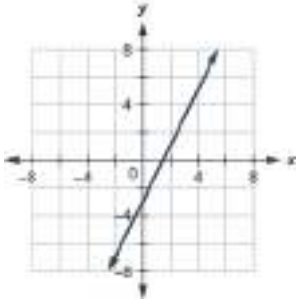
3.2.



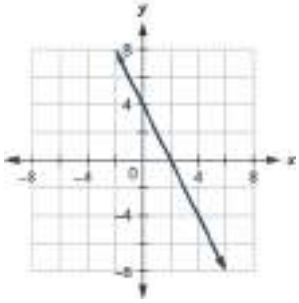
3.3. ⓐ yes, yes ⓑ yes, yes

3.4. ⓐ no, no ⓑ yes, yes

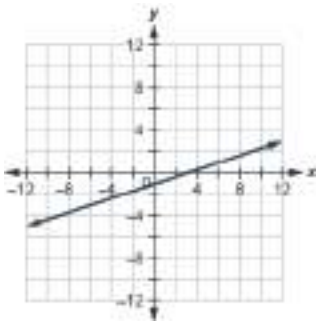
3.5.



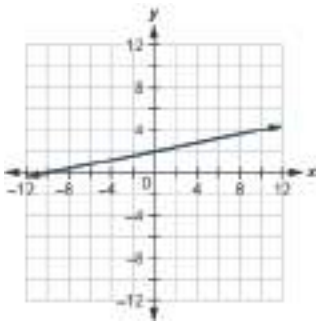
3.6.



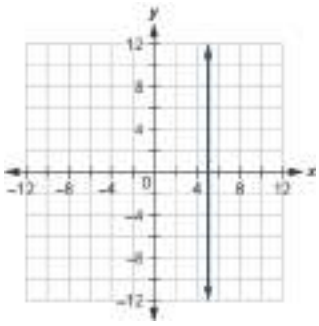
3.7.



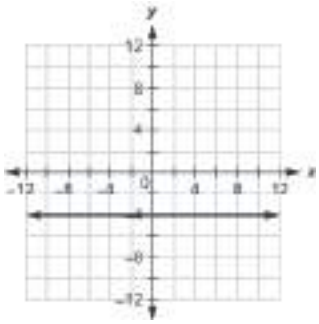
3.8.



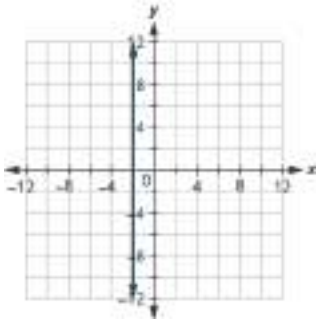
3.9. (a)



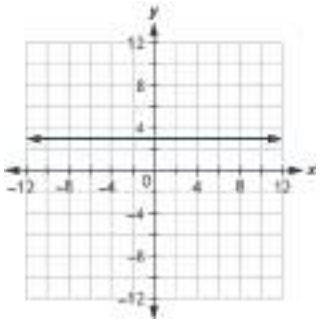
(b)



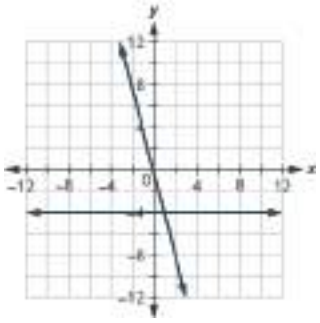
3.10. (a)



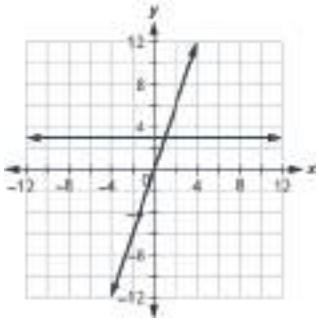
(b)



3.11.



3.12.



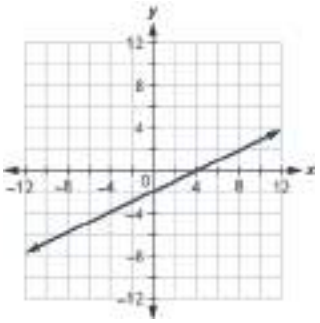
3.13. x-intercept: $(2, 0)$, y-intercept: $(0, -2)$

3.14. x-intercept: $(3, 0)$, y-intercept: $(0, 2)$

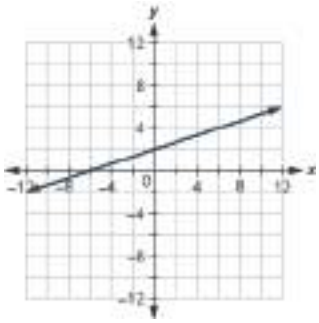
3.15. x-intercept: $(4, 0)$, y-intercept: $(0, 12)$

3.16. x-intercept: $(8, 0)$, y-intercept: $(0, 2)$

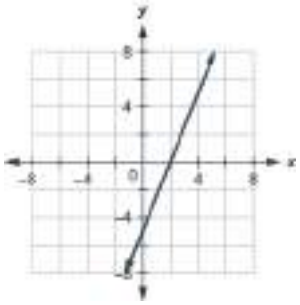
3.17.



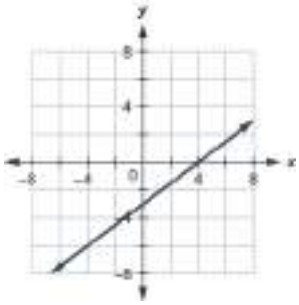
3.18.



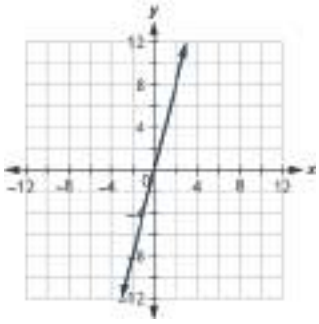
3.19.



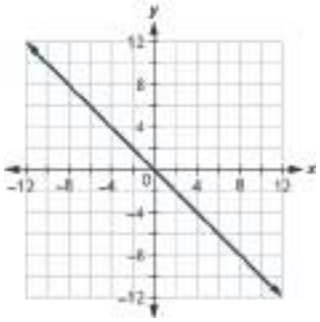
3.20.



3.21.



3.22.



3.23. $-\frac{4}{3}$

3.24. $-\frac{3}{5}$

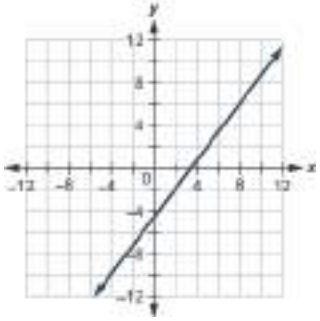
3.25. undefined

3.26. 0

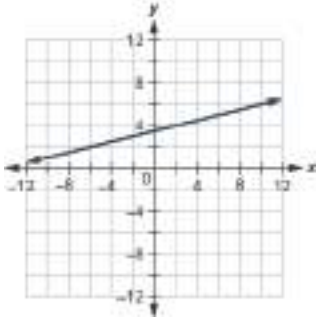
3.27. -1

3.28. 10

3.29.



3.30.



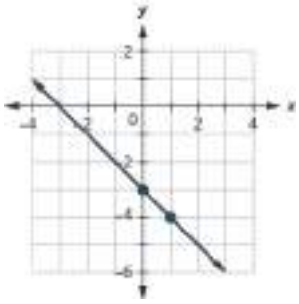
3.31. ⓐ $m = \frac{2}{5}$; $(0, -1)$

ⓑ $m = -\frac{1}{4}$; $(0, 2)$

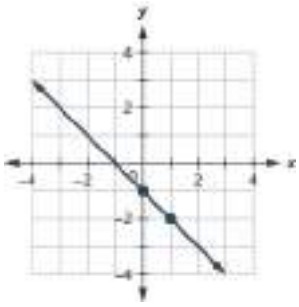
3.32. ⓐ $m = -\frac{4}{3}$; $(0, 1)$

ⓑ $m = -\frac{3}{2}$; $(0, 6)$

3.33.



3.34.



3.35. ⓐ intercepts ⓑ horizontal line ⓒ slope-intercept ⓓ vertical line

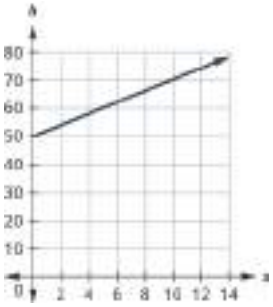
3.36. ⓐ vertical line ⓑ slope-intercept ⓒ horizontal line ⓓ intercepts

3.37. ⓐ 50 inches

ⓑ 66 inches

ⓒ The slope, 2, means that the height, h, increases by 2 inches when the shoe size, s, increases by 1. The h-intercept means that when the shoe size is 0, the height is 50 inches.

ⓓ

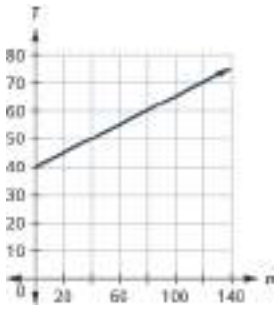


3.38. ⓐ 40 degrees

ⓑ 65 degrees

ⓒ The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (F) increases 1 degree when the number of chirps, n, increases by 4. The T-intercept means that when the number of chirps is 0, the temperature is 40°.

ⓓ

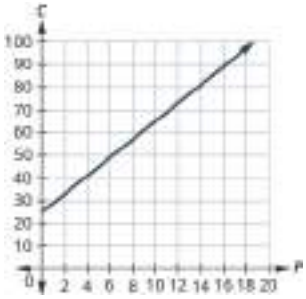


3.39. ⓐ $25

ⓑ $85

ⓒ The slope, 4, means that the weekly cost, C, increases by $4 when the number of pizzas sold, p, increases by 1. The C-intercept means that when the number of pizzas sold is 0, the weekly cost is $25.

ⓓ

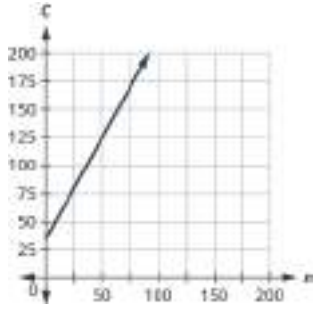


3.40. ⓐ $35

ⓑ $170

ⓒ The slope, 1.8, means that the weekly cost, C, increases by $\$1.80$ when the number of invitations, n, increases by 1. The C-intercept means that when the number of invitations is 0, the weekly cost is $35.

ⓓ



3.41. ⓐ parallel ⓑ not parallel; same line

3.42. ⓐ parallel ⓑ not parallel; same line

3.43. ⓐ parallel ⓑ parallel

3.44. ⓐ parallel ⓑ parallel

3.45. ⓐ perpendicular ⓑ not perpendicular

3.46. ⓐ perpendicular ⓑ not perpendicular

3.47. $y = \frac{2}{5}x + 4$

3.48. $y = -x - 3$

3.49. $y = \frac{3}{5}x + 1$

3.50. $y = \frac{4}{3}x - 5$

3.51. $y = -\frac{2}{5}x - 1$

3.52. $y = -\frac{3}{4}x - 4$

3.53. $y = 8$

3.54. $y = 4$

3.55. $y = \frac{1}{3}x - \frac{10}{3}$

3.56. $y = -\frac{2}{5}x - \frac{23}{5}$

3.57. $x = 5$

3.58. $x = -4$

3.59. $y = 3x - 10$

3.60. $y = \frac{1}{2}x + 1$

3.61. $y = -\frac{1}{3}x + \frac{10}{3}$

3.62. $y = -2x + 16$

3.63. $y = -5$

3.64. $y = -1$

3.65. $x = -5$

3.66. $x = -4$

3.67. ⓐ yes ⓑ yes ⓒ yes ⓓ yes ⓔ no

3.68. ⓐ yes ⓑ yes ⓒ no ⓓ no ⓔ yes

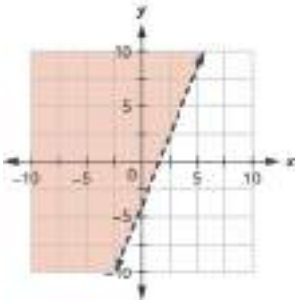
3.69. $y \geq -2x + 3$

3.70. $y \leq \frac{1}{2}x - 4$

3.71. $x - 4y \leq 8$

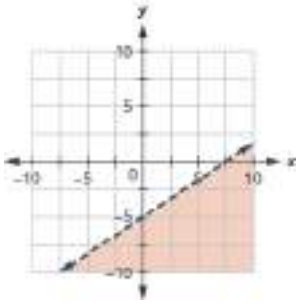
3.72. $3x - y \geq 6$

3.73.



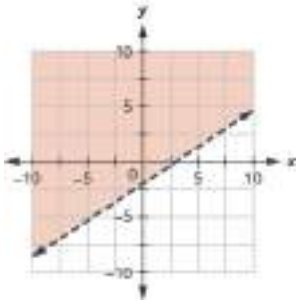
All points in the shaded region and on the boundary line, represent the solutions to $y > \frac{5}{2}x - 4$.

3.74.



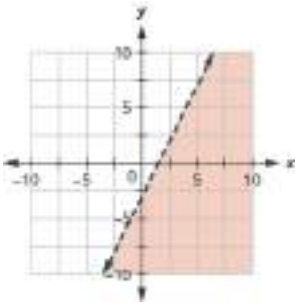
All points in the shaded region, but not those on the boundary line, represent the solutions to $y < \frac{2}{3}x - 5$.

3.75.



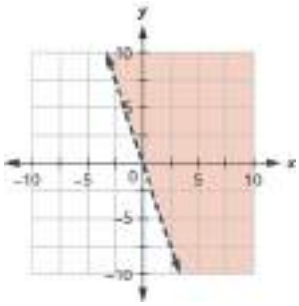
All points in the shaded region, but not those on the boundary line, represent the solutions to $2x - 3y < 6$.

3.76.



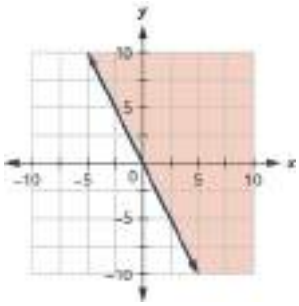
All points in the shaded region, but not those on the boundary line, represent the solutions to $2x - y > 3$.

3.77.



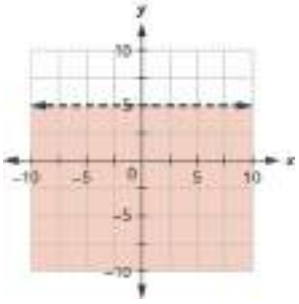
All points in the shaded region, but not those on the boundary line, represent the solutions to $y > -3x$.

3.78.



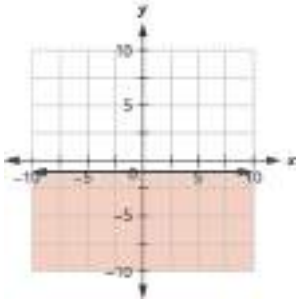
All points in the shaded region and on the boundary line, represent the solutions to $y \geq -2x$.

3.79.



All points in the shaded region, but not those on the boundary line, represent the solutions to $y < 5$.

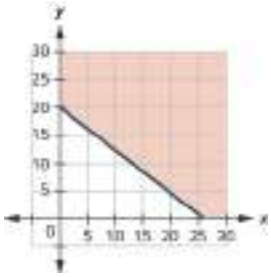
3.80.



All points in the shaded region and on the boundary line represent the solutions to $y \leq -1$.

3.81. ⓐ $10x + 13y \geq 260$

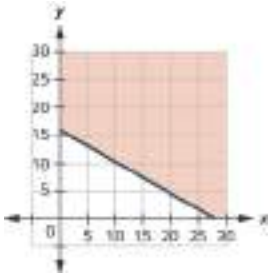
ⓑ



ⓒ Answers will vary.

3.82. ⓐ $10x + 17.5y \geq 280$

ⓑ



ⓒ Answers will vary.

3.83. ⓐ $\{1, 2, 3, 4, 5\}$

ⓑ $\{1, 8, 27, 64, 125\}$

3.84. ⓐ $\{1, 2, 3, 4, 5\}$

ⓑ $\{3, 6, 9, 12, 15\}$

3.85. ⓐ (Khanh Nguyen, kn68413), (Abigail Brown, ab56781), (Sumantha Mishal, sm32479), (Jose Hern and ez, jh47983) ⓑ {Khanh Nguyen, Abigail Brown, Sumantha Mishal, Jose Hern and ez} ⓒ {kn68413, ab56781, sm32479, jh47983}

3.86. ⓐ (Maria, November 6), (Arm and o, January 18), (Cynthia, December 8), (Kelly, March 15), (Rachel, November 6) ⓑ {Maria, Arm and o, Cynthia, Kelly, Rachel} ⓒ {November 6, January 18, December 8, March 15}

3.87. ⓐ $(-3, 3), (-2, 2), (-1, 0), (0, -1), (2, -2), (4, -4)$

ⓑ $\{-3, -2, -1, 0, 2, 4\}$

ⓒ $\{3, 2, 0, -1, -2, -4\}$

3.88. ⓐ $(-3, 0), (-3, 5), (-3, -6), (-1, -2), (1, 2), (4, -4)$

ⓑ $\{-3, -1, 1, 4\}$

ⓒ $\{-6, 0, 5, -2, 2, -4\}$

3.89. ⓐ Yes; $\{-3, -2, -1, 0, 1, 2, 3\}$; $\{-6, -4, -2, 0, 2, 4, 6\}$

ⓑ No; $\{0, 2, 4, 8\}$; $\{-4, -2, -1, 0, 1, 2, 4\}$

3.90. ⓐ No; $\{0, 1, 8, 27\}$; $\{-3, -2, -1, 0, 2, 2, 3\}$

ⓑ Yes; $\{7, -5, 8, 0, -6, -2, -1\}$; $\{-3, -4, 0, 4, 2, 3\}$

3.91. ⓐ no ⓑ {NBC, HGTV, HBO}

ⓒ {Ellen Degeneres Show, Law and Order, Tonight Show, Property Brothers, House Hunters, Love it or List it, Game of Thrones, True Detective, Sesame Street}

3.92. ⓐ No ⓑ {Neal, Krystal, Kelvin, George, Christa, Mike} ⓒ {123-567-4839 work, 231-378-5941 cell, 743-469-9731 cell, 567-534-2970 work, 684-369-7231 cell, 798-367-8541 cell, 639-847-6971 cell}

3.93. ⓐ yes ⓑ no ⓒ yes

3.94. ⓐ no ⓑ yes ⓒ yes

3.95. ⓐ $f(3) = 22$ ⓑ $f(-1) = 6$

ⓒ $f(t) = 3t^2 - 2t - 1$

3.96. ⓐ $(2) = 13$ ⓑ $f(-3) = 3$

ⓒ $f(h) = 2h^2 + 4h - 3$

3.97. ⓐ $4m^2 - 7$ ⓑ $4x - 19$

ⓒ $x - 12$

3.98. ⓐ $2k^2 + 1$ ⓑ $2x + 3$

ⓒ $2x + 4$

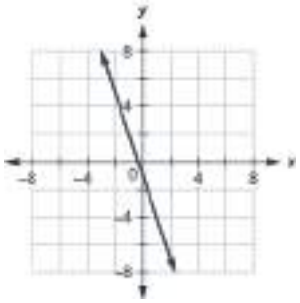
3.99. ⓐ t IND; N DEP ⓑ 205; the number of unread emails in Bryan's account on the seventh day.

3.100. ⓐ t IND; N DEP ⓑ 460; the number of unread emails in Anthony's account on the fourteenth day

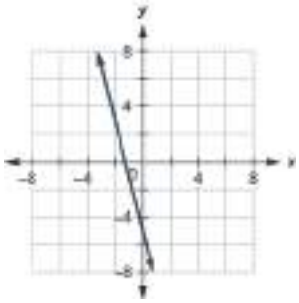
3.101. ⓐ yes ⓑ no

3.102. ⓐ no ⓑ yes

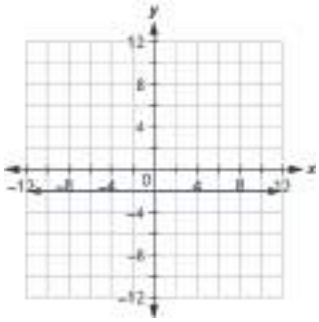
3.103.



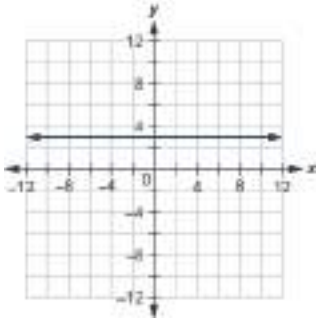
3.104.



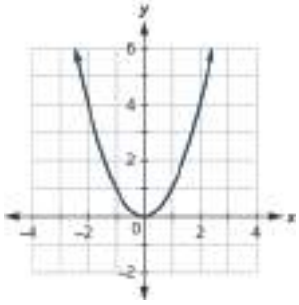
3.105.



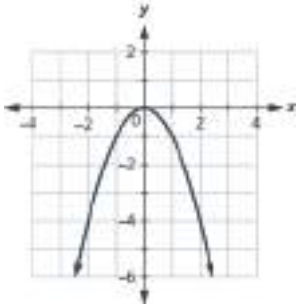
3.106.



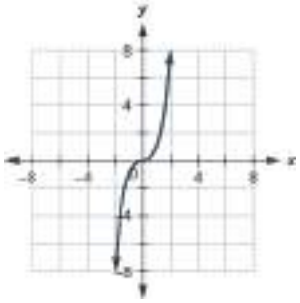
3.107.



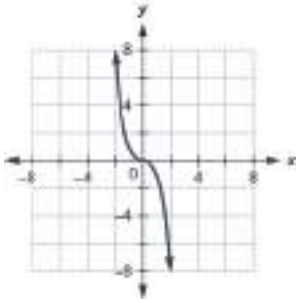
3.108.



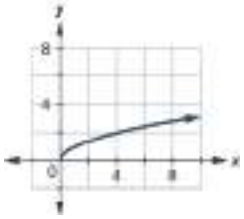
3.109.



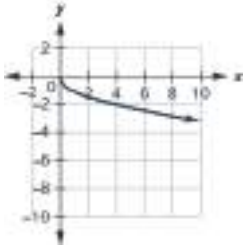
3.110.



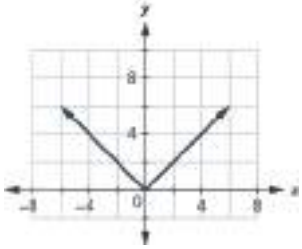
3.111.



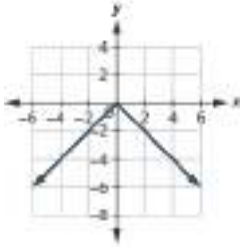
3.112.



3.113.



3.114.



3.115. The domain is $[-5, 1]$. The range is $[-4, 2]$.

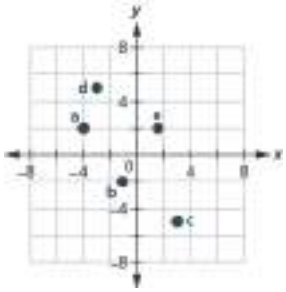
3.116. The domain is $[-2, 4]$. The range is $[-5, 3]$.

3.117. ⓐ $f(0) = 0$ ⓑ $f = \left(\frac{\pi}{2}\right) = 2$ ⓒ $f = \left(\frac{-3\pi}{2}\right) = 2$ ⓓ $f(x) = 0$ for $x = -2\pi, -\pi, 0, \pi, 2\pi$ ⓔ $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$ ⓕ $(0, 0)$ ⓖ $[-2\pi, 2\pi]$ ⓗ $[-2, 2]$

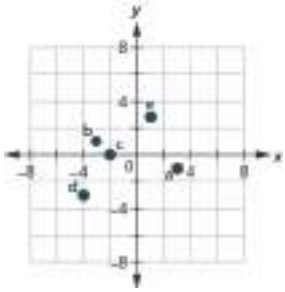
3.118. ⓐ $f(0) = 1$ ⓑ $f(\pi) = -1$ ⓒ $f(-\pi) = -1$ ⓓ $f(x) = 0$ for $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ ⓔ $(-2pi, 0), (-pi, 0), (0, 0), (pi, 0), (2pi, 0)$ ⓕ $(0, 1)$ ⓖ $[-2pi, 2pi]$ ⓗ $[-1, 1]$

Section Exercises

1.



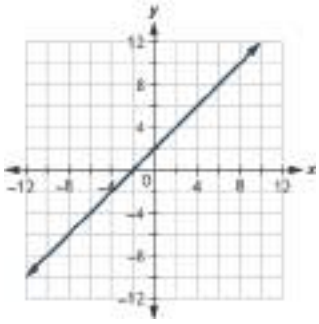
3.



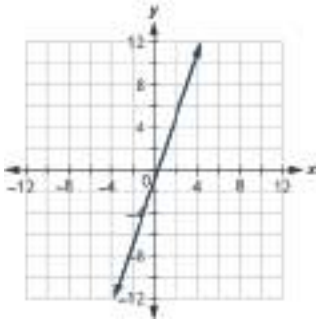
5. ⓐ A: yes, B: no, C: yes, D: yes ⓑ A: yes, B: no, C: yes, D: yes

7. ⓐ A: yes, B: yes, C: yes, D: no ⓑ A: yes, B: yes, C: yes, D: no

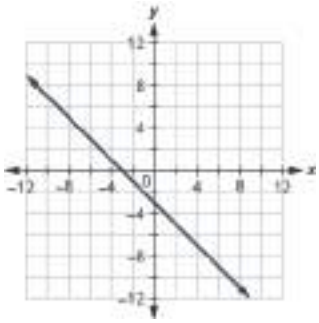
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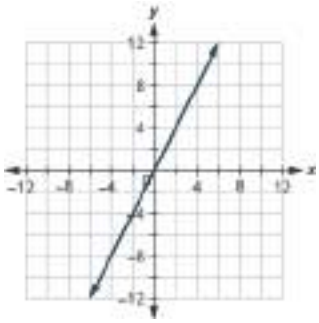
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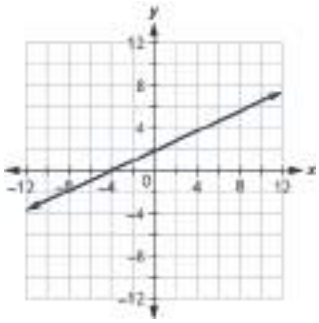
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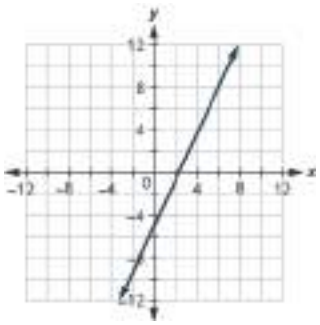
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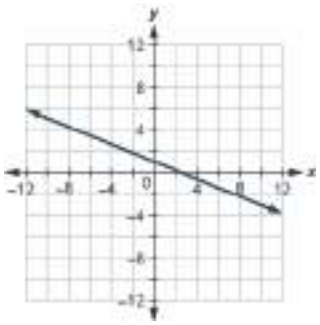
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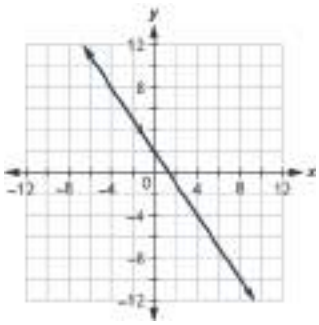
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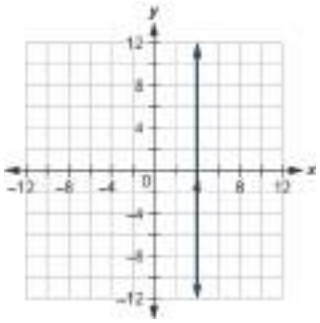
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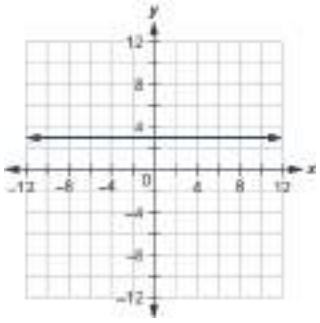
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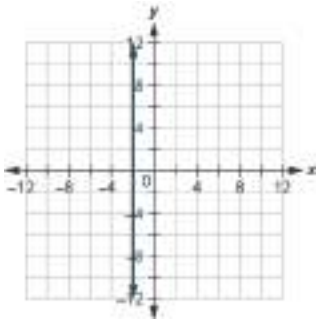
25. ⓐ



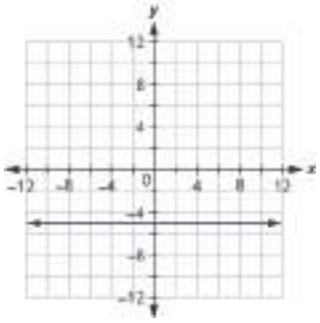
ⓑ



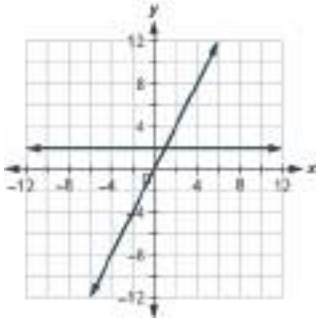
27. ⓐ



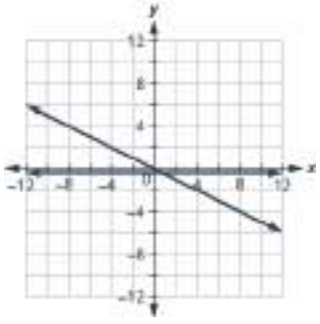
ⓑ



29.



31.



33. (3, 0), (0, 3)

35. (5, 0), (0, −5)

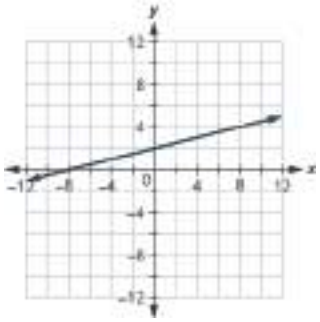
37. (5, 0), (0, −5)

39. (2, 0), (0, 6)

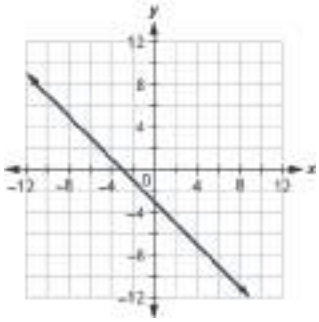
41. (2, 0), (0, −8)

43. (5, 0), (0, 2)

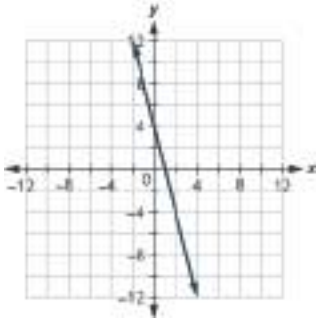
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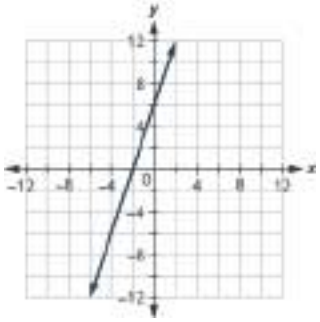
47.



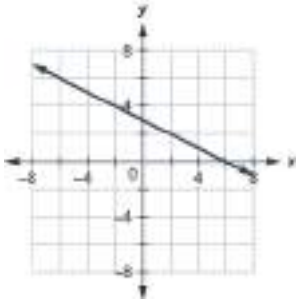
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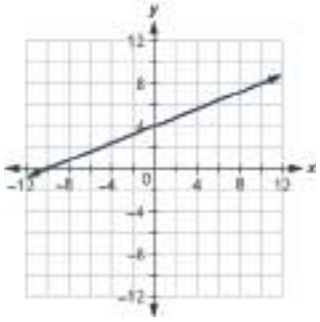
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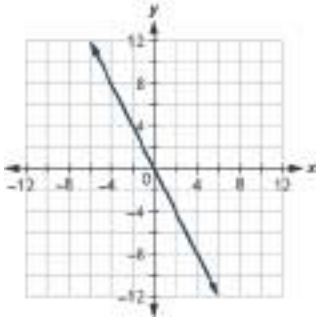
53.



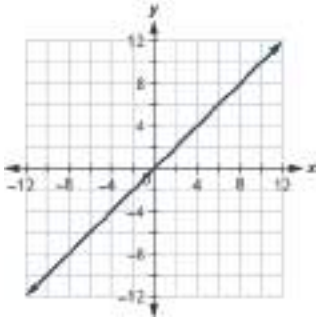
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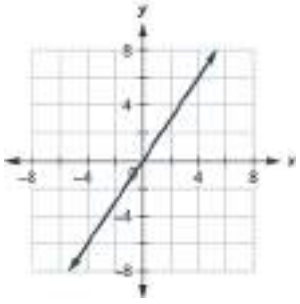
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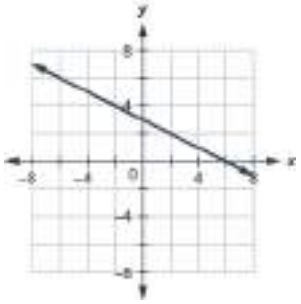
59.



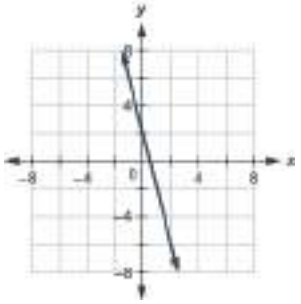
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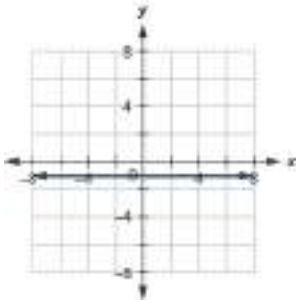
63.



65.



67.



69. Answers will vary.

71. Answers will vary.

73. $\frac{2}{5}$

75. $\frac{5}{4}$

77. $-\frac{1}{3}$

79. $-\frac{5}{2}$

81. 0

83. undefined

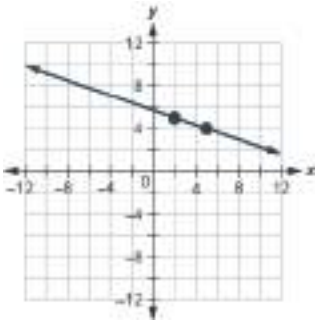
85. $-\frac{5}{2}$

87. $-\frac{8}{5}$

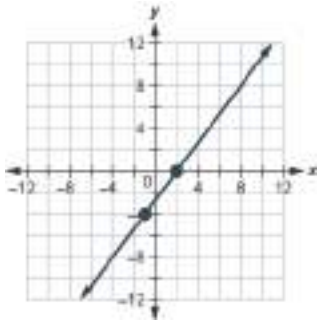
89. $\frac{7}{3}$

91. -1

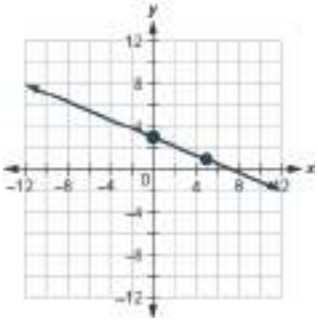
93.



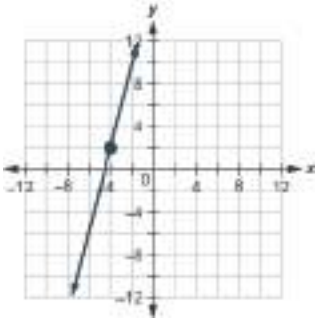
95.



97.



99.



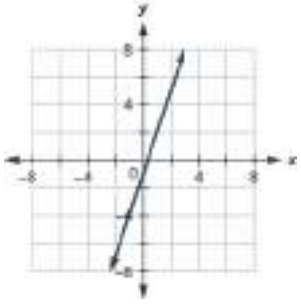
101. $m = -7$; $(0, 3)$

103. $m = -3$; $(0, 5)$

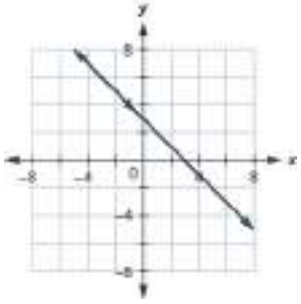
105. $m = -\frac{3}{2}$; $(0, 3)$

107. $m = \frac{5}{2}$; $(0, -3)$

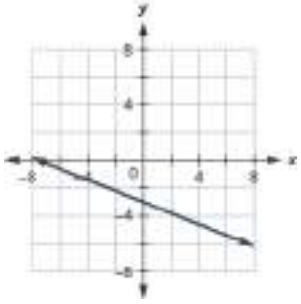
109.



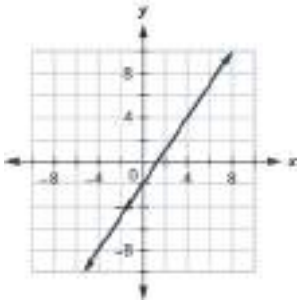
111.



113.



115.



117. vertical line

119. slope-intercept

121. intercepts

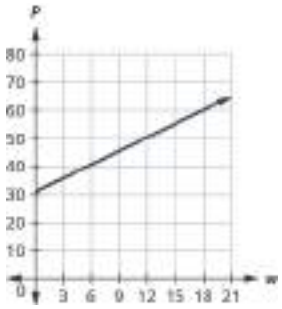
123. intercepts

125. ⓐ $31

ⓑ $52

ⓒ The slope, 1.75, means that the payment, P, increases by $\$1.75$ when the number of units of water used, w, increases by 1. The P-intercept means that when the number units of water Tuyet used is 0, the payment is $31.

ⓓ

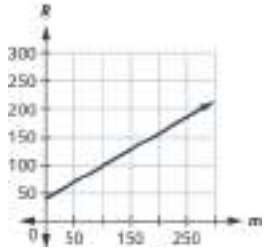


127. ⓐ $42

ⓑ $168.50

ⓒ The slope, 0.575 means that the amount he is reimbursed, R, increases by $0.575 when the number of miles driven, m, increases by 1. The R-intercept means that when the number miles driven is 0, the amount reimbursed is $42.

ⓓ

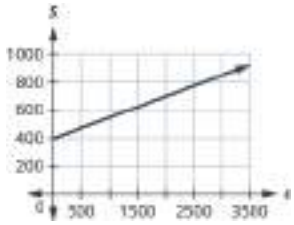


129. ⓐ $400

ⓑ $940

ⓒ The slope, 0.15, means that Cherie's salary, S, increases by $0.15 for every $1 increase in her sales. The S-intercept means that when her sales are $0, her salary is $400.

ⓓ

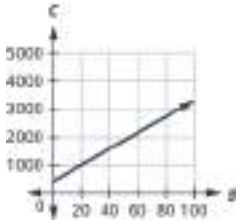


131. ⓐ $1570

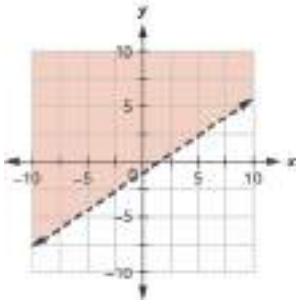
ⓑ $5690

ⓒ The slope gives the cost per guest. The slope, 28, means that the cost, C, increases by $28 when the number of guests increases by 1. The C-intercept means that if the number of guests was 0, the cost would be $450.

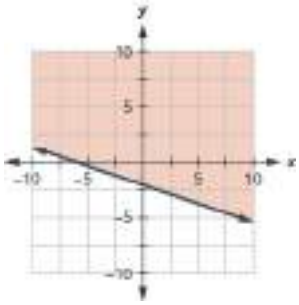
ⓓ



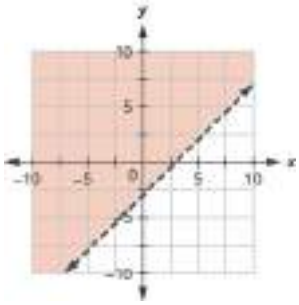
133. parallel
135. neither
137. parallel
139. perpendicular
141. neither
143. perpendicular
145. perpendicular
147. neither
149. parallel
151. Answers will vary.
153. Answers will vary.
155. $y = 3x + 5$
157. $y = -3x - 1$
159. $y = \frac{1}{5}x - 5$
161. $y = -1$
163. $y = 3x - 5$
165. $y = \frac{1}{2}x - 3$
167. $y = -\frac{4}{3}x + 3$
169. $y = -2$
171. $y = \frac{5}{8}x - 2$
173. $y = -\frac{3}{5}x + 1$
175. $y = -\frac{3}{2}x + 9$
177. $y = -7x - 10$
179. $y = 5$
181. $y = -7$
183. $y = -x + 8$
185. $y = \frac{1}{4}x - \frac{13}{4}$
187. $y = 2x + 5$
189. $y = -\frac{7}{2}x + 4$
191. $x = 7$
193. $y = -4$
195. $y = 4x - 2$
197. $y = 2x - 6$
199. $x = -3$
201. $y = -2$
203. $y = \frac{1}{2}x + 1$
205. $y = -\frac{4}{3}x$
207. $y = -\frac{3}{2}x + 5$
209. $y = \frac{5}{2}x$
211. $y = 4$
213. $y = -4$
215. $x = -2$
217. $y = 4$
219. $y = -\frac{1}{2}x + 5$
221. $y = \frac{1}{6}x$
223. $y = -\frac{4}{3}x - 3$
225. $y = -\frac{3}{4}x + 1$
227. $x = -2$
229. $x = -2$
231. $y = -\frac{1}{5}x - \frac{23}{5}$
233. $y = -2x - 2$
235. Answers will vary.
237. ⓐ yes ⓑ yes ⓒ no ⓓ no ⓔ no
239. ⓐ no ⓑ no ⓒ yes ⓓ yes ⓔ no
241. ⓐ yes ⓑ no ⓒ no ⓓ no ⓔ no
243. $y < 3x - 4$
245. $y \leq -\frac{1}{2}x + 1$
247. $x + y \geq 5$
249. $3x - y < 6$
251.



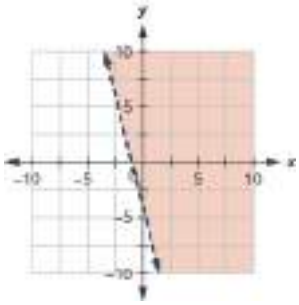
253.



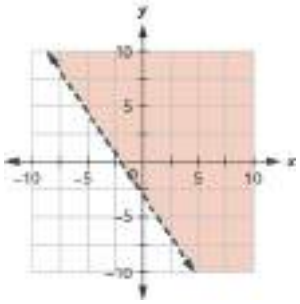
255.



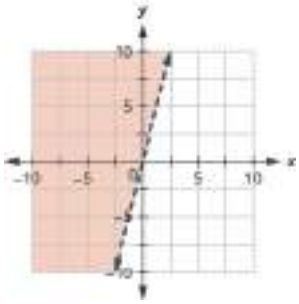
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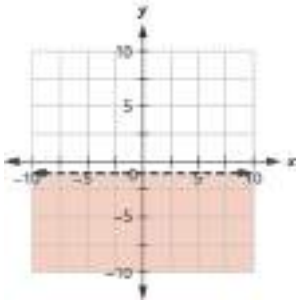
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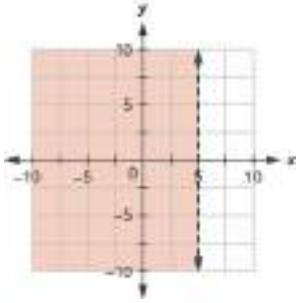
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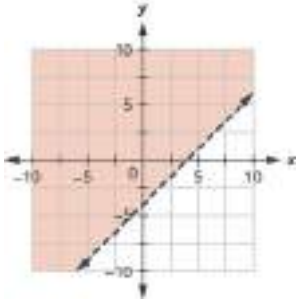
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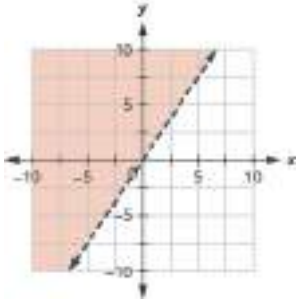
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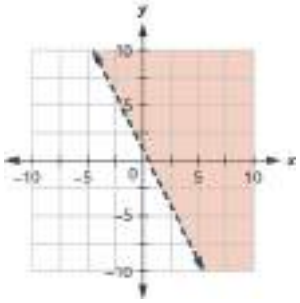
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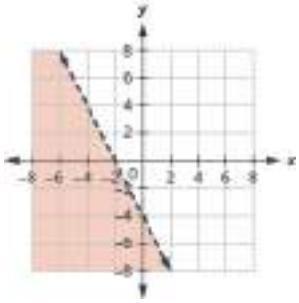
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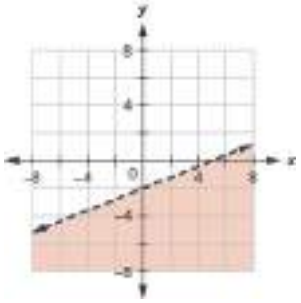
271.



273.

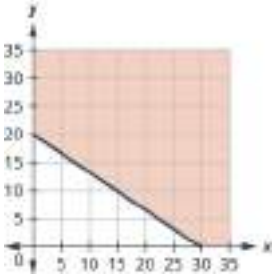


275.



277. ⓐ $11x + 16.5y \geq 330$

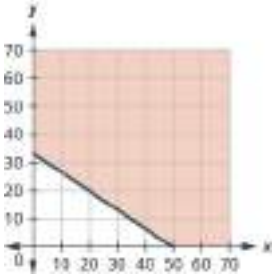
ⓑ



ⓒ Answers will vary.

279. ⓐ $15x + 10y \geq 500$

ⓑ



ⓒ Answers will vary.

281. Answers will vary.

283. ⓐ {1, 2, 3, 4, 5} ⓑ {4, 8, 12, 16, 20}

285. ⓐ {1, 5, 7, −2} ⓑ {7, 3, 9, −3, 8}

287. ⓐ (Rebecca, January 18), (Jennifer, April 1), (John, January 18), (Hector, June 23), (Luis, February 15), (Ebony, April 7), (Raphael, November 6), (Meredith, August 19), (Karen, August 19), (Joseph, July 30)
ⓑ {Rebecca, Jennifer, John, Hector, Luis, Ebony, Raphael, Meredith, Karen, Joseph}
ⓒ {January 18, April 1, June 23, February 15, April 7, November 6, August 19, July 30}

289. ⓐ (+100, 17. 2), (110, 18.9), (120, 20.6), (130, 22.3), (140, 24.0), (150, 25.7), (160, 27.5) ⓑ {+100, 110, 120, 130, 140, 150, 160,} ⓒ {17.2, 18.9, 20.6, 22.3, 24.0, 25.7, 27.5}

291. ⓐ (2, 3), (4, −3), (−2, −1), (−3, 4), (4, −1), (0, −3) ⓑ {−3, −2, 0, 2, 4} ⓒ {−3, −1, 3, 4}

293. ⓐ (1, 4), (1, −4), (−1, 4), (−1, −4), (0, 3), (0, −3) ⓑ {−1, 0, 1} ⓒ {−4, −3, 3,4}

295. ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3} ⓒ {9, 4, 1, 0}

297. ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3} ⓒ 0, 1, 8, 27}

299. ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3} ⓒ {0, 1, 2, 3}

301. ⓐ no ⓑ {Jenny, R and y, Dennis, Emily, Raul} ⓒ {RHern and ez@state.edu, JKim@gmail.com, Raul@gmail.com, ESmith@state.edu, DBroen@aol.com, jenny@aol.cvom, R and y@gmail.com}

303. ⓐ yes ⓑ yes ⓒ no

305. ⓐ yes ⓑ no ⓒ yes

307. ⓐ $f(2) = 7$ ⓑ $f(-1) = -8$ ⓒ $f(a) = 5a - 3$

309. ⓐ $f(2) = -6$ ⓑ $f(-1) = 6$ ⓒ $f(a) = -4a + 2$

311. ⓐ $f(2) = 5$ ⓑ $f(-1) = 5$ ⓒ $f(a) = a^2 - a + 3$

313. ⓐ $f(2) = 9$ ⓑ $f(-1) = 6$ ⓒ $f(a) = 2a^2 - a + 3$

315. ⓐ $g(h^2) = 2h^2 + 1$ ⓑ $g(x + 2) = 4x + 5$ ⓒ $g(x) + g(2) = 2x + 6$

317. ⓐ $g(h^2) = -3h^2 - 2$ ⓑ $g(x + 2) = -3x - 8$ ⓒ $g(x) + g(2) = -3x - 10$

319. ⓐ $g(h^2) = 3 - h^2$ ⓑ $g(x + 2) = 1 - x$ ⓒ $g(x) + g(2) = 4 - x$

321. 2

323. 6

325. 22

327. 4

329. ⓐ t IND; N DEP ⓑ $N(4) = 165$ the number of unwatched shows in Sylvia's DVR at the fourth week.

331. ⓐ x IND; C DEP

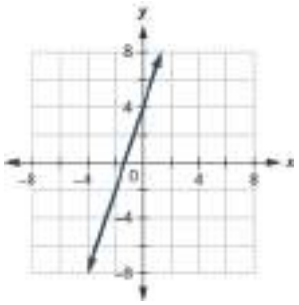
ⓑ $N(0) = 1500$ the daily cost if no books are printed

ⓒ $N(1000) = 4750$ the daily cost of printing 1000 books

337. ⓐ no ⓑ yes

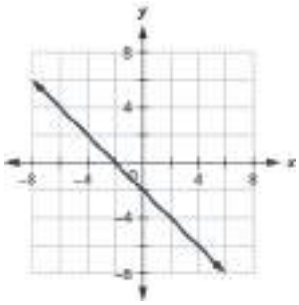
339. ⓐ no ⓑ yes

341. ⓐ



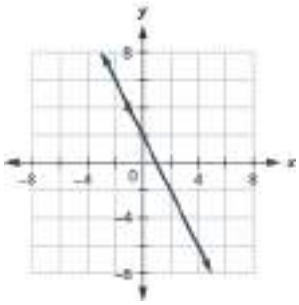
ⓑ D:(-∞,∞), R:(-∞,∞)

343. ⓐ



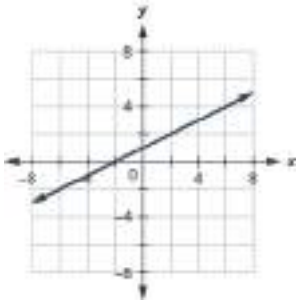
ⓑ D:(-∞,∞), R:(-∞,∞)

345. ⓐ



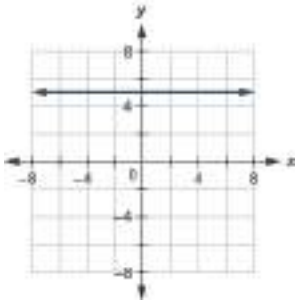
ⓑ D:(-∞,∞), R:(-∞,∞)

347. ⓐ



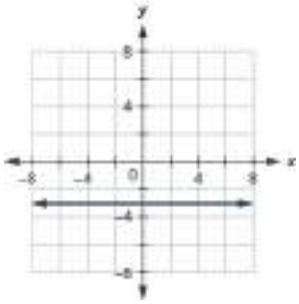
ⓑ D:(-∞,∞), R:(-∞,∞)

349. ⓐ



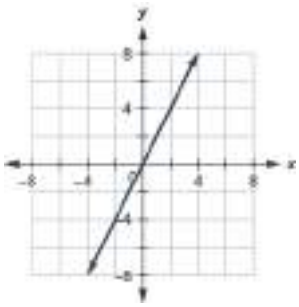
ⓑ D:(-∞,∞), R:{5}

351. ⓐ



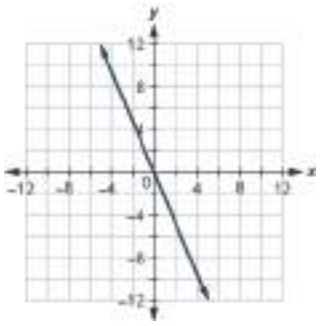
ⓑ D:(-∞,∞), R: $\{-3\}$

353. ⓐ



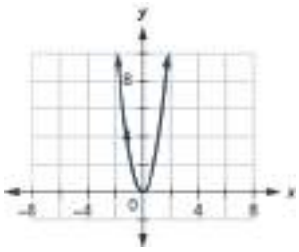
ⓑ D:(-∞,∞), R:(-∞,∞)

355. ⓐ



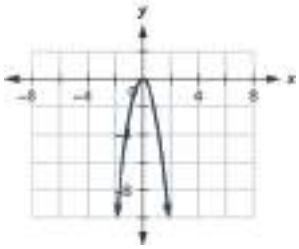
ⓑ D:(-∞,∞), R:(-∞,∞)

357. ⓐ



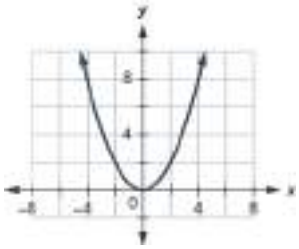
ⓑ D:(-∞,∞), R:(0,∞)

359. ⓐ



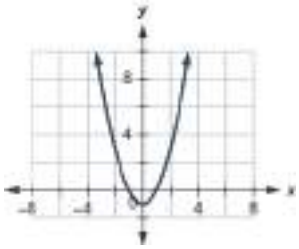
ⓑ (-∞,∞), R:(-∞,0)

361. ⓐ



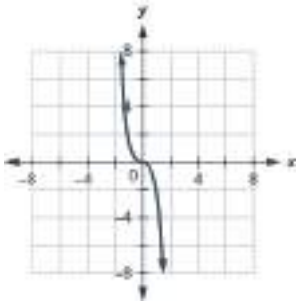
ⓑ (-∞,∞), R:(-∞,0)

363. ⓐ



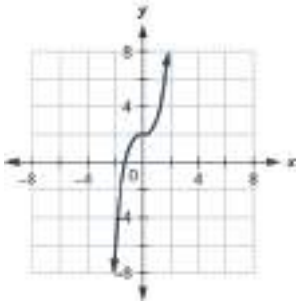
ⓑ (-∞,∞), R:(−1, ∞)

365. ⓐ



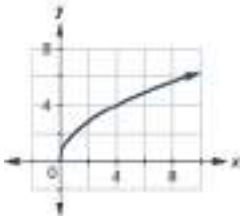
ⓑ D:(-∞,∞), R:(-∞,∞)

367. ⓐ



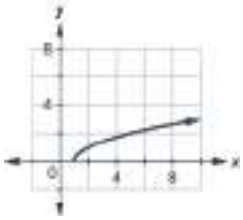
ⓑ D:(-∞,∞), R:(-∞,∞)

369. ⓐ



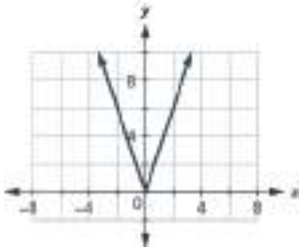
ⓑ D:[0,∞), R:[0,∞)

371. ⓐ



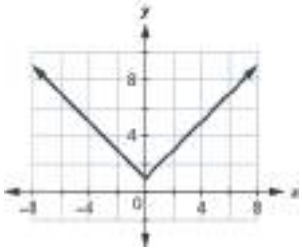
ⓑ D:[1,∞), R:[0,∞)

373. ⓐ



ⓑ D:[−1, ∞), R:[0,∞)

375. ⓐ



ⓑ D:(-∞,∞), R:[1,∞)

377. D: [2,∞), R: [0,∞)

379. D: (-∞,∞), R: [4,∞)

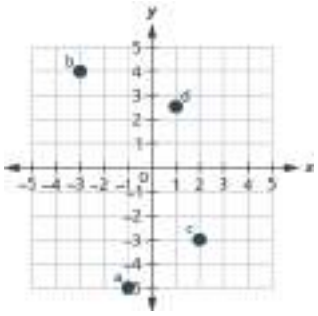
381. D: $[-2, 2]$, R: [0, 2]

383. ⓐ $f(0) = 0$ ⓑ $(\text{pi}/2) = -1$ ⓒ $f(-3\text{pi}/2) = -1$ ⓓ $f(x) = 0$ for $x = -2\text{pi}, -\text{pi}, 0, \text{pi}, 2\text{pi}$ ⓔ $(-2\text{pi}, 0), (-\text{pi}, 0), (0, 0), (\text{pi}, 0), (2\text{pi}, 0)$ $(f)(0, 0)$ ⓖ $[-2\text{pi}, 2\text{pi}]$ ⓗ $[-1, 1]$

385. ⓐ $f(0) = -6$ ⓑ $f(-3) = 3$ ⓒ $f(3) = 3$ ⓓ $f(x) = 0$ for no x ⓔ none ⓕ $y = 6$ ⓖ $[-3, 3]$ ⓗ $[-3, 6]$

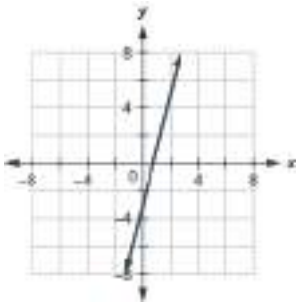
Review Exercises

391.

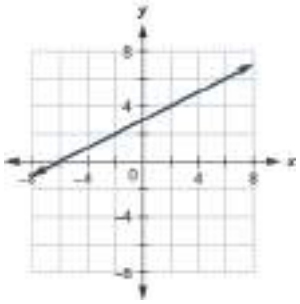


393. ⓑ, ⓒ

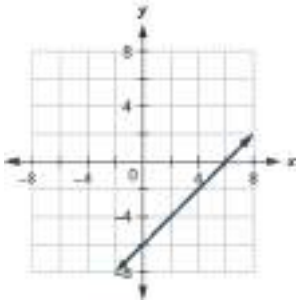
395.



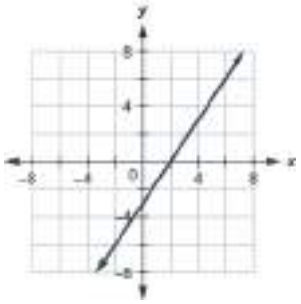
397.



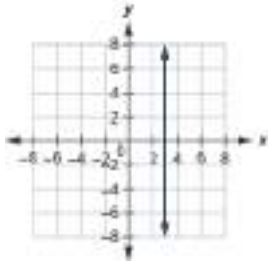
399.



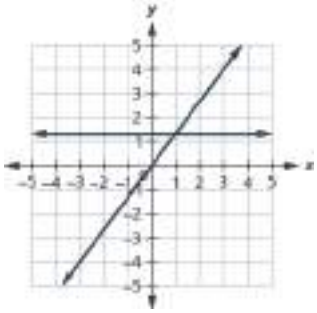
401.



403.



405.

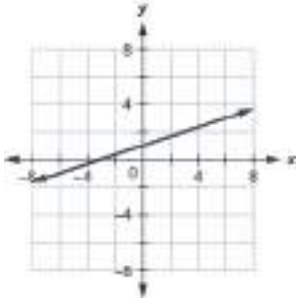


407. (0, 3)(3, 0)

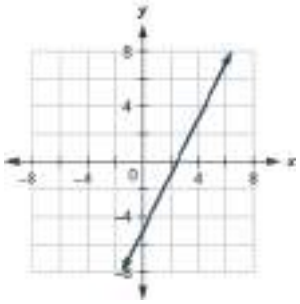
409. (6, 0), (0, 3)

411. (16, 0), (0, −12)

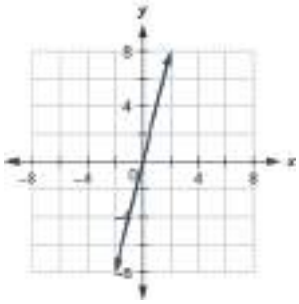
413.



415.



417.



419. 1

421. $-\frac{1}{2}$

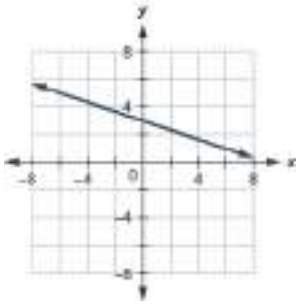
423. undefined

425. 0

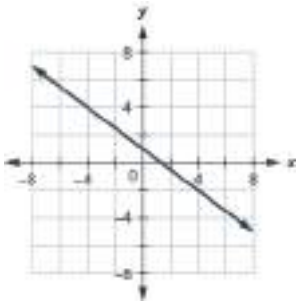
427. -6

429. $\frac{5}{2}$

431.



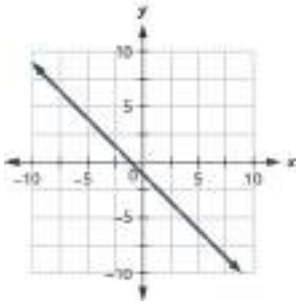
433.



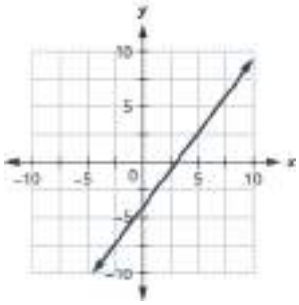
435. $m = \frac{5}{3}$; $(0, -6)$

437. $m = \frac{4}{5}$; $\left(0, -\frac{8}{5}\right)$

439.



441.



443. horizontal line

445. intercepts

447. plotting points

449. ⓐ $-\$250$

ⓑ $450

ⓒ The slope, 35, means that Marjorie's weekly profit, P, increases by $35 for each additional student lesson she teaches.
The P-intercept means that when the number of lessons is 0, Marjorie loses $250.

ⓓ



451. neither

453. not parallel

455. $y = -5x - 3$

457. $y = -2x$

459. $y = -3x + 5$

461. $y = -4$

463. $y = \frac{3}{5}x$

465. $y = -2x - 5$

467. $y = \frac{1}{2}x - \frac{5}{2}$

469. $y = 2$

471. $y = -\frac{2}{5}x + 8$

473. $y = 3$

475. $y = -\frac{3}{2}x - 6$

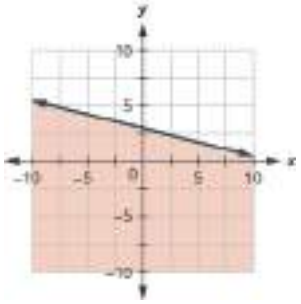
477. $y = 1$

479. ⓐ yes ⓑ no ⓒ yes ⓓ yes; ⓔ no

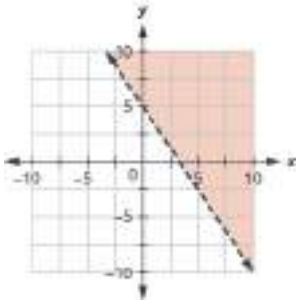
481. $y > \frac{2}{3}x - 3$

483. $x - 2y \geq 6$

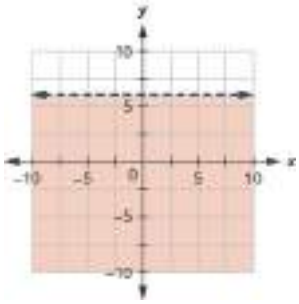
485.



487.

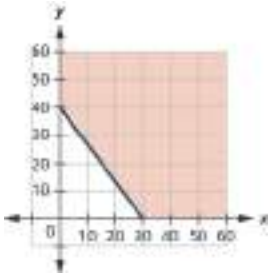


489.



491. ⓐ $20x + 15y \geq 600$

ⓑ



ⓒ Answers will vary.

493. ⓐ D: {−3, −2, −1, 0}

ⓑ R: {7, 3, 9, −3, 8}

495. ⓐ (4, 3), (−2, −3), (−2, −1), (−3, 1), (0, −1), (0, 4),

ⓑ D: {−3, −2, 0, 4}

ⓒ R: {−3, −1, 1, 3, 4}

497. ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3}

ⓒ {0, 1, 8, 27}

499. ⓐ {−3, −2, −1, 0, 1, 2, 3}

ⓑ {−3, −2, −1, 0, 1, 2, 3}

ⓒ {−243, −32, −1, 0, 1, 32, 243}

501. yes

503. yes

505. ⓐ $f(-2) = -10$ ⓑ $f(3) = 5$

ⓒ $f(a) = 3a - 4$

507. ⓐ $f(-2) = 20$ ⓑ $f(3) = 0$

ⓒ $f(a) = a^2 - 5a + 6$

509. 2

511. 18

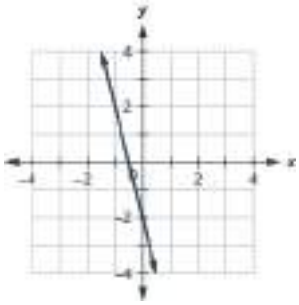
513. yes

515. no

517. yes

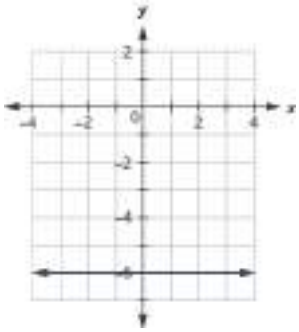
519. no

521. ⓐ



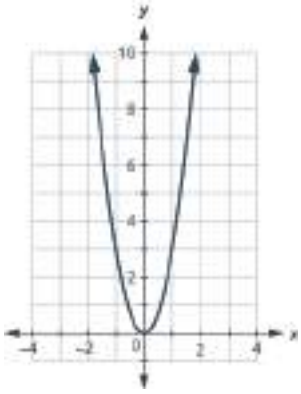
ⓑ D: (-∞,∞), R: (-∞,∞)

523. ⓐ



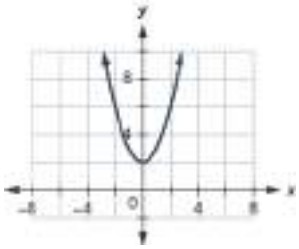
ⓑ D: (-∞,∞), R: (-∞,∞)

525. ⓐ



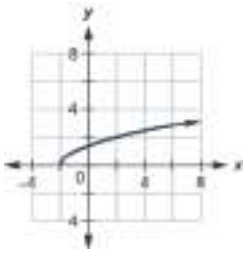
ⓑ D: (-∞,∞), R: (-∞,0]

527. ⓐ



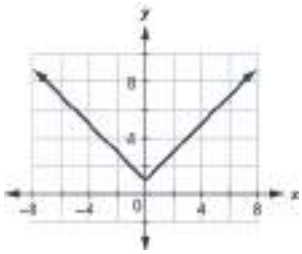
ⓑ D: (-∞,∞), R: (-∞,∞)

529. ⓐ



ⓑ D: $[-2, \infty)$, R: [0,∞)

531. ⓐ



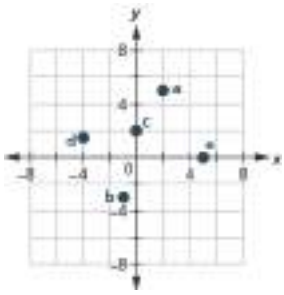
ⓑ D: (-∞,∞), R: [1,∞)

533. D: (-∞,∞), R: [2,∞)

535. ⓐ $f(x) = 0$ ⓑ $f(\pi/2) = 1$ ⓒ $f(-3\pi/2) = 1$ ⓓ $f(x) = 0$ for $x = -2\pi, -\pi, 0, \pi, 2\pi$ ⓔ $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$ $(f)(0, 0)$ ⓖ $[-2\pi, 2\pi]$ ⓗ $[-1, 1]$

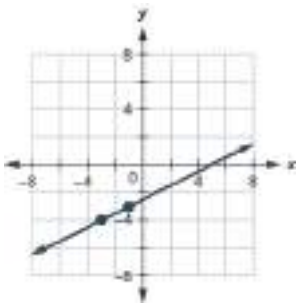
Practice Test

537.

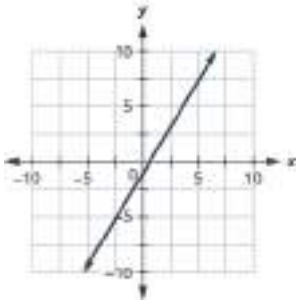


539. ⓐ $-\frac{3}{5}$ ⓑ undefined

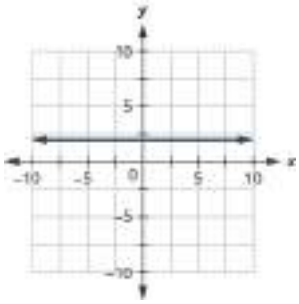
541.



543.



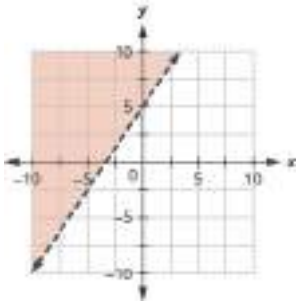
545.



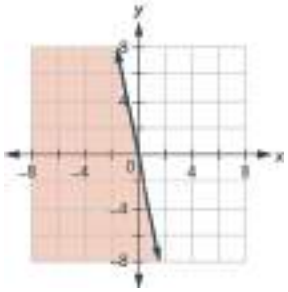
547. $y = 2x + 5$

549. $y = -\frac{4}{5}x - 5$

551.



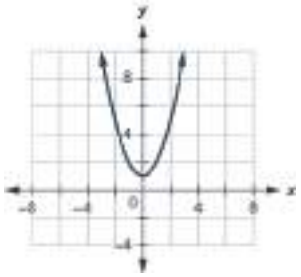
553.



555. ⓐ yes ⓑ $\{-3, -2, -1, 0, 1, 2, 3\}$ ⓒ {0, 1, 8, 27}

557. 12

559. ⓐ



ⓑ D: (-∞,∞), R: [1,∞)

561. ⓐ $x = -2, 2$ ⓑ $y = -4$ ⓒ $f(-1) = -3$ ⓓ $f(1) = -3$ ⓔ D: (-∞,∞) ⓕ R: $[-4, \infty)$

Chapter 4

Try It

4.1. ⓐ yes ⓑ no

4.2. ⓐ no ⓑ yes

4.3. $(3, 2)$

4.4. $(2, 3)$

4.5. $(3, 4)$

4.6. $(5, -4)$

4.7. no solution

4.8. no solution

4.9. infinitely many solutions

4.10. infinitely many solutions

4.11. ⓐ no solution, inconsistent, independent ⓑ one solution, consistent, independent

4.12. ⓐ no solution, inconsistent, independent ⓑ one solution, consistent, independent

4.13. $(6, 1)$

4.14. $(-3, 5)$

4.15. $\left(2, \frac{3}{2}\right)$

4.16. $\left(-\frac{1}{2}, -2\right)$

4.17. $(2, -1)$

4.18. $(-2, 3)$

4.19. $(1, 3)$

4.20. $(4, -3)$

4.21. $(6, 2)$

4.22. $(1, -2)$

4.23. infinitely many solutions

4.24. infinitely many solutions

4.25. ⓐ Since both equations are in standard form, using elimination will be most convenient. ⓑ Since one equation is already solved for x, using substitution will be most convenient.

4.26. ⓐ Since one equation is already solved for y, using substitution will be most convenient. ⓑ Since both equations are in standard form, using elimination will be most convenient.

4.27. 3, 7

4.28. $2, -8$

4.29. 160 policies

4.30. 1000 suits

4.31. Mark burned 11 calories for each minute of yoga and 7 calories for each minute of jumping jacks.

4.32. Erin burned 11 calories for each minute on the rowing machine and 5 calories for each minute of weight lifting.

4.33. The angle measures are 55 and 35.

4.34. The angle measures are 5 and 85.

4.35. The angle measures are 42 and 138.

4.36. The angle measures are 66 and 114.

4.37. 22, 68

4.38. 36, 54

4.39. The length is 60 feet and the width is 35 feet.

4.40. The length is 60 feet and the width is 38 feet.

4.41. It will take Clark 4 hours to catch Mitchell.

4.42. It will take Sally $1\frac{1}{2}$ hours to catch up to Charlie.

4.43. The rate of the boat is 11 mph and the rate of the current is 1 mph.

4.44. The speed of the canoe is 7 mph and the speed of the current is 1 mph.

4.45. The speed of the jet is 235 mph and the speed of the wind is 30 mph.

4.46. The speed of the jet is 408 mph and the speed of the wind is 24 mph.

4.47. 206 adults, 347 children

4.48. 42 adults, 105 children

4.49. 13 dimes and 29 quarters

4.50. 19 quarters and 51 nickels

4.51. 3 pounds peanuts and 2 pounds cashews

4.52. 10 pounds of beans, 10 pounds of ground beef

4.53. 120 ml of 25% solution and 30 ml of 50% solution

4.54. 125 ml of 10% solution and 125 ml of 40% solution

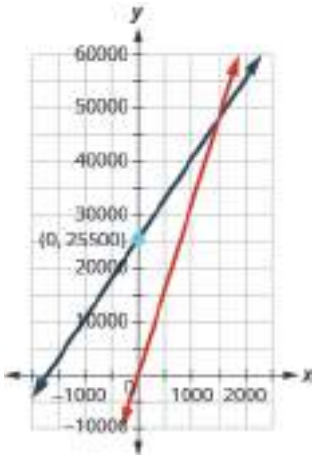
4.55. $42,000 in the stock fund and $8000 in the savings account

4.56. $1750 at 11% and $5250 at 13%

4.57. Bank $4,000; Federal $14,000

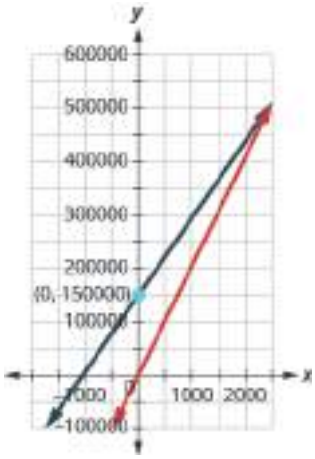
4.58. $41,200 at 4.5%, $24,000 at 7.2%

4.59. ⓐ $C(x) = 15x + 25,\ 500$ ⓑ $R(x) = 32x$ ⓒ



ⓓ 1,500 ; when 1,500 benches are sold, the cost and revenue will be both 48,000

4.60. ⓐ $C(x) = 120x + 150,000$ ⓑ $R(x) = 170x$ ⓒ



ⓓ 3,000 ; when 3,000 benches are sold, the revenue and costs are both $510,000

4.61. ⓐ yes ⓑ no

4.62. ⓐ no ⓑ yes

4.63. $(2, -1, 3)$

4.64. $(-2, 3, 4)$

4.65. $(-3, 4, -2)$

4.66. $(-2, 3, -1)$

4.67. no solution

4.68. no solution

4.69. infinitely many solutions $(x, 3, z)$ where $x = z - 3;\ y = 3;\ z$ is any real number

4.70. infinitely many solutions (x, y, z) where $x = 5z - 2;\ y = 4z - 3;\ z$ is any real number

4.71. The fine arts department sold 75 adult tickets, 200 student tickets, and 75 child tickets.

4.72. The soccer team sold 200 adult tickets, 300 student tickets, and 100 child tickets.

4.73.

ⓐ $\begin{bmatrix} 3 & 8 & -3 \\ 2 & 5 & -3 \end{bmatrix}$

ⓑ $\begin{bmatrix} 2 & -5 & 3 & 8 \\ 3 & -1 & 4 & 7 \\ 1 & 3 & 2 & -3 \end{bmatrix}$

4.74.

ⓐ $\begin{bmatrix} 11 & 9 & -5 \\ 7 & 5 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 5 & -3 & 2 & -5 \\ 2 & -1 & -1 & 4 \\ 3 & -2 & 2 & -7 \end{bmatrix}$

4.75. $\begin{cases} x - y + 2z = 3 \\ 2x + y - 2z = 1 \\ 4x - y + 2z = 0 \end{cases}$

4.76. $\begin{cases} x + y + z = 4 \\ 2x + 3y - z = 8 \\ x + y - z = 3 \end{cases}$

4.77.

ⓐ $\begin{bmatrix} -2 & 3 & 0 & -2 \\ 4 & -1 & -4 & 4 \\ 5 & -2 & -2 & -2 \end{bmatrix}$

ⓑ $\begin{bmatrix} -2 & 3 & 0 & -2 \\ 4 & -1 & -4 & 4 \\ 15 & -6 & -6 & -6 \end{bmatrix}$

ⓒ $\begin{bmatrix} -2 & 3 & 0 & -2 & \\ 3 & 4 & -13 & -16 & -8 \\ 15 & -6 & -6 & -6 & \end{bmatrix}$

4.78.

ⓐ $\begin{bmatrix} 4 & 1 & -3 & 2 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 8 & 2 & -6 & 4 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{bmatrix}$

ⓒ $\begin{bmatrix} 14 & -7 & -12 & -8 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{bmatrix}$

4.79. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & -4 \end{bmatrix}$

4.80. $\begin{bmatrix} 1 & -1 & 3 \\ 0 & -5 & 8 \end{bmatrix}$

4.81. The solution is $(4, -1)$.

4.82. The solution is $(-2, 0)$.

4.83. $(6, -1, -3)$

4.84. $(5, 7, 4)$

4.85. no solution

4.86. no solution

4.87. infinitely many solutions (x, y, z), where $x = z - 3; y = 3; z$ is any real number.

4.88. infinitely many solutions (x, y, z), where $x = 5z - 2; y = 4z - 3; z$ is any real number.

4.89. ⓐ -14; ⓑ -28

4.90. ⓐ 2 ⓑ -15

4.91. ⓐ 3 ⓑ 11 ⓒ 2

4.92. ⓐ -3 ⓑ 2 ⓒ 3

4.93. 37

4.94. 7

4.95. -11

4.96. 8

4.97. $(-\frac{15}{7}, \frac{24}{7})$

4.98. $(-2, 0)$

4.99. $(-9, 3, -1)$

4.100. $(-6, 3, -2)$

4.101. no solution

4.102. infinite solutions

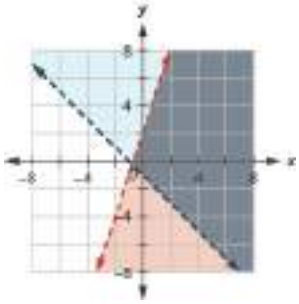
4.103. yes

4.104. yes

4.105. ⓐ no ⓑ yes

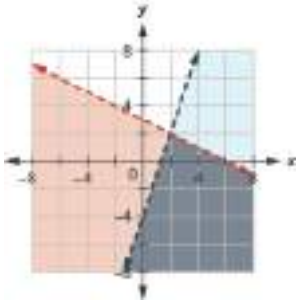
4.106. ⓐ yes ⓑ no

4.107.



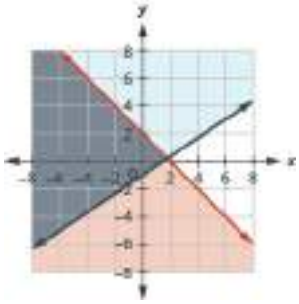
The solution is the grey region.

4.108.



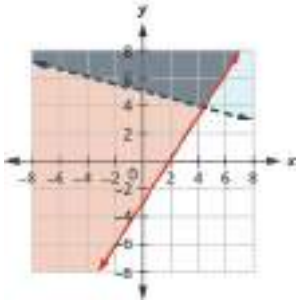
The solution is the grey region.

4.109.



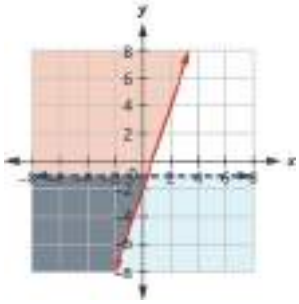
The solution is the grey region.

4.110.



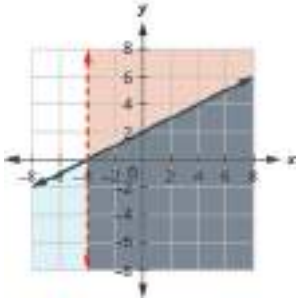
The solution is the grey region.

4.111.



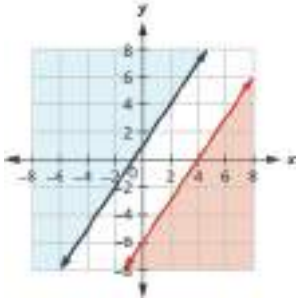
The solution is the grey region.

4.112.



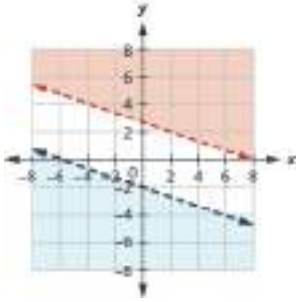
The solution is the grey region.

4.113.



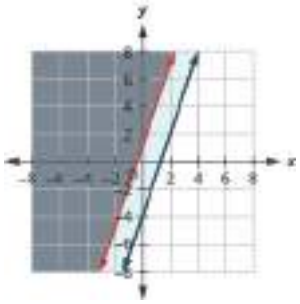
No solution.

4.114.



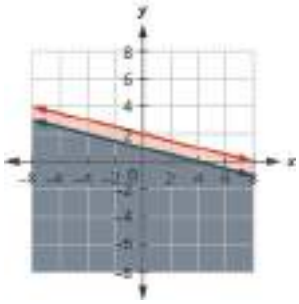
No solution.

4.115.



The solution is the grey region.

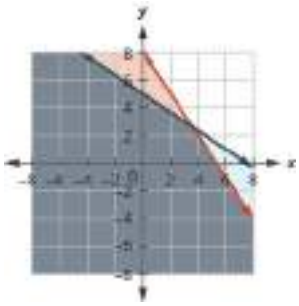
4.116.



The solution is the grey region.

4.117. ⓐ $\begin{cases} 30m + 20p \leq 160 \\ 2m + 3p \leq 15 \end{cases}$

ⓑ

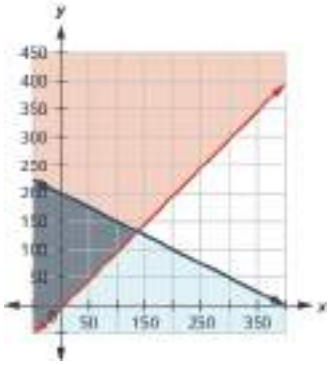


ⓒ yes

ⓓ no

4.118. ⓐ $\begin{cases} a \geq p + 5 \\ a + 2p \leq 400 \end{cases}$

ⓑ

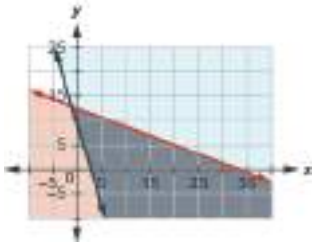


ⓒ no

ⓓ no

4.119. ⓐ $\begin{cases} 0.75d + 2e \leq 25 \\ 360d + 110e \geq 1000 \end{cases}$

ⓑ

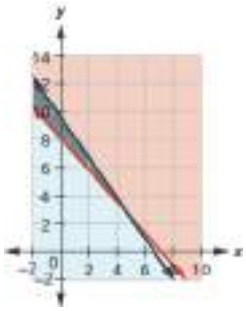


ⓒ yes

ⓓ no

4.120. ⓐ $\begin{cases} 140p + 125j \geq 1000 \\ 1.80p + 1.25j \leq 12 \end{cases}$

ⓑ



ⓒ yes

ⓓ no

Section Exercises

1. ⓐ yes ⓑ no

3. ⓐ yes ⓑ no

5. $(-3, 2)$

7. $(0, 2)$

9. $(2, 4)$

11. $(-2, 2)$

13. $(3, 3)$

15. $(6, -4)$

17. no solution

19. no solution

21. infinite solutions

23. infinite solutions

25. 1 point, consistent and independent

27. 1 point, consistent and independent

29. infinite solutions, consistent, dependent

31. $(1, -4)$

33. $(-3, 2)$

35. $(-1/2, 5/2)$

37. $(-5, 4)$

39. $(0, 10)$

41. $(4, -2)$

43. $(4, 0)$

45. none

47. $(6, 1)$

49. $(2, 3)$

51. $(-9, 3)$

53. $(9, 5)$

55. $(-3, 2)$

57. $(-15/7, 24/7)$

59. infinitely many

61. infinitely many

63. infinitely many

65. ⓐ substitution ⓑ elimination

67. ⓐ elimination ⓑ substituion

69. Answers will vary.

71. Answers will vary.

73. 13 and 17

75. -7 and -19

77. 14 and 23

79. 22 and -67

81. Eighty cable packages would need to be sold to make the total pay the same.

83. Mitchell would need to sell 120 stoves for the companies to be equal.

85. 8 and 40 gallons

87. 1000 calories playing basketball and 400 calories canoeing

89. Oranges cost \$2 per pound and bananas cost \$1 per pound

91. Package of paper \$4, stapler \$7

93. Hot dog 150 calories, cup of cottage cheese 220 calories

95. Owen will need 80 quarts of water and 20 quarts of concentrate to make 100 quarts of lemonade.

97. 53.5 degrees and 36.5 degrees

99. 16 degrees and 74 degrees

101. 134 degrees and 46 degrees

103. 37 degrees and 143 degrees

105. $16°$ and $74°$

107. $45°$ and $45°$

109. Width is 41 feet and length is 118 feet.

111. Width is 10 feet and length is 40 feet.

113. 11 hours

115. 1.5 hour

117. Boat rate is 16 mph and current rate is 4 mph.

119. Boat rate is 18 mph and current rate is 2 mph.

121. Jet rate is 265 mph and wind speed is 22 mph.

123. Jet rate is 415 mph and wind speed is 25 mph.

125. Answers will vary.

127. 110 adult tickets, 190 child tickets

129. 6 good seats, 10 cheap seats

131. 92 adult tickets, 220 children tickets

133. 13 nickels, 3 dimes

135. 42 dimes, 8 quarters

137. 17 \$10 bills, 37 \$20 bills

139. 80 pounds nuts and 40 pounds raisins

141. 9 pounds of Chicory coffee, 3 pounds of Jamaican Blue Mountain coffee

143. 10 bags of M&M's, 15 bags of Reese's Pieces

145. 7.5 liters of each solution

147. 80 liters of the 25% solution and 40 liters of the 10% solution

149. 240 liters of the 90% solution and 120 liters of the 75% solution

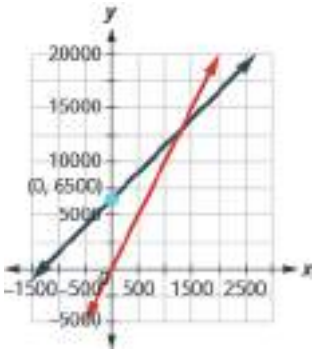
151. \$1600 at 8%, 960 at 6%

153. \$28,000 at 9%, \$36,000 at 5.5%

155. \$8500 CD, \$1500 savings account

157. \$55,000 on loan at 6% and \$30,000 on loan at 4.5%

159. ⓐ $C(x) = 5x + 6500$ ⓑ $R(x) = 10x$ ⓒ



ⓓ 1,500; when 1,500 water bottles are sold, the cost and the revenue equal \$15,000

161. Answers will vary.

163. ⓐ no ⓑ yes

165. ⓐ no ⓑ yes

167. $(4, 5, 2)$

169. $(7, 12, -2)$

171. $(-3, -5, 4)$

173. $(2, -3, -2)$

175. $(6, -9, -3)$

177. $(3, -4, -2)$

179. $(-3, 2, 3)$

181. $(-2, 0, -3)$

183. no solution

185. no solution

187. (x, y, z) where $x = 5z + 2; y = -3z + 1; z$ is any real number

189. (x, y, z) where $x = 5z - 2; y = 4z - 3; z$ is any real number

191. 42, 50, 58

193. 20, 5, 10

195. Answers will vary.

197.

ⓐ $\begin{bmatrix} 2 & 4 & -5 \\ 3 & -2 & 2 \end{bmatrix}$

ⓑ $\begin{bmatrix} 3 & -2 & -1 & -2 \\ -2 & 1 & 0 & 5 \\ 5 & 4 & 1 & -1 \end{bmatrix}$

199.

ⓐ $\begin{bmatrix} 2 & -5 & -3 \\ 4 & -3 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 4 & 3 & -2 & -3 \\ -2 & 1 & -3 & 4 \\ -1 & -4 & 5 & -2 \end{bmatrix}$

201. $\begin{cases} 2x - 4y = -2 \\ 3x - 3y = -1 \end{cases}$

203. $\begin{cases} 2x - 2y = -1 \\ 2y - z = 2 \\ 3x - z = -2 \end{cases}$

205.

ⓐ $\begin{bmatrix} 3 & 2 & 1 \\ 4 & -6 & -3 \end{bmatrix}$

ⓑ $\begin{bmatrix} 12 & 8 & 4 \\ 4 & -6 & -3 \end{bmatrix}$

ⓒ $\begin{bmatrix} 12 & 8 & 4 \\ 24 & -10 & -5 \end{bmatrix}$

207.

ⓐ $\begin{bmatrix} 2 & 1 & -4 & 5 \\ 6 & -5 & 2 & 3 \\ 3 & -3 & 1 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 2 & 1 & -4 & 5 \\ 6 & -5 & 2 & 3 \\ 3 & -3 & 1 & -1 \end{bmatrix}$

ⓒ $\begin{bmatrix} 2 & 1 & -4 & 5 \\ 6 & -5 & 2 & 3 \\ -4 & 7 & -6 & 7 \end{bmatrix}$

209. $\begin{bmatrix} 1 & -2 & 3 & -4 \\ 0 & 5 & -11 & 17 \\ 0 & 1 & -10 & 7 \end{bmatrix}$

211. $(1, -1)$

213. $(3, 3)$

215. $(-2, 5, 2)$

217. $(-3, -5, 4)$

219. $(-3, 2, 3)$

221. $(-2, 0, -3)$

223. no solution

225. no solution

227. infinitely many solutions (x, y, z) where $x = \frac{1}{2}z + 4; y = \frac{1}{2}z - 6; z$ is any real number

229. infinitely many solutions (x, y, z) where $x = 5z + 2; y = -3z + 1; z$ is any real number

231. Answers will vary.

233. 4

235. 10

237. ⓐ 6 ⓑ -14 ⓒ -6

239. ⓐ 9 ⓑ -3 ⓒ 8

241. -77

243. 49

245. -24

247. 25

249. $(7, 6)$

251. $(-2, 0)$

253. $(-3, 2)$

255. $(-9, 3)$

257. $(-3, -5, 4)$

259. $(2, -3, -2)$

261. $(-3, 2, 3)$

263. $(-2, 0, -3)$

265. infinitely many solutions

267. inconsistent

269. inconsistent

271. infinitely many solutions

273. yes

275. no

277. Answers will vary.

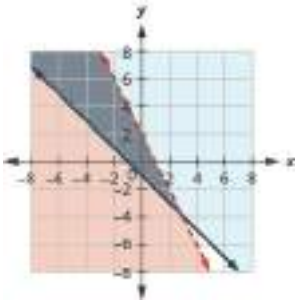
279. Answers will vary.

281. ⓐ false ⓑ true

283. ⓐ false ⓑ true

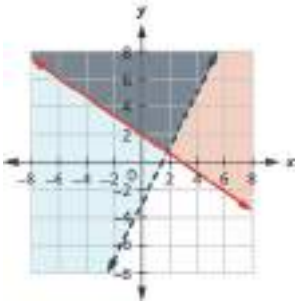
285. ⓐ false ⓑ true

287.



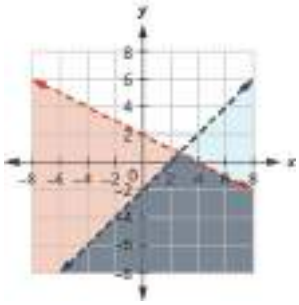
The solution is the grey region.

289.



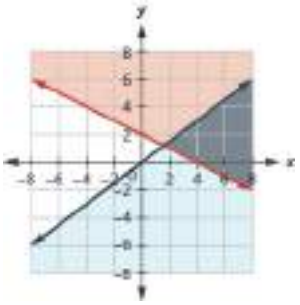
The solution is the grey region.

291.



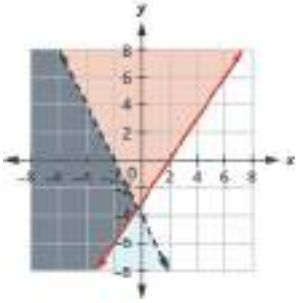
The solution is the grey region.

293.



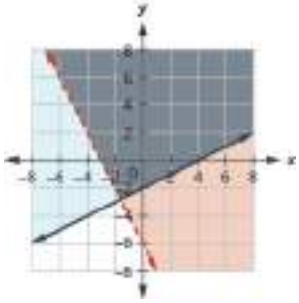
The solution is the grey region.

295.



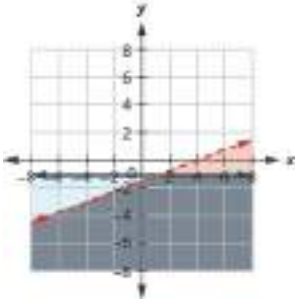
The solution is the grey region.

297.



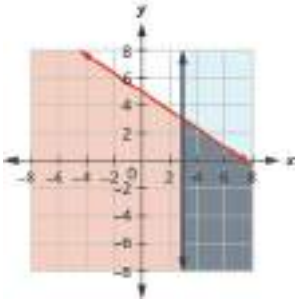
The solution is the grey region.

299.



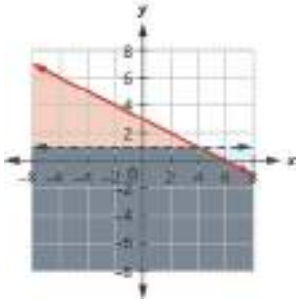
The solution is the grey region.

301.



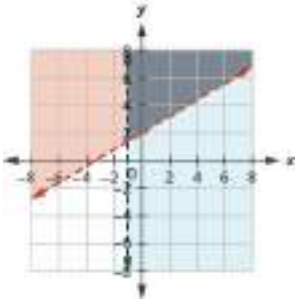
The solution is the grey region.

303.



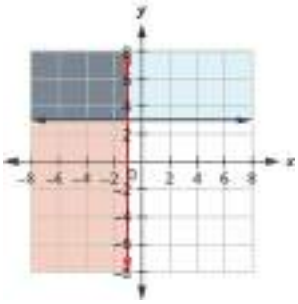
The solution is the grey region.

305.



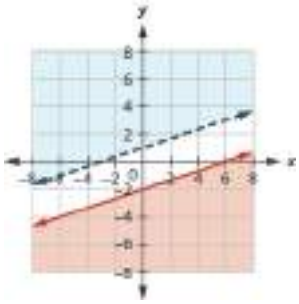
The solution is the grey region.

307.



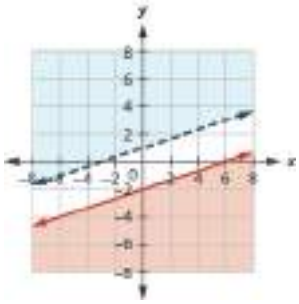
The solution is the grey region.

309.



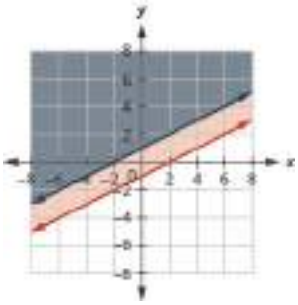
No solution.

311.



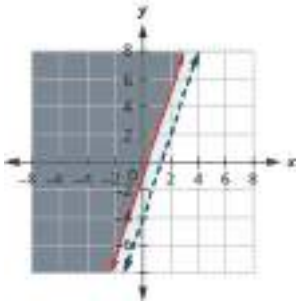
No solution.

313.



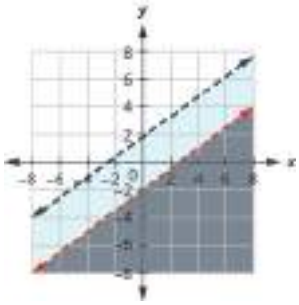
The solution is the grey region.

315.



The solution is the grey region.

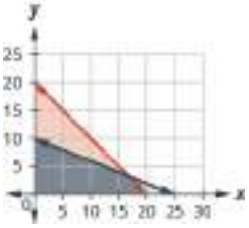
317.



The solution is the grey region.

319. ⓐ $\begin{cases} f \geq 0 \\ p \geq 0 \\ f + p \leq 20 \\ 2f + 5p \leq 50 \end{cases}$

ⓑ

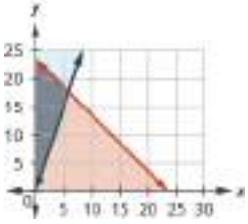


ⓒ yes

ⓓ no

321. ⓐ $\begin{cases} c \geq 0 \\ a \geq 0 \\ c + a \leq 24 \\ a \geq 3c \end{cases}$

ⓑ

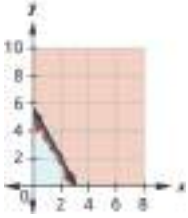


ⓒ yes

ⓓ no

323. ⓐ $\begin{cases} w \geq 0 \\ b \geq 0 \\ 27w + 16b > 80 \\ 3.20w + 1.75b \leq 10 \end{cases}$

ⓑ

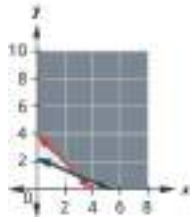


ⓒ no

ⓓ yes

325. ⓐ $\begin{cases} w \geq 0 \\ r \geq 0 \\ w + r \geq 4 \\ 270w + 650r \geq 1500 \end{cases}$

ⓑ



ⓒ no

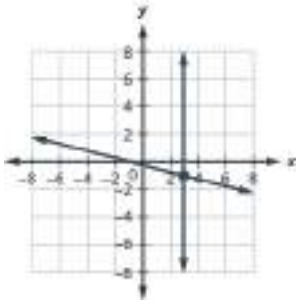
ⓓ yes

327. Answers will vary.

Review Exercises

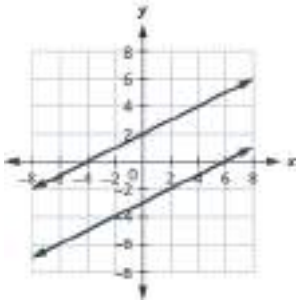
329. ⓐ yes ⓑ no

331.



$(3, -1)$

333.



no solution

335. one solution, consistent system, independent equations

337. $(4, 5)$

339. $(3, 1)$

341. infinitely many solutions

343. $(4, -1)$

345. $(6, 2)$

347. elimination

349. 50 irises and 150 tulips

351. 10 calories jogging and 10 calories cycling

353. 119, 61

355. $35°$ and $55°$

357. the length is 450 feet, the width is 264 feet

359. $\frac{1}{2}$ an hour

361. the rate of the jet is 395 mph, the rate of the wind is 7 mph

363. 41 dimes and 11 pennies

365. $46\frac{2}{3}$ liters of 30% solution, $23\frac{1}{3}$ liters of 60% solution

367. \$29,000 for the federal loan, \$14,000 for the private loan

369. ⓐ no ⓑ yes

371. $(-3, 2, -4)$

373. no solution

375. 25, 20, 15

377. $\begin{bmatrix} 4 & 3 & 0 & -2 \\ 1 & -2 & -3 & 7 \\ 2 & -1 & 2 & -6 \end{bmatrix}$

379. $\begin{cases} x - 3z = -1 \\ x - 2y = -27 \\ -y + 2z = 3 \end{cases}$

381.

ⓐ $\begin{bmatrix} 1 & -3 & -2 & 4 \\ 4 & -2 & -3 & -1 \\ 2 & 2 & -1 & -3 \end{bmatrix}$

ⓑ $\begin{bmatrix} 2 & -6 & -4 & 8 \\ 4 & -2 & -3 & -1 \\ 2 & 2 & -1 & -3 \end{bmatrix}$

ⓒ $\begin{bmatrix} 2 & -6 & -4 & 8 \\ 4 & -2 & -3 & -1 \\ 0 & -6 & -1 & 5 \end{bmatrix}$

383. $(-2, 5, -2)$

385. no solution

387. -4

389. 21

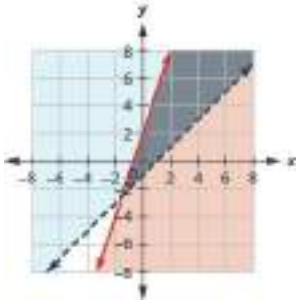
391. $(-3, 2)$

393. $(-3, 2, 3)$

395. inconsistent

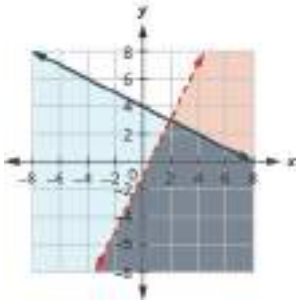
397. ⓐ yes ⓑ no

399.



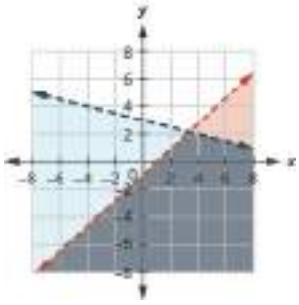
The solution is the grey region.

401.



The solution is the grey region.

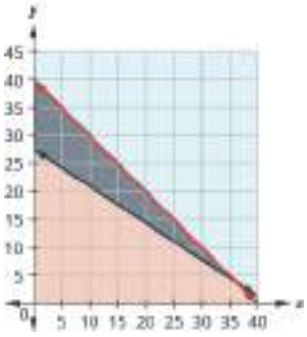
403.



No solution.

405. ⓐ $\begin{cases} b \geq 0 \\ n \geq 0 \\ b + n \leq 40 \\ 12b + 18n \geq 500 \end{cases}$

ⓑ

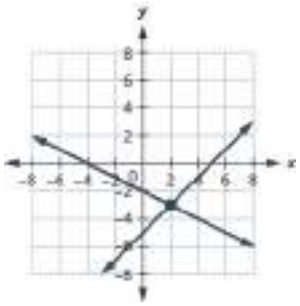


ⓒ yes

ⓓ no

Practice Test

407.



$(2, -3)$

409. $(2, 1)$

411. $(2, -2, 1)$

413. $(5, 7, 4)$

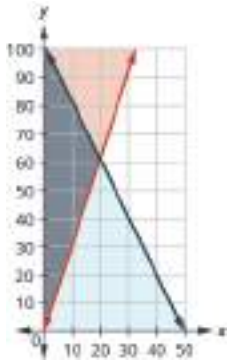
415. 9

417. 15 liters of 1% solution, 5 liters of 5% solution

419. The candy cost \$20; the cookies cost \$5; and the popcorn cost \$10.

421. ⓐ $\begin{cases} C \geq 0 \\ L \geq 0 \\ C + 0.5L \leq 50 \\ L \geq 3C \end{cases}$

ⓑ



ⓒ no

ⓓ yes

Chapter 5

Try It

5.1. ⓐ monomial, 0 ⓑ polynomial, 3 ⓒ trinomial, 3 ⓓ binomial, 2 ⓔ monomial, 10

5.2. ⓐ binomial, 3 ⓑ trinomial, 3 ⓒ monomial, 0 ⓓ polynomial, 4 ⓔ monomial, 7

5.3. ⓐ $21q^2$ ⓑ $13mn^3$

5.4. ⓐ $-7c^2$ ⓑ $-10y^2z^3$

5.5. ⓐ $5y^2 + 3z^2$ ⓑ $m^2n^2 - 8m^2 + 4n^2$

5.6. ⓐ $-4m^2 + n^2$ ⓑ $pq^2 - 6p - 5q^2$

5.7. $8x^2 - 11x + 8$

5.8. $17y^2 + 14y + 1$

5.9. $x^2 + 3x - 5$

5.10. $6b^2 + 3$

5.11. $-5ab + 7b^2$

5.12. $7mn + 4n^2$

5.13. $5x^2 - 5xy + 5y^2$

5.14. $7x^2 - 6xy - 2y^2$

5.15. $x^3 + y^3$

5.16. $p^3 - 3p^2q + q^3$

5.17. ⓐ 18 ⓑ 50 ⓒ -15

5.18. ⓐ 20 ⓑ 2 ⓒ -4

5.19. The height is 150 feet.

5.20. The height is 31 feet.

5.21. ⓐ $(f+g)(x) = 3x^2 - 6x - 3$
ⓑ $(f+g)(3) = 6$
ⓒ $(f-g)(x) = x^2 - 2x + 9$
ⓓ $(f-g)(-2) = 17$

5.22. ⓐ $(f+g)(x) = 6x^2 - x + 7$
ⓑ $(f+g)(3) = 58$
ⓒ $(f-g)(x) = 4x^2 - 7x - 9$
ⓓ $(f-g)(-2) = 21$

5.23. ⓐ b^{17} ⓑ 4^{3x} ⓒ $12p^6$
ⓓ x^{18}

5.24. ⓐ x^{16} ⓑ 10^{x+1} ⓐ $12z^8$
ⓓ b^{19}

5.25. ⓐ x^5 ⓑ 6^9 ⓒ $\frac{1}{x^4}$
ⓓ $\frac{1}{12^{15}}$

5.26. ⓐ y^6 ⓑ 10^8 ⓒ $\frac{1}{m^8}$
ⓓ $\frac{1}{9^{11}}$

5.27. ⓐ 1 ⓑ 1

5.28. ⓐ 1 ⓑ 1

5.29. ⓐ $\frac{1}{z^3}$ ⓑ $\frac{1}{10^7}$ ⓒ p^8 ⓓ 64

5.30. ⓐ $\frac{1}{n^2}$ ⓑ $\frac{1}{10{,}000}$ ⓒ q^7
ⓓ 16

5.31. ⓐ $\frac{81}{16}$ ⓑ $\frac{n^2}{m^2}$

5.32. ⓐ $\frac{125}{27}$ ⓑ $\frac{b^4}{a^4}$

5.33. ⓐ $\frac{1}{z^9}$ ⓑ $\frac{1}{p^3 q^3}$ ⓒ $-\frac{12y^5}{u}$

5.34. ⓐ $\frac{1}{c^{15}}$ ⓑ $\frac{1}{r^2 s^8}$ ⓒ $\frac{30d^3}{c^8}$

5.35. ⓐ b^{35} ⓑ 5^{12} ⓒ a^{48}

5.36. ⓐ z^{54} ⓑ 3^{49} ⓒ q^{29}

5.37. ⓐ $32w^5x^5$ ⓑ 1 ⓒ $\frac{1}{16b^{12}}$
ⓓ $\frac{64}{a^8}$

5.38. ⓐ $-27y^3$ ⓑ 1 ⓒ $\frac{1}{16x^8}$
ⓓ $\frac{8}{c^{12}}$

5.39. ⓐ $\frac{p^4}{10000}$ ⓑ $\frac{n^7}{m^7}$
ⓒ $\frac{81a^4b^{12}}{c^8}$ ⓓ $\frac{27}{x^6y^9}$

5.40. ⓐ $\frac{-8}{q^3}$ ⓑ $\frac{x^4}{w^4}$ ⓒ $\frac{x^2y^6}{9z^4}$
ⓓ $\frac{8n^6}{m^6}$

5.41. ⓐ $81c^{24}d^{30}$ ⓑ $\frac{1}{a^{18}}$
ⓒ y^{15}

5.42. ⓐ $256a^{22}b^{24}$ ⓑ $\frac{1}{p^{39}}$
ⓒ $2x^3y^{10}$

5.43. ⓐ 9.6×10^4 ⓑ 7.8×10^{-3}

5.44. ⓐ 4.83×10^4
ⓑ 1.29×10^{-2}

5.45. ⓐ 1,300 ⓑ −0.00012

5.46. ⓐ −950,000 ⓑ 0.075

5.47. ⓐ −0.006 ⓑ 20,000

5.48. ⓐ −0.009 ⓑ 400,000

5.49. ⓐ $-35y^{11}$ ⓑ $6a^5b^6$

5.50. ⓐ $54b^9$ ⓑ $8r^{11}s^8$

5.51. ⓐ $-15y^3 - 24y^2 + 21y$
ⓑ $12x^4y^2 - 20x^3y^3 + 12x^2y^4$

5.52. ⓐ $8x^4 - 24x^3 + 20x^2$
ⓑ $-18a^5b + 12a^4b^2 - 36a^3b^3$

5.53. ⓐ $x^2 + 17x + 72$
ⓑ $15c^2 + 14c - 8$

5.54. ⓐ $20x^2 + 51x + 27$
ⓑ $30y^2 - 3y - 6$

5.55. ⓐ $x^2 - 2x - 35$
ⓑ $15x^2 + 29x - 14$

5.56. ⓐ $b^2 + 3b - 18$
ⓑ $16y^2 - 20y - 50$

5.57. ⓐ $x^3 - 8x^2 + 6x - 48$
ⓑ $8a^2b^2 + 12ab - 20$

5.58. ⓐ $y^3 - 9y^2 + 7y - 63$
ⓑ $8x^2y^2 + 2xy - 15$

5.59. $15m^2 - 51m + 42$

5.60. $42b^2 - 53b + 15$

5.61. ⓐ $y^3 - 8y^2 + 17y - 6$
ⓑ $y^3 - 8y^2 + 17y - 6$

5.62. ⓐ $2x^3 + 5x^2 - 7x + 20$
ⓑ $y^3 - 8y^2 + 17y - 6$

5.63. ⓐ $x^2 + 18x + 81$
ⓑ $4c^2 - 4cd + d^2$

5.64. ⓐ $y^2 + 22y + 121$
ⓑ $16x^2 - 40xy + 25y^2$

5.65. ⓐ $36x^2 - 25$
ⓑ $16p^2 - 49q^2$

5.66. ⓐ $4x^2 - 49$ ⓑ $9x^2 - y^2$

5.67. ⓐ FOIL; $18b^2 + 77b - 18$
ⓑ Binomial Squares; $81p^2 - 72p + 16$
ⓒ Binomial Squares; $49y^2 + 14y + 1$
ⓓ Product of Conjugates; $16r^2 - 9$

5.68. ⓐ Binomial Squares; $36x^2 + 84x + 49$ ⓑ Product of Conjugates; $9x^2 - 16$ ⓒ FOIL; $10x^2 - 29x + 10$ ⓓ Binomial Squares; $36n^2 - 12n + 1$

5.69. ⓐ $(f \cdot g)(x) = x^3 - 7x^2 + 13x - 15$
ⓑ $(f \cdot g)(2) = -9$

5.70. ⓐ $(f \cdot g)(x) = x^3 + x^2 - 52x - 28$
ⓑ $(f \cdot g)(2) = -120$

5.71. $-\frac{9}{a^5 b}$

5.72. $\frac{-9d}{c^4}$

5.73. $\frac{4y^2}{7x^4}$

5.74. $\frac{5}{8m^5 n^3}$

5.75. $-4a + 2b$

5.76. $8a^5 b + 6a^3 b^2$

5.77. $y + 7$

5.78. $m + 5$

5.79. $x^3 - 3x^2 + 2x + 1 + \frac{3}{x+3}$

5.80. $x^3 - 3x^2 - 2x - 1 - \frac{3}{x+3}$

5.81. $x^2 + 4x + 16$

5.82. $25x^2 + 10x + 4$

5.83. $3x^2 + 4x - 2; 2$

5.84. $4x^2 - 3x + 1; 1$

5.85. $x^3 - 4x^2 + 5; 0$

5.86. $x^3 - 3x^2 + 2; 0$

5.87. ⓐ $\left(\frac{f}{g}\right)(x) = x - 8$
ⓑ $\left(\frac{f}{g}\right)(-3) = -11$

5.88. ⓐ $\left(\frac{f}{g}\right)(x) = x - 9$
ⓑ $\left(\frac{f}{g}\right)(-5) = -14$

5.89. -1

5.90. 6

5.91. yes

5.92. yes

Section Exercises

1. ⓐ trinomial, 5 ⓑ polynomial, 3 ⓒ binomial, 1 ⓓ monomial, 1 ⓔ binomial, 1

3. ⓐ binomial ⓑ trinomial ⓒ polynomial ⓓ trinomial ⓔ monomial

5. ⓐ 2^0 ⓑ 3^0 ⓒ 1^0 ⓓ 3^0 ⓔ 0^0

7. ⓐ 1^0 ⓑ 2^0 ⓒ 3^0 ⓓ 3^0 ⓔ 0^0

9. ⓐ $12x^2$ ⓑ $-5a$

11. ⓐ $6w$ ⓑ $19x^2 y$

13. $12x^2 - 5a$

15. $6w + 19x^2 y$

17. ⓐ $-22b$ ⓑ $16xy$

19. ⓐ $-10a + 5b$
ⓑ $pq^2 - 4p - 3q^2$

21. ⓐ $-4a^2 + b^2$
ⓑ $x^2 y - 3x + 7xy^2$

23. ⓐ $xy^2 - 5x - 5y^2$
ⓑ $19y + 5z$

25. $x - 3y + a^2 b - 4a - 5ab^2$

27. $-u^2 + 4v^2 + 12a + 8b$

29. $-4a - 3b$

31. $-7x^6$

33. $11y^2 + 4y + 11$

35. $-3x^2 + 17x - 1$

37. $11x^2 - 5x + 5$

39. $6a^2 - 4a - 1$

41. $2m^2 - 7m + 4$

43. $5a + 3$

45. $12s^2 - 14s + 9$

47. $3x^2 - x + 4$

49. $w^2 + 3w + 4$

51. $11w - 64$

53. $10x^2 - 7xy + 6y^2$

55. $10m^2 + 3mn - 8n^2$

57. $-3ab + 3b^2$

59. $p^3 - 6p^2q + pq^2 + 4q^3$

61. $x^3 + 2x^2y - 5xy^2 + y^3$

63. (a) 187 (b) 40 (c) 2

65. (a) -104 (b) 4 (c) 40

67. The height is 11 feet.

69. The revenue is $10,800.

71. The cost is $456.

73. (a) $(f+g)(x) = 7x^2 + 4x + 4$

(b) $(f+g)(2) = 40$

(c) $(f-g)(x) = -3x^2 - 12x - 2$

(d) $(f-g)(-3) = 7$

75.

(a) $(f+g)(x) = 6x^3 - x^2 - 9x + 3$

(b) $(f+g)(2) = 29$

(c) $(f-g)(x) = -x^2 + 5x + 3$

(d) $(f-g)(-3) = -21$

77. Answers will vary.

79. Answers will vary.

81. (a) d^9 (b) 4^{14x} (c) $8y^4$ (d) w^6

83. (a) n^{31} (b) 3^{x+6} (c) $56w^6$ (d) a^{16}

85. m^{x+3}

87. y^{a+b}

89. (a) x^{15} (b) 5^9 (c) $\frac{1}{q^{18}}$ (d) $\frac{1}{10}$

91. (a) p^{14} (b) 4^{12} (c) $\frac{1}{b^8}$ (d) $\frac{1}{4^5}$

93. (a) 1 (b) 1

95. (a) -1 (b) -1

97. (a) $\frac{1}{a^2}$ (b) $\frac{1}{1000}$ (c) c^5 (d) 9

99. (a) $\frac{1}{r^3}$ (b) $\frac{1}{100{,}000}$ (c) q^{10} (d) 1,000

101. (a) $\frac{64}{25}$ (b) $\frac{a^2}{b^2}$

103. (a) $\frac{729}{64}$ (b) $-\frac{v^5}{u^5}$

105. (a) $\frac{1}{25}$ (b) $\frac{1}{25}$ (c) 25 (d) -25

107. (a) $\frac{3}{5}$ (b) $\frac{1}{15}$

109. (a) $\frac{1}{b^4}$ (b) $\frac{w^2}{x^9}$ (c) $-12cd^4$

111. (a) 1 (b) $\frac{1}{u^4v^5}$ (c) $-36\frac{r^2}{j^5}$

113. $\frac{1}{p}$

115. (a) m^8 (b) 10^{18} (c) $\frac{1}{x^{12}}$

117. (a) y^{3x} (b) 5^{xy} (c) $\frac{1}{q^{48}}$

119. (a) $9x^2y^2$ (b) 1 (c) $\frac{1}{25x^4}$ (d) $\frac{16}{y^6}$

121. (a) $-125a^3b^3$ (b) 1 (c) $\frac{1}{36x^6}$ (d) $\frac{9}{y^8}$

123. (a) $\frac{p^5}{32}$ (b) $\frac{x^6}{y^6}$ (c) $\frac{8x^3y^6}{z^3}$ (d) $\frac{16}{p^6q^4}$

125. (a) $\frac{a^4}{81b^4}$ (b) $\frac{16m^2}{25}$ (c) $\frac{8x^3y^6}{z^3}$ (d) $\frac{16}{p^6q^4}$

127. (a) $1125t^8$ (b) $\frac{1}{t^{19}}$ (c) $\frac{y^4}{3x^2}$

129. (a) $16m^8n^{22}$ (b) $\frac{4}{p^6}$

131. (a) $\frac{7}{n}$ (b) $\frac{1}{7n}$ (c) $-\frac{1}{7n}$

133. (a) $\frac{1}{9p^2}$ (b) $\frac{3}{p^2}$ (c) $\frac{-3}{p^2}$

135. x^{14}

137. x^{30}

139. $2m^{18}$

141. $1{,}000x^6y^3$

143. $16a^{12}b^8$

145. $\frac{8}{27}x^6y^3$

147. $1{,}024a^{10}$

149. $25{,}000p^{24}$

151. $x^{18}y^{18}$

153. $144m^8n^{22}$

155. (a) $45x^3$ (b) $48y^4$

157. (a) $\frac{1}{2r^4}$ (b) $\frac{1}{3}x^{11}$

159. $\frac{1}{j^3}$

161. $-\frac{4000}{n^{12}}$

163. (a) 34×10^4 (b) 41×10^{-3}

165. (a) 1.29×10^6 (b) 103×10^{-8}

167. (a) -830 (b) 0.038

169. (a) 16,000,000,000 (b) 0.00000843

171. (a) 0.02 (b) 500,000,000

173. (a) 0.0000056 (b) 20,000,000

175. Answers will vary.

177. Answers will vary.

179. (a) $30x^8$ (b) $15x^8y^2$

181. ⓐ $72c^5$ ⓑ $\frac{1}{3}m^5n^5$

183. ⓐ $-5t^3-15t^2+90t$
ⓑ $9sr^5-27s^2r^4+45s^3r^3$

185. ⓐ $-5m^3-15m^2+90m$
ⓑ $-21x^4y^2-30x^3y^3+3x^2y^4$

187. $y^2+12y+27$

189. $21q^2-44q-32$

191. $y^2-8y+12$

193. $6p^2+11p+5$

195. $m^2+7m-44$

197. $33r^2-85r-8$

199. $y^3+3y^2-4y-12$

201. $6x^2y^2+13xy+6$

203. y^4-11y^2+28

205. $9r^2s^2-33rs+28$

207. $u^3+7u^2+14u+8$

209. $3a^3+31a^2+5a-50$

211. $6a^4-13a^3+15a^2+35a-50$

213. $p^3-10p^2+33p-36$

215. $6r^3-41r^2-61r-9$

217. $q^2+24q+144$

219. $4y^2-12yz+9z^2$

221. $x^2+\frac{4}{3}x+\frac{4}{9}$

223. $\frac{1}{64}x^2-\frac{1}{36}xy+\frac{1}{81}y^2$

225. $25u^4+90u^2+81$

227. $64p^6-48p^3+9$

229. $64j^2-16$

231. $81c^2-25$

233. $49w^2-100x^2$

235. $p^2-\frac{16}{25}q^2$

237. x^2y^2-81

239. $225m^4-64n^8$

241. $t^2-18t+81$

243. $2x^2-3xy-2y^2$

245. $9p^2-64$

247. $k^2-12k+36$

249. $8x^3-x^2y^2+64xy-8y^3$

251. $y^8+4y^4z+4z^2$

253. $m^6-16m^3n+64n^2$

255. $r^5+r^2s^2-r^3s^3-s^5$

257. $18p-9$

259. $-10j-15$

261. $9d^7$

263. $72m^5n^8$

265. $5q^5-10q^4+30q^3$

267. y^3-y^2-2y

269. $6k^3-11k^2-26k+4$

271. $121-b^2$

273. $4x^4-9y^8$

275. $9d^2+6d+1$

277. $9z^2-\frac{6}{5}z+\frac{1}{25}$

279.
ⓐ $(f\cdot g)(x)=4x^3-x^2-8x+5$
ⓑ $(f\cdot g)(-2)=-15$

281. ⓐ $(f\cdot g)(x)=49x^2-64$
ⓑ $(f\cdot g)(-2)=187$

283.
ⓐ
$(f\cdot g)(x)=x^4+6x^3+9x^2+10x-12$
ⓑ $(f\cdot g)(1)=14$

285. Answers will vary.

287. Answers will vary.

289. $\frac{2m^3}{3n^5}$

291. $\frac{-3y^3}{4x^3}$

293. $\frac{-3q^5}{p^5}$

295. $\frac{5v^5}{u^2}$

297. $4x^2+3x$

299. $-6y^2+3y$

301. $6r^3+11r^2s-8rs^2$

303. $-5y-3+\frac{1}{4y}$

305. $a-7$

307. $4x+3$

309. $p+3-\frac{8}{p+8}$

311. $2n^2-6n+8$

313. $m^2-10m+100$

315. $25y^2+20x+16$

317. $x^2-5x+6;0$

319. $2x^2-3x+4;4$

321. $x^3-2x^2+5x-4;-2$

323. $3x^3-2x^2-4x-2;0$

325. ⓐ $\left(\frac{f}{g}\right)(x)=x-6$
ⓑ $\left(\frac{f}{g}\right)(-5)=-11$

327. ⓐ $\left(\frac{f}{g}\right)(x)=x^2+5x-4$
ⓑ $\left(\frac{f}{g}\right)(0)=-4$

329.
ⓐ
$(f\cdot g)(x)=x^4+6x^3+9x^2+10x-12;$
ⓑ $(f\cdot g)(1)=14$

331. -9

333. -6

335. no

337. yes

339. answer will vary

341. Answers will vary.

Review Exercises

343. binomial

345. other polynomial

347. $-13y^3$

349. $6m^2 + 19m - 4$

351. $5u^2 + 4u + 7$

353. $2x^2 - 2x + 23$

355. $-7b - 3a$

357. $8a^2 - 5a - 2$

359. $-3m + 3$

361. $3y^2 - 8y - 14$

363. $x^3 + 2x^2y - 4xy^2$

365. ⓐ 165 ⓑ 39 ⓒ 5

367. The height is 64 feet .

369. ⓐ $(f+g)(x) = 4x^2 - 5x - 2$

ⓑ $(f+g)(3) = 19$

ⓒ $(f-g)(x) = -3x - 12$

ⓓ $(f-g)(-2) = -6$

371. p^{13}

373. a^6

375. y^{a+b}

377. a^5

379. $\frac{1}{x^4}$

381. 1

383. -3

385. $-\frac{1}{1000}$

387. $\frac{1}{8n}$

389. $\frac{1}{1000}$

391. 36

393. -125

395. $-\frac{1}{125}$

397. $\frac{x^3}{27}$

399. 3^{10}

401. x^5

403. $\frac{1}{u^3v^5}$

405. $\frac{1}{m^5}$

407. $\frac{1}{q^{16}}$

409. n^2

411. 1

413. $\frac{9}{y^8}$

415. $\frac{81x^4y^8}{z^4}$

417. $27x^7y^{17}$

419. $\frac{3y^4}{4x^4}$

421. 5.3×10^6

423. 29,000

425. 0.00009413

427. 0.00072

429. 9,000

431. $10c^{10}$

433. $\frac{m^7n^{10}}{9}$

435. $a^4 - 9a^3 - 36a^2$

437. $8n^4 - 10n^3$

439. $y^2 + 8y - 48$

441. $18p^2 - 93p + 110$

443. $k^2 - 3k - 54$

445. $10y^2 - 59y + 63$

447. $x^2 + x - 72$

449. $30a^2 - 33a + 3$

451. $15b^3 - b^2 - 47b + 18$

453. $24y^2 - 54y^2 + 32y - 5$

455. $x^2 + \frac{3}{2}x + \frac{9}{16}$

457. $25p^2 + 70pq + 49q^2$

459. $36x^2 - y^2$

461. $144x^6 - 49y^4$

463. $9p^9$

465. $-\frac{3}{y^4}$

467. $\frac{4a^5}{b^2}$

469. $\frac{4m^9}{n^4}$

471. $-9y^2 + 4y$

473. $-3x - 1 + \frac{3}{4x}$

475. $y - 3 + \frac{33}{q+6}$

477. $a^2 + a + 1$

479. $2x^2 - 5x - 4; 0$

481. ⓐ $\left(\frac{f}{g}\right)(x) = x - 6$

ⓑ $\left(\frac{f}{g}\right)(-2) = -8$

483. -9

485. no

Practice Test

487. ⓐ trinomial ⓑ 4

489. $6x^2 - 3x + 11$

491. x

493. 1

495. $\frac{1}{8y^3}$

497. x^9

499. $\frac{4r}{s^6}$

501. $-48x^5y^9$

503. $21m^2 - 19m - 6$

505. $16x^2 - 24x + 9$

507. $3y^2 - 7x$

509. yes

511. 4.4×10^3

513. ⓐ 36 ⓑ 21

515. ⓐ $\left(\frac{f}{g}\right)(x) = 3x + 4$ ⓑ $\left(\frac{f}{g}\right)(3) = 13$

Chapter 6

Try It

6.1. $5m^2$

6.2. $7x$

6.3. $3y^2\left(3x + 2x^2 + 7y\right)$

6.4. $3p\left(p^2 - 2pq + 3q^2\right)$

6.5. $2x^2(x + 6)$

6.6. $3y^2(2y - 5)$

6.7. $3xy\left(5x^2 - xy + 2y^2\right)$

6.8. $2ab\left(4a^2 + ab - 3b^2\right)$

6.9. $-4b\left(b^2 - 4b + 2\right)$

6.10. $-7a\left(a^2 - 3a + 2\right)$

6.11. $(m + 3)(4m - 7)$

6.12. $(n - 4)(8n + 5)$

6.13. $(x + 8)(y + 3)$

6.14. $(a + 7)(b + 8)$

6.15. ⓐ $(x - 5)(x + 2)$ ⓑ $(5x - 4)(4x - 3)$

6.16. ⓐ $(y + 4)(y - 7)$ ⓑ $(7m - 3)(6m - 5)$

6.17. $(q + 4)(q + 6)$

6.18. $(t + 2)(t + 12)$

6.19. $(u - 3)(u - 6)$

6.20. $(y - 7)(y - 9)$

6.21. $(m + 3)(m + 6)$

6.22. $(n - 3)(n - 4)$

6.23. $(a - b)(a - 10b)$

6.24. $(m - n)(m - 12n)$

6.25. prime

6.26. prime

6.27. $5x(x - 1)(x + 4)$

6.28. $6y(y - 2)(y + 5)$

6.29. $(a + 1)(2a + 3)$

6.30. $(b + 1)(4b + 1)$

6.31. $(2x - 3)(4x - 1)$

6.32. $(2y - 7)(5y - 1)$

6.33. $(3x + 2y)(6x - 5y)$

6.34. $(3x + y)(10x - 21y)$

6.35. $5n(n - 4)(3n - 5)$

6.36. $8q(q + 6)(7q - 2)$

6.37. $(x + 2)(6x + 1)$

6.38. $(2y + 1)(2y + 3)$

6.39. $4(2x - 3)(2x - 1)$

6.40. $3(3w - 2)(2w - 3)$

6.41. $\left(h^2 - 2\right)\left(h^2 + 6\right)$

6.42. $\left(y^2 + 4\right)\left(y^2 - 5\right)$

6.43. $(x - 3)(x - 1)$

6.44. $(y - 1)(y + 1)$

6.45. $(2x + 3)^2$

6.46. $(3y + 4)^2$

6.47. $(8y - 5)^2$

6.48. $(4z - 9)^2$

6.49. $(7x + 6y)^2$

6.50. $(8m + 7n)^2$

6.51. $2y(2x - 3)^2$

6.52. $3q(3p + 5)^2$

6.53. $(11m - 1)(11m + 1)$

6.54. $(9y - 1)(9y + 1)$

6.55. $(16m - 5n)(16m + 5n)$

6.56. $(11p - 3q)(11p + 3q)$

6.57. $2y^2(x - 2)(x + 2)\left(x^2 + 4\right)$

6.58. $7c^2(a - b)(a + b)\left(a^2 + b^2\right)$

6.59. $(x - 5 - y)(x - 5 + y)$

6.60. $(x + 3 - 2y)(x + 3 + 2y)$

6.61. $(x + 3)\left(x^2 - 3x + 9\right)$

6.62. $(y + 2)\left(y^2 - 2y + 4\right)$

6.63. $(2x - 3y)\left(4x^2 - 6xy + 9y^2\right)$

6.64. $(10m - 5n)\left(100m^2 - 50mn + 25n^2\right)$

6.65. $4(5p + q)\left(25p^2 - 5pq + q^2\right)$

6.66. $2(6c + 7d)\left(36c^2 - 42cd + 49d^2\right)$

6.67. $(-2y + 1)\left(13y^2 + 5y + 1\right)$

6.68. $(-4n + 3)\left(31n^2 + 21n + 9\right)$

6.69. $8y(y - 1)(y + 3)$

6.70. $5y(y - 9)(y + 6)$

6.71. $4x(2x - 3)(2x + 3)$

6.72. $3(3y - 4)(3y + 4)$

6.73. $(2x + 5y)^2$

6.74. $(3x - 4y)^2$

6.75. $2xy\left(25x^2 + 36\right)$

6.76. $3xy(9y^2+16)$

6.77. $2(5m+6n)(25m^2-30mn+36n^2)$

6.78. $2(p+3q)(p^2-3pq+9q^2)$

6.79. $4ab(a^2+4)(a-2)(a+2)$

6.80. $7xy(y^2+1)(y-1)(y+1)$

6.81. $6(x+b)(x-2c)$

6.82. $2(4x-1)(2x+3y)$

6.83. $4q(p-3)(p-1)$

6.84. $3p(2q+1)(q-2)$

6.85. $(2x-3y-5)(2x-3y+5)$

6.86. $(4x-3y-8)(4x-3y+8)$

6.87. $m=\frac{2}{3}$, $m=-\frac{1}{2}$

6.88. $p=-\frac{3}{4}$, $p=\frac{3}{4}$

6.89. $c=2$, $c=\frac{4}{3}$

6.90. $d=3$, $d=-\frac{1}{2}$

6.91. $p=\frac{7}{5}$, $p=-\frac{7}{5}$

6.92. $x=\frac{11}{6}$, $x=-\frac{11}{6}$

6.93. $m=1$, $m=\frac{3}{2}$

6.94. $k=3$, $k=-3$

6.95. $a=-\frac{5}{2}$, $a=\frac{2}{3}$

6.96. $b=-2$, $b=-\frac{1}{20}$

6.97. $x=0$, $x=\frac{3}{2}$

6.98. $y=0$, $y=\frac{1}{4}$

6.99. ⓐ $x=-3$ or $x=5$
ⓑ $(-3,\ 7)$ $(5,\ 7)$

6.100. ⓐ $x=1$ or $x=7$
ⓑ $(1,\ -4)$ $(7,\ -4)$

6.101. ⓐ $x=1$ or $x=\frac{5}{2}$
ⓑ $(1,\ 0)$, $\left(\frac{5}{2},\ 0\right)$ ⓒ $(0,\ 5)$

6.102. ⓐ $x=-3$ or $x=\frac{5}{6}$
ⓑ $(-3,\ 0)$, $\left(\frac{5}{6},\ 0\right)$ ⓒ $(0,\ -15)$

6.103. $-15,\ -17$ and 15, 17

6.104. $-23,\ -21$ and 21, 23

6.105. The width is 5 feet and length is 6 feet.

6.106. The length of the patio is 12 feet and the width 15 feet.

6.107. 5 feet and 12 feet

6.108. 24 feet and 25 feet

6.109. ⓐ 5 ⓑ 0;3 ⓒ 196

6.110. ⓐ 4 ⓑ 0;2 ⓒ 144

Section Exercises

1. $2pq$

3. $6m^2n^3$

5. $2a$

7. $5x^3y$

9. $3(2m+3)$

11. $9(n-7)$

13. $3(x^2+2x-3)$

15. $2(p^2+4p+1)$

17. $8y^2(y+2)$

19. $5x(x^2-3x+4)$

21. $3x(8x^2-4x+5)$

23. $6y^2(2x+3x^2-5y)$

25. $4xy(5x^2-xy+3y^2)$

27. $-2(x+4)$

29. $-2x(x^2-9x+4)$

31. $-4pq(p^2+3pq-4q)$

33. $(x+1)(5x+3)$

35. $(b-2)(3b-13)$

37. $(b+5)(a+3)$

39. $(y+5)(8y+1)$

41. $(u+2)(v-9)$

43. $(u-1)(u+6)$

45. $(3p-5)(3p+4)$

47. $(n-6)(m-4)$

49. $(x-7)(2x-5)$

51. $-9xy(3x+2y)$

53. $(x^2+2)(3x-7)$

55. $(x+y)(x+5)$

57. Answers will vary.

59. Answers will vary.

61. $(p+5)(p+6)$

63. $(n+3)(n+16)$

65. $(a+5)(a+20)$

67. $(x-2)(x-6)$

69. $(y-3)(y-15)$

71. $(x-1)(x-7)$

73. $(p-1)(p+6)$

75. $(x-4)(x-2)$

77. $(x-12)(x+1)$

79. $(x+8y)(x-10y)$

81. $(m+n)(m-65n)$

83. $(a+8b)(a-3b)$

85. Prime

87. Prime

89. $p(p-10)(p+2)$

91. $3m(m-5)(m-2)$

93. $5x^2(x-3)(x+5)$

95. $(2t+5)(t+1)$

97. $(11x+1)(x+3)$

99. $(4w-1)(w-1)$

101. $(4q+1)(q-2)$

103. $(2p-5q)(3p-2q)$

105. $(4a-3b)(a+5b)$

107. $-16(x-1)(x-1)$

109. $10q(3q+2)(q+4)$

111. $(5n+1)(n+4)$

113. $(2k-3)(2k-5)$

115. $(3y+5)(2y-3)$

117. $(2n+3)(n-15)$

119. $10(6y-1)(y+5)$

121. $3z(8z+3)(2z-5)$

123. $8(2s+3)(s+1)$

125. $12(4y-3)(y+1)$

127. $\left(x^2+1\right)\left(x^2-7\right)$

129. $(x^2-7)(x^2+4)$

131. $(x-12)(x+1)$

133. $(3y-4)(3y-1)$

135. $(u-6)(u-6)$

137. $(r-4s)(r-16s)$

139. $(4y-7)(3y-2)$

141. $(2n-1)(3n+4)$

143. $13\left(z^2+3z-2\right)$

145. $3p(p+7)$

147. $6(r+2)(r+3)$

149. $4(2n+1)(3n+1)$

151. $\left(x^2+2\right)\left(x^2-6\right)$

153. $(x-9)(x+6)$

155. Answers will vary.

157. Answers will vary.

159. $(4y+3)^2$

161. $(6s+7)^2$

163. $(10x-1)^2$

165. $(5n-12)^2$

167. $(7x+2y)^2$

169. $(50y-1)(2y-1)$

171. $10j(k+4)^2$

173. $3u^2(5u-v)^2$

175. $(5v-1)(5v+1)$

177. $(7x-2)(7x+2)$

179. $6p^2(q-3)(q+3)$

181. $6\left(4p^2+9\right)$

183. $(11x-12y)(11x+12y)$

185. $(13c-6d)(13c+6d)$

187. $(2z-1)(2z+1)(4z^2+1)$

189. $2b^2(3a-2)(3a+2)\left(9a^2+4\right)$

191. $(x-8-y)(x-8+y)$

193. $(a+3-3b)(a+3+3b)$

195. $(x+5)(x^2-5x+25)$

197. $(z^2-3)(z^4+3z^2+9)$

199. $(2-7t)(4+14t+49t^2)$

201. $(2y-5z)(4y^2+10yz+25z^2)$

203. $(6a+5b)(36a^2-30ab+25b^2)$

205. $7(k+2)(k^2-2k+4)$

207. $2x^2(1-2y)(1+2y+4y^2)$

209. $-9(x+1)\left(x^2+3\right)$

211. $-(3y+5)\left(21y^2-30y+25\right)$

213. $(8a-5)(8a+5)$

215. $3(3q-1)(3q+1)$

217. $(4x-9)^2$

219. $2\left(4p^2+1\right)$

221. $(5-2y)\left(25+10y+4y^2\right)$

223. $5(3n+2)^2$

225. $(x+y-5)(x-y-5)$

227. $(3x+1)\left(3x^2+1\right)$

229. Answers will vary.

231. Answers will vary.

233. $(2n-1)(n+7)$

235. $a^3(a^2+9)$

237. $(11r-s)(11r+s)$

239. $8(m-2)(m+2)$

241. $(5w-6)^2$

243. $(m+7n)^2$

245. $7(b+3)(b-2)$

247. $3(x-3)\left(x^2+3x+9\right)$

249. $(k-2)(k+2)(k^2+4)$

251. $5xy^2\left(x^2+4\right)(x+2)(x-2)$

253. $3(5p+4)(q-1)$

255. $4(x+3)(x+7)$

257. $u^2(u+1)\left(u^2-u+1\right)$

259. prime

261. $10(m-5)(m+5)\left(m^2+25\right)$

263. $3y(3x+2)(4x-1)$

265. $(2x-3y)\left(4x^2+6xy+9y^2\right)$

267. $(y+1)(y-1)\left(y^2-y+1\right)\left(y^2+y+1\right)$

269. $(3x-y+7)(3x-y-7)$

271. $\left(9x^2-12x+4\right)$

273. Answers will vary.

275. Answers will vary.

277. $a=10/3,\ a=7/2$

279. $m=0,\ m=5/12$

281. $x=1/2$

283. $a=-5/4,\ a=6$

285. $m=5/4,\ m=3$

287. $a=-1,\ a=0$

289. $m=12/7,\ m=-12/7$

291. $y=-9/4,\ y=9/4$

293. $n=-6/11,$ n $=6/11$

295. $x=2,\ x=-5$

297. $x=3/2,\ x=-1$

299. $x=3/2,\ x=-1$

301. $x=-2/3$

303. $x=2,\ x=-4/3$

305. $x=-3/2,\ x=1/3$

307. $p=0,\ p=¾$

309. $x=0,\ x=6$

311. $x=0,\ x=1/3$

313. ⓐ $x=2$ or $x=6$ ⓑ $(2,-4)$ $(6,-4)$

315. ⓐ $x=\frac{3}{2}$ or $x=\frac{3}{4}$ ⓑ $\left(\frac{3}{2},-4\right)$ $\left(\frac{3}{4},-4\right)$

317. ⓐ $x=\frac{2}{3}$ or $x=-\frac{2}{3}$ ⓑ $\left(\frac{2}{3},0\right),\left(-\frac{2}{3},0\right)$ ⓒ $(0,-4)$

319. ⓐ $x=\frac{5}{3}$ or $x=-\frac{1}{2}$ ⓑ $\left(\frac{5}{3},0\right),\left(-\frac{1}{2},0\right)$ ⓒ $(0,-5)$

321. $-13,-11$ and 11, 13

323. $-14,-12$ and 12, 14

325. -4 and 7
327. 5, 11
329. 6, 8
331. 8, 15, 17
333. ⓐ 0, 2 ⓑ 1
335. Answers will vary.

Review Exercises

337. $3ab^2$
339. $3y$
341. $7(5y+12)$
343. $3x(6x^2-5)$
345. $4x(x^2-3x+4)$
347. $-3x\left(x^2-9x+4\right)$
349. $(a+b)(x-y)$
351. $(x-3)(x+7)$
353. $(m^2+1)(m+1)$
355. $(a+3)(a+11)$
357. $(m+9)(m-6)$
359. $(x+5y)(x+7y)$
361. $(a+7b)(a-3b)$
363. Prime
365. $3y(y-5)(y-2)$
367. $(5y+9)(y+1)$
369. $(5y+1)(2y-11)$
371. $-9(9a-1)(a+2)$
373. $(3a-1)(6a-1)$
375. $(3x-1)(5x+2)$
377. $3(x+4)(x-3)$
379. $3(2a-7)(3a+1)$
381. $(x^2-15)(x^2+2)$
383. $(5x+3)^2$
385. $10(2x+9)^2$
387. $3u^2(5u-v)^2$
389. $(13m+n)(13m-n)$
391. $(3+11y)(3-11y)$
393. $n(13n+1)(13n-1)$
395. $6\left(4p^2+9\right)$
397. $(2z-1)(2z+1)(4z^2+1)$
399. $(a+3-3b)(a+3+3b)$
401. $(a-5)\left(a^2+5a+25\right)$
403. $2(m+3)\left(m^2-3m+9\right)$
405. $4x^2(6x+11)$
407. $(4n-7m)^2$
409. $5u^2(u+3)(u-3)$
411. prime
413. $(b-4)\left(b^2+4b+16\right)$
415. $(2b+5c)(b-c)$
417. $5(q+3)(q-6)$
419. $10(m-5)(m+5)\left(m^2+25\right)$
421. $(4x-3y+8)(4x-3y-8)$
423. $b=-1/5,\ b=-1/6$
425. $x=1/2$
427. $x=-4,\ x=-5$
429. $p=-\frac{5}{2},\ p=8$
431. $m=\frac{5}{12},\ m=-\frac{5}{12}$
433. $x=2,\ x=-5$
435. $p=0,\ p=¾$
437. ⓐ $x=-7$ or $x=-4$
ⓑ $(-7,-8)$ $(-4,-8)$
439. ⓐ $x=\frac{7}{8}$ or $x=-\frac{7}{8}$
ⓑ $\left(\frac{7}{8},0\right),\ \left(-\frac{7}{8},0\right)$ ⓒ $(0,-49)$
441. The numbers are -21 and -19 or 19 and 21.
443. The lengths are 8, 15, and 17 ft.

Practice Test

445. $40a^2(2+3a)$
447. $(x+7)(x+6)$
449. $(x-8)(y+7)$
451. $(3s-2)^2$
453. $3(x+5y)(x-5y)$
455. $(x+5)\left(x^2-5x+25\right)$
457. $\left(3x^2-5\right)\left(2x^2-3\right)$
459. $a=4/5,\ a=-6$
461. The width is 12 inches and the length is 14 inches.
463. ⓐ $x=3$ or $x=4$ ⓑ $(3,-7)$ $(4,-7)$

Chapter 7

Try It

7.1. ⓐ $x=0$ ⓑ $n=-\frac{1}{3}$
ⓒ $a=-1,\ a=-3$
7.2. ⓐ $q=0$ ⓑ $y=-\frac{2}{3}$
ⓒ $m=2,\ m=-3$
7.3. $\frac{x+1}{x-1},\ x\neq 2,\ x\neq 1$
7.4. $\frac{x-5}{x-1},\ x\neq -2,\ x\neq 1$
7.5. $\frac{2(x-3y)}{3(x+3y)}$
7.6. $\frac{5(x-y)}{2(x+5y)}$

7.7. $-\frac{x+1}{x+5}$

7.8. $-\frac{x+2}{x+1}$

7.9. $\frac{x-2}{2(x+3)}$

7.10. $\frac{3(x-6)}{x+5}$

7.11. $\frac{x-4}{x-5}$

7.12. $-\frac{(b+2)(b-1)}{(1+b)(b+4)}$

7.13. $\frac{2\left(x^2+2x+4\right)}{(x+2)\left(x^2-2x+4\right)}$

7.14. $\frac{2z}{z-1}$

7.15. $\frac{x+2}{4}$

7.16. $\frac{2}{y+5}$

7.17. $\frac{2(m+1)(m+2)}{3(m+4)(m-3)}$

7.18. $\frac{(n+5)(n+9)}{2(n+6)(2n+3)}$

7.19. The domain of $R(x)$ is all real numbers where $x \neq 5$ and $x \neq -1$.

7.20. The domain of $R(x)$ is all real numbers where $x \neq 4$ and $x \neq -2$.

7.21. $R(x) = 2$

7.22. $R(x) = \frac{1}{3}$

7.23. $R(x) = \frac{x-2}{4(x-8)}$

7.24. $R(x) = \frac{x(x-2)}{x-1}$

7.25. $x+2$

7.26. $x+3$

7.27. $\frac{x-11}{x-2}$

7.28. $\frac{x-3}{x+9}$

7.29. $\frac{y+3}{y+2}$

7.30. $\frac{3n-2}{n-1}$

7.31. ⓐ $(x-4)(x+3)(x+4)$

ⓑ $\frac{2x+8}{(x-4)(x+3)(x+4)}$, $\frac{x+3}{(x-4)(x+3)(x+4)}$

7.32. ⓐ $(x+2)(x-5)(x+1)$

ⓑ $\frac{3x^2+3x}{(x+2)(x-5)(x+1)}$, $\frac{5x-25}{(x+2)(x-5)(x+1)}$

7.33. $\frac{7x-4}{(x-2)(x+3)}$

7.34. $\frac{7m+25}{(m+3)(m+4)}$

7.35. $\frac{5m^2-9m+2}{(m+1)(m-2)(m+2)}$

7.36. $\frac{2n^2+12n-30}{(n+2)(n-5)(n+3)}$

7.37. $\frac{1}{x-2}$

7.38. $\frac{-3}{z-3}$

7.39. $\frac{5x+1}{(x-6)(x+1)}$

7.40. $\frac{y+3}{y+4}$

7.41. $\frac{1}{(b+1)(b-1)}$

7.42. $\frac{1}{(x+2)(x+1)}$

7.43. $\frac{v+3}{v+1}$

7.44. $\frac{3w}{w+7}$

7.45. $\frac{x-7}{x-4}$

7.46. $\frac{x^2-3x+18}{(x+3)(x-3)}$

7.47. $\frac{2}{3(x-1)}$

7.48. $\frac{1}{2(x-3)}$

7.49. $\frac{14}{11}$

7.50. $\frac{10}{23}$

7.51. $\frac{y+x}{y-x}$

7.52. $\frac{ab}{b-a}$

7.53. $\frac{b(b+2)(b-5)}{3b-5}$

7.54. $\frac{3}{c+3}$

7.55. $\frac{7}{3}$

7.56. $\frac{10}{3}$

7.57. $\frac{b+a}{a^2+b^2}$

7.58. $\frac{y-x}{xy}$

7.59. $\frac{3(x-2)}{5x+7}$

7.60. $\frac{x+21}{6x-43}$

7.61. $\frac{3}{5x+22}$

7.62. $\frac{2\left(2y^2+13y+5\right)}{3y}$

7.63. $\frac{x}{x+4}$

7.64. $\frac{x(x+1)}{3(x-1)}$

7.65. $y = -\frac{7}{15}$

7.66. $x = \frac{13}{15}$

7.67. $x = -3,\ x = 5$

7.68. $y = -2,\ y = 6$

7.69. $x = \frac{2}{3}$

7.70. $y = 2$

7.71. There is no solution.

7.72. There is no solution.

7.73. $x = 3$

7.74. $y = 7$

7.75. There is no solution.

7.76. There is no solution.

7.77. There is no solution.

7.78. There is no solution.

7.79. ⓐ The domain is all real numbers except $x \neq 3$ and $x \neq 4$. ⓑ $x = 2, x = \frac{14}{3}$ ⓒ $(2, 3), \left(\frac{14}{3}, 3\right)$

7.80. ⓐ The domain is all real numbers except $x \neq 1$ and $x \neq 5$. ⓑ $x = \frac{21}{4}$ ⓒ $\left(\frac{21}{4}, 4\right)$

7.81. $y = mx - 4m + 5$

7.82. $y = mx + 5m + 1$

7.83. $a = \frac{b}{cb - 1}$

7.84. $y = \frac{3x}{x + 6}$

7.85. $y = 33$

7.86. $z = 14$

7.87. The pediatrician will prescribe 12 ml of acetaminophen to Emilia.

7.88. The pediatrician will prescribe 180 mg of fever reducer to Isabella.

7.89. The distance is 150 miles.

7.90. The distance is 350 miles.

7.91. The telephone pole is 40 feet tall.

7.92. The pine tree is 60 feet tall.

7.93. Link's biking speed is 15 mph.

7.94. The speed of Danica's boat is 17 mph.

7.95. Dennis's uphill speed was 10 mph and his downhill speed was 5 mph.

7.96. Joon's rate on the country roads is 50 mph.

7.97. Kayla's biking speed was 15 mph.

7.98. Victoria jogged 6 mph on the flat trail.

7.99. When the two gardeners work together it takes 2 hours and 24 minutes.

7.100. When Daria and her mother work together it takes 2 hours and 6 minutes.

7.101. Kristina can paint the room in 12 hours.

7.102. It will take Jordan 6 hours.

7.103. ⓐ $c = 4.8t$ ⓑ He would burn 432 calories.

7.104. ⓐ $d = 50t$ ⓑ It would travel 250 miles.

7.105. ⓐ $h = \frac{130}{t}$ ⓑ $1\frac{2}{3}$ hours

7.106. ⓐ $x = \frac{3500}{p}$ ⓑ 500 units

7.107. $(-\infty, -4) \cup [2, \infty)$

7.108. $(-\infty, -2] \cup (4, \infty)$

7.109. $\left(-\frac{3}{2}, 3\right)$

7.110. $(-8, 4)$

7.111. $(-\infty, -4) \cup (2, \infty)$

7.112. $(-\infty, -4) \cup (3, \infty)$

7.113. $(2, 4)$

7.114. $(3, 6)$

7.115. $(-4, 2]$

7.116. $[-1, 4)$

7.117. ⓐ $c(x) = \frac{20x + 6000}{x}$ ⓑ More than 150 items must be produced to keep the average cost below \$60 per item.

7.118. ⓐ $c(x) = \frac{5x + 900}{x}$ ⓑ More than 60 items must be produced to keep the average cost below \$20 per item.

Section Exercises

1. ⓐ $z = 0$ ⓑ $p = \frac{5}{6}$ ⓒ $n = -4, n = 2$

3. ⓐ $y = 0$ ⓑ $x = -\frac{1}{2}$ ⓒ $u = -4, u = 7$

5. $-\frac{4}{5}$

7. $\frac{2m^2}{3n}$

9. $\frac{8}{3}$

11. $\frac{x + 5}{x - 1}$

13. $\frac{a + 2}{a + 8}$

15. $\frac{p^2 + 4}{p - 2}$

17. $\frac{4b(b - 4)}{(b + 5)(b - 8)}$

19. $\frac{3(m + 5n)}{4(m - 5n)}$

21. -1

23. $-\frac{5}{y + 4}$

25. $\frac{w^2-6w+36}{w-6}$

27. $-\frac{z-5}{4+z}$

29. $\frac{3}{10}$

31. $\frac{x^3}{8y}$

33. $\frac{p(p-4)}{2(p-9)}$

35. $\frac{y-5}{3(y+5)}$

37. $-\frac{4(b+9)}{3(b+7)}$

39. $\frac{(c+2)(c+2)}{(c-2)(c-3)}$

41. $-\frac{(m-2)(m-3)}{(3+m)(m+4)}$

43. $-\frac{1}{v+5}$

45. $\frac{3s}{s+4}$

47. $\frac{4(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

49. $\frac{x-2}{8x}$

51. $\frac{2a-7}{5}$

53. $3(3c-5)$

55. $\frac{4(m+8)(m+7)}{3(m-4)(m+2)}$

57. $\frac{(4p+1)(p-7)}{3p(p+9)(p-1)}$

59. $x \neq 5$ and $x \neq -5$

61. $x \neq 2$ and $x \neq -3$

63. $R(x) = 2$

65. $R(x) = \frac{x+5}{2x(x+2)}$

67. $R(x) = \frac{3x(x+7)}{x-7}$

69. $R(x) = \frac{x(x-5)}{x-6}$

71. Answers will vary.

73. Answers will vary.

75. $\frac{3}{5}$

77. $\frac{3c+5}{4c-5}$

79. $r+8$

81. $\frac{2w}{w-4}$

83. $3a+7$

85. $\frac{m-2}{2}$

87. $\frac{p+3}{p+5}$

89. $\frac{r+9}{r+7}$

91. 4

93. $x+2$

95. $\frac{z+4}{z-5}$

97. $\frac{4b-3}{b-7}$

99. ⓐ $(x+2)(x-4)(x+3)$

ⓑ $\frac{5x+15}{(x+2)(x-4)(x+3)}$, $\frac{2x^2+4x}{(x+2)(x-4)(x+3)}$

101. ⓐ $(z-2)(z+4)(z-4)$

ⓑ $\frac{9z-36}{(z-2)(z+4)(z-4)}$, $\frac{4z^2-8z}{(z-2)(z+4)(z-4)}$

103. ⓐ $(b+3)(b+3)(b-5)$

ⓑ $\frac{4b-20}{(b+3)(b+3)(b-5)}$, $\frac{2b^2+6b}{(b+3)(b+3)(b-5)}$

105. ⓐ $(d+5)(3d-1)(d-6)$

ⓑ $\frac{2d-12}{(d+5)(3d-1)(d-6)}$, $\frac{5d^2+25d}{(d+5)(3d-1)(d-6)}$

107. $\frac{21y+8x}{30x^2y^2}$

109. $\frac{5r-7}{(r+4)(r-5)}$

111. $\frac{11w+1}{(3w-2)(w+1)}$

113. $\frac{2y^2+y+9}{(y+3)(y-1)}$

115. $\frac{b(5b+10+2a^2)}{a^2(b-2)(b+2)}$

117. $-\frac{m}{m+4}$

119. $\frac{3(r^2+6r+18)}{(r+1)(r+6)(r+3)}$

121. $\frac{2(7t-6)}{(t-6)(t+6)}$

123. $\frac{4a^2+25a-6}{(a+3)(a+6)}$

125. $\frac{-6}{m-6}$

127. $\frac{p+2}{p+3}$

129. $\frac{3}{r-2}$

131. $\frac{4(8x+1)}{10x-1}$

133. $\frac{x-5}{(x-4)(x+1)(x-1)}$

135. $\frac{1}{(x-1)(x+1)}$

137. $\frac{5a^2+7a-36}{a(a-2)}$

139. $\frac{c-5}{c+2}$

141. $\frac{3(d+1)}{d+2}$

143. ⓐ $R(x) = -\frac{(x+8)(x+1)}{(x-2)(x+3)}$

ⓑ $R(x) = \frac{x+1}{x+3}$

145. ⓐ $\frac{3(3x+8)}{(x-8)(x+8)}$

ⓑ $R(x)=\frac{3}{x+8}$

147. Answers will vary.

149. ⓐ Answers will vary.

ⓑ Answers will vary.

ⓒ Answers will vary.

ⓓ $\frac{x+y}{xy}$

151. $\frac{a-4}{2a}$

153. $\frac{1}{2(c-2)}$

155. $\frac{12}{13}$

157. $\frac{20}{57}$

159. $\frac{n^2+m}{m-n^2}$

161. $\frac{rt}{t-r}$

163. $\frac{(x+1)(x-3)}{2}$

165. $\frac{4}{a+1}$

167. $\frac{11}{8}$

169. 19

171. $\frac{c^2+c}{c-d^2}$

173. $\frac{pq}{q-p}$

175. $\frac{2x-10}{3x+16}$

177. $\frac{3z-19}{3z+8}$

179. $\frac{4}{3a-7}$

181. $\frac{2c+29}{5c}$

183. $\frac{2p-5}{5}$

185. $\frac{m(m-5)}{(4m-19)(m+5)}$

187. $\frac{13}{24}$

189. $2(a-4)$

191. $\frac{3mn}{n-m}$

193. $\frac{(x-1)(x-2)}{6}$

195. Answers will vary.

197. $a=10$

199. $v=\frac{40}{21}$

201. $m=-2,\ m=4$

203. $p=-5,\ p=-4$

205. $v=14$

207. $x=-\frac{4}{5}$

209. $z=-145$

211. $q=-18,\ q=-1$

213. no solution

215. no solution

217. $b=-8$

219. $d=2$

221. $m=1$

223. no solution

225. $s=\frac{5}{4}$

227. $x=-\frac{4}{3}$

229. no solution

231.

ⓐ The domain is all real numbers except $x\neq -2$ and $x\neq -4$.

ⓑ $x=-3,\ x=-\frac{14}{5}$

ⓒ $(-3,\ 5),\left(-\frac{14}{5},\ 5\right)$

233.

ⓐ The domain is all real numbers except $x\neq 2$ and $x\neq 5$.

ⓑ $x=\frac{9}{2}$,

ⓒ $\left(\frac{9}{2},\ 2\right)$

235. $r=\frac{C}{2\pi}$

237. $w=2v+7$

239. $c=\frac{b+3+2a}{a}$

241. $p=\frac{q}{4q-2}$

243. $w=\frac{15v}{10+v}$

245. $n=\frac{5m+23}{4}$

247. $c=\frac{E}{m^2}$

249. $y=\frac{20x}{12-x}$

251. Answers will vary.

253. $x=49$

255. $p=-11$

257. $a=16$

259. $m=60$

261. $p=30$

263. ⓐ 162 beats per minute

ⓑ yes

265. 9 ml

267. 159 calories

269. 325 Canadian dollars

271. 3 cups

273. 4 bags

275. ⓐ 6 ⓑ 12

277. 950 miles

279. 680 miles

281. $\frac{2}{3}$ foot (8 in.)

283. 247.3 feet

285. 160 mph

287. 29 mph

289. 30 mph

291. 20 mph

293. 4 mph

295. 60 mph
297. 650 mph
299. 50 mph
301. 50 mph
303. 4.2 mph
305. 2 hours
307. 2 hours and 44 minutes
309. 7 hours and 30 minutes
311. 10 min
313. $y = \frac{14}{3}x$
315. $p = 3.2q$
317. ⓐ $P = 2.5g$ ⓑ $82.50
319. ⓐ $m = 8v$ ⓑ 16 liters
321. ⓐ $L = 3d^2$ ⓑ 300 pounds
323. $y = \frac{20}{x}$
325. $v = \frac{3}{w}$
327. ⓐ $g = \frac{92{,}400}{w}$ ⓑ 16.8 mpg
329. ⓐ $t = \frac{1000}{r}$ ⓑ 2.5 hours
331. ⓐ $c = \frac{2}{t}$ ⓑ 1 cavity
333. ⓐ $c = 2.5m$ ⓑ $55
335. Answers will vary.
337. Answers will vary.
339. $(-\infty, -4) \cup [3, \infty)$
341. $[-1, 3)$
343. $(-\infty, 1) \cup (7, \infty)$
345. $(-5, 6)$
347. $\left(-\frac{5}{2}, 5\right)$
349. $(-\infty, -3) \cup (6, \infty)$
351. $[-9, 6)$
353. $(-\infty, -6] \cup (4, \infty)$
355. $(-\infty, -3) \cup (-4, \infty)$
357. $(1, 4)$
359. $(-\infty, -3) \cup (\frac{5}{2}, \infty)$
361. $(-\infty, \frac{2}{3}) \cup (\frac{3}{2}, \infty)$
363. $(-\infty, 0) \cup (0, 4) \cup (6, \infty)$
365. $[-2, 0) \cup (0, 4]$
367. $(-4, 4)$
369. $[-10, -1) \cup (2, \infty)$
371. $(2, 5]$
373. $(-\infty, -2) \cup [6, \infty)$
375. Answers will vary.

Review Exercises

377. $a \neq \frac{2}{3}$
379. $y \neq 0$
381. $\frac{3}{4}$
383. $\frac{x+3}{x+4}$
385. $\frac{1}{6}$
387. $\frac{-3x}{2}$
389. $\frac{3x}{(x+6)(x+6)}$
391. $\frac{1}{11+w}$
393. $\frac{5}{c+4}$
395. $R(x) = 3$
397. 1
399. $y + 5$
401. $x + 4$
403. $\frac{q-3}{q+5}$
405. $\frac{15w+2}{6w-1}$
407. $\frac{3b-2}{b+7}$
409. $(a+2)(a-5)(a+4)$
411. $(3p+1)(p+6)(p+8)$
413. $\frac{11c-12}{(c-2)(c+3)}$
415. $\frac{5x^2+26x}{(x+4)(x+4)(x+6)}$
417. $\frac{2(y^2+10y-2)}{(y+2)(y+8)}$
419. $\frac{2m-7}{m+2}$
421. $\frac{4}{a-8}$
423. $R(x) = \frac{x+8}{x+5}$
425. $R(x) = \frac{2}{x+11}$
427. $\frac{x-2}{2x}$
429. $\frac{(x-8)(x-5)}{2}$
431. $\frac{11}{8}$
433. $\frac{z-5}{23z+21}$
435. $x = \frac{6}{7}$
437. $b = \frac{3}{2}$
439. no solution
441. ⓐ The domain is all real numbers except $x \neq 2$ and $x \neq 4$. ⓑ $x = 1, x = 6$ ⓒ $(1, 1), (6, 1)$
443. $l = \frac{V}{hw}$
445. $z = \frac{y+5+7x}{x}$
447. $\frac{12}{5}$
449. 15
451. 1161 calories
453. $b = 9; x = 2\frac{1}{3}$
455. 23 feet
457. 45 mph
459. 16 mph
461. $\frac{4}{5}$ hour
463. 12 days
465. 7
467. 301 mph
469. 288 feet
471. 97 tickets

473. $(-4, 3]$

475. $[-6, 4)$

477. $(-\infty, -2] \cup [4, \infty)$

479. $(-\infty, 2) \cup [5, \infty)$

481.

ⓐ $c(x) = \frac{150x + 100000}{x}$

ⓑ More than 10,000 items must be produced to keep the average cost below $\$160$ per item.

Practice Test

483. $\frac{a}{3b}$

485. $\frac{x+3}{3x}$

487. $\frac{x-3}{x+9}$

489. $\frac{3n-2}{n-1}$

491. $\frac{n-m}{m+n}$

493. $z = \frac{1}{2}$

495. $[-3, 6)$

497. $(-\infty, 0) \cup (0, 4] \cup [6, \infty)$

499. $R(x) = \frac{1}{(x+2)(x+2)}$

501. $(2, 5]$

503. $y = \frac{81}{16}$

505. Oliver's dad would take $4\frac{4}{5}$ hours to split the logs himself.

507. The distance between Dayton and Columbus is 64 miles.

Chapter 8

Try It

8.1. ⓐ -8 ⓑ 15

8.2. ⓐ 10 ⓑ -11

8.3. ⓐ not a real number ⓑ -9

8.4. ⓐ -7 ⓑ not a real number

8.5. ⓐ 3 ⓑ 4 ⓒ 3

8.6. ⓐ 10 ⓑ 2 ⓒ 3

8.7. ⓐ -3 ⓑ not real ⓒ -2

8.8. ⓐ -6 ⓑ not real ⓒ -4

8.9. ⓐ $6 < \sqrt{38} < 7$

ⓑ $4 < \sqrt[3]{93} < 5$

8.10. ⓐ $9 < \sqrt{84} < 10$

ⓑ $5 < \sqrt[3]{152} < 6$

8.11. ⓐ ≈ 3.32 ⓑ ≈ 4.14

ⓒ ≈ 3.36

8.12. ⓐ ≈ 3.61 ⓑ ≈ 4.38

ⓒ ≈ 3.15

8.13. ⓐ $|b|$ ⓑ w ⓒ $|m|$ ⓓ q

8.14. ⓐ $|y|$ ⓑ p ⓒ $|z|$ ⓓ q

8.15. ⓐ $\left|y^9\right|$ ⓑ z^6

8.16. ⓐ m^2 ⓑ $\left|b^5\right|$

8.17. ⓐ $\left|u^3\right|$ ⓑ v^5

8.18. ⓐ c^4 ⓑ d^4

8.19. ⓐ $8|x|$ ⓑ $-10|p|$

8.20. ⓐ $13|y|$ ⓑ $-11|y|$

8.21. ⓐ $3x^9$ ⓑ $3\left|q^7\right|$

8.22. ⓐ $5p^3$ ⓑ $3q^5$

8.23. ⓐ $10|ab|$ ⓑ $12p^6q^{10}$

ⓒ $2x^{10}y^4$

8.24. ⓐ $15|mn|$ ⓑ $13\left|x^5y^7\right|$

ⓒ $3w^{12}z^5$

8.25. $4\sqrt{3}$

8.26. $3\sqrt{5}$

8.27. ⓐ $12\sqrt{2}$ ⓑ $3\sqrt[3]{3}$ ⓒ $2\sqrt[4]{4}$

8.28. ⓐ $12\sqrt{3}$ ⓑ $5\sqrt[3]{5}$ ⓒ $3\sqrt[4]{9}$

8.29. ⓐ $b^2\sqrt{b}$ ⓑ $|y|\sqrt[4]{y^2}$ ⓒ $z\sqrt[3]{z^2}$

8.30. ⓐ $p^4\sqrt{p}$ ⓑ $p\sqrt[5]{p^3}$

ⓒ $q^2\sqrt[6]{q}$

8.31. ⓐ $4y^2\sqrt{2y}$ ⓑ $3p^3\sqrt[3]{2p}$

ⓒ $2q^2\sqrt[4]{4q^2}$

8.32. ⓐ $5a^4\sqrt{3a}$ ⓑ $4m^3\sqrt[3]{2m^2}$

ⓒ $3|n|\sqrt[4]{2n^3}$

8.33. ⓐ $7\left|a^3\right|b^2\sqrt{2ab}$

ⓑ $2xy\sqrt[3]{7x^2y}$ ⓒ $2|x|y^2\sqrt[4]{2x}$

8.34. ⓐ $6m^4\left|n^5\right|\sqrt{5mn}$

ⓑ $2x^2y\sqrt[3]{9y^2}$ ⓒ $2|xy|\sqrt[4]{5x^3}$

8.35. ⓐ -4 ⓑ no real number

8.36. ⓐ $-5\sqrt[3]{5}$ ⓑ no real number

8.37. ⓐ $5+5\sqrt{3}$ ⓑ $2-\sqrt{3}$

8.38. ⓐ $2+7\sqrt{2}$ ⓑ $2-\sqrt{5}$

8.39. ⓐ $\frac{5}{4}$ ⓑ $\frac{3}{5}$ ⓒ $\frac{2}{3}$

8.40. ⓐ $\frac{7}{9}$ ⓑ $\frac{2}{5}$ ⓒ $\frac{1}{3}$

8.41. ⓐ $|a|$ ⓑ $|x|$ ⓒ y^3

8.42. ⓐ x^2 ⓑ m^2 ⓒ n^2

8.43. $\frac{2|p|\sqrt{6p}}{7}$

8.44. $\frac{2x^2\sqrt{3x}}{5}$

8.45. ⓐ $\frac{4|m|\sqrt{5m}}{|n^3|}$ ⓑ $\frac{3c^3\sqrt[3]{4c}}{d^2}$ ⓒ $\frac{2x^2\sqrt[4]{5x^2}}{|y|}$

8.46. ⓐ $\frac{3u^3\sqrt{6u}}{v^4}$ ⓑ $\frac{2r\sqrt[3]{5}}{s^2}$ ⓒ $\frac{3|m^3|\sqrt[4]{2m^2}}{|n^3|}$

8.47. ⓐ $\frac{5|y|\sqrt{x}}{6}$ ⓑ $\frac{2xy\sqrt[3]{y^2}}{3}$ ⓒ $\frac{|ab|\sqrt[4]{a}}{2}$

8.48. ⓐ $\frac{2|m|\sqrt{3}}{5|n^3|}$ ⓑ $\frac{3xy\sqrt[3]{x^2}}{5}$ ⓒ $\frac{2|ab|\sqrt[4]{a^2}}{3}$

8.49. ⓐ $7z^2$ ⓑ $-5\sqrt[3]{2}$ ⓒ $3|m|\sqrt[4]{2m^2}$

8.50. ⓐ $8m^4$ ⓑ -4 ⓒ $3|n|\sqrt[4]{2}$

8.51. ⓐ $\sqrt{t}$ ⓑ $\sqrt[3]{m}$ ⓒ $\sqrt[4]{r}$

8.52. ⓐ $\sqrt[6]{b}$ ⓑ $\sqrt[5]{z}$ ⓒ $\sqrt[4]{p}$

8.53. ⓐ $(10m)^{\frac{1}{2}}$ ⓑ $(3n)^{\frac{1}{5}}$ ⓒ $3(6y)^{\frac{1}{4}}$

8.54. ⓐ $(3k)^{\frac{1}{7}}$ ⓑ $(5j)^{\frac{1}{4}}$ ⓒ $8(2a)^{\frac{1}{3}}$

8.55. ⓐ 6 ⓑ 2 ⓒ 2

8.56. ⓐ 10 ⓑ 3 ⓒ 3

8.57. ⓐ No real solution ⓑ -8 ⓒ $\frac{1}{8}$

8.58. ⓐ No real solution ⓑ -4 ⓒ $\frac{1}{4}$

8.59. ⓐ $x^{\frac{5}{2}}$ ⓑ $(3y)^{\frac{3}{4}}$ ⓒ $\left(\frac{2m}{3n}\right)^{\frac{5}{2}}$

8.60. ⓐ $a^{\frac{2}{5}}$ ⓑ $(5ab)^{\frac{5}{3}}$ ⓒ $\left(\frac{7xy}{z}\right)^{\frac{3}{2}}$

8.61. ⓐ 9 ⓑ $\frac{1}{729}$ ⓒ $\frac{1}{8}$

8.62. ⓐ 8 ⓑ $\frac{1}{9}$ ⓒ $\frac{1}{125}$

8.63. ⓐ -64 ⓑ $-\frac{1}{64}$ ⓒ not a real number

8.64. ⓐ -729 ⓑ $-\frac{1}{729}$ ⓒ not a real number

8.65. ⓐ $x^{\frac{3}{2}}$ ⓑ x^8 ⓒ $\frac{1}{x}$

8.66. ⓐ $y^{\frac{11}{8}}$ ⓑ m^2 ⓒ $\frac{1}{d}$

8.67. ⓐ $8x^{\frac{1}{5}}$ ⓑ $x^{\frac{1}{2}}y^{\frac{1}{3}}$

8.68. ⓐ $729n^{\frac{3}{5}}$ ⓑ $a^2b^{\frac{2}{3}}$

8.69. ⓐ m^2 ⓑ $\frac{5n}{m^{\frac{1}{4}}}$

8.70. ⓐ u^3 ⓑ $3x^{\frac{1}{5}}y^{\frac{1}{3}}$

8.71. ⓐ $-\sqrt{2}$ ⓑ $11\sqrt[3]{x}$ ⓒ $3\sqrt[4]{x}-5\sqrt[4]{y}$

8.72. ⓐ $-4\sqrt{3}$ ⓑ $8\sqrt[3]{y}$ ⓒ $5\sqrt[4]{m}-2\sqrt[3]{m}$

8.73. ⓐ $-2\sqrt{7x}$ ⓑ $-\sqrt[4]{5xy}$

8.74. ⓐ $-\sqrt{3y}$ ⓑ $3\sqrt[3]{7mn}$

8.75. ⓐ $9\sqrt{2}$ ⓑ $2\sqrt[3]{2}$ ⓒ $\sqrt[3]{3}$

8.76. ⓐ $7\sqrt{3}$ ⓑ $-10\sqrt[3]{5}$ ⓒ $-3\sqrt[3]{2}$

8.77. ⓐ $-m^3\sqrt{2m}$ ⓑ $x^2\sqrt[3]{5x}$

8.78. ⓐ $-p\sqrt{3p}$ ⓑ $4y\sqrt[3]{4y^2}-2n\sqrt[3]{4n^2}$

8.79. ⓐ $12\sqrt{15}$ ⓑ $-18\sqrt[3]{2}$

8.80. ⓐ $27\sqrt{2}$ ⓑ $-36\sqrt[3]{2}$

8.81. ⓐ $36x^3\sqrt{5}$ ⓑ $8y\sqrt[4]{3y^2}$

8.82. ⓐ $144y^2\sqrt{5y}$ ⓑ $-36\sqrt[4]{3a}$

8.83. ⓐ $18+\sqrt{6}$ ⓑ $-2\sqrt[3]{4}-2\sqrt[3]{3}$

8.84. ⓐ $-40+4\sqrt{2}$ ⓑ $-3-\sqrt[3]{18}$

8.85. ⓐ $-66+15\sqrt{7}$ ⓑ $\sqrt[3]{x^2}-5\sqrt[3]{x}+6$

8.86. ⓐ $41-14\sqrt{11}$ ⓑ $\sqrt[3]{x^2}+4\sqrt[3]{x}+3$

8.87. $1+9\sqrt{21}$

8.88. $-12-20\sqrt{3}$

8.89. ⓐ $102+20\sqrt{2}$ ⓑ $55+6\sqrt{6}$

8.90. ⓐ $41-12\sqrt{5}$ ⓑ $121-36\sqrt{10}$

8.91. -11

8.92. -159

8.93. ⓐ $\frac{5s}{8}$ ⓑ $\frac{2}{a}$

8.94. ⓐ $\frac{5q^2}{6}$ ⓑ $\frac{2}{b}$

8.95. ⓐ $\frac{9x^2}{y^2}$ ⓑ $\frac{-4x}{y}$

8.96. ⓐ $\frac{10n^3}{m}$ ⓑ $\frac{-3p}{q^2}$

8.97. $4xy\sqrt{2x}$

8.98. $4ab\sqrt{3b}$

8.99. ⓐ $\frac{5\sqrt{3}}{3}$ ⓑ $\frac{\sqrt{6}}{8}$ ⓒ $\frac{\sqrt{2x}}{x}$

8.100. ⓐ $\frac{6\sqrt{5}}{5}$ ⓑ $\frac{\sqrt{14}}{6}$ ⓒ $\frac{\sqrt{5x}}{x}$

8.101. ⓐ $\frac{\sqrt[3]{49}}{7}$ ⓑ $\frac{\sqrt[3]{90}}{6}$ ⓒ $\frac{5\sqrt[3]{3y^2}}{3y}$

8.102. ⓐ $\frac{\sqrt[3]{4}}{2}$ ⓑ $\frac{\sqrt[3]{150}}{10}$ ⓒ $\frac{2\sqrt[3]{5n^2}}{5n}$

8.103. ⓐ $\frac{\sqrt[4]{27}}{3}$ ⓑ $\frac{\sqrt[4]{12}}{4}$ ⓒ $\frac{3\sqrt[4]{5x^3}}{5x}$

8.104. ⓐ $\frac{\sqrt[4]{125}}{5}$ ⓑ $\frac{\sqrt[4]{224}}{8}$ ⓒ $\frac{\sqrt[4]{64x^3}}{x}$

8.105. $-\frac{3(1+\sqrt{5})}{4}$

8.106. $\frac{4+\sqrt{6}}{5}$

8.107. $\frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}$

8.108. $\frac{\sqrt{10}(\sqrt{y}+\sqrt{3})}{y-3}$

8.109. $\frac{(\sqrt{p}+\sqrt{2})^2}{p-2}$

8.110. $\frac{(\sqrt{q}-\sqrt{10})^2}{q-10}$

8.111. $m=\frac{23}{3}$

8.112. $z=\frac{3}{10}$

8.113. no solution

8.114. no solution

8.115. $x=2,\ x=3$

8.116. $y=5,\ y=6$

8.117. $x=-6$

8.118. $x=-9$

8.119. $x=8$

8.120. $x=6$

8.121. $m=7$

8.122. $n=3$

8.123. $a=63$

8.124. $b=311$

8.125. $x=3$

8.126. $x=-\frac{6}{5}$

8.127. $x=4$

8.128. $x=9$

8.129. $x=5$

8.130. $x=0\ x=4$

8.131. 9 seconds

8.132. 3.5 seconds

8.133. 42.7 feet

8.134. 54.1 feet

8.135. ⓐ $f(6)=4$ ⓑ no value at $x=0$

8.136. ⓐ $g(4)=5$ ⓑ no value at $f(-3)$

8.137. ⓐ $g(4)=2$ ⓑ $g(1)=-1$

8.138. ⓐ $h(2)=2$ ⓑ $h(-5)=-3$

8.139. ⓐ $f(4)=2$ ⓑ $f(-1)=1$

8.140. ⓐ $g(16)=3$ ⓑ $g(3)=2$

8.141. $\left[\frac{5}{6},\ \infty\right)$

8.142. $\left(-\infty,\ \frac{4}{5}\right]$

8.143. $(-3,\ \infty)$

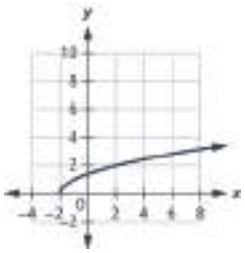
8.144. $(5,\ \infty)$

8.145. $(-\infty,\ \infty)$

8.146. $(-\infty,\ \infty)$

8.147. ⓐ domain: $[-2,\ \infty)$

ⓑ



ⓒ range: $[0,\ \infty)$

8.148. ⓐ domain: $[2, \infty)$

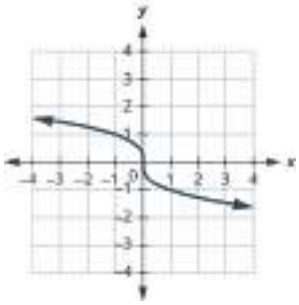
ⓑ



ⓒ range: $[0, \infty)$

8.149. ⓐ domain: $(-\infty, \infty)$

ⓑ



ⓒ range: $(-\infty, \infty)$

8.150. ⓐ domain: $(-\infty, \infty)$

ⓑ



ⓒ range: $(-\infty, \infty)$

8.151. ⓐ $9i$ ⓑ $\sqrt{5}i$ ⓒ $3\sqrt{2}i$

8.152. ⓐ $6i$ ⓑ $\sqrt{3}i$ ⓒ $3\sqrt{3}i$

8.153. $6\sqrt{2}i$

8.154. $7\sqrt{3}i$

8.155. ⓐ $6 + 5i$ ⓑ $6 - 3i$

8.156. ⓐ $-2 - 6i$ ⓑ $2 + 9i$

8.157. $12 + 20i$

8.158. $12 + 6i$

8.159. $-11 - 7i$

8.160. $-5 - 10i$

8.161. $-21 - 20i$

8.162. $9 - 40i$

8.163. -14

8.164. -54

8.165. $-12 - 22\sqrt{3}i$

8.166. $6 + 12\sqrt{2}i$

8.167. 25

8.168. 29

8.169. 109

8.170. 41

8.171. i

8.172. i

8.173. $\frac{4}{17} + \frac{16}{17}i$

8.174. $\frac{2}{5} + \frac{4}{5}i$

8.175. $\frac{3}{2} - \frac{3}{2}i$

8.176. $\frac{4}{5} - \frac{2}{5}i$

8.177. $-i$

8.178. 1

Section Exercises

1. ⓐ 8 ⓑ -9

3. ⓐ 14 ⓑ -1

5. ⓐ $\frac{2}{3}$ ⓑ -0.1

7. ⓐ not real number ⓑ -17

9. ⓐ -15 ⓑ not real number

11. ⓐ 6 ⓑ 4

13. ⓐ 8 ⓑ 3 ⓒ 1

15. ⓐ -2 ⓑ not real ⓒ -2

17. ⓐ -5 ⓑ not real ⓒ -4

19. ⓐ $8 < \sqrt{70} < 9$

21. ⓐ $14 < \sqrt{200} < 15$

23. ⓐ 4.36 ⓑ ≈ 4.46

ⓑ $4 < \sqrt[3]{71} < 5$

ⓑ $5 < \sqrt[3]{137} < 6$

ⓒ ≈ 3.14

25. ⓐ 7.28 ⓑ ≈ 5.28

27. ⓐ u ⓑ $|v|$

29. ⓐ $|y|$ ⓑ m

ⓒ ≈ 4.61

31. ⓐ $\left|x^3\right|$ ⓑ y^8

33. ⓐ x^{12} ⓑ $\left|y^{11}\right|$

35. ⓐ x^3 ⓑ $\left|y^3\right|$

37. ⓐ m^2 ⓑ n^4

39. ⓐ $7|x|$ ⓑ $-9\left|x^9\right|$

41. ⓐ $11m^{10}$ ⓑ $-8|a|$

43. ⓐ $2x^2$ ⓑ $2y^2$

45. ⓐ $6a^2$ ⓑ $2b^4$

47. ⓐ $12|xy|$ ⓑ $13w^4\left|y^5\right|$

ⓒ $2a^{17}b^2$

49. ⓐ $11|ab|$ ⓑ $3c^4d^6$

51. Answers will vary.

53. Answers will vary.

ⓒ $4x^5y^{22}$

55. $3\sqrt{3}$

57. $5\sqrt{5}$

59. $7\sqrt{3}$

61. $20\sqrt{2}$

63. ⓐ $2\sqrt[4]{2}$ ⓑ $2\sqrt[5]{2}$

65. ⓐ $2\sqrt[5]{2}$ ⓑ $4\sqrt[3]{4}$

67. ⓐ $|y^5|\sqrt{y}$ ⓑ $r\sqrt[3]{r^2}$ ⓒ $s^2\sqrt[4]{s^2}$

69. ⓐ $n^{10}\sqrt{n}$ ⓑ $q^2\sqrt[3]{q^2}$ ⓒ $|n|\sqrt[8]{n^2}$

71. ⓐ $5r^6\sqrt{5r}$ ⓑ $3x\sqrt[3]{4x^2}$ ⓒ $2|y|\sqrt[4]{3y^2}$

73. ⓐ $11|m^{11}|\sqrt{2m}$ ⓑ $3m^2\sqrt[4]{5m^2}$ ⓒ $2n\sqrt[5]{5n^3}$

75. ⓐ $7|m^3n^5|\sqrt{3mn}$ ⓑ $2x^2y^2\sqrt[3]{6y}$ ⓒ $2|xy|\sqrt[4]{2x}$

77. ⓐ $8|qr^3|\sqrt{3qr}$ ⓑ $3m^3n^3\sqrt[3]{2n}$ ⓒ $3a^2b^2\sqrt[4]{a}$

79. ⓐ $-6\sqrt[3]{4}$ ⓑ not real

81. ⓐ -2 ⓑ not real

83. ⓐ $5+2\sqrt{3}$ ⓑ $5-\sqrt{6}$

85. ⓐ $1+3\sqrt{5}$ ⓑ $1+\sqrt{10}$

87. ⓐ $\frac{3}{4}$ ⓑ $\frac{2}{3}$ ⓒ $\frac{1}{3}$

89. ⓐ $\frac{5}{3}$ ⓑ $\frac{3}{5}$ ⓒ $\frac{1}{4}$

91. ⓐ x^2 ⓑ p^3 ⓒ $|q|$

93. ⓐ $\frac{1}{y^2}$ ⓑ u^2 ⓒ $|v^3|$

95. $\frac{4|x^3|\sqrt{6x}}{11}$

97. $\frac{10m^2\sqrt{3m}}{8}$

99. $\frac{7r^2\sqrt{2r}}{10}$

101. $\frac{2|q^3|\sqrt{7}}{15}$

103. ⓐ $\frac{5r^4\sqrt{3r}}{s^4}$ ⓑ $\frac{3a^2\sqrt[3]{2a^2}}{|b|}$ ⓒ $\frac{2|c|\sqrt[4]{4c}}{|d|}$

105. ⓐ $\frac{2|p^3|\sqrt{7p}}{|q|}$ ⓑ $\frac{3s^2\sqrt[3]{3s^2}}{t}$ ⓒ $\frac{2|p^3|\sqrt[4]{4p^3}}{|q^3|}$

107. ⓐ $\frac{4|xy|}{3}$ ⓑ $\frac{y^2\sqrt[3]{x}}{2}$ ⓒ $\frac{|ab|\sqrt[4]{a}}{4}$

109. ⓐ $\frac{1}{2|pq|}$ ⓑ $\frac{2cd\sqrt[5]{2d^2}}{5}$ ⓒ $\frac{|mn|\sqrt[6]{2}}{2}$

111. ⓐ $\frac{3p^4\sqrt{p}}{|q|}$ ⓑ $2\sqrt[4]{2}$ ⓒ $2x\sqrt[5]{2x}$

113. ⓐ $5|m^3|$ ⓑ $5\sqrt[3]{5}$ ⓒ $3|y|\sqrt[4]{3y^2}$

115. Answers will vary.

117. Answers will vary.

119. ⓐ $\sqrt{x}$ ⓑ $\sqrt[3]{y}$ ⓒ $\sqrt[4]{z}$

121. ⓐ $\sqrt[5]{u}$ ⓑ $\sqrt[9]{v}$ ⓒ $\sqrt[20]{w}$

123. ⓐ $\frac{1}{x^7}$ ⓑ $\frac{1}{y^9}$ ⓒ $f^{\frac{1}{5}}$

125. ⓐ $(7c)^{\frac{1}{4}}$ ⓑ $(12d)^{\frac{1}{7}}$ ⓒ $2(6b)^{\frac{1}{4}}$

127. ⓐ $(21p)^{\frac{1}{2}}$ ⓑ $(8q)^{\frac{1}{4}}$ ⓒ $4(36r)^{\frac{1}{6}}$

129. ⓐ 9 ⓑ 5 ⓒ 8

131. ⓐ 2 ⓑ 4 ⓒ 5

133. ⓐ -6 ⓑ -6 ⓒ $\frac{1}{6}$

135. ⓐ not real ⓑ -3 ⓒ $\frac{1}{3}$

137. ⓐ not real ⓑ -6 ⓒ $\frac{1}{6}$

139. ⓐ not real ⓑ -10 ⓒ $\frac{1}{10}$

141. ⓐ $m^{\frac{5}{2}}$ ⓑ $(3y)^{\frac{7}{3}}$ ⓒ $\left(\frac{4x}{5y}\right)^{\frac{3}{5}}$

143. ⓐ $u^{\frac{2}{5}}$ ⓑ $(6x)^{\frac{5}{3}}$ ⓒ $\left(\frac{18a}{5b}\right)^{\frac{7}{4}}$

145. ⓐ 32,768 ⓑ $\frac{1}{729}$ ⓒ 9

147. ⓐ 4 ⓑ $\frac{1}{9}$ ⓒ not real

149. ⓐ -27 ⓑ $-\frac{1}{27}$ ⓒ not real

151. ⓐ $c^{\frac{7}{8}}$ ⓑ p^9 ⓒ $\frac{1}{r}$

153. ⓐ $y^{\frac{5}{4}}$ ⓑ x^8 ⓒ $\frac{1}{m}$

155. ⓐ $81q^2$ ⓑ $a^{\frac{1}{2}}b$

157. ⓐ $8u^{\frac{1}{4}}$ ⓑ $8p^{\frac{1}{2}}q^{\frac{3}{4}}$

159. ⓐ $r^{\frac{7}{2}}$ ⓑ $\frac{6s}{t}$

161. ⓐ c^2 ⓑ $\frac{2x}{3y}$

163. Answers will vary.

165. ⓐ $3\sqrt{2}$ ⓑ $7\sqrt[3]{m}$ ⓒ $6\sqrt[4]{m}$

167. ⓐ $9\sqrt{5}$ ⓑ $12\sqrt[3]{a}$ ⓒ $6\sqrt[4]{2z}$

169. ⓐ $4\sqrt{2a}$ ⓑ 0

171. ⓐ $\sqrt{3c}$ ⓑ $\sqrt[3]{4pq}$

173. ⓐ $4\sqrt{3}$ ⓑ $-2\sqrt[3]{5}$ ⓒ $3\sqrt[3]{2}$

175. ⓐ $7\sqrt{3}$ ⓑ $7\sqrt[3]{2}$ ⓒ $3\sqrt[4]{5}$

177. ⓐ $a^2\sqrt{2a}$ ⓑ 0

179. ⓐ $2c^3\sqrt{5c}$ ⓑ $14r^2\sqrt[4]{2r^2}$

181. $4y\sqrt{2}$

183. ⓐ $-18\sqrt{6}$ ⓑ $-64\sqrt[3]{9}$

185. ⓐ $-30\sqrt{2}$) ⓑ $6\sqrt[4]{2}$

187. ⓐ $72z^2\sqrt{3}$ ⓑ $45x^2\sqrt[3]{2}$

189. ⓐ $-42z^5\sqrt{2z}$ ⓑ $-8y\sqrt[4]{6y}$

191. ⓐ $14+5\sqrt{7}$ ⓑ $4\sqrt[3]{6}+3\sqrt[3]{4}$

193. ⓐ $44 - 3\sqrt{11}$ ⓑ $3\sqrt[4]{2} + \sqrt[4]{54}$

195. $60 + 2\sqrt{3}$

197. ⓐ $30 + 18\sqrt{2}$ ⓑ $\sqrt[3]{x^2} - 2\sqrt[3]{x} - 3$

199. ⓐ $-54 + 13\sqrt{10}$ ⓑ $2\sqrt[3]{x^2} + 8\sqrt[3]{x} + 6$

201. $23 + 3\sqrt{30}$

203. $-439 - 2\sqrt{77}$

205. ⓐ $14 + 6\sqrt{5}$ ⓑ $79 - 20\sqrt{3}$

207. ⓐ $87 - 18\sqrt{6}$ ⓑ $163 + 60\sqrt{7}$

209. 14

211. -227

213. 19

215. $\sqrt[3]{9x^2} - 4$

217. $5\sqrt{3}$

219. $9\sqrt{2}$

221. $-\sqrt[4]{5}$

223. $10c^2\sqrt{3} - 9c^3\sqrt{3}$

225. $2\sqrt{3}$

227. $17q^2$

229. $3\sqrt{7}$

231. $-42\sqrt[3]{9}$

233. 29

235. $29 - 7\sqrt{17}$

237. $72 - 36\sqrt{2}$

239. $6 + 3\sqrt[3]{2}$

241. Answers will vary.

243. Answers will vary.

245. ⓐ $\frac{4}{3}$ ⓑ $\frac{4}{3}$

247. ⓐ $\frac{10m^2}{7}$ ⓑ $\frac{3}{y}$

249. ⓐ $\frac{5}{6r^2}$ ⓑ $\frac{2x}{3}$

251. ⓐ $\frac{6p}{q^2}$ ⓑ $-\frac{2a^2}{b}$

253. ⓐ $\frac{8m^4}{3n^4}$ ⓑ $-\frac{x^2}{2y^2}$

255. $4x^4\sqrt{7y}$

257. $2ab\sqrt[3]{2a}$

259. ⓐ $\frac{5\sqrt{6}}{3}$ ⓑ $\frac{2\sqrt{3}}{9}$ ⓒ $\frac{2\sqrt{5x}}{x}$

261. ⓐ $\frac{6\sqrt{7}}{7}$ ⓑ $\frac{2\sqrt{10}}{15}$ ⓒ $\frac{4\sqrt{3p}}{p}$

263. ⓐ $\frac{\sqrt[3]{25}}{5}$ ⓑ $\frac{\sqrt[3]{45}}{6}$ ⓒ $\frac{2\sqrt[3]{6a^2}}{3a}$

265. ⓐ $\frac{\sqrt[3]{121}}{11}$ ⓑ $\frac{\sqrt[3]{28}}{6}$ ⓒ $\frac{\sqrt[3]{9x}}{x}$

267. ⓐ $\frac{\sqrt[4]{343}}{7}$ ⓑ $\frac{\sqrt[4]{40}}{4}$ ⓒ $\frac{2\sqrt[4]{4x^2}}{x}$

269. ⓐ $\frac{\sqrt[4]{9}}{3}$ ⓑ $\frac{\sqrt[4]{50}}{4}$ ⓒ $\frac{2\sqrt[4]{3a^2}}{a}$

271. $-2(1 + \sqrt{5})$

273. $3(3 + \sqrt{7})$

275. $\frac{\sqrt{3}(\sqrt{m} + \sqrt{5})}{m - 5}$

277. $\frac{\sqrt{2}(\sqrt{x} + \sqrt{6})}{x - 6}$

279. $\frac{(\sqrt{r} + \sqrt{5})^2}{r - 5}$

281. $\frac{(\sqrt{x} + 2\sqrt{2})^2}{x - 8}$

283. Answers will vary.

285. Answers will vary.

287. $m = 14$

289. no solution

291. $x = -4$

293. $m = 14$

295. $v = 17$

297. $m = \frac{7}{2}$

299. no solution

301. $u = 3, u = 4$

303. $r = 1, r = 2$

305. $x = 10$

307. $x = -8$

309. $x = 8$

311. $x = -4$

313. $x = 7$

315. $x = 3$

317. $z = 21$

319. $x = 42$

321. $r = 3$

323. $u = 3$

325. $r = -2$

327. $x = 1$

329. $x = -8, x = 2$

331. $a = 0$

333. $u = \frac{9}{4}$

335. $a = 4$

337. $x = 0\, x = 4$

339. $x = 1\, x = 5$

341. $x = 9$

343. 8.7 feet

345. 4.7 seconds

347. 72 feet

349. Answers will vary.

351. ⓐ $f(5) = 4$ ⓑ no value at $x = 0$

353. ⓐ $g(4) = 5$ ⓑ $g(8) = 7$

355. ⓐ $F(1) = 1$ ⓑ $F(-11) = 5$

357. ⓐ $G(5) = 2\sqrt{6}$ ⓑ $G(2) = 3$

359. ⓐ $g(6) = 2$ ⓑ $g(-2) = -2$

361. ⓐ $h(-2) = 0$ ⓑ $h(6) = 2\sqrt[3]{4}$

363. ⓐ $f(0) = 0$ ⓑ $f(2) = 2$

365. ⓐ $g(1) = 0$ ⓑ $g(-3) = 2$

367. $\left[\frac{1}{3}, \infty\right)$

369. $\left(-\infty, \frac{2}{3}\right]$

371. $(2, \infty)$

373. $(-\infty, -3] \cup (2, \infty)$

375. $(-\infty, \infty)$

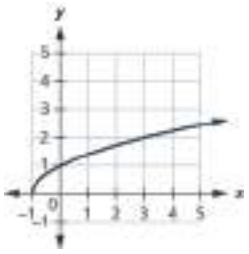
377. $(-\infty, \infty)$

379. $\left[-\frac{3}{8}, \infty\right)$

381. $(-\infty, \infty)$

383. ⓐ domain: $[-1, \infty)$

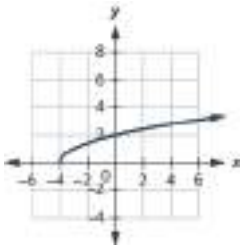
ⓑ



ⓒ $[0, \infty)$

385. ⓐ domain: $[-4, \infty)$

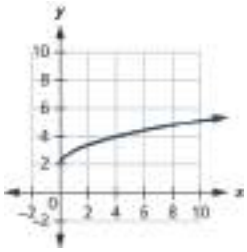
ⓑ



ⓒ $[0, \infty)$

387. ⓐ domain: $[0, \infty)$

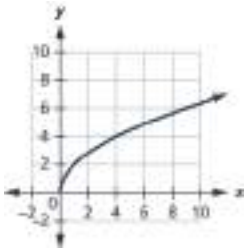
ⓑ



ⓒ $[2, \infty)$

389. ⓐ domain: $[0, \infty)$

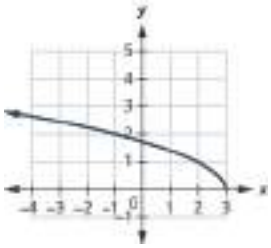
ⓑ



ⓒ $[0, \infty)$

391. ⓐ domain: $(-\infty, 3]$

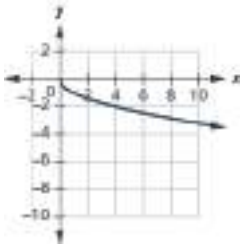
ⓑ



ⓒ $[0, \infty)$

393. ⓐ domain: $[0, \infty)$

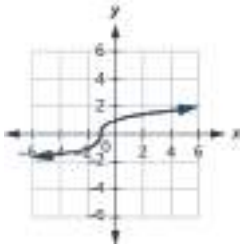
ⓑ



ⓒ $(-\infty, 0]$

395. ⓐ domain: $(-\infty, \infty)$

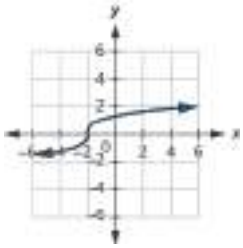
ⓑ



ⓒ $(-\infty, \infty)$

397. ⓐ domain: $(-\infty, \infty)$

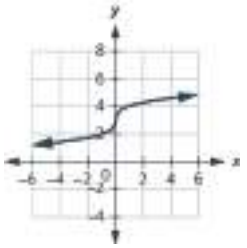
ⓑ



ⓒ $(-\infty, \infty)$

399. ⓐ domain: $(-\infty, \infty)$

ⓑ



ⓒ $(-\infty, \infty)$

401. ⓐ domain: $(-\infty, \infty)$

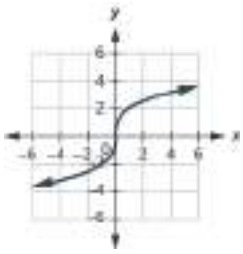
ⓑ



ⓒ $(-\infty, \infty)$

403. ⓐ domain: $(-\infty, \infty)$

ⓑ



ⓒ $(-\infty, \infty)$

405. Answers will vary.

407. Answers will vary.

409. ⓐ $4i$ ⓑ $\sqrt{11}i$ ⓒ $2\sqrt{2}i$

411. ⓐ $10i$ ⓑ $\sqrt{13}i$ ⓒ $3\sqrt{5}i$

413. $9\sqrt{3}i$

415. $8\sqrt{2}i$

417. $8+7i$

419. $14+2i$

421. $-2+2i$

423. $8+5i$

425. $7-13i$

427. $25-2\sqrt{2}i$

429. $12+20i$

431. $-12+18i$

433. $-38+\ +9i$

435. $27+15i$

437. $-7+24i$

439. $-5-12i$

441. -11

443. -30

445. $-44+4\sqrt{3}i$

447. $-20-2\sqrt{2}i$

449. 5

451. 53

453. 50

455. 85

457. i

459. $\frac{2}{25}+\frac{11}{25}i$

461. $\frac{6}{13}+\frac{9}{13}i$

463. $-\frac{12}{13}-\frac{8}{13}i$

465. $\frac{4}{3}-\frac{1}{3}i$

467. $-\frac{3}{4}+\frac{1}{2}i$

469. i

471. -1

473. 1

475. i

477. Answers will vary.

479. Answers will vary.

Review Exercises

481. ⓐ 15 ⓑ -4

483. ⓐ 2 ⓑ 3 ⓒ 3

485. ⓐ $8<\sqrt{68}<9$

ⓑ $4<\sqrt[3]{84}<5$

487. ⓐ a ⓑ $|b|$

489. ⓐ m^2 ⓑ n^4

491. ⓐ $6a^2$ ⓑ $2b^4$

493. $5\sqrt{5}$

495. ⓐ $5\sqrt[3]{5}$ ⓑ $2\sqrt[6]{2}$

497. ⓐ $4|s^7|\sqrt{5s}$ ⓑ $2a\sqrt[5]{3a^2}$

ⓒ $2|b|\sqrt[6]{2b}$

499. ⓐ -2 ⓑ not real

501. ⓐ $\frac{6}{7}$ ⓑ $\frac{2}{3}$ ⓒ $\frac{1}{2}$

503. $\frac{10m^2\sqrt{3m}}{8}$

505. ⓐ $\frac{1}{2|pq|}$ ⓑ $\frac{2cd\sqrt[5]{2d^2}}{5}$

ⓒ $\frac{|mn|\sqrt[6]{2}}{2}$

507. ⓐ $\sqrt{r}$ ⓑ $\sqrt[3]{s}$ ⓒ $\sqrt[4]{t}$

509. ⓐ 5 ⓑ 3 ⓒ 2

511. ⓐ -2 ⓑ $\frac{1}{3}$ ⓒ -5

513. ⓐ 125 ⓑ $\frac{1}{27}$ ⓒ 16

515. ⓐ 6^3 ⓑ b^9 ⓒ $\frac{1}{w}$

517. ⓐ $4\sqrt{2}$ ⓑ $9\sqrt[3]{p}$ ⓒ $2\sqrt[3]{x}$

519. ⓐ $7\sqrt{3}$ ⓑ $7\sqrt[3]{2}$ ⓒ $3\sqrt[4]{5}$

521. $37y\sqrt{3}$

523. ⓐ $126x^2\sqrt{2}$ ⓑ $48a\sqrt[3]{a^2}$

525. ⓐ $71-22\sqrt{7}$

ⓑ $\sqrt[3]{x^2}-8\sqrt[3]{x}+15$

527. ⓐ $27+8\sqrt{11}$ ⓑ $29-12\sqrt{5}$

529. $\sqrt[3]{9x^2}-4$

531. ⓐ $\frac{8m^4}{3n^4}$ ⓑ $-\frac{x^2}{2y^2}$

533. ⓐ $\frac{\sqrt[3]{121}}{11}$ ⓑ $\frac{\sqrt[3]{28}}{6}$ ⓒ $\frac{\sqrt[3]{9x}}{x}$

535. $-\frac{7(2+\sqrt{6})}{2}$

537. $\frac{(\sqrt{x}+2\sqrt{2})^2}{x-8}$

539. no solution

541. $u=3,\ u=4$

543. $x=-4$

545. $r=3$

547. $x=-8,\ x=2$

549. $x=3$

551. 64.8 feet

553. ⓐ $G(5)=2\sqrt{6}$ ⓑ $G(2)=3$

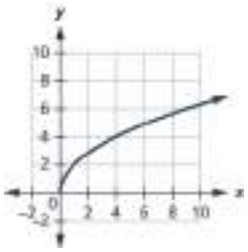
555. ⓐ $g(1)=0$ ⓑ $g(-3)=2$

557. $(2,\ \infty)$

559. $\left[\frac{7}{10},\ \infty\right)$

561. ⓐ domain: $[0,\ \infty)$

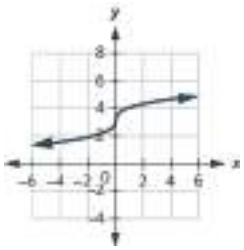
ⓑ



ⓒ range: $[0,\ \infty)$

563. ⓐ domain: $(-\infty,\ \infty)$

ⓑ



ⓒ range: $(-\infty,\ \infty)$

565. $8\sqrt{2}i$

567. $8+5i$

569. $23+14i$

571. -6

573. $-5-12i$

575. $\frac{2}{25}+\frac{11}{25}i$

577. 1

Practice Test

579. $5x^3$

581. $2x^2y\sqrt[3]{9x^2y}$

583. ⓐ $\frac{1}{4}$ ⓑ -343

585. $x^{\frac{7}{4}}$

587. $-x^2\sqrt{3x}$

589. $36x^4\sqrt{2}$

591. $2-7\sqrt{3}$

593. $\frac{7x^5}{3y^7}$

595. $3(2-\sqrt{3})$

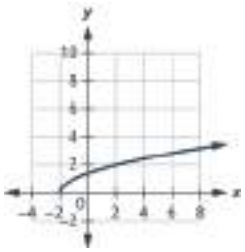
597. $-12+8i$

599. $-i$

601. $x=4$

603. ⓐ domain: $[-2,\ \infty)$

ⓑ



ⓒ range: $[0,\ \infty)$

Chapter 9

Try It

9.1. $x=4\sqrt{3},\ x=-4\sqrt{3}$

9.2. $y=3\sqrt{3},\ y=-3\sqrt{3}$

9.3. $x=7,\ x=-7$

9.4. $m=4,\ m=-4$

9.5. $c=2\sqrt{3}i,\ \ c=-2\sqrt{3}i$

9.6. $c=2\sqrt{6}i,\ \ c=-2\sqrt{6}i$

9.7. $x = 2\sqrt{10}, \quad x = -2\sqrt{10}$

9.8. $y = 2\sqrt{7}, \quad y = -2\sqrt{7}$

9.9. $r = \frac{6\sqrt{5}}{5}, \quad r = -\frac{6\sqrt{5}}{5}$

9.10. $t = \frac{8\sqrt{3}}{3}, \quad t = -\frac{8\sqrt{3}}{3}$

9.11. $a = 3 + 3\sqrt{2}, \quad a = 3 - 3\sqrt{2}$

9.12. $b = -2 + 2\sqrt{10}, \quad b = -2 - 2\sqrt{10}$

9.13. $x = \frac{1}{2} + \frac{\sqrt{5}}{2}, x = \frac{1}{2} - \frac{\sqrt{5}}{2}$

9.14. $y = -\frac{3}{4} + \frac{\sqrt{7}}{4}, \quad y = -\frac{3}{4} - \frac{\sqrt{7}}{4}$

9.15. $a = 5 + 2\sqrt{5}, \quad a = 5 - 2\sqrt{5}$

9.16. $b = -3 + 4\sqrt{2}, \quad b = -3 - 4\sqrt{2}$

9.17. $r = -\frac{4}{3} + \frac{2\sqrt{2}i}{3}, \quad \mathrm{r} = -\frac{4}{3} - \frac{2\sqrt{2}i}{3}$

9.18. $t = 4 + \frac{\sqrt{10}i}{2}, \quad \mathrm{t} = 4 - \frac{\sqrt{10}i}{2}$

9.19. $m = \frac{7}{3}, \quad m = -1$

9.20. $n = -\frac{3}{4}, \quad n = -\frac{7}{4}$

9.21. ⓐ $(a - 10)^2$ ⓑ $\left(b - \frac{5}{2}\right)^2$ ⓒ $\left(p + \frac{1}{8}\right)^2$

9.22. ⓐ $(b - 2)^2$ ⓑ $\left(n + \frac{13}{2}\right)^2$ ⓒ $\left(q - \frac{1}{3}\right)^2$

9.23. $x = -5, x = -1$

9.24. $y = 1, y = 9$

9.25. $y = 5 + \sqrt{15}i, \quad y = 5 - \sqrt{15}i$

9.26. $z = -4 + \sqrt{3}i, \ \mathrm{z} = -4 - \sqrt{3}i$

9.27. $x = 8 + 4\sqrt{3}, \quad x = 8 - 4\sqrt{3}$

9.28. $y = -4 + 3\sqrt{3}, \ \mathrm{y} = -4 - 3\sqrt{3}$

9.29. $a = -7, a = 3$

9.30. $b = -10, b = 2$

9.31. $p = \frac{5}{2} + \frac{\sqrt{61}}{2}, \quad p = \frac{5}{2} - \frac{\sqrt{61}}{2}$

9.32. $q = \frac{7}{2} + \frac{\sqrt{37}}{2}, \ \mathrm{q} = \frac{7}{2} - \frac{\sqrt{37}}{2}$

9.33. $c = -9, c = 3$

9.34. $d = 11, d = -7$

9.35. $m = -7, m = -1$

9.36. $n = -2, n = 8$

9.37. $r = -\frac{7}{3}, r = 3$

9.38. $t = -\frac{5}{2}, t = 2$

9.39. $x = -\frac{3}{8} + \frac{\sqrt{41}}{8}, \quad x = -\frac{3}{8} - \frac{\sqrt{41}}{8}$

9.40. $y = \frac{5}{3} + \frac{\sqrt{10}}{3}, \quad y = \frac{5}{3} - \frac{\sqrt{10}}{3}$

9.41. $y = 1, y = \frac{2}{3}$

9.42. $z = 1, z = -\frac{3}{2}$

9.43. $a = -3, a = 5$

9.44. $b = -6, b = -4$

9.45. $m = \frac{-6 + \sqrt{15}}{3}, m = \frac{-6 - \sqrt{15}}{3}$

9.46. $n = \frac{-2 + 2\sqrt{6}}{5}, n = \frac{-2 - 2\sqrt{6}}{5}$

9.47. $a = \frac{1}{4} + \frac{\sqrt{31}}{4}i, \quad a = \frac{1}{4} - \frac{\sqrt{31}}{4}i$

9.48. $b = -\frac{1}{5} + \frac{\sqrt{19}}{5}i, \quad b = -\frac{1}{5} - \frac{\sqrt{19}}{5}i$

9.49. $x = -1 + \sqrt{6}, x = -1 - \sqrt{6}$

9.50. $y = 1 + \sqrt{2}, y = 1 - \sqrt{2}$

9.51. $c = \frac{2 + \sqrt{7}}{3}, \quad c = \frac{2 - \sqrt{7}}{3}$

9.52. $d = \frac{9 + \sqrt{33}}{4}, \ \mathrm{d} = \frac{9 - \sqrt{33}}{4}$

9.53. $r = -5$

9.54. $t = \frac{4}{5}$

9.55. ⓐ 2 complex solutions; ⓑ 2 real solutions; ⓒ 1 real solution

9.56. ⓐ 2 real solutions; ⓑ 2 complex solutions; ⓒ 1 real solution

9.57. ⓐ factoring; ⓑ Square Root Property; ⓒ Quadratic Formula

9.58. ⓐ Quadratic Forumula; ⓑ Factoring or Square Root Property ⓒ Square Root Property

9.59. $x = \sqrt{2}, x = -\sqrt{2}, x = 2, x = -2$

9.60. $x = \sqrt{7}, x = -\sqrt{7}, x = 2, x = -2$

9.61. $x = 3, x = 1$

9.62. $y = -1, y = 1$

9.63. $x = 9, x = 16$

9.64. $x = 4, x = 16$

9.65. $x = -8, x = 343$

9.66. $x = 81, x = 625$

9.67. $x = \frac{4}{3}x = 2$

9.68. $x = \frac{2}{5}, x = \frac{3}{4}$

9.69. The two consecutive odd integers whose product is 99 are 9, 11, and −9, −11

9.70. The two consecutive even integers whose product is 128 are 12, 14 and −12, −14.

9.71. The height of the triangle is 12 inches and the base is 76 inches.

9.72. The height of the triangle is 11 feet and the base is 20 feet.

9.73. The length of the garden is approximately 18 feet and the width 11 feet.

9.74. The length of the tablecloth is approximatel 11.8 feet and the width 6.8 feet.

9.75. The length of the flag pole's shadow is approximately 6.3 feet and the height of the flag pole is 18.9 feet.

9.76. The distance between the opposite corners is approximately 7.2 feet.

9.77. The arrow will reach 180 feet on its way up after 3 seconds and again on its way down after approximately 3.8 seconds.

9.78. The ball will reach 48 feet on its way up after approximately .6 second and again on its way down after approximately 5.4 seconds.

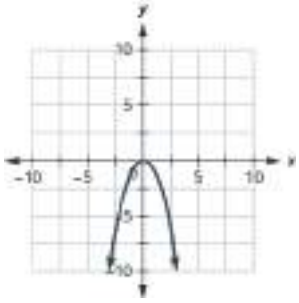
9.79. The speed of the jet stream was 100 mph.

9.80. The speed of the jet stream was 50 mph.

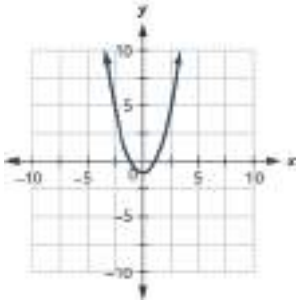
9.81. Press #1 would take 12 hours, and Press #2 would take 6 hours to do the job alone.

9.82. The red hose take 6 hours and the green hose take 3 hours alone.

9.83.



9.84.



9.85. ⓐ up; ⓑ down

9.86. ⓐ down; ⓑ up

9.87. ⓐ $x = 2$; ⓑ $(2, -7)$

9.88. ⓐ $x = 1$; ⓑ $(1, -5)$

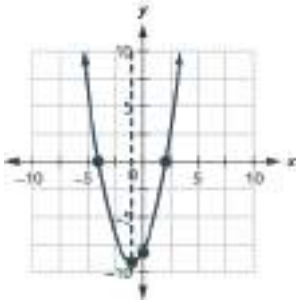
9.89. y-intercept: $(0, -8)$
x-intercepts $(-4, 0), (2, 0)$

9.90. y-intercept: $(0, -12)$
x-intercepts $(-2, 0), (6, 0)$

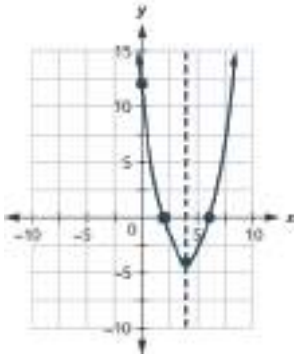
9.91. y-intercept: $(0, 4)$ no x-intercept

9.92. y-intercept: $(0, -5)$
x-intercepts $(-1, 0), (5, 0)$

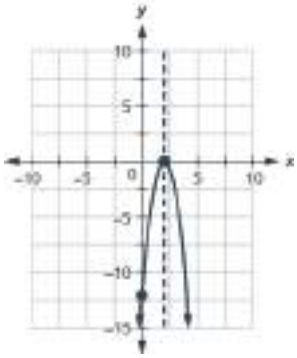
9.93.



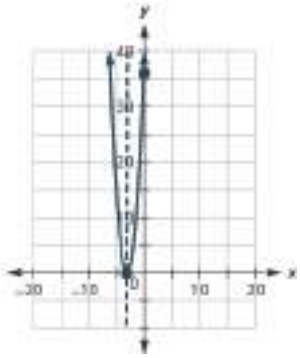
9.94.



9.95.



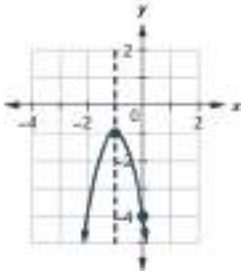
9.96.



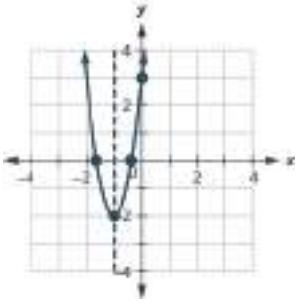
9.97.



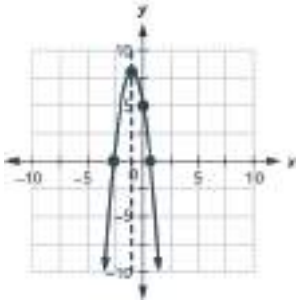
9.98.



9.99.



9.100.



9.101. The minimum value of the quadratic function is −4 and it occurs when $x = 4$.

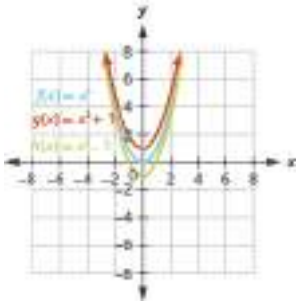
9.102. The maximum value of the quadratic function is 5 and it occurs when $x = 2$.

9.103. It will take 4 seconds for the stone to reach its maximum height of 288 feet.

9.104. It will 6.5 seconds for the rocket to reach its maximum height of 676 feet.

9.105.

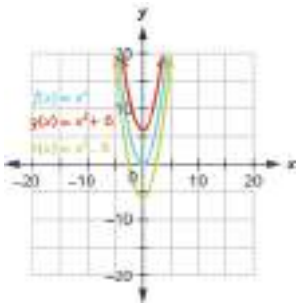
ⓐ



ⓑ The graph of $g(x) = x^2 + 1$ is the same as the graph of $f(x) = x^2$ but shifted up 1 unit. The graph of $h(x) = x^2 - 1$ is the same as the graph of $f(x) = x^2$ but shifted down 1 unit.

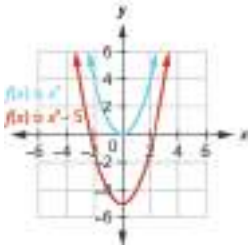
9.106.

ⓐ

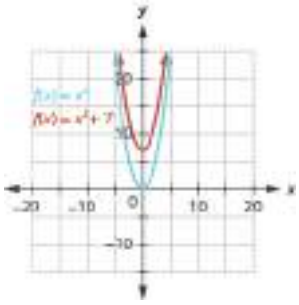


ⓑ The graph of $h(x) = x^2 + 6$ is the same as the graph of $f(x) = x^2$ but shifted up 6 units. The graph of $h(x) = x^2 - 6$ is the same as the graph of $f(x) = x^2$ but shifted down 6 units.

9.107.

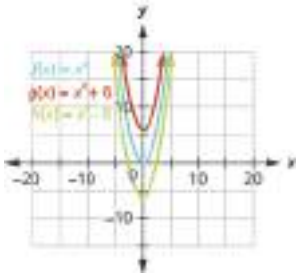


9.108.



9.109.

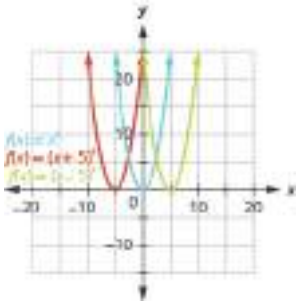
ⓐ



ⓑ The graph of $g(x) = (x+2)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 2 units. The graph of $h(x) = (x-2)^2$ is the same as the graph of $f(x) = x^2$ but shift right 2 units.

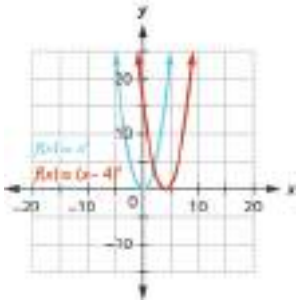
9.110.

ⓐ

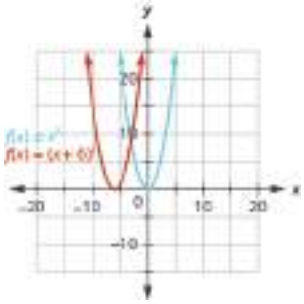


ⓑ The graph of $g(x) = (x+5)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 5 units. The graph of $h(x) = (x-5)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 5 units.

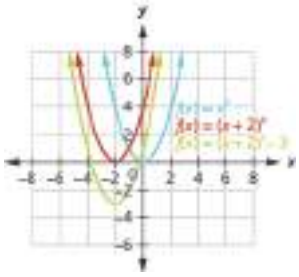
9.111.



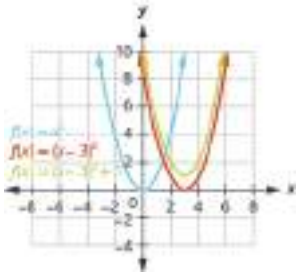
9.112.



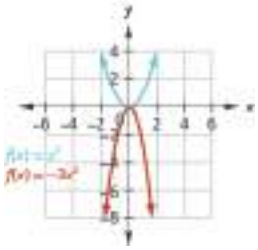
9.113.



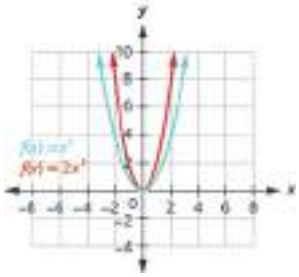
9.114.



9.115.



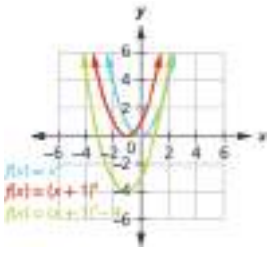
9.116.



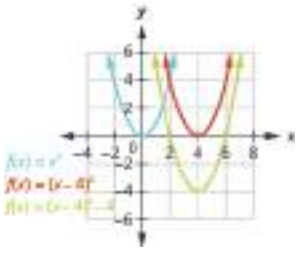
9.117. $f(x) = -4(x+1)^2 + 5$

9.118. $f(x) = 2(x-2)^2 - 5$

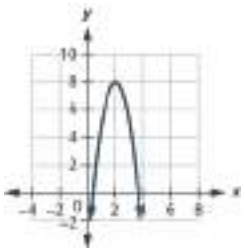
9.119.



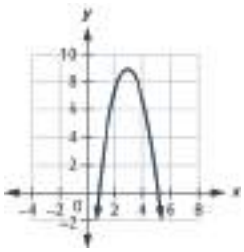
9.120.



9.121.



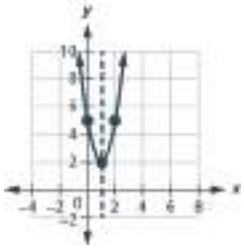
9.122.



9.123.

ⓐ $f(x) = 3(x-1)^2 + 2$

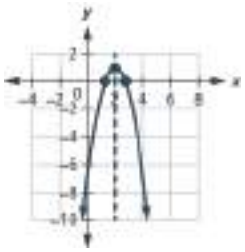
ⓑ



9.124.

ⓐ $f(x) = -2(x-2)^2 + 1$

ⓑ

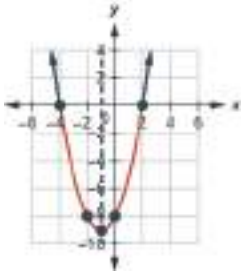


9.125. $f(x) = (x-3)^2 - 4$

9.126. $f(x) = (x+3)^2 - 1$

9.127.

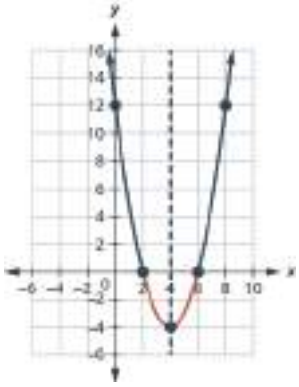
ⓐ



ⓑ $(-4, -2)$

9.128.

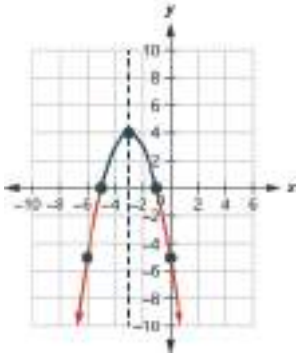
ⓐ



ⓑ $(-\infty, 2] \cup [6, \infty)$

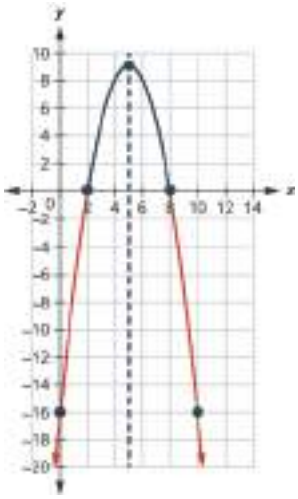
9.129.

ⓐ



ⓑ $(-1, 5)$

9.130.
ⓐ



ⓑ $(-\infty, 2] \cup [8, \infty)$

9.131. $(-\infty, -4] \cup [2, \infty)$

9.132. $[-3, 5]$

9.133. $\left[-1-\sqrt{2}, -1+\sqrt{2}\right]$

9.134.
$\left(-\infty, 4-\sqrt{2}\right) \cup \left(4+\sqrt{2}, \infty\right)$

9.135. ⓐ $(-\infty, \infty)$
ⓑ no solution

9.136. ⓐ no solution
ⓑ $(-\infty, \infty)$

Section Exercises

1. $a = \pm 7$

3. $r = \pm 2\sqrt{6}$

5. $u = \pm 10\sqrt{3}$

7. $m = \pm 3$

9. $x = \pm 6$

11. $x = \pm 5i$

13. $x = \pm 3\sqrt{7}i$

15. $x = \pm 9$

17. $a = \pm 2\sqrt{5}$

19. $p = \pm \frac{4\sqrt{7}}{7}$

21. $y = \pm \frac{4\sqrt{10}}{5}$

23. $u = 14, u = -2$

25. $m = 6 \pm 2\sqrt{5}$

27. $r = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

29. $y = -\frac{2}{3} \pm \frac{2\sqrt{2}}{9}$

31. $a = 7 \pm 5\sqrt{2}$

33. $x = -3 \pm 2\sqrt{2}$

35. $c = -\frac{1}{5} \pm \frac{3\sqrt{3}}{5}i$

37. $x = \frac{3}{4} \pm \frac{\sqrt{7}}{2}i$

39. $m = 2 \pm 2\sqrt{2}$

41. $x = 3 + 2\sqrt{3}, \quad x = 3 - 2\sqrt{3}$

43. $x = -\frac{3}{5}, x = \frac{9}{5}$

45. $x = -\frac{7}{6}, x = \frac{11}{6}$

47. $r = \pm 4$

49. $a = 4 \pm 2\sqrt{7}$

51. $w = 1, w = \frac{5}{3}$

53. $a = \pm 3\sqrt{2}$

55. $p = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

57. no real solution

59. $u = 7 \pm 6\sqrt{2}$

61. $m = 4 \pm 2\sqrt{3}$

63. $x = -3, x = -7$

65. $c = \pm \frac{5\sqrt{6}}{6}$

67. no real solution

69. Answers will vary.

71. ⓐ $(m-12)^2$ ⓑ $\left(x - \frac{11}{2}\right)^2$
ⓒ $\left(p - \frac{1}{6}\right)^2$

73. ⓐ $(p-11)^2$ ⓑ $\left(y+\frac{5}{2}\right)^2$ ⓒ $\left(m+\frac{1}{5}\right)^2$

75. $u=-3,\ u=1$

77. $x=-1,\ x=21$

79. $m=-2\pm 2\sqrt{10}i$

81. $r=-3\pm\sqrt{2}i$

83. $a=5\pm 2\sqrt{5}$

85. $x=-\frac{5}{2}\pm\frac{\sqrt{33}}{2}$

87. $u=1,\ u=13$

89. $r=-2,\ r=6$

91. $v=\frac{9}{2}\pm\frac{\sqrt{89}}{2}$

93. $x=5\pm\sqrt{30}$

95. $x=-7,\ x=3$

97. $x=-5,\ x=-1$

99. $m=-11,\ m=1$

101. $n=1\pm 2\sqrt{3}$

103. $c=-2,\ c=\frac{3}{2}$

105. $x=-5,\ x=\frac{3}{2}$

107. $p=-\frac{7}{4}\pm\frac{\sqrt{161}}{4}$

109. $x=\frac{3}{10}\pm\frac{\sqrt{191}}{10}i$

111. Answers will vary.

113. $m=-1,\ m=\frac{3}{4}$

115. $p=\frac{1}{3},\ p=2$

117. $p=-4,\ p=-3$

119. $r=-3,\ r=11$

121. $u=\frac{-7\pm\sqrt{73}}{6}$

123. $a=\frac{3\pm\sqrt{3}}{2}$

125. $x=-4\pm 2\sqrt{5}$

127. $y=-\frac{2}{3},\ y=-1$

129. $x=-\frac{3}{4}\pm\frac{\sqrt{15}}{4}i$

131. $x=\frac{3}{8}\pm\frac{\sqrt{7}}{8}i$

133. $v=2\pm 2\sqrt{2}$

135. $y=-4,\ y=7$

137. $m=-1,\ m=\frac{3}{4}$

139. $b=\frac{-2\pm\sqrt{11}}{6}$

141. $c=-\frac{3}{4}$

143. $q=-\frac{3}{5}$

145. ⓐ no real solutions ⓑ 1 ⓒ 2

147. ⓐ 1 ⓑ no real solutions ⓒ 2

149. ⓐ factor ⓑ square root ⓒ Quadratic Formula

151. ⓐ Quadratic Formula ⓑ square root ⓒ factor

153. Answers will vary.

155. $x=\pm\sqrt{3},\ x=\pm 2$

157. $x=\pm\sqrt{15},\ x=\pm\sqrt{2}i$

159. $x=\pm 1,\ x=\frac{\pm\sqrt{6}}{2}$

161. $x=\pm\sqrt{3},\ x=\pm\frac{\sqrt{2}}{2}$

163. $x=-1,\ x=12$

165. $x=-\frac{5}{3},\ x=0$

167. $x=0,\ x=\pm\sqrt{3}$

169. $x=\pm\frac{11}{2},\ x=\pm\frac{\sqrt{22}}{2}$

171. $x=25$

173. $x=4$

175. $x=\frac{1}{4}$

177. $x=\frac{1}{25}$

179. $x=-1,\ x=-512$

181. $x=8,\ x=-216$

183. $x=\frac{27}{8},\ x=-\frac{64}{27}$

185. $x=27,\ x=64{,}000$

187. $x=1,\ x=49$

189. $x=-2,\ x=-\frac{3}{5}$

191. $x=-2,\ x=\frac{4}{3}$

193. Answers will vary.

195. Two consecutive odd numbers whose product is 255 are 15 and 17, and −15 and −17.

197. The first and second consecutive odd numbers are 24 and 26, and −26 and −24.

199. Two consecutive odd numbers whose product is 483 are 21 and 23, and −21 and −23.

201. The width of the triangle is 5 inches and the height is 18 inches.

203. The base is 24 feet and the height of the triangle is 10 feet.

205. The length of the driveway is 15.0 feet and the width is 3.3 feet.

207. The length of table is 8 feet and the width is 3 feet.

209. The length of the legs of the right triangle are 3.2 and 9.6 cm.

211. The length of the diagonal fencing is 7.3 yards.

213. The ladder will reach 24.5 feet on the side of the house.

215. The arrow will reach 400 feet on its way up in 2.8 seconds and on the way down in 11 seconds.

217. The bullet will take 70 seconds to hit the ground.

219. The speed of the wind was 49 mph.

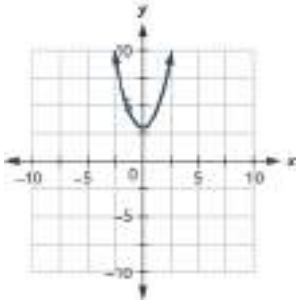
221. The speed of the current was 4.3 mph.

223. The less experienced painter takes 6 hours and the experienced painter takes 3 hours to do the job alone.

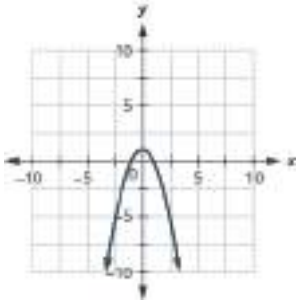
225. Machine #1 takes 3.6 hours and Machine #2 takes 4.6 hours to do the job alone.

227. Answers will vary.

229.



231.



233. ⓐ down ⓑ up

235. ⓐ down ⓑ up

237. ⓐ $x = -4$; ⓑ $(-4, -17)$

239. ⓐ $x = 1$; ⓑ $(1, 2)$

241. y-intercept: $(0, 6)$;
x-intercept $(-1, 0), (-6, 0)$

243. y-intercept: $(0, 12)$;
x-intercept $(-2, 0), (-6, 0)$

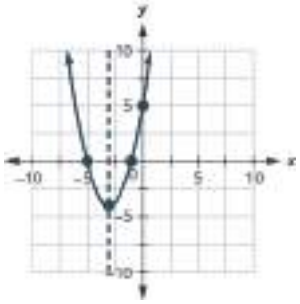
245. y-intercept: $(0, -19)$;
x-intercept: none

247. y-intercept: $(0, 13)$;
x-intercept: none

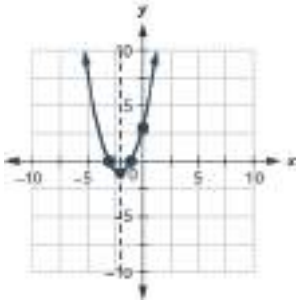
249. y-intercept: $(0, -16)$;
x-intercept $\left(\frac{5}{2}, 0\right)$

251. y-intercept: $(0, 9)$;
x-intercept $(-3, 0)$

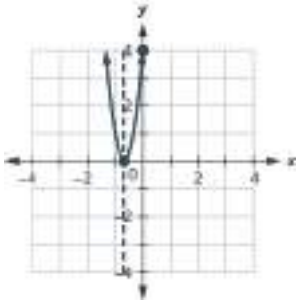
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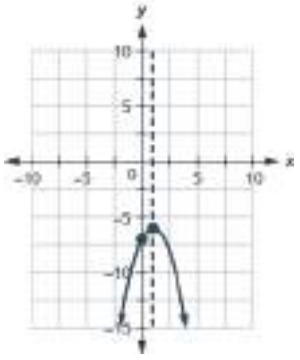
255.



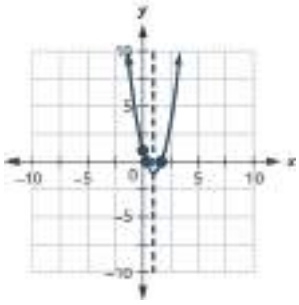
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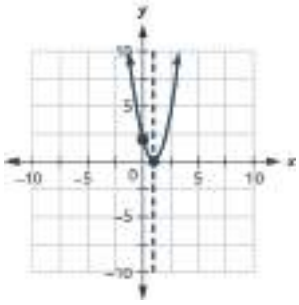
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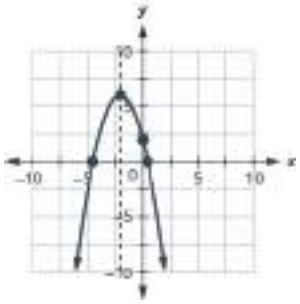
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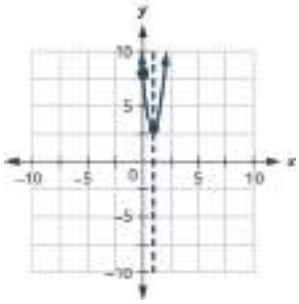
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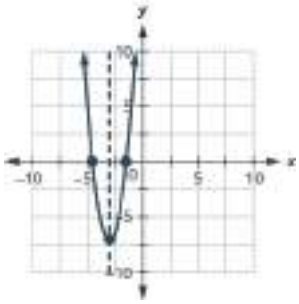
265.



267.



269.



271. The minimum value is $-\frac{9}{8}$ when $x = -\frac{1}{4}$.

273. The maximum value is 6 when $x = 3$.

275. The maximum value is 16 when $x = 0$.

277. In 5.3 sec the arrow will reach maximum height of 486 ft.

279. In 3.4 seconds the ball will reach its maximum height of 185.6 feet.

281. 20 computers will give the maximum of $400 in receipts.

283. He will be able to sell 35 pairs of boots at the maximum revenue of $1,225.

285. The length of the side along the river of the corral is 120 feet and the maximum area is 7,200 square feet.

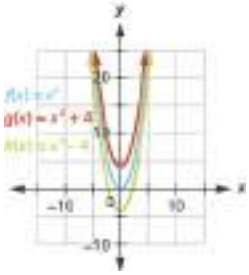
287. The maximum area of the patio is 800 feet.

289. Answers will vary.

291. Answers will vary.

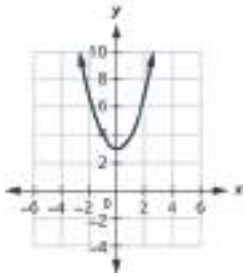
293.

ⓐ

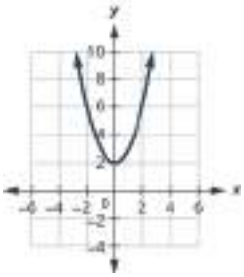


ⓑ The graph of $g(x) = x^2 + 4$ is the same as the graph of $f(x) = x^2$ but shifted up 4 units. The graph of $h(x) = x^2 - 4$ is the same as the graph of $f(x) = x^2$ but shift down 4 units.

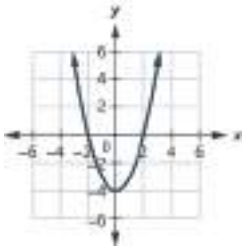
295.



297.

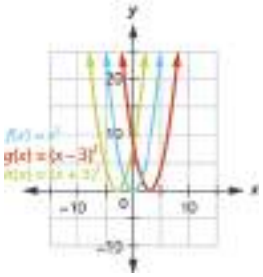


299.



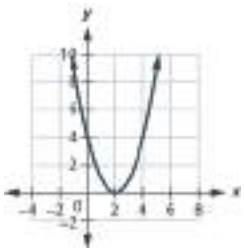
301.

ⓐ

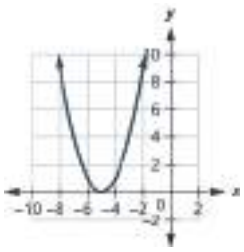


ⓑ The graph of $g(x) = (x-3)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 3 units. The graph of $h(x) = (x+3)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 3 units.

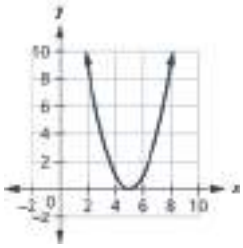
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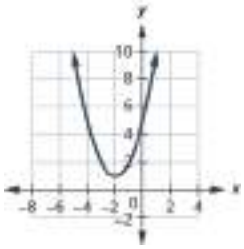
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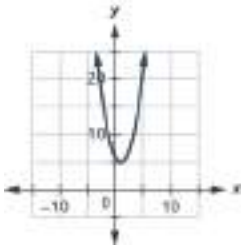
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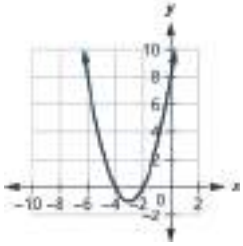
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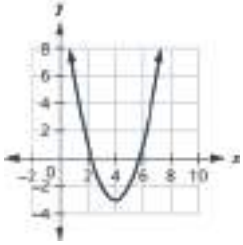
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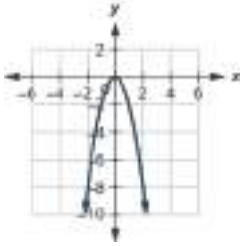
313.



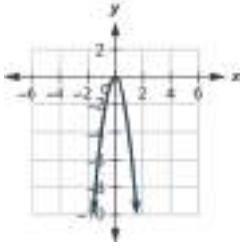
315.



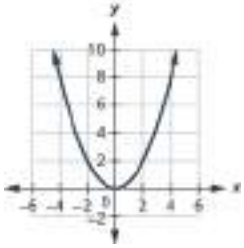
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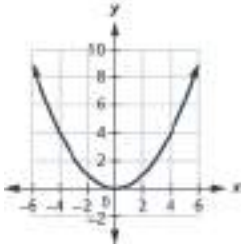
319.



321.



323.

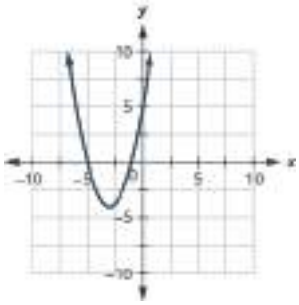


325. $f(x) = -3(x+2)^2 + 7$

327. $f(x) = 3(x+1)^2 - 4$

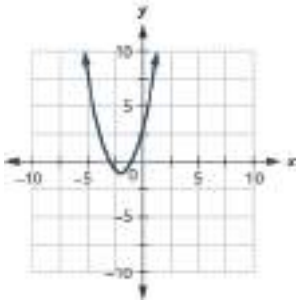
329. ⓐ $f(x) = (x+3)^2 - 4$

ⓑ



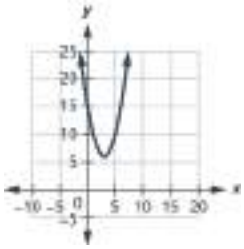
331. ⓐ $f(x) = (x+2)^2 - 1$

ⓑ



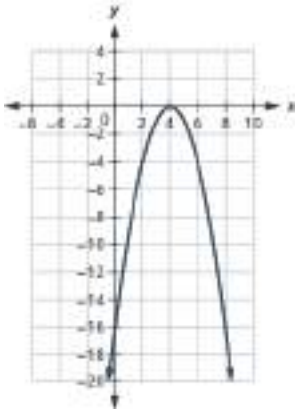
333. ⓐ $f(x) = (x-3)^2 + 6$

ⓑ



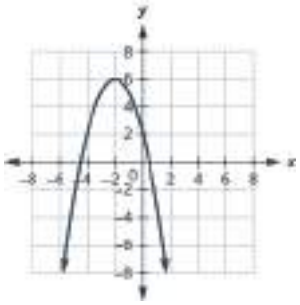
335. ⓐ $f(x) = -(x-4)^2 + 0$

ⓑ



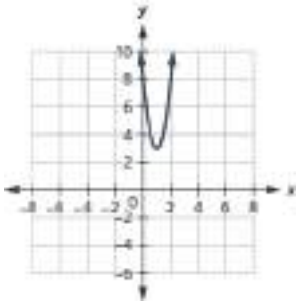
337. ⓐ $f(x) = -(x+2)^2 + 6$

ⓑ



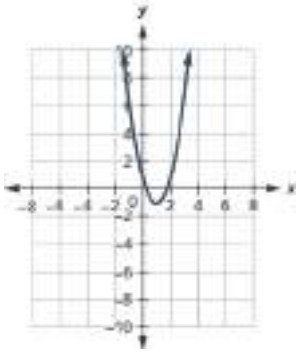
339. ⓐ $f(x) = 5(x-1)^2 + 3$

ⓑ



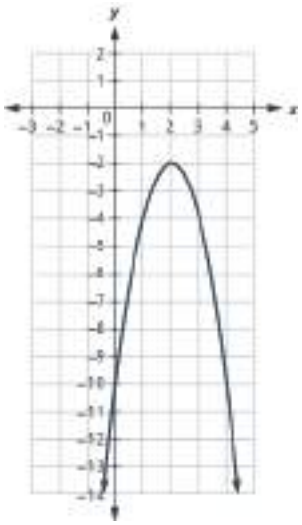
341. ⓐ $f(x) = 2(x-1)^2 - 1$

ⓑ



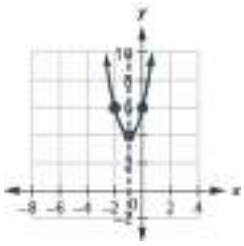
343. ⓐ $f(x) = -2(x-2)^2 - 2$

ⓑ



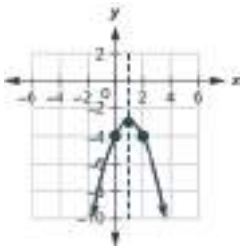
345. ⓐ $f(x) = 2(x+1)^2 + 4$

ⓑ



347. ⓐ $f(x) = -(x-1)^2 - 3$

ⓑ



349. ⓒ

351. ⓔ

353. ⓓ

355. ⓖ

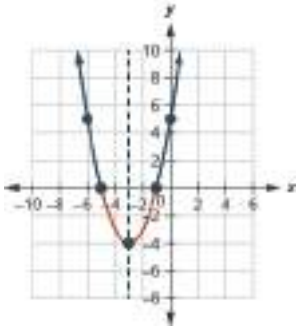
357. $f(x) = (x+1)^2 - 5$

359. $f(x) = 2(x-1)^2 - 3$

361. Answers will vary.

363.

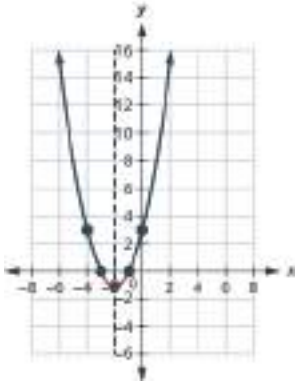
ⓐ



ⓑ $(-\infty, -5) \cup (-1, \infty)$

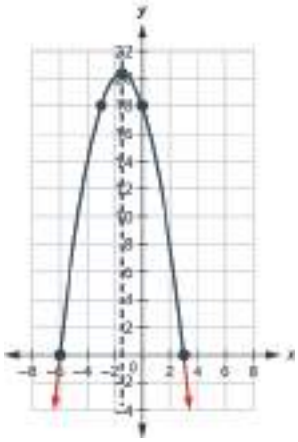
365.

ⓐ



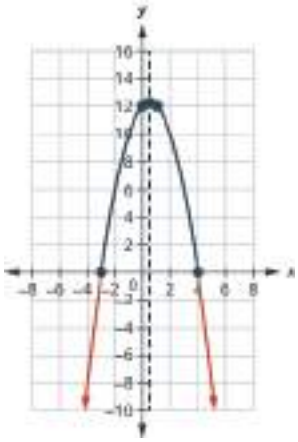
ⓑ $[-3, -1]$

367.
ⓐ



ⓑ $(-\infty, -6] \cup [3, \infty)$

369.
ⓐ



ⓑ $[-3, 4]$

371. $(-\infty, -4] \cup [1, \infty)$

373. $(2, 5)$

375. $(-\infty, -5) \cup (-3, \infty)$

377. $\left[2-\sqrt{2}, 2+\sqrt{2}\right]$

379. $\left(-\infty, 5-\sqrt{6}\right) \cup \left(5+\sqrt{6}, \infty\right)$

381. $\left(-\infty, -\frac{5}{2}\right] \cup \left[-\frac{2}{3}, \infty\right)$

383. $\left(-\infty, -\frac{1}{2}\right] \cup [4, \infty)$

385. $(-\infty, \infty)$.

387. no solution

389. $(-\infty, \infty)$.

391. Answers will vary.

393. Answers will vary.

Review Exercises

395. $y = \pm 12$

397. $a = \pm 5$

399. $r = \pm 4\sqrt{2}i$

401. $w = \pm 5\sqrt{3}$

403. $p = -1, 9$

405. $x = \frac{1}{4} \pm \frac{\sqrt{3}}{4}$

407. $n = 4 \pm 10\sqrt{2}$

409. $n = -5 \pm 2\sqrt{3}$

411. $(x+11)^2$

413. $\left(a-\frac{3}{2}\right)^2$

415. $d = -13, -1$

417. $m = -3 \pm 10i$

419. $v = 7 \pm 3\sqrt{2}$

421. $m = -9, -1$

423. $a = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$

425. $u = -6 \pm 2\sqrt{2}$

427. $p = 0, 6$

429. $y = -\frac{1}{2}, 2$

431. $c = -\frac{1}{3} \pm \frac{2\sqrt{7}}{3}$

433. $x = \frac{3}{2} \pm \frac{1}{2}i$

435. $x = \frac{1}{4}, 1$

437. $r = -6, 7$

439. $v = \frac{-1 \pm \sqrt{21}}{8}$

441. $m = \frac{-4 \pm \sqrt{10}}{3}$

443. $a = \frac{5}{12} \pm \frac{\sqrt{23}}{12}i$

445. $u = 5 \pm \sqrt{21}$

447. $p = \frac{4 \pm \sqrt{5}}{5}$

449. $c = -\frac{1}{2}$

451. ⓐ 1 ⓑ 2 ⓒ 2 ⓓ 2

453. ⓐ factor ⓑ Quadratic Formula ⓒ square root

455. $x = \pm\sqrt{2}, \quad x = \pm 2\sqrt{3}$

457. $x = \pm 1, \quad x = \pm\frac{1}{2}$

459. $x = 16$

461. $x = 64, x = 216$

463. $x = -2, x = \frac{4}{3}$

465. Two consecutive even numbers whose product is 624 are 24 and 26, and −24 and −26.

467. The height is 14 inches and the width is 10 inches.

469. The length of the diagonal is 3.6 feet.

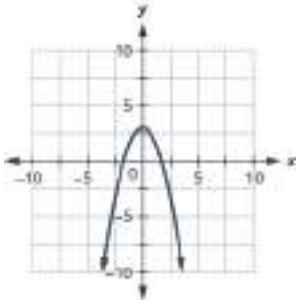
471. The width of the serving table is 4.7 feet and the length is 16.1 feet.



473. The speed of the wind was 30 mph.

475. One man takes 3 hours and the other man 6 hours to finish the repair alone.

477.



479. ⓐ up ⓑ down

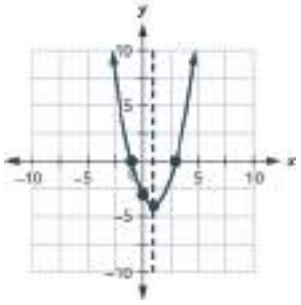
481. $x = 2$; $(2, -7)$

483. y : $(0, 15)$
x : $(3, 0)$, $(5, 0)$

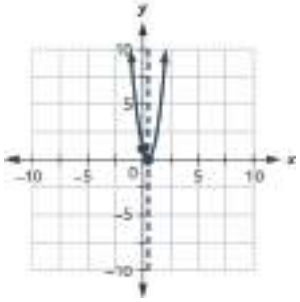
485. y : $(0, -46)$
x : none

487. y : $(0, -64)$
x : $(-8, 0)$

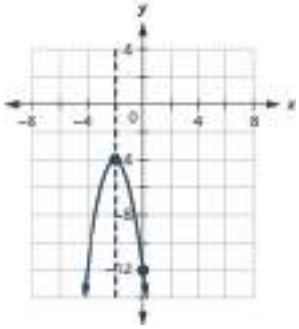
489.



491.



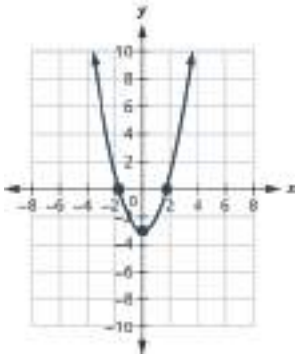
493.



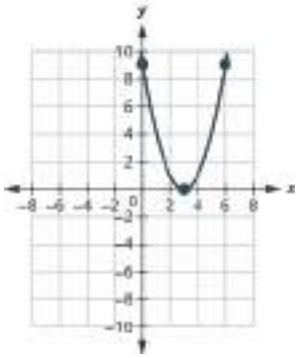
495. The maximum value is 2 when $x = 2$.

497. The length adjacent to the building is 90 feet giving a maximum area of 4,050 square feet.

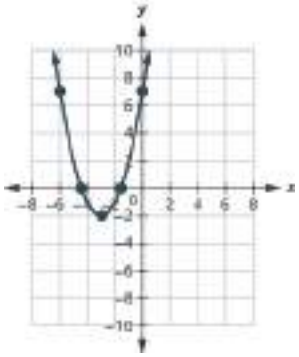
499.



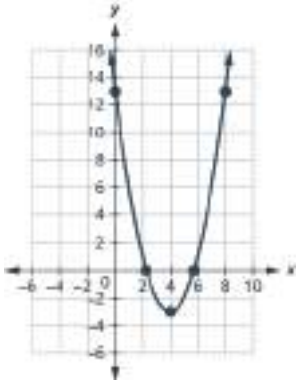
501.



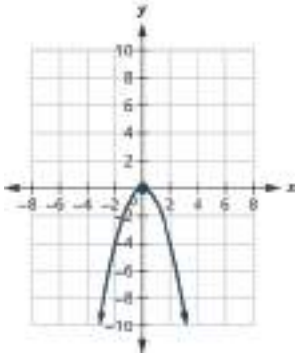
503.



505.



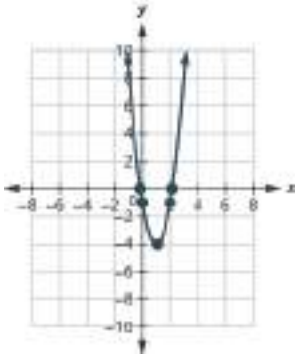
507.



509. $f(x) = 2(x-1)^2 - 6$

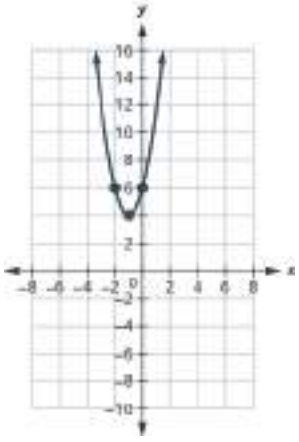
511. ⓐ $f(x) = 3(x-1)^2 - 4$

ⓑ



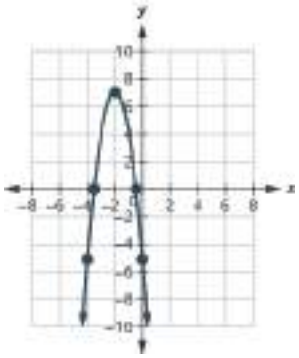
513. ⓐ $f(x) = 2(x+1)^2 + 4$

ⓑ



515. ⓐ $f(x) = -3(x+2)^2 + 7$

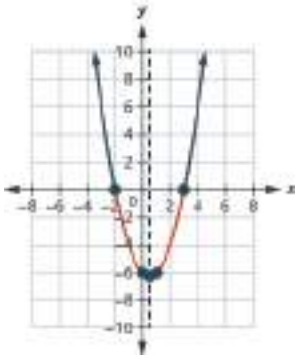
ⓑ



517. $f(x) = (x+1)^2 - 5$

519.

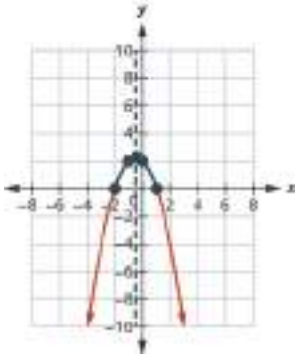
ⓐ



ⓑ $(-\infty, -2) \cup (3, \infty)$

521.

ⓐ



$[-2, 1]$

523. $(2, 4)$

525. $\left[3 - \sqrt{5}, 3 + \sqrt{5}\right]$

527. no solution

Practice Test

529. $w = -2, w = -8$

531. $m = 1, m = \frac{3}{2}$

533. $y = \frac{2}{3}$

535. 2 complex

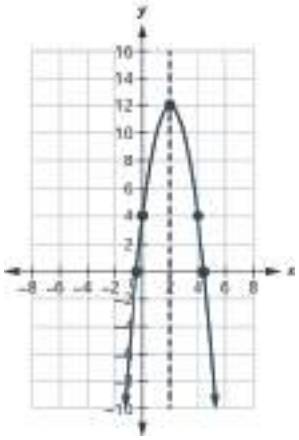
537. $y = 1, y = -27$

539. ⓐ down ⓑ $x = -4$

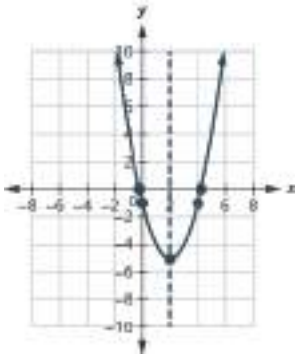
ⓒ $(-4, 0)$ ⓓ y: $(0, 16)$; x: $(-4, 0)$

ⓔ minimum value of -4 when $x = 0$

541.



543.



$f(x) = 2(x-1)^2 - 6$

545. $\left(-\infty, -\frac{5}{2}\right) \cup (2, \infty)$

547. The diagonal is 3.8 units long.

Chapter 10

Try It

10.1. ⓐ $15x + 1$ ⓑ $15x - 9$ ⓒ $15x^2 - 7x - 2$

10.2. ⓐ $24x - 23$ ⓑ $24x - 23$ ⓒ $24x^2 - 38x + 15$

10.3. ⓐ -8 ⓑ 5 ⓒ 40

10.4. ⓐ 65 ⓑ 10 ⓒ 5

10.5. ⓐ One-to-one function ⓑ Function; not one-to-one

10.6. ⓐ Not a function ⓑ Function; not one-to-one

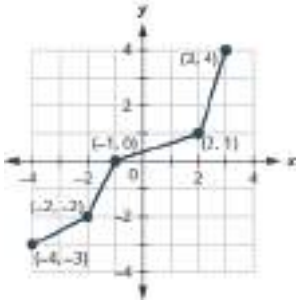
10.7. ⓐ Not a function ⓑ One-to-one function

10.8. ⓐ Function; not one-to-one ⓑ One-to-one function

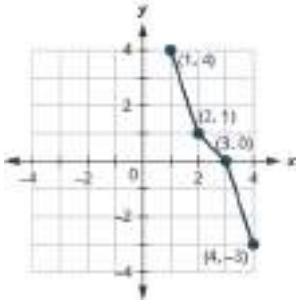
10.9. Inverse function: $\{(4, 0), (7, 1), (10, 2), (13, 3)\}$. Domain: $\{4, 7, 10, 13\}$. Range: $\{0, 1, 2, 3\}$.

10.10. Inverse function: $\{(4, -1), (1, -2), (0, -3), (2, -4)\}$. Domain: $\{0, 1, 2, 4\}$. Range: $\{-4, -3, -2, -1\}$.

10.11.



10.12.



10.13. $g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

10.14. $g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

10.15. $f^{-1}(x) = \frac{x+3}{5}$

10.16. $f^{-1}(x) = \frac{x-5}{8}$

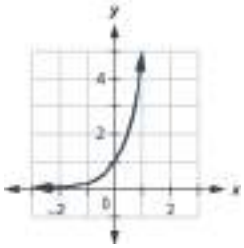
10.17. $f^{-1}(x) = \frac{x^5+2}{3}$

10.18. $f^{-1}(x) = \frac{x^4+7}{6}$

10.19.



10.20.



10.21.



10.22.



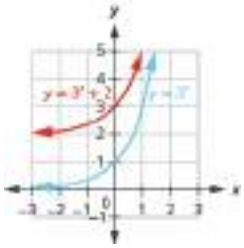
10.23.



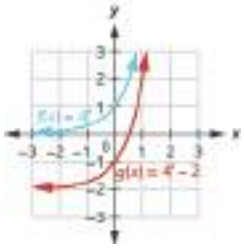
10.24.



10.25.



10.26.



10.27. $x = 2$

10.28. $x = 4$

10.29. $x = -1,\ x = 2$

10.30. $x = -2,\ x = 3$

10.31. ⓐ $22,332.96 ⓑ $22,362.49 ⓒ $22,377.37

10.32. ⓐ $21,071.81 ⓑ $21,137.04 ⓒ $21,170.00

10.33. She will find 166 bacteria.

10.34. She will find 1,102 viruses.

10.35. ⓐ $\log_3 9 = 2$ ⓑ $\log_7 \sqrt{7} = \frac{1}{2}$ ⓒ $\log_{\frac{1}{3}} \frac{1}{27} = x$

10.36. ⓐ $\log_4 64 = 3$ ⓑ $\log_4 \sqrt[3]{4} = \frac{1}{3}$ ⓒ $\log_{\frac{1}{2}} \frac{1}{32} = x$

10.37. ⓐ $64 = 4^3$ ⓑ $1 = x^0$ ⓒ $\frac{1}{100} = 10^{-2}$

10.38. ⓐ $27 = 3^3$ ⓑ $1 = x^0$ ⓒ $\frac{1}{10} = 10^{-1}$

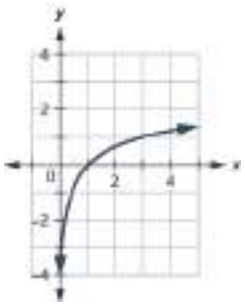
10.39. ⓐ $x = 8$ ⓑ $x = 125$ ⓒ $x = 2$

10.40. ⓐ $x = 9$ ⓑ $x = 243$ ⓒ $x = 3$

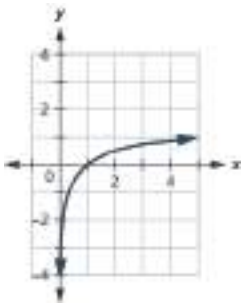
10.41. ⓐ 2 ⓑ $\frac{1}{2}$ ⓒ -5

10.42. ⓐ 2 ⓑ $\frac{1}{3}$ ⓒ -2

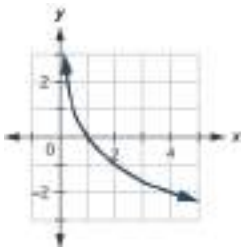
10.43.



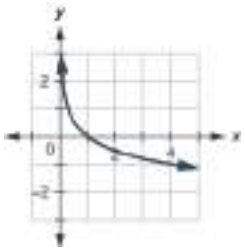
10.44.



10.45.



10.46.



10.47.

ⓐ

$a = 11$

ⓑ $x = e^7$

10.48.

ⓐ

$a = 4$

ⓑ $x = e^9$

10.49.

ⓐ

$x = 13$

ⓑ $x = 2$

10.50.

ⓐ

$x = 6$

ⓑ $x = 1$

10.51. The quiet dishwashers have a decibel level of 50 dB.

10.52. The decibel level of heavy traffic is 90 dB.

10.53. The intensity of the 1906 earthquake was about 8 times the intensity of the 1989 earthquake.

10.54. The intensity of the earthquake in Chile was about 1,259 times the intensity of the earthquake in Los Angeles.

10.55. ⓐ 0 ⓑ 1

10.56. ⓐ 0 ⓑ 1

10.57. ⓐ 15 ⓑ 4

10.58. ⓐ 8 ⓑ 15

10.59. ⓐ $1 + \log_3 x$

ⓑ $3 + \log_2 x + \log_2 y$

10.60. ⓐ $1 + \log_9 x$

ⓑ $3 + \log_3 x + \log_3 y$

10.61. ⓐ $\log_4 3 - 1$ ⓑ $\log x - 3$

10.62. ⓐ $\log_2 5 - 2$ ⓑ $1 - \log y$

10.63. ⓐ $4\log_7 5$ ⓑ $100 \cdot \log x$

10.64. ⓐ $7\log_2 3$ ⓑ $20 \cdot \log x$

10.65. $\log_2 5 + 4\log_2 x + 2\log_2 y$

10.66. $\log_3 7 + 5\log_3 x + 3\log_3 y$

10.67. $\frac{1}{5}\left(4\log_4 x - \frac{1}{2} - 3\log_4 y - 2\log_4 z\right)$

10.68. $\frac{1}{3}(2\log_3 x - \log_3 5 - \log_3 y - \log_3 z)$

10.69. $\log_2 \frac{5x}{y}$

10.70. $\log_3 \frac{6}{xy}$

10.71. $\log_2 x^3 (x-1)^2$

10.72. $\log x^2 (x+1)^2$

10.73. 3.402

10.74. 2.379

10.75. $x = 6$

10.76. $x = 4$

10.77. $x = 4$

10.78. $x = 8$

10.79. $x = 3$

10.80. $x = 8$

10.81. $x = \frac{\log 43}{\log 7} \approx 1.933$

10.82. $x = \frac{\log 98}{\log 8} \approx 2.205$

10.83. $x = \ln 9 + 2 \approx 4.197$

10.84. $x = \frac{\ln 5}{2} \approx 0.805$

10.85. $r \approx 9.3\%$

10.86. $r \approx 11.9\%$

10.87. There will be 62,500 bacteria.

10.88. There will be 5,870,061 bacteria.

10.89. There will be 6.43 mg left.

10.90. There will be 31.5 mg left.

Section Exercises

1. ⓐ $8x + 23$ ⓑ $8x + 11$ ⓒ $8x^2 + 26x + 15$

3. ⓐ $24x + 1$ ⓑ $24x - 19$ ⓒ $24x^2 + 19x - 5$

5. ⓐ $6x^2 - 9x$ ⓑ $18x^2 - 9x$ ⓒ $6x^3 - 9x^2$

7. ⓐ $2x^2 + 3$ ⓑ $4x^2 - 4x + 3$ ⓒ $2x^3 - x^2 + 4x - 2$

9. ⓐ 245 ⓑ 104 ⓒ 53

11. ⓐ 250 ⓑ 14 ⓒ 77

13. Function; not one-to-one

15. One-to-one function

17. ⓐ Not a function ⓑ Function; not one-to-one

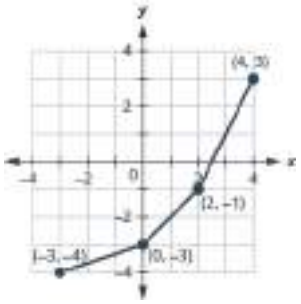
19. ⓐ One-to-one function ⓑ Function; not one-to-one

21. Inverse function: $\{(1, 2), (2, 4), (3, 6), (4, 8)\}$. Domain: $\{1, 2, 3, 4\}$. Range: $\{2, 4, 6, 8\}$.

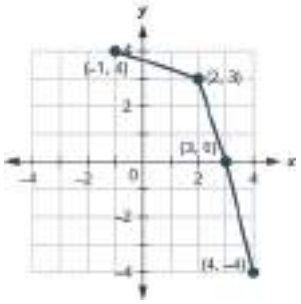
23. Inverse function: $\{(-2, 0), (3, 1), (7, 2), (12, 3)\}$. Domain: $\{-2, 3, 7, 12\}$. Range: $\{0, 1, 2, 3\}$.

25. Inverse function: $\{(-3, -2), (-1, -1), (1, 0), (3, 1)\}$. Domain: $\{-3, -1, 1, 3\}$. Range: $\{-2, -1, 0, 1\}$.

27.



29.



31. $g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

33. $g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

35. $g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

37. $g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses (for nonnegative x).

39. $f^{-1}(x) = x + 12$

41. $f^{-1}(x) = \frac{x}{9}$

43. $f^{-1}(x) = 6x$

45. $f^{-1}(x) = \frac{x + 7}{6}$

47. $f^{-1}(x) = \frac{x - 5}{-2}$

49. $f^{-1}(x) = \sqrt{x - 6}$

51. $f^{-1}(x) = \sqrt[3]{x + 4}$

53. $f^{-1}(x) = \frac{1}{x} - 2$

55. $f^{-1}(x) = x^2 + 2,\ x \geq 0$

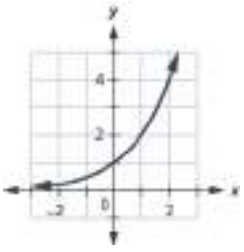
57. $f^{-1}(x) = x^3 + 3$

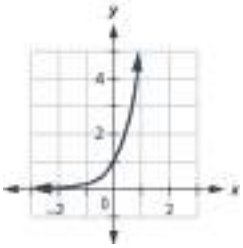
59. $f^{-1}(x) = \frac{x^4 + 5}{9},\ x \geq 0$

61. $f^{-1}(x) = \frac{x^5 - 5}{-3}$

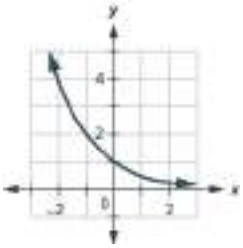
63. Answers will vary.

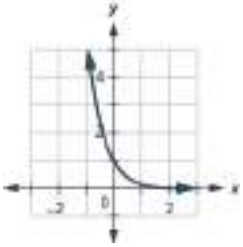
65.



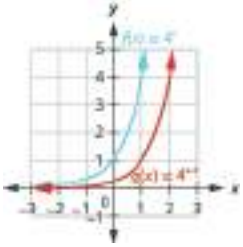
67. 

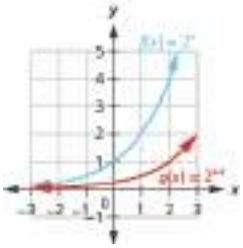
69. 

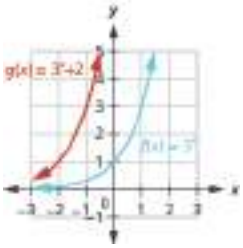
71. 

73. 

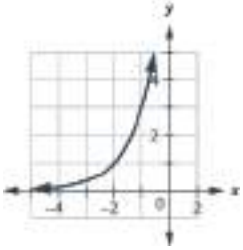
75. 

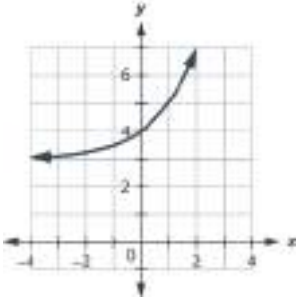
77. 

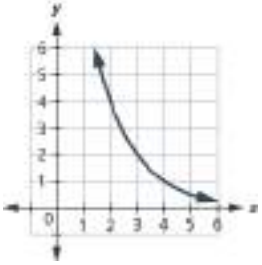
79. 

81. 

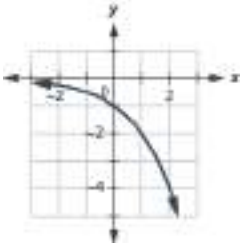
83. 

85. 

87. 

89. 

91. 

93. 

95. $x = 4$

97. $x = -1$

99. $x = -1, x = 1$

101. $x = 1$

103. $x = -1$

105. $x = 2$

107. $x = -1, x = 2$

109. (f)

111. (a)

113. (e)

115. (a) $7,387.28 (b) $7,434.57 (c) $7,459.12

117. $36,945.28

119. 223 bacteria

121. 288,929,825

123. Answers will vary.

125. Answers will vary.

127. $\log_2 32 = 5$

129. $\log_5 125 = 3$

131. $\log \frac{1}{100} = -2$

133. $\log_x \sqrt[3]{6} = \frac{1}{3}$

135. $\log_{17} \sqrt[5]{17} = x$

137. $\log_{\frac{1}{3}} \frac{1}{81} = 4$

139. $\log_4 \frac{1}{64} = -3$

141. $\ln x = 3$

143. $64 = 2^6$

145. $32 = x^5$

147. $1 = 7^0$

149. $9 = 9^1$

151. $1{,}000 = 10^3$

153. $43 = e^x$

155. $x = 11$

157. $x = 4$

159. $x = 125$

161. $x = \frac{1}{243}$

163. $x = 2$

165. $x = -2$

167. 2

169. 0

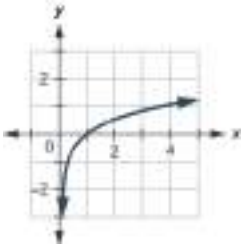
171. $\frac{1}{3}$

173. -2

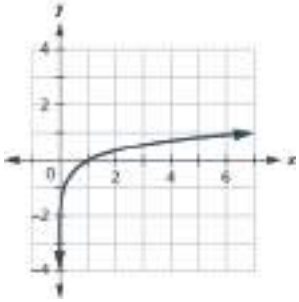
175. -3

177. -2

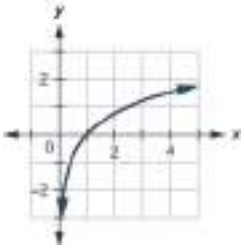
179.



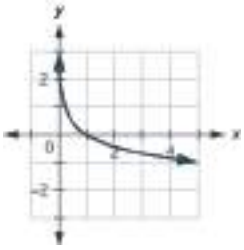
181.



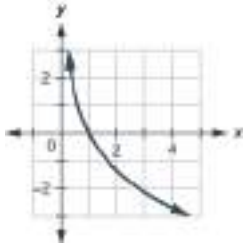
183.



185.



187.



189. $a = 9$

191. $a = 3$

193. $a = \sqrt[3]{24}$

195. $x = e^4$

197. $x = 5$

199. $x = 17$

201. $x = 6$

203. $x = 3$

205. $x = -5\sqrt{5}, x = 5\sqrt{5}$

207. $x = -5, x = 5$

209. A whisper has a decibel level of 20 dB.

211. The sound of a garbage disposal has a decibel level of 100 dB.

213. The intensity of the 1994 Northridge earthquake in the Los Angeles area was about 40 times the intensity of the 2014 earthquake.

215. Answers will vary.

217. Answers will vary.

219. ⓐ 0 ⓑ 1

221. ⓐ 10 ⓑ 10

223. ⓐ 15 ⓑ -4

225. ⓐ $\sqrt{3}$ ⓑ -1

227. ⓐ 3 ⓑ 7

229. $\log_5 8 + \log_5 y$

231. $4 + \log_3 x + \log_3 y$

233. $3 + \log y$

235. $\log_6 5 - 1$

237. $3 - \log_5 x$

239. $4 - \log y$

241. $4 - \ln 16$

243. $5\log_2 x$

245. $-3\log x$

247. $\frac{1}{3}\log_5 x$

249. $\sqrt[3]{4}\ln x$

251. $\log_2 3 + 5\log_2 x + 3\log_2 y$

253. $\frac{1}{4}\log_5 21 + 3\log_5 y$

255. $\log_5 4 + \log_5 a + 3\log_5 b + 4\log_5 c - 2\log_5 d$

257. $\frac{2}{3}\log_3 x - 3 - 4\log_3 y$

259. $\frac{1}{2}\log_3\left(3x + 2y^2\right) - \log_3 5 - 2\log_3 z$

261. $\frac{1}{3}(\log_5 3 + 2\log_5 x - \log_5 4 - 3\log_5 y - \log_5 z)$

263. 2

265. 2

267. $\log_2 \frac{5}{x-1}$

269. $\log_5 \frac{2}{xy}$

271. $\log_3 x^6 y^9$

273. 0

275. $\ln \frac{x^3 y^4}{z^2}$

277. $\log(2x+3)^2 \cdot \sqrt{x+1}$

279. 2.379

281. 1.674

283. 5.542

285. Answers will vary.

287. Answers will vary.

289. $x = 7$

291. $x = 4$

293. $x = 1, \quad x = 3$

295. $x = 8$

297. $x = 3$

299. $x = 20$

301. $x = 3$

303. $x = 6$

305. $x = \frac{5}{3}$

307. $x = \frac{\log 74}{\log 2} \approx 6.209$

309. $x = \frac{\log 112}{\log 4} \approx 3.404$

311. $x = \ln 8 \approx 2.079$

313. $x = \frac{\log 8}{\log \frac{1}{3}} \approx -1.893$

315. $x = \ln 3 - 2 \approx -0.901$

317. $x = \frac{\ln 16}{3} \approx 0.924$

319. $x = \ln 6 \approx 1.792$

321. $x = \ln 8 + 1 \approx 3.079$

323. $x = 5$

325. $x = -4, \; x = 5$

327. $a = 3$

329. $x = e^9$

331. $x = 7$

333. $x = 3$

335. $x = 2$

337. $x = 6$

339. $x = 5$

341. $x = \frac{\log 10}{\log \frac{1}{2}} \approx -3.322$

343. $x = \ln 7 - 5 \approx -3.054$

345. 6.9%

347. 13.9 years

349. 122,070 bacteria

351. 8 times as large as the original population

353. 0.03 ml

355. Answers will vary.

Review Exercises

357. ⓐ $4x^2 + 12x$ ⓑ $16x^2 + 12x$ ⓒ $4x^3 + 12x^2$

359. ⓐ -123 ⓑ 356 ⓒ 41

361. Function; not one-to-one

363. ⓐ Function; not one-to-one ⓑ Not a function

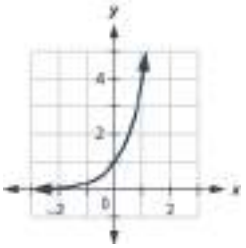
365. Inverse function: $\{(10, -3), (5, -2), (2, -1), (1, 0)\}$. Domain: $\{1, 2, 5, 10\}$. Range: $\{-3, -2, -1, 0\}$.

367. $g(f(x)) = x, \quad$ and $f(g(x)) = x, \quad$ so they are inverses.

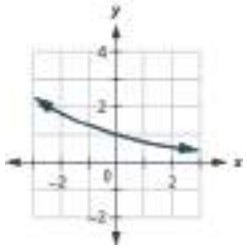
369. $f^{-1}(x) = \frac{x+11}{6}$

371. $f^{-1}(x) = \frac{1}{x} - 5$

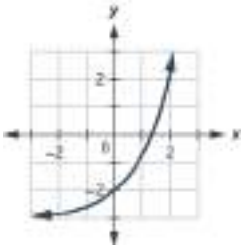
373.



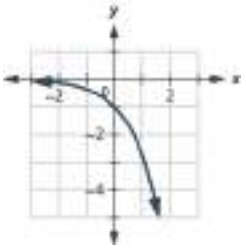
375.



377.



379.



381. $x = -2,\ x = 2$

383. $x = -1$

385. $x = -3,\ x = 5$

387. \$163,323.40

389. 330,259,000

391. $\log\frac{1}{1{,}000} = -3$

393. $\ln 16 = y$

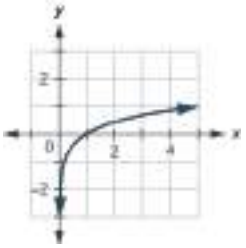
395. $100000 = 10^5$

397. $x = 5$

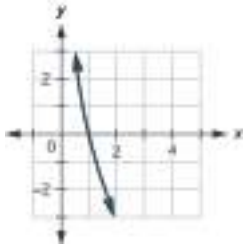
399. $x = 4$

401. 0

403.



405.



407. $x = e^{-3}$

409. $x = 8$

411. 90 dB

413. ⓐ 13 ⓑ −9

415. ⓐ 8 ⓑ 5

417. $4 + \log m$

419. $5 - \ln 2$

421. $\frac{1}{7}\log_4 z$

423. $\log_5 8 + 2\log_5 a + 6\log_5 b + \log_5 c - 3\log_5 d$

425. $\frac{1}{3}(\log_6 7 + 2\log_6 x - 1 - 3\log_6 y - 5\log_6 z)$

427. $\log_3 x^3 y^7$

429. $\log\frac{\sqrt[4]{y}}{(y-3)^2}$

431. 5.047

433. $x = 4$

435. $x = 3$

437. $x = \frac{\log 101}{\log 2} \approx 6.658$

439. $x = \frac{\log 7}{\log\frac{1}{3}} \approx -1.771$

441. $x = \ln 15 + 4 \approx 6.708$

443. 11.6 years

445. 12.7 months

Practice Test

447. ⓐ $48x - 17$ ⓑ $48x + 5$ ⓒ $48x^2 - 10x - 3$

449. ⓐ Not a function ⓑ One-to-one function

451. $f^{-1}(x) = \sqrt[5]{x+9}$

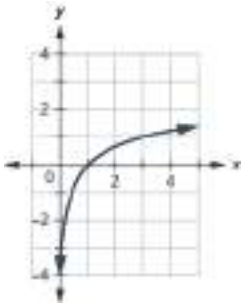
453. $x = 5$

455. ⓐ \$31,250.74 ⓑ \$31,302.29 ⓒ \$31,328.32

457. $343 = 7^3$

459. 0

461.



463. 40 dB

465. $2 + \log_5 a + \log_5 b$

467. $\frac{1}{4}(\log_2 5 + 3\log_2 x - 4 - 2\log_2 y - 7\log_2 z)$

469. $\log\frac{\sqrt[6]{x}}{(x+5)^3}$

471. $x = 6$

473. $x = \ln 8 + 4 \approx 6.079$

475. 1,921 bacteria

Chapter 11

Try It

11.1. $d = 5$

11.2. $d = 10$

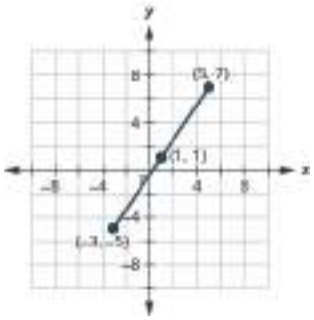
11.3. $d = 15$

11.4. $d = 13$

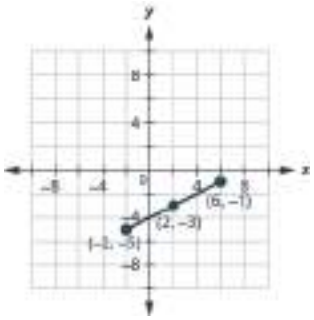
11.5. $d = \sqrt{130}$, $d \approx 11.4$

11.6. $d = \sqrt{2}$, $d \approx 1.4$

11.7.



11.8.



11.9. $x^2 + y^2 = 36$

11.10. $x^2 + y^2 = 64$

11.11. $(x-2)^2 + (y+4)^2 = 49$

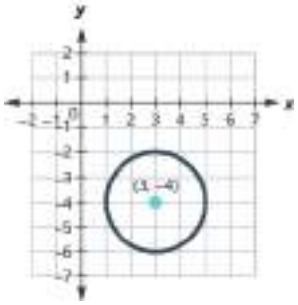
11.12. $(x+3)^2 + (y+5)^2 = 81$

11.13. $(x-2)^2+(y-1)^2=25$

11.14. $(x-7)^2+(y-1)^2=100$

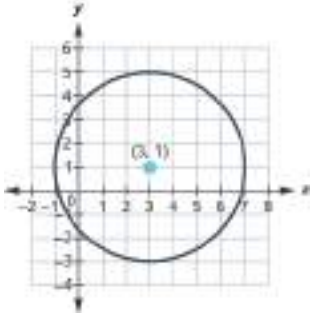
11.15. ⓐ The circle is centered at $(3,\ -4)$ with a radius of 2.

ⓑ



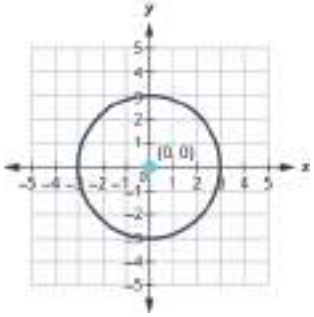
11.16. ⓐ The circle is centered at $(3,\ 1)$ with a radius of 4.

ⓑ



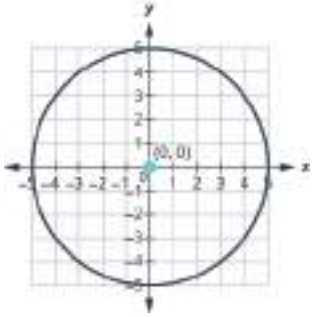
11.17. ⓐ The circle is centered at $(0,\ 0)$ with a radius of 3.

ⓑ



11.18. ⓐ The circle is centered at $(0,\ 0)$ with a radius of 5.

ⓑ



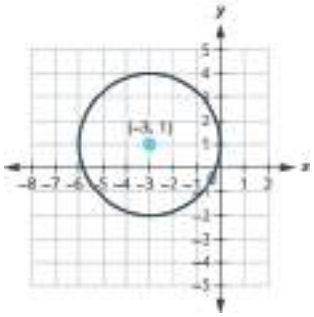
11.19. ⓐ The circle is centered at $(3,\ 4)$ with a radius of 4.

ⓑ

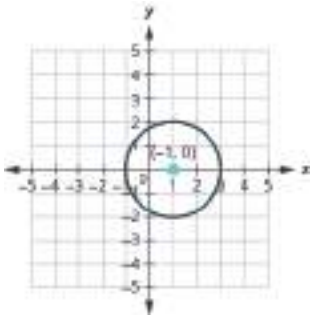


11.20. ⓐ The circle is centered at $(-3,\ 1)$ with a radius of 3.

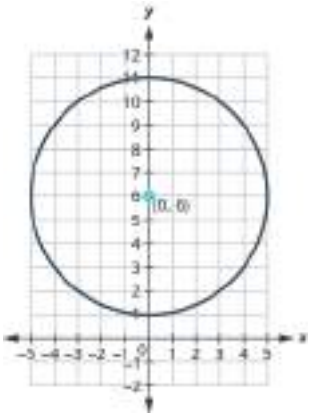
ⓑ



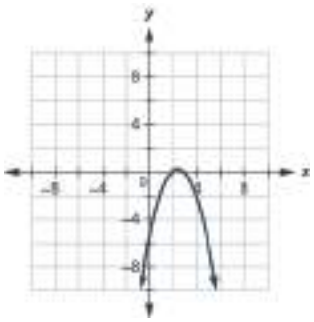
11.21. ⓐ The circle is centered at $(-1,\ 0)$ with a radius of 2.



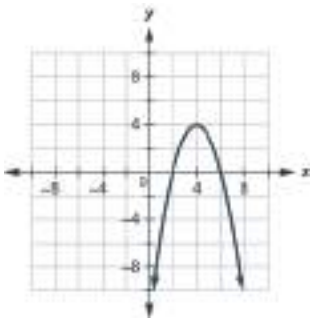
11.22. ⓐ The circle is centered at $(0, 6)$ with a radius of 5.



11.23.

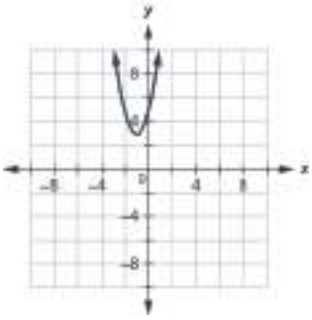


11.24.



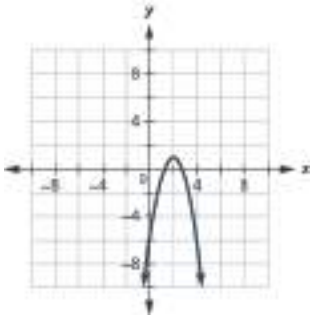
11.25. ⓐ $y = 2(x + 1)^2 + 3$

ⓑ

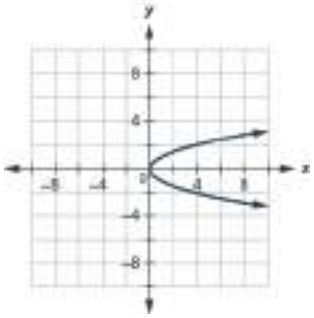


11.26. ⓐ $y = -2(x - 2)^2 + 1$

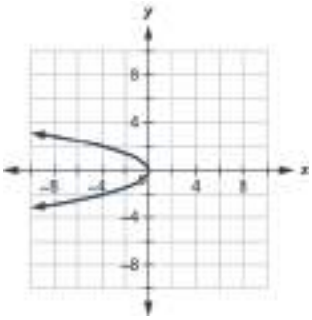
ⓑ



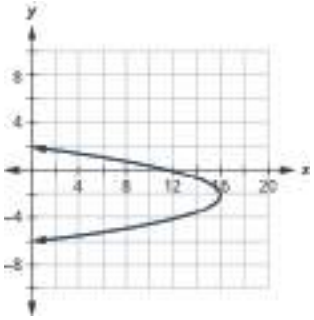
11.27.



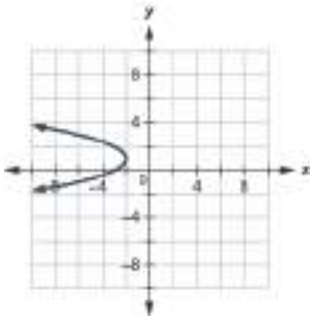
11.28.



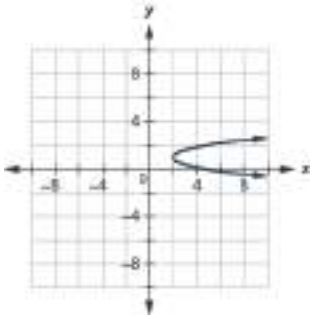
11.29.



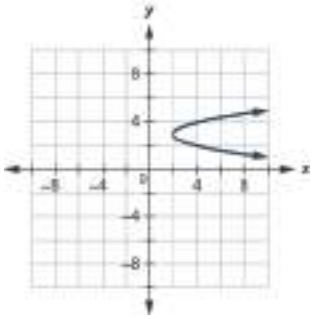
11.30.



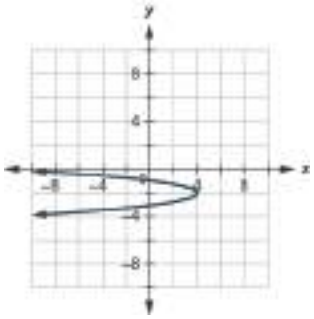
11.31.



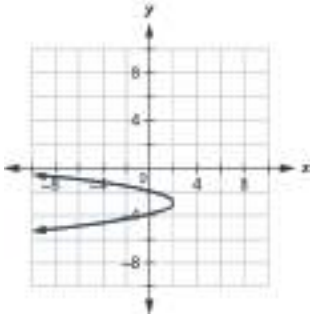
11.32.



11.33.

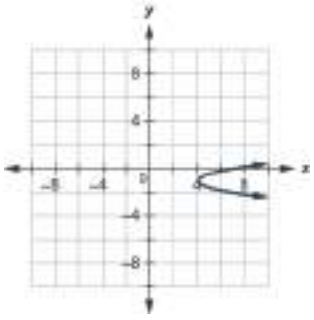


11.34.



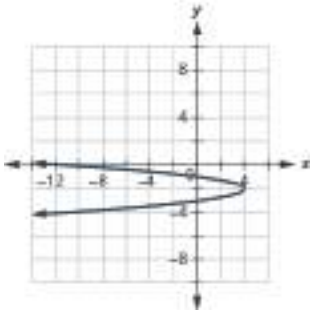
11.35. ⓐ $x = 3(y+1)^2 + 4$

ⓑ



11.36. ⓐ $x = -4(y+2)^2 + 4$

ⓑ

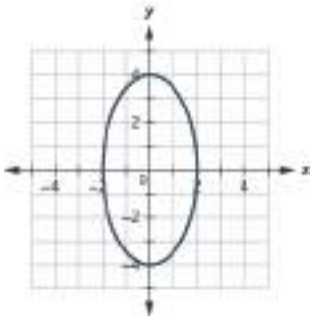


11.37. $y = -\frac{1}{20}(x-20)^2 + 20$

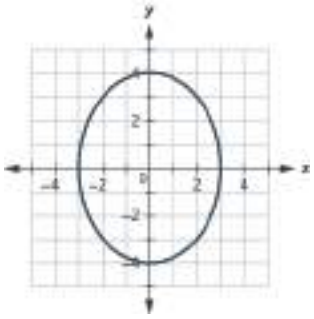
11.38. $y = -\frac{1}{5}x^2 + 2x$

$y = -\frac{1}{5}(x-5)^2 + 5$

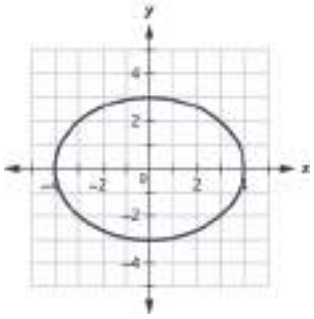
11.39.



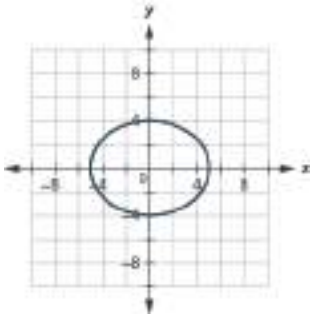
11.40.



11.41.



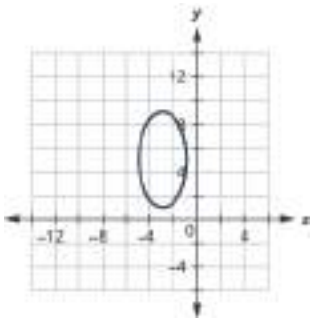
11.42.



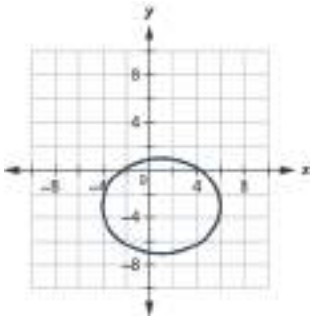
11.43. $\frac{x^2}{4}+\frac{y^2}{25}=1$

11.44. $\frac{x^2}{9}+\frac{y^2}{4}=1$

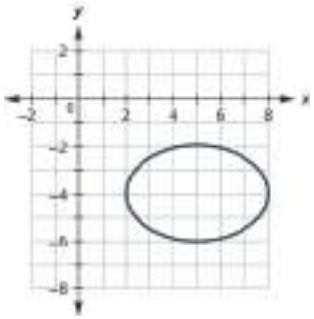
11.45.



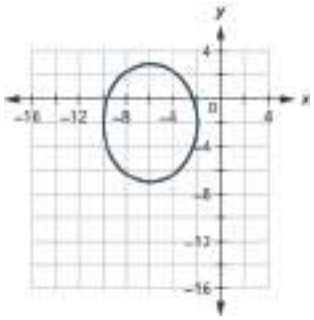
11.46.



11.47.

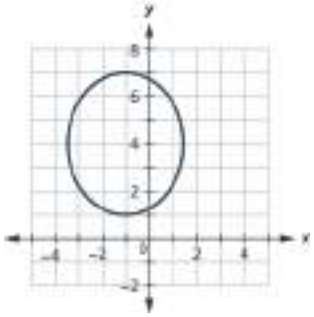


11.48.



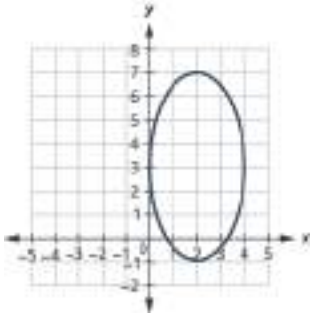
11.49. ⓐ $\frac{(x+1)^2}{6}+\frac{(y-4)^2}{9}=1$

ⓑ



11.50. ⓐ $\frac{(x-2)^2}{4}+\frac{(y-3)^2}{16}=1$

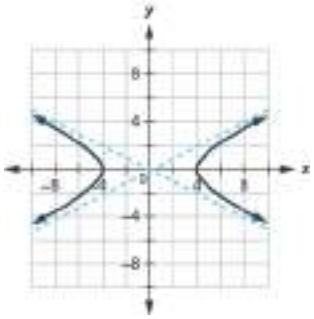
ⓑ



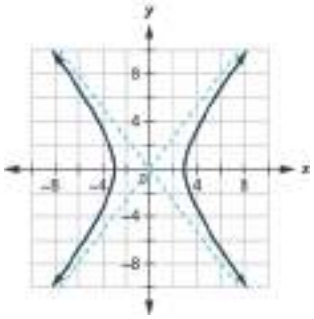
11.51. $\frac{x^2}{625}+\frac{y^2}{600}=1$

11.52. $\frac{x^2}{1225}+\frac{y^2}{1000}=1$

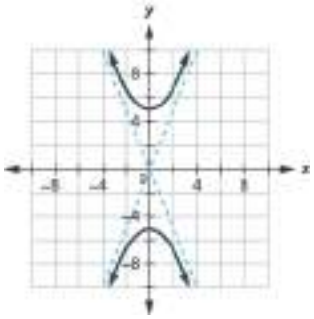
11.53.



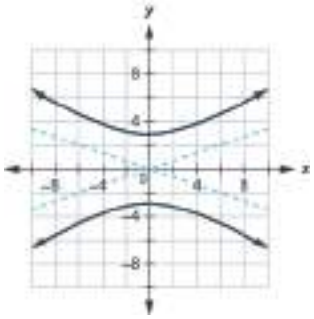
11.54.



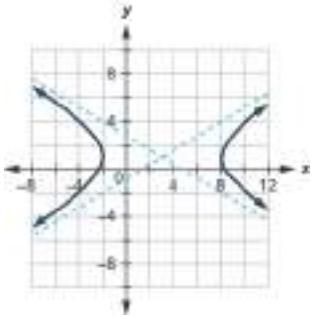
11.55.



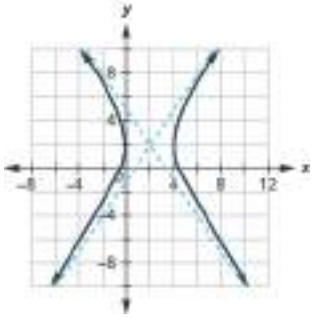
11.56.



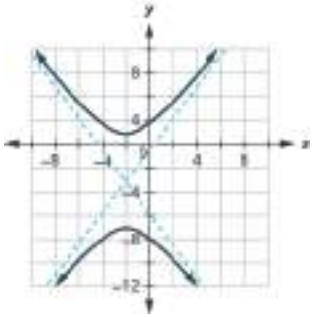
11.57.



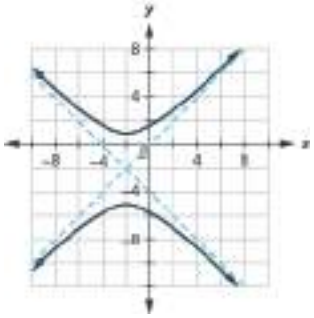
11.58.



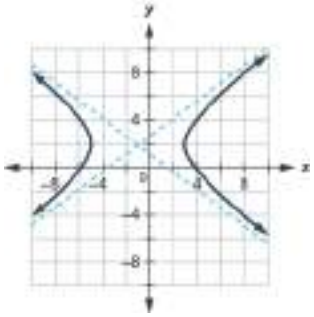
11.59.



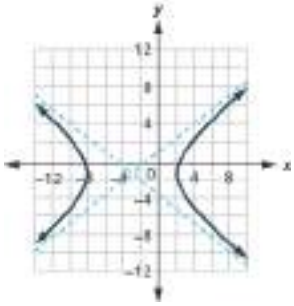
11.60.



11.61. ⓐ $\frac{(x+1)^2}{16} - \frac{(y-2)^2}{9} = 1$ ⓑ



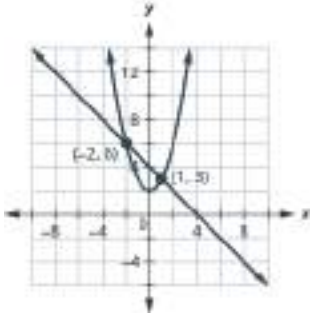
11.62. ⓐ $\frac{(x+3)^2}{25} - \frac{(y+1)^2}{16} = 1$ ⓑ



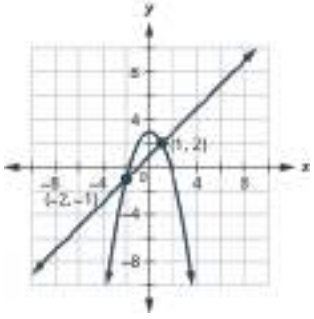
11.63. ⓐ circle ⓑ ellipse ⓒ parabola ⓓ hyperbola

11.64. ⓐ ellipse ⓑ parabola ⓒ circle ⓓ hyperbola

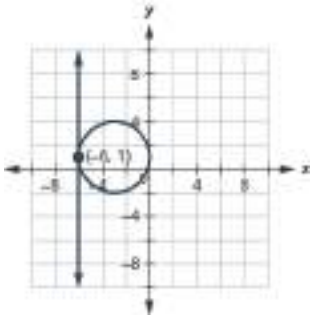
11.65.



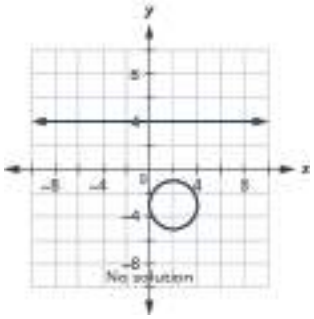
11.66.



11.67.



11.68.



11.69. No solution

11.70. $\left(-\frac{4}{5}, \frac{6}{5}\right)$, $(0, 2)$

11.71. No solution

11.72. $\left(\frac{4}{9}, -\frac{2}{3}\right)$, $(1, 1)$

11.73.
$(-3, 0)$, $(3, 0)$, $\left(-2\sqrt{2}, -1\right)$, $\left(2\sqrt{2}, -1\right)$

11.74. $(-1, 0)$, $(0, 1)$, $(0, -1)$

11.75.
$(-3, -4)$, $(-3, 4)$, $(3, -4)$, $(3, 4)$

11.76. $(-2, 0)$, $(2, 0)$

11.77. 4 and 6

11.78. -18 and 17

11.79. If the length is 12 inches, the width is 16 inches. If the length is 16 inches, the width is 12 inches.

11.80. If the length is 12 inches, the width is 9 inches. If the length is 9 inches, the width is 12 inches.

Section Exercises

1. $d = 5$

3. 13

5. 5

7. 13

9. 76. $d = 3\sqrt{5}$, $d \approx 6.7$

11. $d = \sqrt{68}$, $d \approx 8.2$

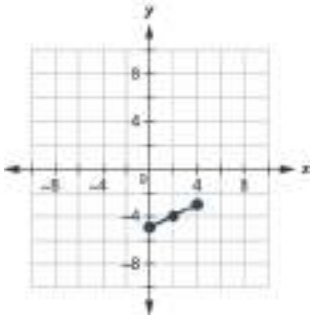
13. ⓐ Midpoint: $(2, -4)$

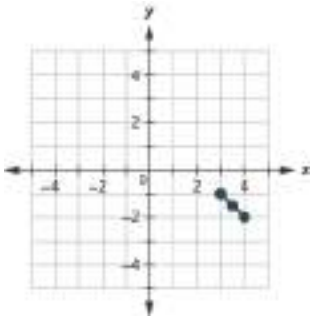
ⓑ

15. ⓐ Midpoint: $\left(3\frac{1}{2}, -1\frac{1}{2}\right)$

ⓑ

17. $x^2 + y^2 = 49$





19. $x^2 + y^2 = 2$

21. $(x - 3)^2 + (y - 5)^2 = 1$

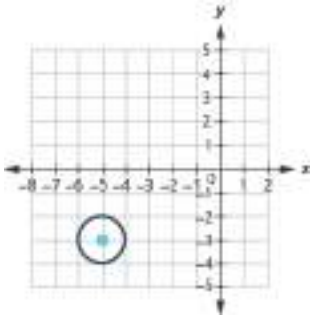
23. $(x - 1.5)^2 + (y + 3.5)^2 = 6.25$

25. $(x-3)^2+(y+2)^2=64$

27. $(x-4)^2+(y-4)^2=8$

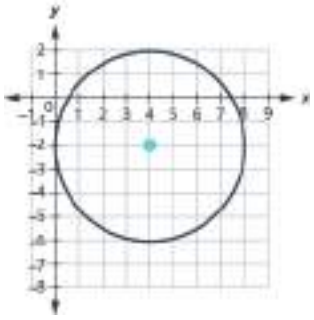
29. ⓐ The circle is centered at $(-5,\ -3)$ with a radius of 1.

ⓑ



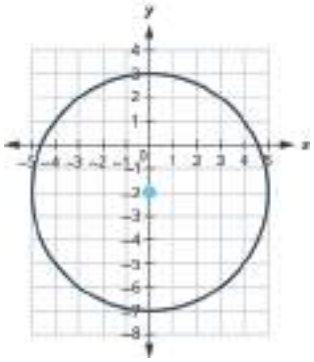
31. ⓐ The circle is centered at $(4,\ -2)$ with a radius of 4.

ⓑ



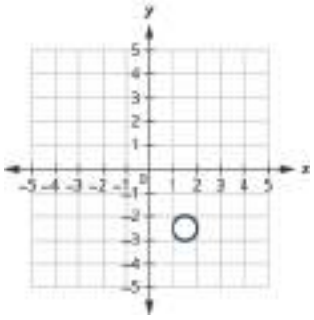
33. ⓐ The circle is centered at $(0,\ -2)$ with a radius of 5.

ⓑ



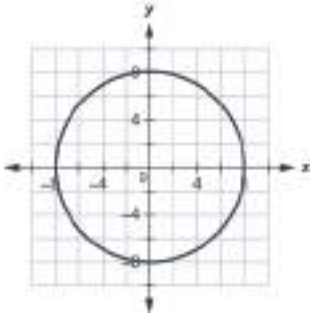
35. ⓐ The circle is centered at $(1.5,\ 2.5)$ with a radius of 0.5.

ⓑ



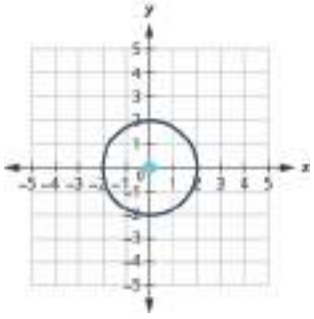
37. ⓐ The circle is centered at $(0,\ 0)$ with a radius of 8.

ⓑ



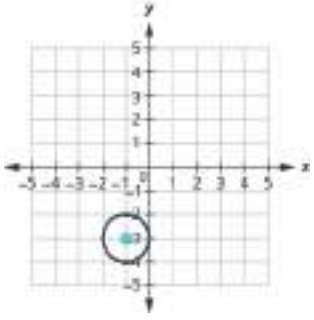
39. ⓐ The circle is centered at $(0,\ 0)$ with a radius of 2.

ⓑ



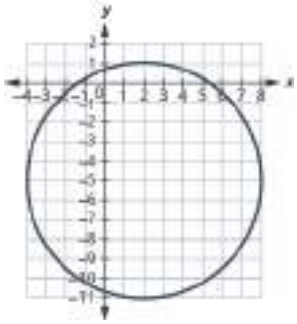
41. ⓐ Center: $(-1,\ -3)$, radius: 1

ⓑ



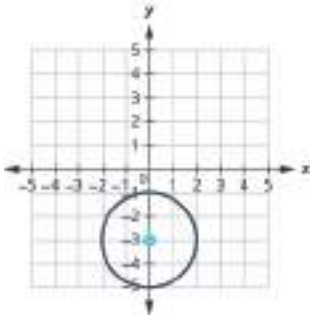
43. ⓐ Center: $(2, -5)$, radius: 6

ⓑ



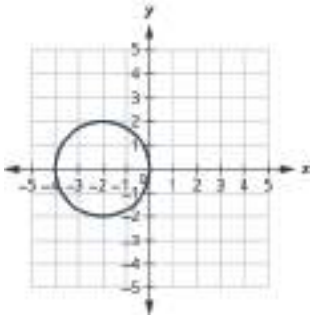
45. ⓐ Center: $(0, -3)$, radius: 2

ⓑ



47. ⓐ Center: $(-2, 0)$, radius:

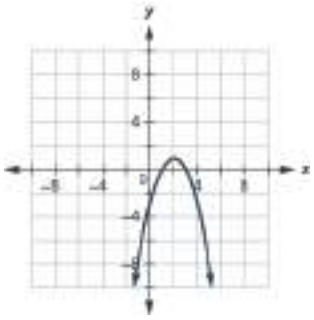
ⓑ



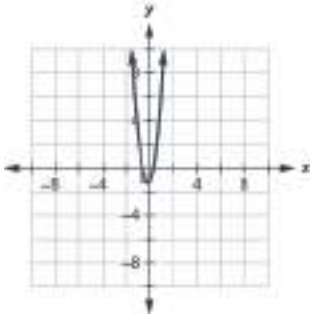
49. Answers will vary.

51. Answers will vary.

53.

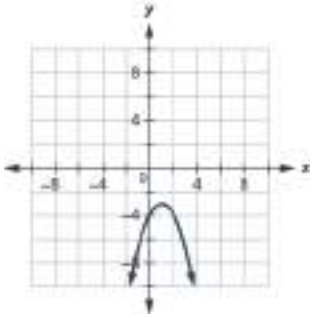


55.



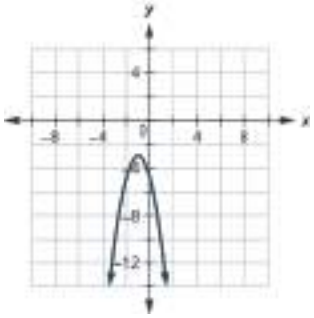
57. ⓐ $y = -(x-1)^2 - 3$

ⓑ

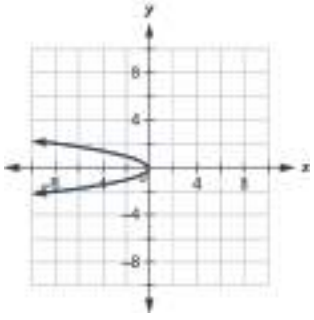


59. ⓐ $y = -2(x+1)^2 - 3$

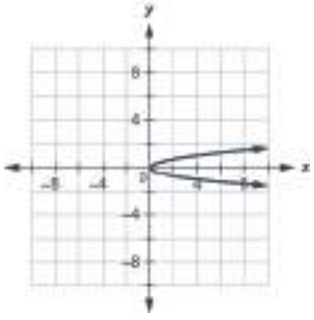
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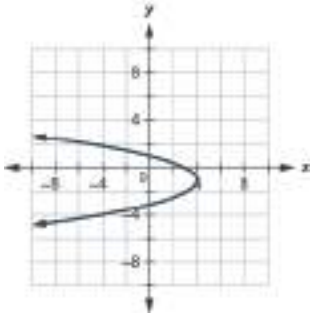
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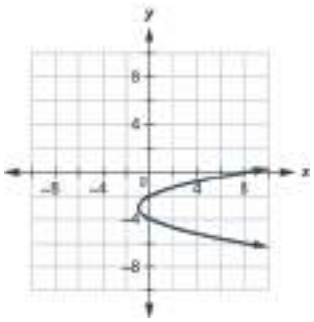
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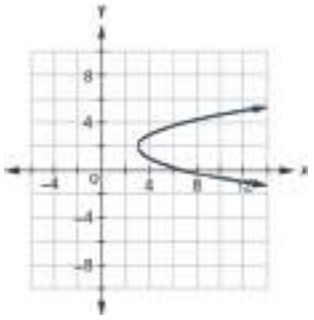
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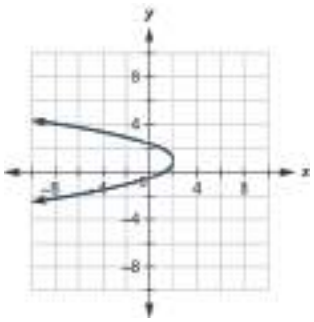
67.



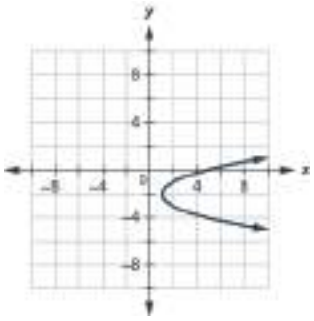
69.



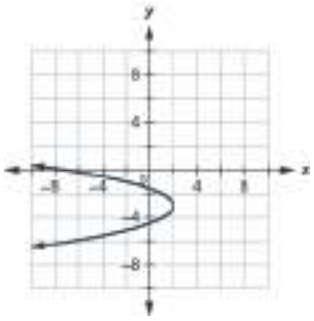
71.



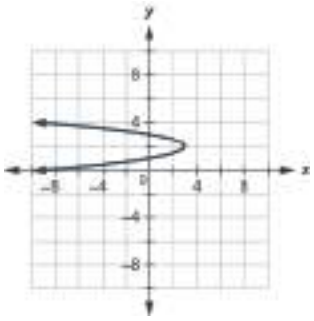
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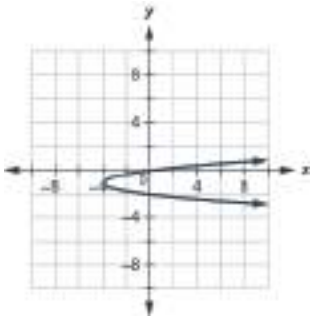
75.



77.

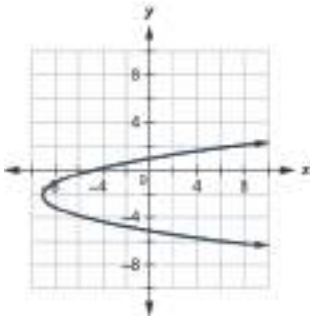


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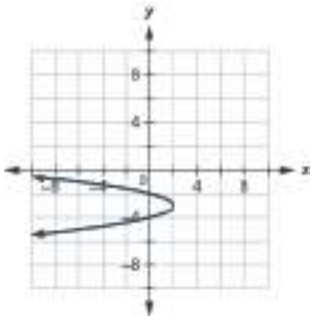
81. ⓐ $x = (y+2)^2 - 9$

ⓑ



83. ⓐ $x = -2(y+3)^2 + 2$

ⓑ



85. ⓐ

87. ⓑ

89. ⓓ

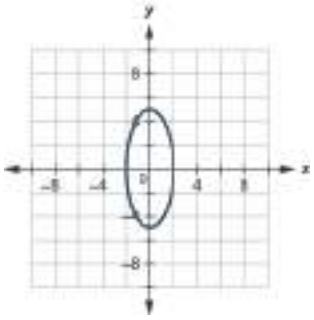
91. $y = -\frac{1}{15}(x-15)^2 + 15$

93. $y = -\frac{1}{10}(x-30)^2 + 90$

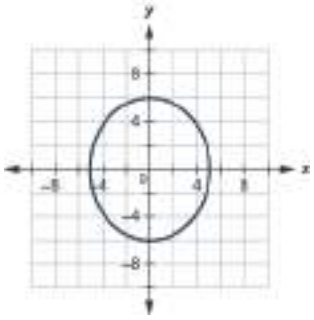
95. Answers will vary.

97. Answers will vary.

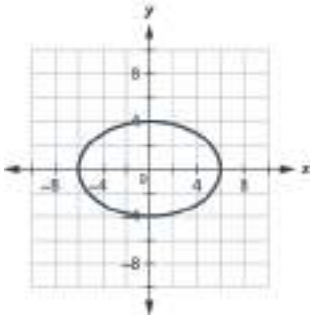
99.



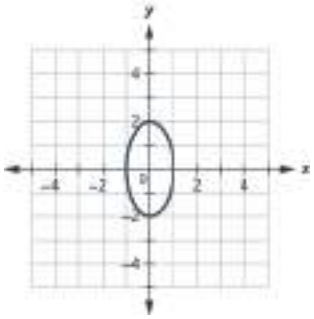
101.



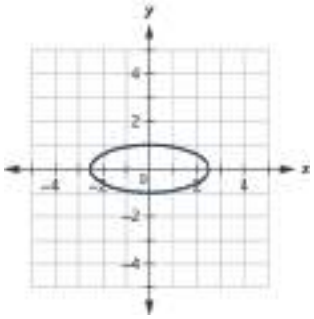
103.



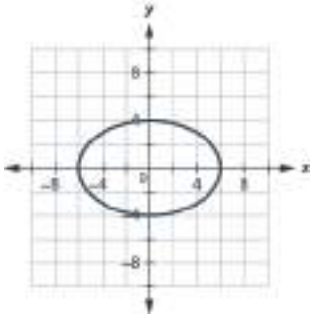
105.



107.



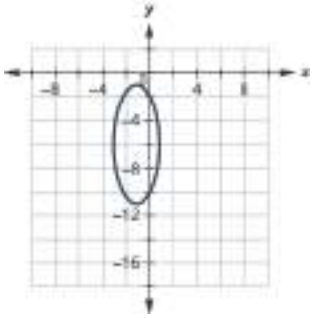
109.



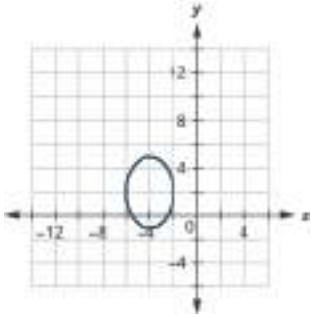
111. $\frac{x^2}{9}+\frac{y^2}{25}=1$

113. $\frac{x^2}{9}+\frac{y^2}{16}=1$

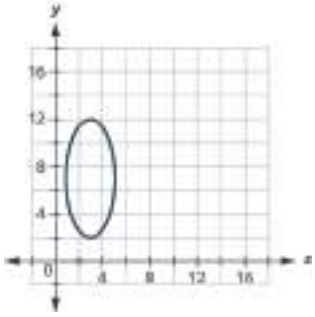
115.



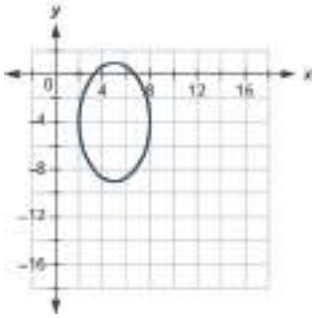
117.



119.

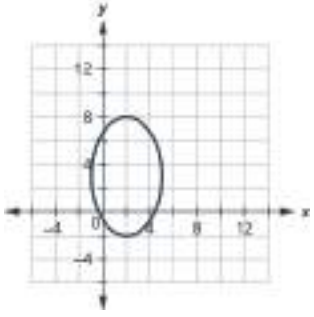


121.



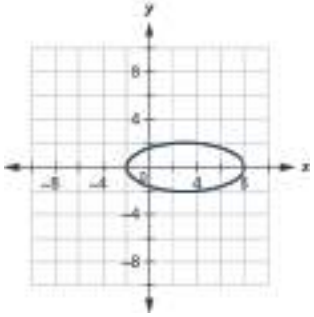
123. ⓐ $\frac{(x-2)^2}{9}+\frac{(y-3)^2}{25}=1$

ⓑ

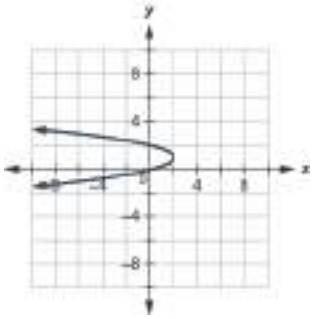


125. ⓐ $\frac{y^2}{4}+\frac{(x-3)^2}{25}=1$

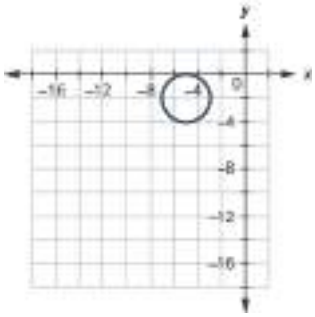
ⓑ



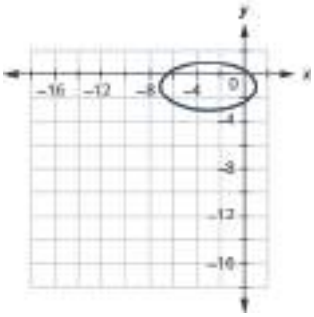
127.



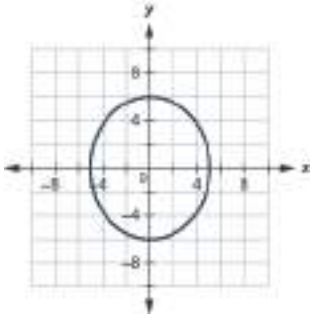
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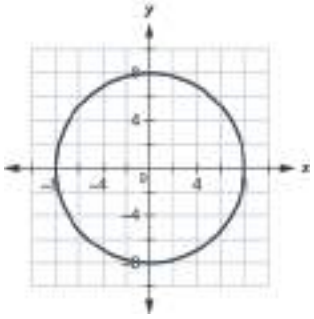
131.



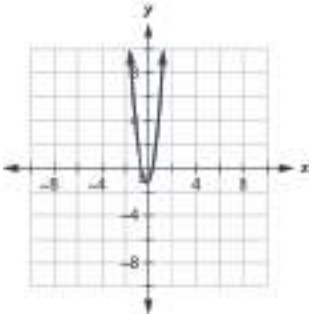
133.



135.



137.



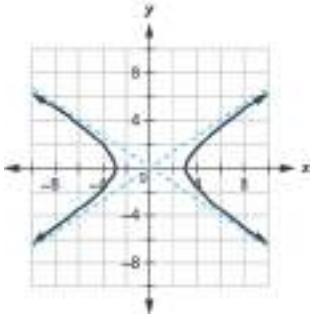
139. $\frac{x^2}{400}+\frac{y^2}{300}=1$

141. $\frac{x^2}{2500}+\frac{y^2}{1275}=1$

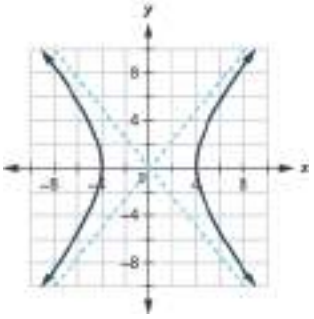
143. Answers will vary.

145. Answers will vary.

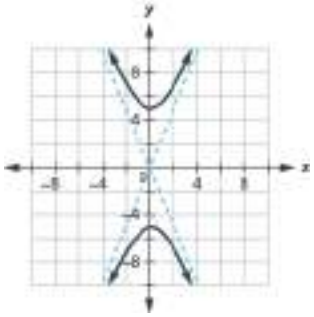
147.



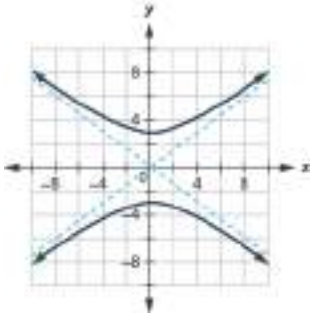
149.



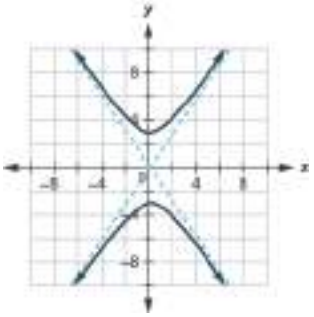
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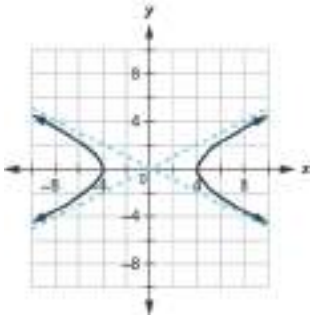
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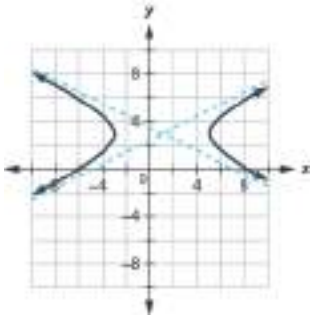
155.



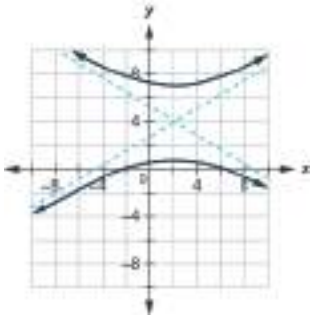
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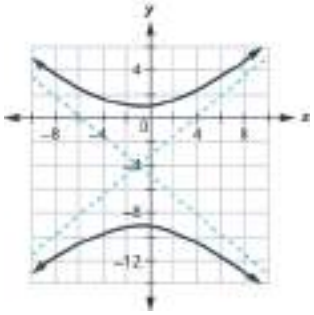
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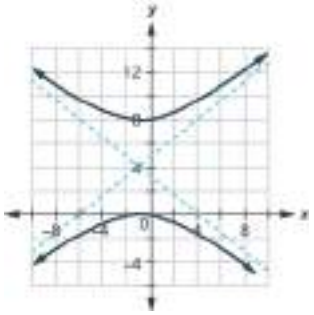
161.



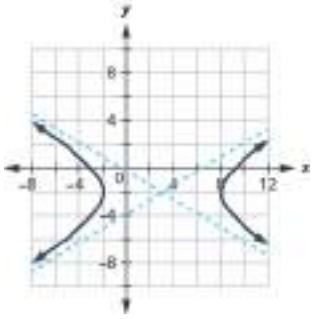
163.



165.

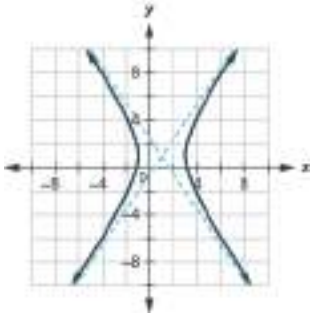


167.



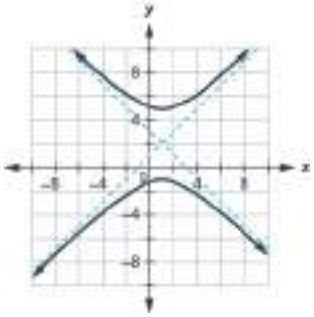
169. ⓐ $\frac{(x-1)^2}{4}-\frac{(y-1)^2}{9}=1$

ⓑ



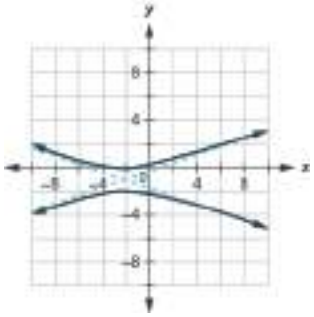
171. ⓐ $\frac{(y-2)^2}{9}-\frac{(x-1)^2}{9}=1$

ⓑ



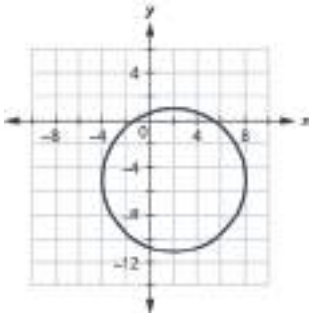
173. ⓐ $\frac{(y+1)^2}{1}-\frac{(x+2)^2}{9}=1$

ⓑ

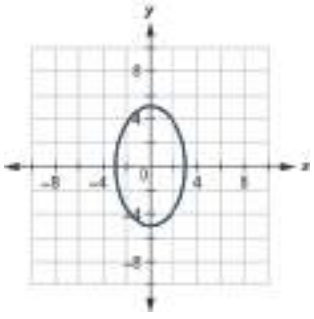


175. ⓐ parabola ⓑ circle ⓒ hyperbola ⓓ ellipse

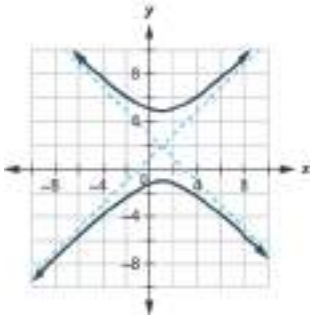
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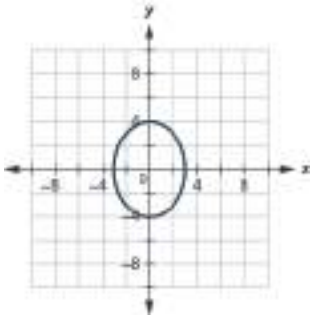
179.



181.



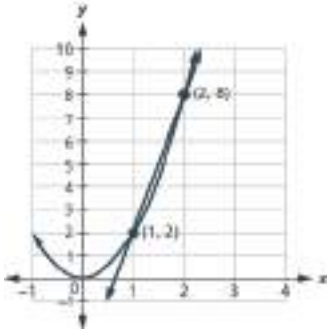
183.



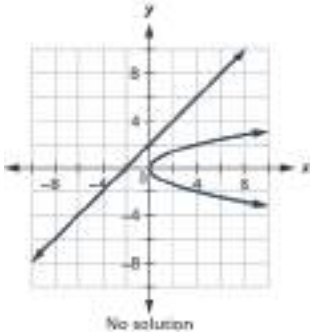
185. Answers will vary.

187. Answers will vary.

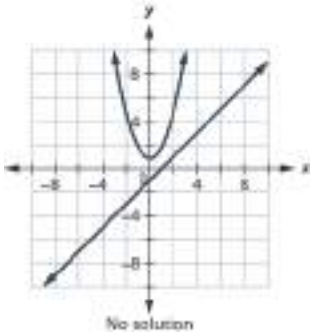
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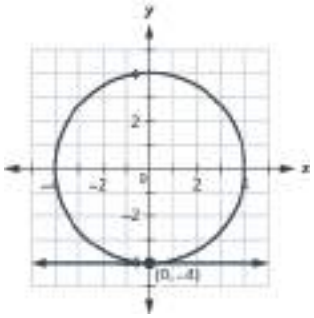
191.



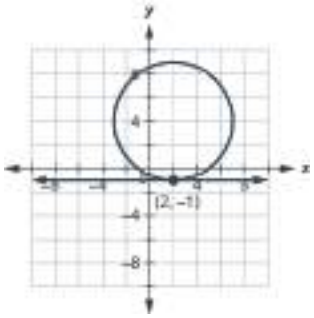
193.



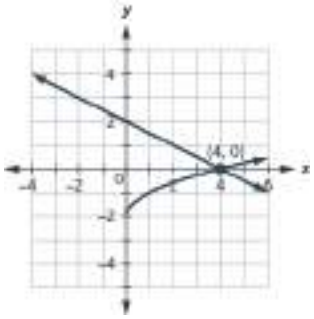
195.



197.



199.



201. $(-1, 0), (0, 3)$

203. $(2, 0)$

205. $(12, -5), (12, 5)$

207. No solution

209. $(0, -4), (1, -3)$

211. $(3, 4), (5, 0)$

213. $(0, -4), (-\sqrt{7}, 3), (\sqrt{7}, 3)$

215. $(0, -2), (-\sqrt{3}, 1), (\sqrt{3}, 1)$

217. $(-2, 0), (1, -\sqrt{3}), (1, \sqrt{3})$

219. $(-2, -4), (-2, 4), (2, -4), (2, 4)$

221. $(-4, 0), (4, 0)$

223. $(-\sqrt{3}, 0), (\sqrt{3}, 0)$

225. $(-2, -3), (-2, 3), (2, -3), (2, 3)$

227. $(-1, -3), (-1, 3), (1, -3), (1, 3)$

229. -3 and 14

231. -7 and -8 or 8 and 7

233. -6 and -4 or -6 and 4 or 6 and -4 or 6 and 4

235. If the length is 11 cm, the width is 15 cm. If the length is 15 cm, the width is 11 cm.

237. If the length is 10 inches, the width is 24 inches. If the length is 24 inches, the width is 10 inches.

239. The length is 40 inches and the width is 30 inches. The TV will not fit Donnette's entertainment center.

241. Answers will vary.

243. Answers will vary.

Review Exercises

245. $d = 3$

247. $d = \sqrt{17}, d \approx 4.1$

249. $(4, 4)$

251. $\left(\frac{3}{2}, -\frac{7}{2}\right)$

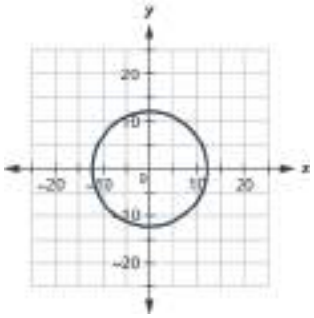
253. $x^2 + y^2 = 7$

255. $(x + 2)^2 + (y + 5)^2 = 49$

257. $(x - 2)^2 + (y - 2)^2 = 8$

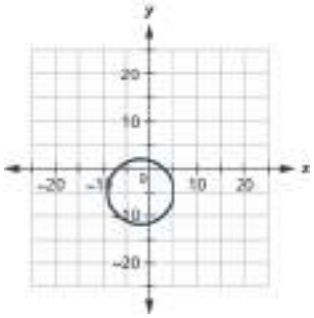
259. ⓐ radius: 12, center: $(0, 0)$

ⓑ



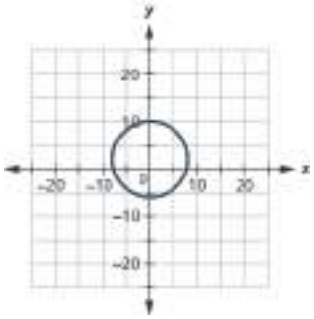
261. ⓐ radius: 7, center: $(-2, -5)$

ⓑ

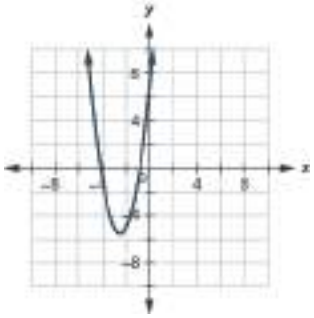


263. ⓐ radius: 8, center: $(0, 2)$

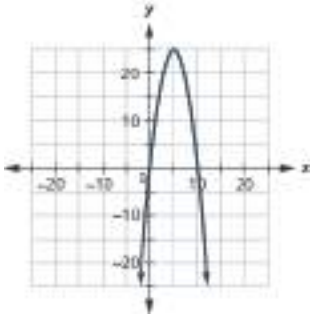
ⓑ



265.

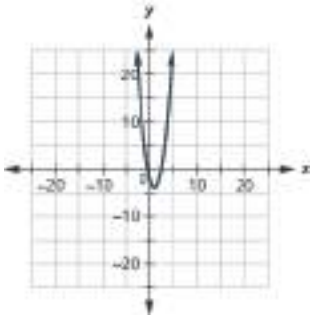


267.



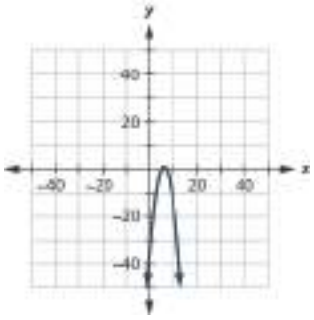
269. ⓐ $y = 2(x-1)^2 - 4$

ⓑ

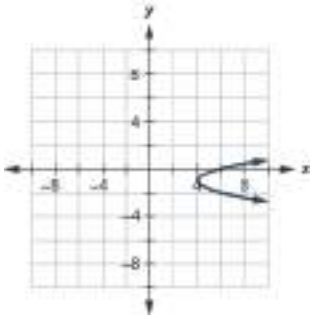


271. ⓐ $y = -(x-6)^2 + 1$

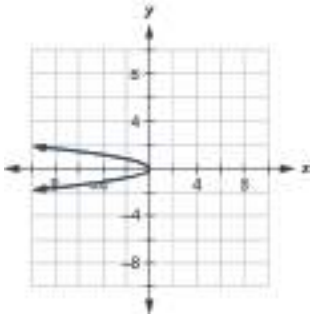
ⓑ



273.

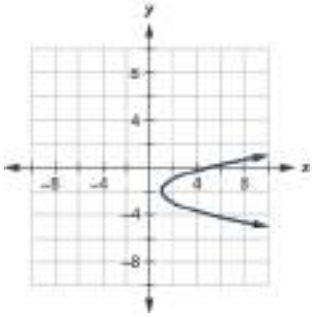


275.



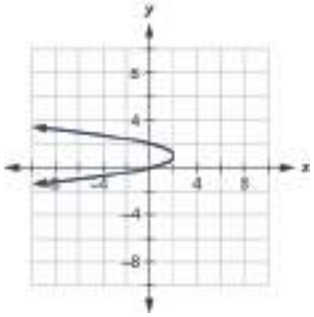
277. ⓐ $x = (y+2)^2 + 1$

ⓑ



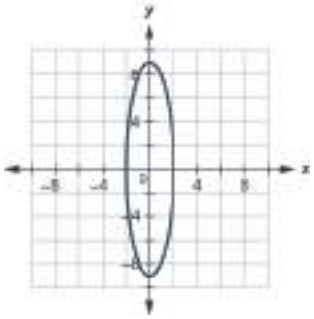
279. ⓐ $x = -2(y-1)^2 + 2$

ⓑ

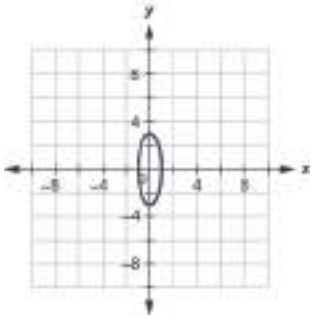


281. $y = -\frac{1}{9}x^2 + \frac{10}{3}x$

283.

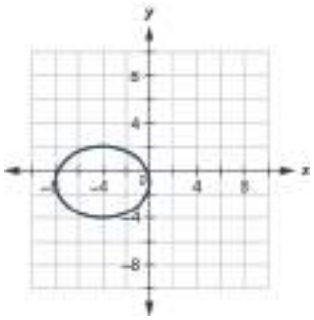


285.

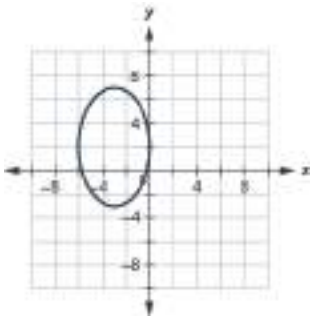


287. $\frac{x^2}{36} + \frac{y^2}{64} = 1$

289.

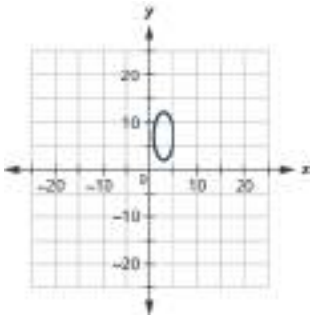


291.



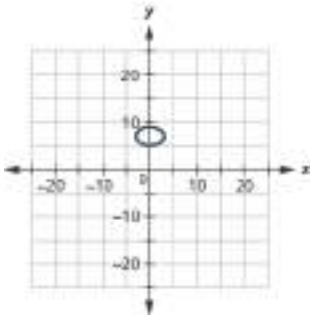
293. ⓐ $\frac{(x-3)^2}{4}+\frac{(y-7)^2}{25}=1$

ⓑ

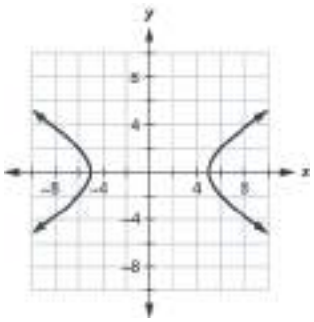


295. ⓐ $\frac{x^2}{9}+\frac{(y-7)^2}{4}=1$

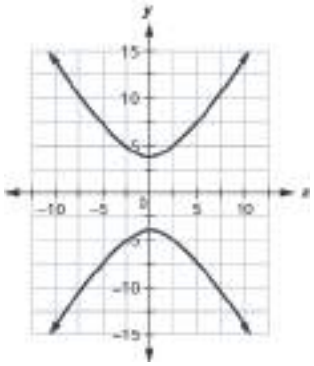
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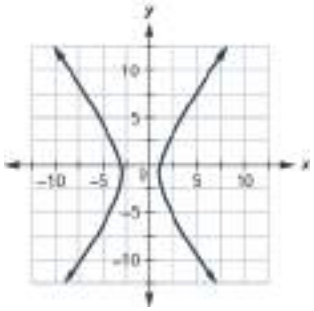
297.



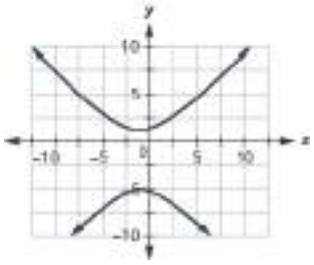
299.



301.

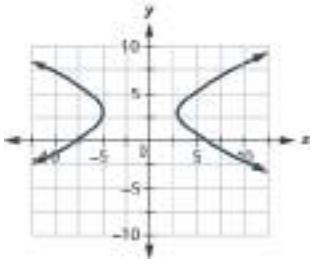


303.



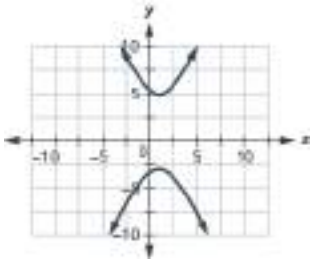
305. ⓐ $\frac{(x+1)^2}{16}-\frac{(y-3)^2}{4}=1$

ⓑ



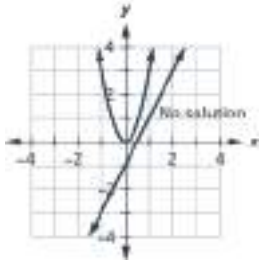
307. ⓐ $\frac{(y-1)^2}{16}-\frac{(x-1)^2}{4}=1$

ⓑ

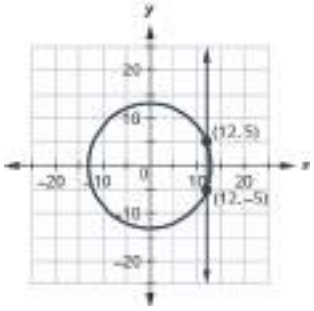


309. ⓐ hyperbola ⓑ circle ⓒ parabola ⓓ ellipse

311.



313.



315. $(-1, 4)$

317. No solution

319. $(-\sqrt{7}, 3), (\sqrt{7}, 3)$

321. $(-3, 0), (0, -2), (0, 2)$

323. -3 and -4 or 4 and 3

325. If the length is 14 inches, the width is 15 inches. If the length is 15 inches, the width is 14 inches.

Practice Test

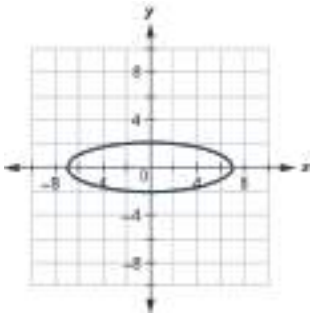
327. distance: 10, midpoint: $(-7, -7)$

329. $x^2 + y^2 = 121$

331. $(x+2)^2 + (y-3)^2 = 52$

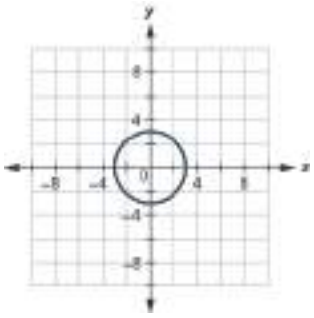
333. ⓐ ellipse

ⓑ



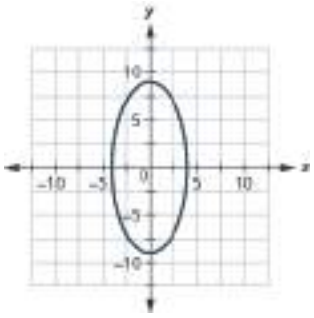
335. ⓐ circle

ⓑ



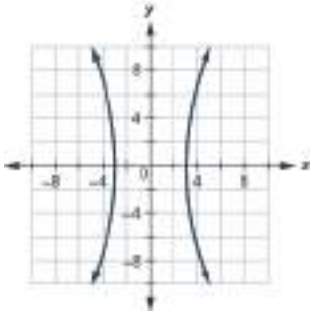
337. ⓐ ellipse

ⓑ



339. ⓐ hyperbola

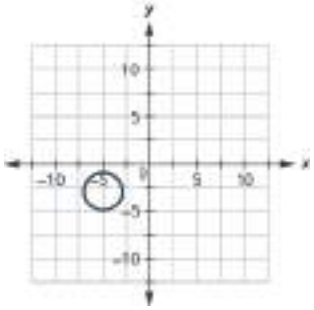
ⓑ



341. ⓐ circle

ⓑ $(x+5)^2 + (y+3)^2 = 4$

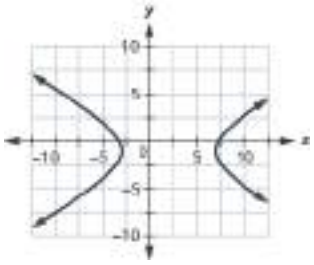
ⓒ



343. ⓐ hyperbola

ⓑ $\frac{(x-2)^2}{25} - \frac{(y+1)^2}{9} = 1$

ⓒ



345. No solution

347. $(0, -3), (0, 3)$

349. $\frac{x^2}{2025} + \frac{y^2}{1400} = 1$

351. The length is 44 inches and the width is 33 inches. The TV will fit Olive's entertainment center.

Chapter 12

Try It

12.1. $-1, 2, 5, 8, 11$

12.2. $-3, -1, 1, 3, 5$

12.3. $7, 13, 31, 85, 247$

12.4. $-3, -1, 3, 11, 27$

12.5. $-1, 4, -9, 16, -25$

12.6. $1, -8, 27, -64, 125$

12.7. $a_n = 3n$

12.8. $a_n = 5n$

12.9. $a_n = (-1)^n 3^n$

12.10. $a_n = (-1)^{n+1} n^2$

12.11. $a_n = \frac{1}{2^n}$

12.12. $a_n = \frac{1}{n^2}$

12.13. $2, 1, \frac{1}{3}, \frac{1}{12}, \frac{1}{60}$

12.14. $3, \frac{3}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{40}$

12.15. $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}$

12.16. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$

12.17. 45

12.18. 60

12.19. $\frac{16}{3}$

12.20. 8

12.21. $\sum_{n=1}^{5} \frac{1}{2^n}$

12.22. $\sum_{n=1}^{5} \frac{1}{n^2}$

12.23. $\sum_{n=1}^{5} (-1)^{n+1} n^2$

12.24. $\sum_{n=1}^{5} (-1)^n 2n$

12.25. ⓐ The sequence is arithmetic with common difference $d = 11$. ⓑ The sequence is arithmetic with common difference $d = -6$. ⓒ The sequence is not arithmetic as all the differences between the consecutive terms are not the same.

12.26. ⓐ The sequence is not arithmetic as all the differences between the consecutive terms are not the same. ⓑ The sequence is arithmetic with common difference $d = 2$. ⓒ The sequence is arithmetic with common difference $d = -5$.

12.27. $7, 3, -1, -5, -9, \ldots$

12.28. $11, 3, -5, -13, -21, \ldots$

12.29. 241

12.30. -106

12.31. $a_{11} = 2$. The general term is $a_n = -3n + 35$.

12.32. $a_{19} = -55$. The general term is $a_n = -4n + 21$.

12.33. $a_1 = 5, \quad d = 4$. The general term is $a_n = 4n + 1$.

12.34. $a_1 = 8, \quad d = -3$. The general term is $a_n = -3n + 11$.

12.35. 1,890

12.36. 1,515

12.37. 2,300

12.38. 5,250

12.39. 2,670

12.40. 3,045

12.41. ⓐ The sequence is geometric with common ratio $r = 3$. ⓑ The sequence is geometric with common ratio $d = \frac{1}{4}$. ⓒ The sequence is not geometric. There is no common ratio.

12.42. ⓐ The sequence is not geometric. There is no common ratio. ⓑ The sequence is geometric with common ratio $r = 2$. ⓒ The sequence is geometric with common ratio $r = \frac{1}{2}$.

12.43. $7, -21, 63, -189, 567$

12.44. $6, -24, 96, -384, 1536$

12.45. $\frac{1}{6{,}561}$

12.46. $\frac{1}{16{,}384}$

12.47. $a_9 = 39{,}366$. The general term is $a_n = 6(3)^{n-1}$.

12.48. $a_{11} = 7{,}168$. The general term is $a_n = 7(2)^{n-1}$.

12.49. 3,145,725

12.50. 10,460,353,200

12.51. 393,204

12.52. 10,230

12.53. 96

12.54. $\frac{256}{3}$

12.55. $\frac{4}{9}$

12.56. $\frac{8}{9}$

12.57. \$10,000

12.58. \$3, 333.33

12.59. \$88,868.36

12.60. \$698,201.57

12.61. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

12.62. $p^7 + 7p^6q + 21p^5q^2 + 35p^4q^3 + 35p^3q^4 + 21p^2q^5 + 7pq^6 + q^7$

12.63. $x^4 + 8x^3 + 24x^2 + 32x + 16$

12.64. $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

12.65. $16x^4 - 96x^3 + 216x^2 - 216x + 81$

12.66. $64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$

12.67. ⓐ 6 ⓑ 1 ⓒ 1 ⓓ 35

12.68. ⓐ 2 ⓑ 1 ⓒ 1 ⓓ 6

12.69. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

12.70. $m^6 + 6m^5n + 15m^4n^2 + 20m^3n^3 + 15m^2n^4 + 6mn^5 + n^6$

12.71. $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

12.72. $y^6 - 6y^5 + 15y^4 - 20y^3 + 15y^2 - 6y + 1$

12.73. $243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$

12.74. $256x^4 - 768x^3y + 864x^2y^2 - 432xy^3 + 81y^4$

12.75. $15x^4y^2$

12.76. $8ab^7$

12.77. 7,168

12.78. 280

Section Exercises

1. $-5, -3, -1, 1, 3$

3. $4, 7, 10, 13, 16$

5. $5, 7, 11, 19, 35$

7. $1, 5, 21, 73, 233$

9. $2, 1, \frac{8}{9}, 1, \frac{32}{25}$

11. $1, \frac{3}{2}, \frac{5}{4}, \frac{7}{8}, \frac{9}{16}$

13. $-2, 4, -6, 8, -10$

15. $1, -4, 9, -16, 25$

17. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}$

19. $a_n = 8n$

21. $a_n = n + 5$

23. $a_n = e^{n+2}$

25. $a_n = (-1)^n 5n$

27. $a_n = (-1)^n n^3$

29. $a_n = (-1)^n 2n$

31. $a_n = \frac{1}{4^n}$

33. $a_n = -\frac{n}{n+1}$

35. $-\frac{5}{2^n}$

37. $4, 2, \frac{2}{3}, \frac{1}{6}, \frac{1}{30}$

39. $3, 6, 18, 72, 360$

41. $2, 24, 720, 40320, 3628800$

43. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

45. $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \frac{24}{5}$

47. $2, \frac{3}{2}, \frac{8}{3}, \frac{15}{2}, \frac{144}{5}$

49. $1 + 4 + 9 + 16 + 25 = 55$

51. $5 + 7 + 9 + 11 + 13 + 15 = 60$

53. $2 + 4 + 8 + 16 = 30$

55. $4 + 4 + 3 + \frac{2}{3} = 11\frac{2}{3}$

57. $2 + 6 + 12 + 20 + 30 = 70$

59. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} = \frac{71}{20}$

61. $\sum_{n=1}^{5} \frac{1}{3^n}$

63. $\sum_{n=1}^{5} \frac{1}{n^3}$

65. $\sum_{n=1}^{5} \frac{2}{n}$

67. $\sum_{n=1}^{5} (-1)^{n+1} 3n$

69. $\sum_{n=1}^{10} (-1)^n 2n$

71. $\sum_{n=1}^{7} (2n + 12)$

73. Answers will vary.

75. Answers will vary.

77. The sequence is arithmetic with common difference $d = 8$.

79. The sequence is not arithmetic.

81. The sequence is arithmetic with common difference $d = -3$.

83. $11, 18, 25, 32, 39$

85. $-7, -3, 1, 5, 9$

87. $14, 5, -4, -13, -22$

89. 163

91. 131

93. -79

95. $a_{20} = -34$. The general term is $a_n = -2n + 6$.

97. $a_{11} = 59$. The general term is $a_n = 5n + 4$.

99. $a_8 = -13$. The general term is $a_n = -5n + 27$.

101. $a_1 = 11, \quad d = 3$. The general term is $a_n = 3n + 8$.

103. $a_1 = 21, \quad d = -8$. The general term is $a_n = -8n + 29$.

105. $a_1 = -15, \quad d = 3$. The general term is $a_n = 3n - 18$.

107. 1,635

109. $-1,065$

111. 360

113. 6,325

115. −3,575

117. 6,280

119. 4,125

121. $-3,850$

123. Answers will vary.

125. Answers will vary.

127. The sequence is geometric with common ratio $r = 4$.

129. The sequence is geometric with common ratio $r = \frac{1}{2}$.

131. The sequence is not geometric. There is no common ratio.

133. The sequence is geometric with common ratio $r = \frac{1}{2}$.

135. The sequence is arithmetic with common difference $d = 5$.

137. The sequence is geometric with common ratio $r = \frac{1}{2}$.

139. $4,\ 12,\ 36,\ 108,\ 324$

141. $-4,\ 8,\ -16,\ 32,\ -64$

143. $27,\ 9,\ 3,\ 1,\ \frac{1}{3}$

145. 472,392

147. 3,072

149. 0.0001

151. $a_9 = 2{,}304$. The general term is $a_n = 9(2)^{n-1}$.

153. $a_{15} = -\frac{2}{19,\ 683}$. The general term is $a_n = -486\left(-\frac{1}{3}\right)^{n-1}$.

155. $a_{10} = 0.000000001$. The general term is $a_n = (0.1)^{n-1}$.

157. 57,395,624

159. $-65{,}538$

161. $\frac{7{,}174{,}453}{59{,}049} \approx 121.5$

163. 65,534

165. 4088

167. $\frac{29{,}524}{6561} \approx 4.5$

169. $\frac{3}{2}$

171. $\frac{9}{2}$

173. no sum as $r \geq 1$

175. 2,048

177. $\frac{1}{3}$

179. $\frac{7}{9}$

181. $\frac{5}{11}$

183. ⓐ $6666.67 ⓑ $4000 ⓒ $15,000 ⓒ $7500

185. $295,581.88

187. $14,234.10

189. Answers will vary.

191. Answers will vary.

193. $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$

195. $p^9 + 9p^8q + 36p^7q^2 + 84p^6q^3 + 126p^5q^4 + 126p^4q^5 + 84p^3q^6 + 36p^2q^7 + 9pq^8 + q^9$

197. $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

199. $x^3 + 15x^2 + 75x + 125$

201. $y^7 + 7y^6 + 21y^5 + 35y^4 + 35y^3 + 21y^2 + 7y + 1$

203. $z^6 - 12z^5 + 60z^4 - 160z^3 + 240z^2 - 192z + 64$

205. $243x^5 - 405x^4 + 270x^3 - 90x^2 + 15x - 1$

207. $27x^3 - 135x^2 + 225x - 125$

209. $27x^3 + 135x^2y + 225xy^2 + 125y^3$

211. ⓐ 7 ⓑ 1 ⓒ 1 ⓓ 45

213. ⓐ 4 ⓑ 1 ⓒ 1 ⓓ 55

215. $m^5 + 5m^4n + 10m^3n^2 + 10m^2n^3 + 5mn^4 + n^5$

217. $s^7 + 7s^6t + 21s^5t^2 + 35s^4t^3 + 35s^3t^4 + 21s^2t^5 + 7st^6 + t^7$

219. $y^4 - 12y^3 + 54y^2 - 108y + 81$

221. $q^3 - 12q^2 + 48q - 64$

223. $625x^4 - 1000x^3y + 600x^2y^2 - 160xy^3 + 16y^4$

225. $243x^5 + 1620x^4y + 4320x^3y^2 + 5760x^2y^3 + 3840xy^4 + 1024y^5$

227. $126a^5b^4$

229. $462x^5y^6$

231. 112

233. 324

235. 30,618

237. Answers will vary.

239. Answers will vary.

Review Exercises

241. $7,\ 13,\ 31,\ 85,\ 247$

243. $\frac{3}{4},\ \frac{5}{16},\ \frac{7}{64},\ \frac{9}{256},\ \frac{11}{1024}$

245. $a_n = 9n$

247. $a_n = e^{n-4}$

249. $a_n = -\frac{n}{n+2}$

251. $\frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}$

253. $-3+(-1)+1+3+5+7+9=21$

255. $4+4+2+\frac{2}{3}+\frac{1}{6}=\frac{65}{6}$

257. $\sum_{n=1}^{5}(-1)^n\frac{1}{3^n}$

259. $\sum_{n=1}^{5}\frac{4}{n}$

261. The sequence is arithmetic with common difference $d=6$.

263. 5, 8, 11, 14, 17

265. $-13, -7, -1, 5, 11$

267. -129

269. $a_{18}=103$. The general term is $a_n = 7n-23$.

271. $a_1 = 1, \quad d = 4$. The general term is $a_n = 4n-3$.

273. -430

275. 585

277. 4850

279. 980

281. The sequence is not geometric.

283. The sequence is geometric with common ratio $r=-2$.

285. $128, 32, 8, 2, \frac{1}{2}$

287. 1,536

289. $a_{12} = -25, 165, 824$. The general term is $a_n = 6(-4)^{n-1}$.

291. 5,460

293. ≈ 3906.25

295. $\frac{189}{8} = 23.625$

297. $\frac{343}{6} \approx 57.167$

299. $\frac{4}{11}$

301. $1,634,421.27

303. $x^4-4x^3y+6x^2y^2-4xy^3+y^4$

305. $32y^5-240y^4+720y^3-1080y^2+810y-243$

307. ⓐ 11 ⓑ 1 ⓒ 1 ⓓ 56

309. ⓐ 1 ⓑ 1 ⓒ 1 ⓓ 55

311. $t^9-9t^8+36t^7-84t^6+126t^5-126t^4+84t^3-36t^2+9t-1$

313. $256x^4+768x^3y+864x^2y^2+432xy^3+81y^4$

315. $84a^6b^3$

317. 135

319. 280

Practice Test

321. $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$

323. $-4+16-64+256=204$

325. $-13, -10, -7, -4, -1$

327. $a_{23}=59$. The general term is $a_n = 3n-10$.

329. 1,325

331. 3,260

333. The sequence is geometric with common ratio $r=\frac{1}{3}$.

335. 5,242,880

337. 797,162

339. $\frac{5}{6}$

341. $1,409,344.19

343. ⓐ 8 ⓑ 1 ⓒ 1 ⓓ 210

INDEX

A

absolute value, 24, 83, 198, 750
absolute value equation, 199
absolute value inequality, 202
additive identity, 73, 83
additive inverse, 74, 83
amount of discount, 124
annuity, 1195, 1215
area, 134
arithmetic sequence, 1176, 1215
Associative Property, 72
asymptote, 1008, 1057
augmented matrix, 437
axis of symmetry, 924, 959

B

base, 12
binomial, 554
binomial coefficient, 1206
Binomial Theorem, 1208, 1210
boundary line, 297, 347
break-even point, 415, 475

C

Change-of-Base Formula, 1043
circle, 1070, 1074, 1077, 1079, 1150
coefficient, 16, 83, 101
coincident, 374
coincident lines, 475
common difference, 1176, 1215
common logarithmic function, 1029, 1057
common ratio, 1186, 1215
Commutative Property, 72
complementary angles, 393, 475
completing the square, 872, 875, 880
complex conjugate pair, 839, 847
complex fraction, 44, 83
complex number, 835, 847
complex number system, 835, 847
complex rational expression, 670, 732
composite number, 83
composition, 990, 1037
compound inequality, 187, 191, 193, 210
compound interest, 1015
conditional equation, 102, 210
conjugate pair, 532, 554
consecutive even integers, 118
consecutive integers, 117
consecutive odd integers, 118
consistent and inconsistent systems, 475
consistent system of equations, 374
constant, 10, 83
contradiction, 103, 210
cost function, 415, 475
critical point, 721
critical point of a rational inequality, 732
critical points, 972

D

degree of a constant, 488, 554
degree of a polynomial, 488, 554
degree of a term, 554
degree of the polynomial equation, 615, 630
degree one, 615
degree two, 615
denominator, 41, 83
determinant, 446, 475
difference of squares, 839
discount rate, 124
discriminant, 895, 930, 978
Distributive Property, 78, 793, 837
divisible, 6
divisible by a number, 83
domain, 314, 828, 829
domain of a relation, 347
double inequality, 190

E

earthquake intensity, 1032
ellipse, 1070, 1105, 1107, 1109, 1150
equal, 10
equal sign, 101
equation, 12, 83
equivalent fractions, 41, 83
evaluate an expression, 15, 83
exponent, 12
exponential decay, 1016, 1051
exponential function, 1006, 1057
exponential growth, 1016
expression, 11, 83
extraneous solution, 816
extraneous solution to a rational equation, 682, 732

F

factorial notation, 1167, 1206
factoring, 565, 630
factoring by substitution, 586
factors, 7, 83
finite sequence, 1164, 1215
FOIL, 574, 576, 837
fraction, 41, 83
function, 317, 347, 992

G

GCF, 570, 579, 595
General Strategy for Solving Linear Equations, 100, 106
general term, 1164
general term of a sequence, 1215
geometric sequence, 1186, 1215
gravity, 820
greatest common factor, 566, 630

H

half-life, 1052
horizontal line, 236, 347
horizontal line test, 993
hyperbola, 1070, 1122, 1150

I

identity, 103, 210
Identity Property of Addition, 73
Identity Property of Multiplication, 73
imaginary unit, 834, 847
inconsistent system of equations, 374
infinite geometric series, 1191, 1215
infinite sequence, 1164, 1215
integers, 26, 83
intercepts of a line, 240, 347
interest, 125
interval notation, 191
inverse function, 999
Inverse Property of Addition, 74
Inverse Property of Multiplication, 74
irrational number, 65, 83

L

leading coefficient, 881

least common denominator, 45, 83, 105, 109
least common multiple, 9, 83
like radicals, 789, 847
like terms, 16, 83
linear equation, 99, 210, 229, 347, 420
linear inequality, 295, 347
list price, 124
logarithmic function, 1057

M
mapping, 315, 347
mark-up, 124
matrix, 433, 475
maximum, 940
minimum, 940
minor of an entry, 447
minor of an entry in a 3×3 determinant, 475
monomial, 488, 554
multiple, 6
multiple of a number, 83
multiplicative identity, 73, 83
multiplicative inverse , 74, 83

N
natural base, 1013, 1057
natural exponential function, 1013, 1057
natural logarithmic function, 1029, 1057
negative numbers, 24, 83
number line, 11
numerator, 41, 83

O
one-to-one, 992
one-to-one function, 1057
One-to-One Property of Logarithmic Equations, 1047
opposite, 24, 83
order of operations, 13, 83, 762
ordered pair, 226, 347
ordered triple, 423
origin, 226, 347
original cost, 124

P
parabola, 921, 1070, 1084, 1088, 1090, 1150
parallel lines, 268, 347
partial sum, 1169, 1215
Pascal's Triangle, 1203, 1209
percent, 62, 83
percent change, 123
perfect square, 594, 747
perimeter, 138, 139
perpendicular lines, 271, 347
point-slope form, 281, 347
polynomial, 488, 554, 567
polynomial equation, 615, 630
polynomial function, 492, 554
Power Property, 509, 554, 782
Power Property for Exponents, 774
Power Property of Logarithms, 1039
prime, 570, 578, 592, 605
prime factorization, 8, 83
prime number, 7, 83
principal, 125
principal square root, 64, 83, 744
product, 565
Product of Binomial Squares Pattern, 795, 838
Product of Conjugates Pattern, 807, 839
Product Property, 501, 508, 554, 783
Product Property of Logarithms, 1037
Product Property of Roots, 757, 791
Product to a Power, 510, 554
Product to a Power Property, 782
Properties of Exponents, 518
Properties of Negative Exponents, 506, 554
proportion, 697, 732
Pythagorean Theorem, 136, 912

Q
quadratic equation, 615, 616, 618, 630, 860, 875, 882, 888, 968
quadratic form, 900
Quadratic Formula, 888, 890, 894, 911, 939
quadratic function, 921, 978
quadratic inequality, 968, 978
Quotient Property, 503, 504, 554, 764, 769, 783
Quotient Property of Logarithms, 1038
Quotient Property of Radical Expressions, 800
Quotient to a Negative Exponent, 554
Quotient to a Negative Power, 507
Quotient to a Power Property, 512, 554, 765

R
radical, 890
radical equation, 812, 847
radical expression, 757, 803
radical function, 826, 847
radicand, 891
range, 828
range of a relation, 347
rate of interest, 125
rational equation, 682, 732
rational expression, 640, 648, 732
rational function, 648, 732
rational inequality, 721, 732
rational number, 65, 83
rationalize the denominator, 807, 864
rationalizing the denominator, 802, 847
real number, 66, 83
reciprocal, 43, 84
relation, 314, 347
revenue, 415, 475
right triangle, 135
row-echelon form, 437, 475

S
scientific notation, 515
sequence, 1164, 1215
similar figures, 699, 732
simple interest, 125
simplified rational expression, 641, 732
simplify an expression, 13, 84
solution, 98, 229
solution of a linear equation in two variables, 347
solution of an equation, 210
solution to a linear inequality, 295, 347
solutions of a system of equations, 368, 475
solutions of a system of linear equations with three variables, 475
solutions of the system of three linear equations with three variables, 423
speed, 822
square, 744
square matrix, 446, 475
square of a number, 64, 84,

847
square root, 744
square root of a number, 64, 84, 847
Square Root Property, 860, 861, 864, 872, 875
standard form, 617, 618, 847
standard form of a linear equation, 229, 347
standard form of a polynomial, 489, 554
summation notation, 1169
supplementary angles, 393, 475
surface area, 133
system of linear equations, 368, 475
system of linear inequalities, 460, 475
system of nonlinear equations, 1136, 1150
systems of equations, 420

T
temperature, 133
term, 16, 84
trinomial, 488, 554, 574, 592, 686, 894

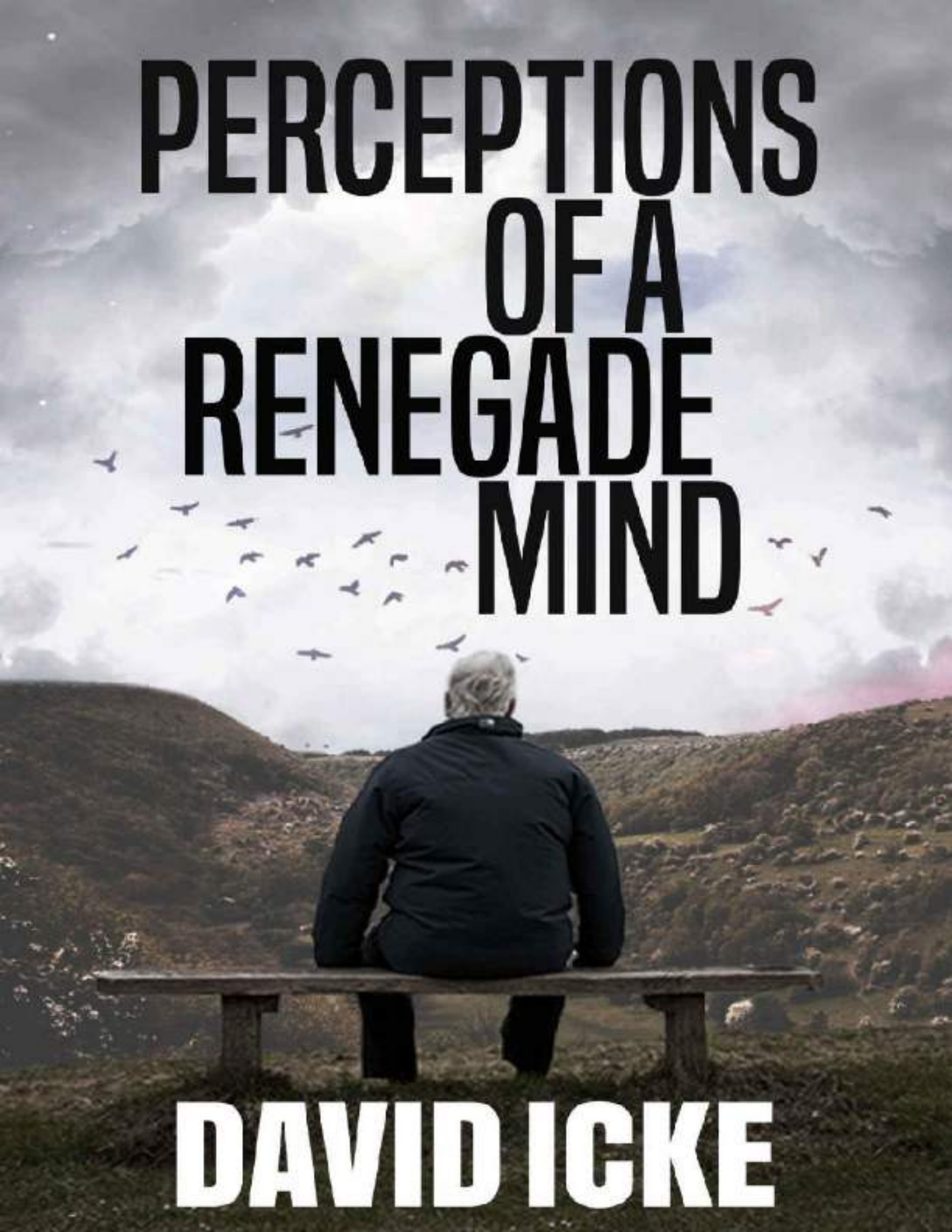
U
uniform motion, 155
union symbol, 191

V
variable, 10, 84, 101
vertex, 925, 940
vertical line, 236, 347
vertical line test, 993
volume, 132

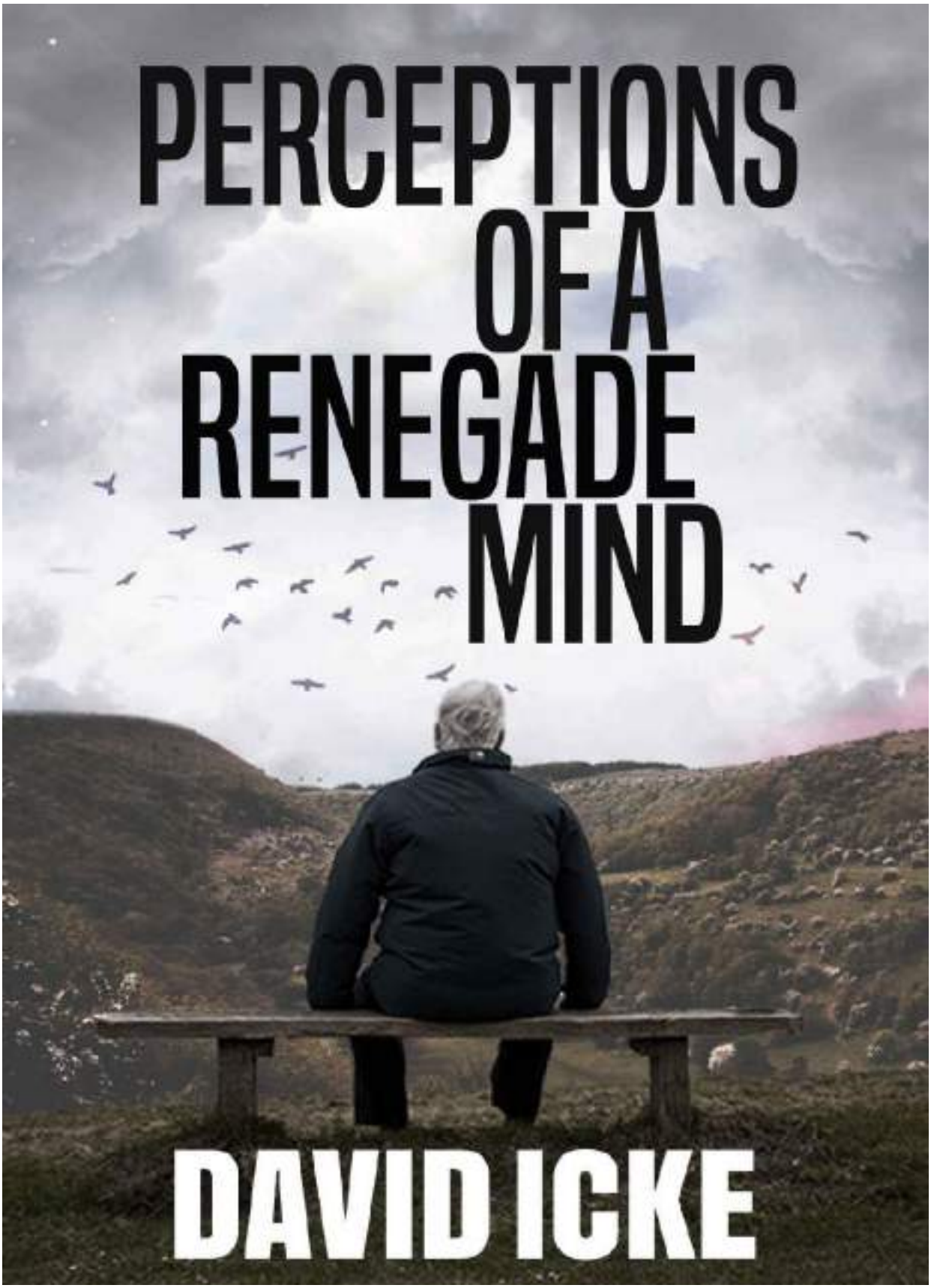
X
x-intercept, 620

Z
Zero Exponent Property, 504, 554
zero of the function, 620, 630
Zero Product Property, 615, 618, 619, 619, 620, 630, 880, 894

PERCEPTIONS OF A RENEGADE MIND



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DAVID ICKE

Dedication:

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Renegade:
Adjective
‘Having rejected tradition: Unconventional.’
Merriam-Webster Dictionary

Acquiescence to tyranny is the death of the spirit

You may be 38 years old, as I happen to be. And one day, some great opportunity stands before you and calls you to stand up for some great principle, some great issue, some great cause. And you refuse to do it because you are afraid … You refuse to do it because you want to live longer … You're afraid that you will lose your job, or you are afraid that you will be criticised or that you will lose your popularity, or you're afraid that somebody will stab you, or shoot at you or bomb your house; so you refuse to take the stand.

Well, you may go on and live until you are 90, but you're just as dead at 38 as you would be at 90. And the cessation of breathing in your life is but the belated announcement of an earlier death of the spirit.

Martin Luther King

How the few control the many and always have – the many do whatever they're told

'Forward, the Light Brigade!'
Was there a man dismayed?
Not though the soldier knew
Someone had blundered.
Theirs not to make reply,
Theirs not to reason why,
Theirs but to do and die.
Into the valley of Death
Rode the six hundred.

Cannon to right of them,
Cannon to left of them,
Cannon in front of them
Volleyed and thundered;
Stormed at with shot and shell,
Boldly they rode and well,
Into the jaws of Death,
Into the mouth of hell
Rode the six hundred

Alfred Lord Tennyson (1809-1892)

The mist is lifting slowly
I can see the way ahead
And I've left behind the empty streets
That once inspired my life
And the strength of the emotion
Is like thunder in the air
'Cos the promise that we made each other
Haunts me to the end

The secret of your beauty
And the mystery of your soul
I've been searching for in everyone I meet
And the times I've been mistaken
It's impossible to say
And the grass is growing
Underneath our feet

The words that I remember
From my childhood still are true
That there's none so blind
As those who will not see
And to those who lack the courage
And say it's dangerous to try
Well they just don't know
That love eternal will not be denied

I know you're out there somewhere
Somewhere, somewhere
I know you're out there somewhere

Somewhere you can hear my voice
I know I'll find you somehow
Somehow, somehow
I know I'll find you somehow
And somehow I'll return again to you

The Moody Blues

Are you a gutless wonder - or a Renegade Mind?

Monuments put from pen to paper,
Turns me into a gutless wonder,
And if you tolerate this,
Then your children will be next.
Gravity keeps my head down,
Or is it maybe shame ...

Manic Street Preachers

Rise like lions after slumber
In unvanquishable number.
Shake your chains to earth like dew
Which in sleep have fallen on you.
Ye are many – they are few.

Percy Shelley

Contents

CHAPTER 1 'I'm thinking' – Oh, but *are* you?
CHAPTER 2 Renegade perception
CHAPTER 3 The Pushbacker sting
CHAPTER 4 'Covid': The calculated catastrophe
CHAPTER 5 There *is no* 'virus'
CHAPTER 6 Sequence of deceit
CHAPTER 7 War on your mind
CHAPTER 8 'Reframing' insanity
CHAPTER 9 We must have it? So what is it?
CHAPTER 10 Human 2.0
CHAPTER 11 Who controls the Cult?
CHAPTER 12 Escaping Wetiko

POSTSCRIPT
APPENDIX Cowan-Kaufman-Morell Statement on Virus Isolation
BIBLIOGRAPHY
INDEX

CHAPTER ONE

I'm thinking' – Oh, but *are* you?

Think for yourself and let others enjoy the privilege of doing so too
Voltaire

French-born philosopher, mathematician and scientist René Descartes became famous for his statement in Latin in the 17th century which translates into English as: 'I think, therefore I am.'

On the face of it that is true. Thought reflects perception and perception leads to both behaviour and self-identity. In that sense 'we' are what we think. But who or what is doing the thinking and is thinking the only route to perception? Clearly, as we shall see, 'we' are not always the source of 'our' perception, indeed with regard to humanity as a whole this is rarely the case; and thinking is far from the only means of perception. Thought is the village idiot compared with other expressions of consciousness that we all have the potential to access and tap into. This has to be true when we *are* those other expressions of consciousness which are infinite in nature. We have forgotten this, or, more to the point, been manipulated to forget.

These are not just the esoteric musings of the navel. The whole foundation of human control and oppression is control of perception. Once perception is hijacked then so is behaviour which is dictated by perception. Collective perception becomes collective behaviour and collective behaviour is what we call human society. Perception is all and those behind human control know that which is

why perception is the target 24/7 of the psychopathic manipulators that I call the Global Cult. They know that if they dictate perception they will dictate behaviour and collectively dictate the nature of human society. They are further aware that perception is formed from information received and if they control the circulation of information they will to a vast extent direct human behaviour. Censorship of information and opinion has become globally Nazi-like in recent years and never more blatantly than since the illusory 'virus pandemic' was triggered out of China in 2019 and across the world in 2020. Why have billions submitted to house arrest and accepted fascistic societies in a way they would have never believed possible? Those controlling the information spewing from government, mainstream media and Silicon Valley (all controlled by the same Global Cult networks) told them they were in danger from a 'deadly virus' and only by submitting to house arrest and conceding their most basic of freedoms could they and their families be protected. This monumental and provable lie became the *perception* of the billions and therefore the *behaviour* of the billions. In those few words you have the whole structure and modus operandi of human control. Fear is a perception – **F**alse **E**motion **A**ppearing **R**eal – and fear is the currency of control. In short … get them by the balls (or give them the impression that you have) and their hearts and minds will follow. Nothing grips the dangly bits and freezes the rear-end more comprehensively than fear.

World number 1

There are two 'worlds' in what appears to be one 'world' and the prime difference between them is knowledge. First we have the mass of human society in which the population is maintained in coldly-calculated ignorance through control of information and the 'education' (indoctrination) system. That's all you really need to control to enslave billions in a perceptual delusion in which what are perceived to be *their* thoughts and opinions are ever-repeated mantras that the system has been downloading all their lives through 'education', media, science, medicine, politics and academia

in which the personnel and advocates are themselves overwhelmingly the perceptual products of the same repetition. Teachers and academics in general are processed by the same programming machine as everyone else, but unlike the great majority they never leave the 'education' program. It gripped them as students and continues to grip them as programmers of subsequent generations of students. The programmed become the programmers – the programmed programmers. The same can largely be said for scientists, doctors and politicians and not least because as the American writer Upton Sinclair said: 'It is difficult to get a man to understand something when his salary depends upon his not understanding it.' If your career and income depend on thinking the way the system demands then you will – bar a few free-minded exceptions – concede your mind to the Perceptual Mainframe that I call the Postage Stamp Consensus. This is a tiny band of perceived knowledge and possibility 'taught' (downloaded) in the schools and universities, pounded out by the mainstream media and on which all government policy is founded. Try thinking, and especially speaking and acting, outside of the 'box' of consensus and see what that does for your career in the Mainstream Everything which bullies, harasses, intimidates and ridicules the population into compliance. Here we have the simple structure which enslaves most of humanity in a perceptual prison cell for an entire lifetime and I'll go deeper into this process shortly. Most of what humanity is taught as fact is nothing more than programmed belief. American science fiction author Frank Herbert was right when he said: 'Belief can be manipulated. Only knowledge is dangerous.' In the 'Covid' age belief is promoted and knowledge is censored. It was always so, but never to the extreme of today.

World number 2

A 'number 2' is slang for 'doing a poo' and how appropriate that is when this other 'world' is doing just that on humanity every minute of every day. World number 2 is a global network of secret societies and semi-secret groups dictating the direction of society via

governments, corporations and authorities of every kind. I have spent more than 30 years uncovering and exposing this network that I call the Global Cult and knowing its agenda is what has made my books so accurate in predicting current and past events. Secret societies are secret for a reason. They want to keep their hoarded knowledge to themselves and their chosen initiates and to hide it from the population which they seek through ignorance to control and subdue. The whole foundation of the division between World 1 and World 2 is *knowledge*. What number 1 knows number 2 must not. Knowledge they have worked so hard to keep secret includes (a) the agenda to enslave humanity in a centrally-controlled global dictatorship, and (b) the nature of reality and life itself. The latter (b) must be suppressed to allow the former (a) to prevail as I shall be explaining. The way the Cult manipulates and interacts with the population can be likened to a spider's web. The 'spider' sits at the centre in the shadows and imposes its will through the web with each strand represented in World number 2 by a secret society, satanic or semi-secret group, and in World number 1 – the world of the seen – by governments, agencies of government, law enforcement, corporations, the banking system, media conglomerates and Silicon Valley (Fig 1 overleaf). The spider and the web connect and coordinate all these organisations to pursue the same global outcome while the population sees them as individual entities working randomly and independently. At the level of the web governments *are* the banking system *are* the corporations *are* the media *are* Silicon Valley *are* the World Health Organization working from their inner cores as one unit. Apparently unconnected countries, corporations, institutions, organisations and people are on the *same team* pursuing the same global outcome. Strands in the web immediately around the spider are the most secretive and exclusive secret societies and their membership is emphatically restricted to the Cult inner-circle emerging through the generations from particular bloodlines for reasons I will come to. At the core of the core you would get them in a single room. That's how many people are dictating the direction of human society and its transformation

through the ‘Covid’ hoax and other means. As the web expands out from the spider we meet the secret societies that many people will be aware of – the Freemasons, Knights Templar, Knights of Malta, Opus Dei, the inner sanctum of the Jesuit Order, and such like. Note how many are connected to the Church of Rome and there is a reason for that. The Roman Church was established as a revamp, a rebranding, of the relocated ‘Church’ of Babylon and the Cult imposing global tyranny today can be tracked back to Babylon and Sumer in what is now Iraq.



Figure 1: The global web through which the few control the many. (Image Neil Hague.)

Inner levels of the web operate in the unseen away from the public eye and then we have what I call the cusp organisations located at the point where the hidden meets the seen. They include a series of satellite organisations answering to a secret society founded in London in the late 19th century called the Round Table and among them are the Royal Institute of International Affairs (UK, founded in 1920); Council on Foreign Relations (US, 1921); Bilderberg Group (worldwide, 1954); Trilateral Commission (US/worldwide, 1972); and the Club of Rome (worldwide, 1968) which was created to exploit environmental concerns to justify the centralisation of global power to ‘save the planet’. The Club of Rome instigated with others the human-caused climate change hoax which has led to all the ‘green

new deals' demanding that very centralisation of control. Cusp organisations, which include endless 'think tanks' all over the world, are designed to coordinate a single global policy between political and business leaders, intelligence personnel, media organisations and anyone who can influence the direction of policy in their own sphere of operation. Major players and regular attenders will know what is happening – or some of it – while others come and go and are kept overwhelmingly in the dark about the big picture. I refer to these cusp groupings as semi-secret in that they can be publicly identified, but what goes on at the inner-core is kept very much 'in house' even from most of their members and participants through a fiercely-imposed system of compartmentalisation. Only let them know what they need to know to serve your interests and no more. The structure of secret societies serves as a perfect example of this principle. Most Freemasons never get higher than the bottom three levels of 'degree' (degree of knowledge) when there are 33 official degrees of the Scottish Rite. Initiates only qualify for the next higher 'compartment' or degree if those at that level choose to allow them. Knowledge can be carefully assigned only to those considered 'safe'. I went to my local Freemason's lodge a few years ago when they were having an 'open day' to show how cuddly they were and when I chatted to some of them I was astonished at how little the rank and file knew even about the most ubiquitous symbols they use. The mushroom technique – keep them in the dark and feed them bullshit – applies to most people in the web as well as the population as a whole. Sub-divisions of the web mirror in theme and structure transnational corporations which have a headquarters somewhere in the world dictating to all their subsidiaries in different countries. Subsidiaries operate in their methodology and branding to the same centrally-dictated plan and policy in pursuit of particular ends. The Cult web functions in the same way. Each country has its own web as a subsidiary of the global one. They consist of networks of secret societies, semi-secret groups and bloodline families and their job is to impose the will of the spider and the global web in their particular country. Subsidiary networks control and manipulate the national political system, finance, corporations, media, medicine, etc. to

ensure that they follow the globally-dictated Cult agenda. These networks were the means through which the 'Covid' hoax could be played out with almost every country responding in the same way.

The 'Yessir' pyramid

Compartmentalisation is the key to understanding how a tiny few can dictate the lives of billions when combined with a top-down sequence of imposition and acquiescence. The inner core of the Cult sits at the peak of the pyramidal hierarchy of human society (Fig 2 overleaf). It imposes its will – its agenda for the world – on the level immediately below which acquiesces to that imposition. This level then imposes the Cult will on the level below them which acquiesces and imposes on the next level. Very quickly we meet levels in the hierarchy that have no idea there even is a Cult, but the sequence of imposition and acquiescence continues down the pyramid in just the same way. 'I don't know why we are doing this but the order came from "on-high" and so we better just do it.' Alfred Lord Tennyson said of the cannon fodder levels in his poem *The Charge of the Light Brigade*: 'Theirs not to reason why; theirs but to do and die.' The next line says that 'into the valley of death rode the six hundred' and they died because they obeyed without question what their perceived 'superiors' told them to do. In the same way the population capitulated to 'Covid'. The whole hierarchical pyramid functions like this to allow the very few to direct the enormous many. Eventually imposition-acquiescence-imposition-acquiescence comes down to the mass of the population at the foot of the pyramid. If they acquiesce to those levels of the hierarchy imposing on them (governments/law enforcement/doctors/media) a circuit is completed between the population and the handful of super-psychopaths in the Cult inner core at the top of the pyramid. Without a circuit-breaking refusal to obey, the sequence of imposition and acquiescence allows a staggeringly few people to impose their will upon the entirety of humankind. We are looking at the very sequence that has subjugated billions since the start of 2020. Our freedom has not been taken from us. Humanity has given it

away. Fascists do not impose fascism because there are not enough of them. Fascism is imposed by the population acquiescing to fascism. Put another way allowing their perceptions to be programmed to the extent that leads to the population giving their freedom away by giving their perceptions – their mind – away. If this circuit is not broken by humanity ceasing to cooperate with their own enslavement then nothing can change. For that to happen people have to critically think and see through the lies and window dressing and then summon the backbone to act upon what they see. The Cult spends its days working to stop either happening and its methodology is systematic and highly detailed, but it can be overcome and that is what this book is all about.

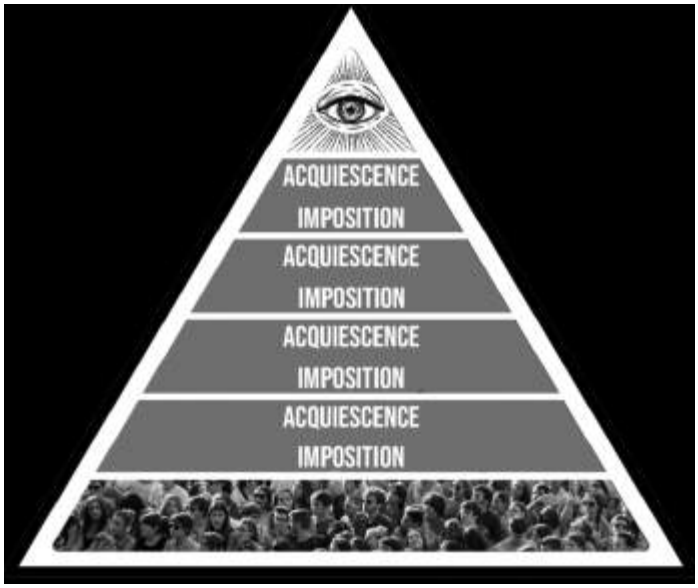


Figure 2: The simple sequence of imposition and compliance that allows a handful of people at the peak of the pyramid to dictate the lives of billions.

The Life Program

Okay, back to world number 1 or the world of the 'masses'. Observe the process of what we call 'life' and it is a perceptual download from cradle to grave. The Cult has created a global structure in which perception can be programmed and the program continually topped-up with what appears to be constant confirmation that the program is indeed true reality. The important word here is 'appears'.

This is the structure, the fly-trap, the Postage Stamp Consensus or Perceptual Mainframe, which represents that incredibly narrow band of perceived possibility delivered by the 'education' system, mainstream media, science and medicine. From the earliest age the download begins with parents who have themselves succumbed to the very programming their children are about to go through. Most parents don't do this out of malevolence and mostly it is quite the opposite. They do what they believe is best for their children and that is what the program has told them is best. Within three or four years comes the major transition from parental programming to full-blown state (Cult) programming in school, college and university where perceptually-programmed teachers and academics pass on their programming to the next generations. Teachers who resist are soon marginalised and their careers ended while children who resist are called a problem child for whom Ritalin may need to be prescribed. A few years after entering the 'world' children are under the control of authority figures representing the state telling them when they have to be there, when they can leave and when they can speak, eat, even go to the toilet. This is calculated preparation for a lifetime of obeying authority in all its forms. Reflex-action fear of authority is instilled by authority from the start. Children soon learn the carrot and stick consequences of obeying or defying authority which is underpinned daily for the rest of their life. Fortunately I daydreamed through this crap and never obeyed authority simply because it told me to. This approach to my alleged 'betters' continues to this day. There can be consequences of pursuing open-minded freedom in a world of closed-minded conformity. I spent a lot of time in school corridors after being ejected from the classroom for not taking some of it seriously and now I spend a lot of time being ejected from Facebook, YouTube and Twitter. But I can tell you that being true to yourself and not compromising your self-respect is far more exhilarating than bowing to authority for authority's sake. You don't have to be a sheep to the shepherd (authority) and the sheep dog (fear of not obeying authority).

The perceptual download continues throughout the formative years in school, college and university while script-reading 'teachers', 'academics' 'scientists', 'doctors' and 'journalists' insist that ongoing generations must be as programmed as they are. Accept the program or you will not pass your 'exams' which confirm your 'degree' of programming. It is tragic to think that many parents pressure their offspring to work hard at school to download the program and qualify for the next stage at college and university. The late, great, American comedian George Carlin said: 'Here's a bumper sticker I'd like to see: We are proud parents of a child who has resisted his teachers' attempts to break his spirit and bend him to the will of his corporate masters.' Well, the best of luck finding many of those, George. Then comes the moment to leave the formal programming years in academia and enter the 'adult' world of work. There you meet others in your chosen or prescribed arena who went through the same Postage Stamp Consensus program before you did. There is therefore overwhelming agreement between almost everyone on the basic foundations of Postage Stamp reality and the rejection, even contempt, of the few who have a mind of their own and are prepared to use it. This has two major effects. Firstly, the consensus confirms to the programmed that their download is really how things are. I mean, everyone knows that, right? Secondly, the arrogance and ignorance of Postage Stamp adherents ensure that anyone questioning the program will have unpleasant consequences for seeking their own truth and not picking their perceptions from the shelf marked: 'Things you must believe without question and if you don't you're a dangerous lunatic conspiracy theorist and a harebrained nutter'.

Every government, agency and corporation is founded on the same Postage Stamp prison cell and you can see why so many people believe the same thing while calling it their own 'opinion'. Fusion of governments and corporations in pursuit of the same agenda was the definition of fascism described by Italian dictator Benito Mussolini. The pressure to conform to perceptual norms downloaded for a lifetime is incessant and infiltrates society right

down to family groups that become censors and condemners of their own 'black sheep' for not, ironically, being sheep. We have seen an explosion of that in the 'Covid' era. Cult-owned global media unleashes its propaganda all day every day in support of the Postage Stamp and targets with abuse and ridicule anyone in the public eye who won't bend their mind to the will of the tyranny. Any response to this is denied (certainly in my case). They don't want to give a platform to expose official lies. Cult-owned-and-created Internet giants like Facebook, Google, YouTube and Twitter delete you for having an unapproved opinion. Facebook boasts that its AI censors delete 97-percent of 'hate speech' before anyone even reports it. Much of that 'hate speech' will simply be an opinion that Facebook and its masters don't want people to see. Such perceptual oppression is widely known as fascism. Even Facebook executive Benny Thomas, a 'CEO Global Planning Lead', said in comments secretly recorded by investigative journalism operation Project Veritas that Facebook is 'too powerful' and should be broken up:

I mean, no king in history has been the ruler of two billion people, but Mark Zuckerberg is … And he's 36. That's too much for a 36-year-old ... You should not have power over two billion people. I just think that's wrong.

Thomas said Facebook-owned platforms like Instagram, Oculus, and WhatsApp needed to be separate companies. 'It's too much power when they're all one together'. That's the way the Cult likes it, however. We have an executive of a Cult organisation in Benny Thomas that doesn't know there is a Cult such is the compartmentalisation. Thomas said that Facebook and Google 'are no longer companies, they're countries'. Actually they are more powerful than countries on the basis that if you control information you control perception and control human society.

I love my oppressor

Another expression of this psychological trickery is for those who realise they are being pressured into compliance to eventually

convince themselves to believe the official narratives to protect their self-respect from accepting the truth that they have succumbed to meek and subservient compliance. Such people become some of the most vehement defenders of the system. You can see them everywhere screaming abuse at those who prefer to think for themselves and by doing so reminding the compliers of their own capitulation to conformity. 'You are talking dangerous nonsense you Covidiot!!' Are you trying to convince me or yourself? It is a potent form of Stockholm syndrome which is defined as: 'A psychological condition that occurs when a victim of abuse identifies and attaches, or bonds, positively with their abuser.' An example is hostages bonding and even 'falling in love' with their kidnappers. The syndrome has been observed in domestic violence, abused children, concentration camp inmates, prisoners of war and many and various Satanic cults. These are some traits of Stockholm syndrome listed at goodtherapy.org:

- Positive regard towards perpetrators of abuse or captor [see 'Covid'].
- Failure to cooperate with police and other government authorities when it comes to holding perpetrators of abuse or kidnapping accountable [or in the case of 'Covid' cooperating with the police to enforce and defend their captors' demands].
- Little or no effort to escape [see 'Covid'].
- Belief in the goodness of the perpetrators or kidnappers [see 'Covid'].
- Appeasement of captors. This is a manipulative strategy for maintaining one's safety. As victims get rewarded – perhaps with less abuse or even with life itself – their appeasing behaviours are reinforced [see 'Covid'].
- Learned helplessness. This can be akin to 'if you can't beat 'em, join 'em'. As the victims fail to escape the abuse or captivity, they may start giving up and soon realize it's just easier for everyone if they acquiesce all their power to their captors [see 'Covid'].

- Feelings of pity toward the abusers, believing they are actually victims themselves. Because of this, victims may go on a crusade or mission to 'save' [protect] their abuser [see the venom unleashed on those challenging the official 'Covid' narrative].
- Unwillingness to learn to detach from their perpetrators and heal. In essence, victims may tend to be less loyal to themselves than to their abuser [*definitely* see 'Covid'].

Ponder on those traits and compare them with the behaviour of great swathes of the global population who have defended governments and authorities which have spent every minute destroying their lives and livelihoods and those of their children and grandchildren since early 2020 with fascistic lockdowns, house arrest and employment deletion to 'protect' them from a 'deadly virus' that their abusers' perceptually created to bring about this very outcome. We are looking at mass Stockholm syndrome. All those that agree to concede their freedom will believe those perceptions are originating in their own independent 'mind' when in fact by conceding their reality to Stockholm syndrome they have by definition conceded any independence of mind. Listen to the 'opinions' of the acquiescing masses in this 'Covid' era and what gushes forth is the repetition of the official version of everything delivered unprocessed, unfiltered and unquestioned. The whole programming dynamic works this way. I must be free because I'm told that I am and so I think that I am.

You can see what I mean with the chapter theme of 'I'm thinking – Oh, but *are* you?' The great majority are not thinking, let alone for themselves. They are repeating what authority has told them to believe which allows them to be controlled. Weaving through this mentality is the fear that the 'conspiracy theorists' are right and this again explains the often hysterical abuse that ensues when you dare to contest the official narrative of anything. Denial is the mechanism of hiding from yourself what you don't want to be true. Telling people what they want to hear is easy, but it's an infinitely greater challenge to tell them what they would rather not be happening.

One is akin to pushing against an open door while the other is met with vehement resistance no matter what the scale of evidence. I don't want it to be true so I'll convince myself that it's not. Examples are everywhere from the denial that a partner is cheating despite all the signs to the reflex-action rejection of any idea that world events in which country after country act in exactly the same way are centrally coordinated. To accept the latter is to accept that a force of unspeakable evil is working to destroy your life and the lives of your children with nothing too horrific to achieve that end. Who the heck wants that to be true? But if we don't face reality the end is duly achieved and the consequences are far worse and ongoing than breaking through the walls of denial today with the courage to make a stand against tyranny.

Connect the dots – but how?

A crucial aspect of perceptual programming is to portray a world in which everything is random and almost nothing is connected to anything else. Randomness cannot be coordinated by its very nature and once you perceive events as random the idea they could be connected is waved away as the rantings of the tinfoil-hat brigade. You can't plan and coordinate random you idiot! No, you can't, but you can hide the coldly-calculated and long-planned behind the *illusion* of randomness. A foundation manifestation of the Renegade Mind is to scan reality for patterns that connect the apparently random and turn pixels and dots into pictures. This is the way I work and have done so for more than 30 years. You look for similarities in people, modus operandi and desired outcomes and slowly, then ever quicker, the picture forms. For instance: There would seem to be no connection between the 'Covid pandemic' hoax and the human-caused global-warming hoax and yet they are masks (appropriately) on the same face seeking the same outcome. Those pushing the global warming myth through the Club of Rome and other Cult agencies are driving the lies about 'Covid' – Bill Gates is an obvious one, but they are endless. Why would the same people be involved in both when they are clearly not connected? Oh, but they

are. Common themes with personnel are matched by common goals. The 'solutions' to both 'problems' are centralisation of global power to impose the will of the few on the many to 'save' humanity from 'Covid' and save the planet from an 'existential threat' (we need 'zero Covid' and 'zero carbon emissions'). These, in turn, connect with the 'dot' of globalisation which was coined to describe the centralisation of global power in every area of life through incessant political and corporate expansion, trading blocks and superstates like the European Union. If you are the few and you want to control the many you have to centralise power and decision-making. The more you centralise power the more power the few at the centre will have over the many; and the more that power is centralised the more power those at the centre have to centralise even quicker. The momentum of centralisation gets faster and faster which is exactly the process we have witnessed. In this way the hoaxed 'pandemic' and the fakery of human-caused global warming serve the interests of globalisation and the seizure of global power in the hands of the Cult inner-circle which is behind 'Covid', 'climate change' *and* globalisation. At this point random 'dots' become a clear and obvious picture or pattern.

Klaus Schwab, the classic Bond villain who founded the Cult's Gates-funded World Economic Forum, published a book in 2020, *The Great Reset*, in which he used the 'problem' of 'Covid' to justify a total transformation of human society to 'save' humanity from 'climate change'. Schwab said: 'The pandemic represents a rare but narrow window of opportunity to reflect, reimagine, and reset our world.' What he didn't mention is that the Cult he serves is behind both hoaxes as I show in my book *The Answer*. He and the Cult don't have to reimagine the world. They know precisely what they want and that's why they destroyed human society with 'Covid' to 'build back better' in their grand design. Their job is not to imagine, but to get humanity to imagine and agree with their plans while believing it's all random. It must be pure coincidence that 'The Great Reset' has long been the Cult's code name for the global imposition of fascism and replaced previous code-names of the 'New World

Order' used by Cult frontmen like Father George Bush and the 'New Order of the Ages' which emerged from Freemasonry and much older secret societies. New Order of the Ages appears on the reverse of the Great Seal of the United States as 'Novus ordo seclorum' underneath the Cult symbol used since way back of the pyramid and all seeing-eye (Fig 3). The pyramid is the hierarchy of human control headed by the illuminated eye that symbolises the force behind the Cult which I will expose in later chapters. The term 'Annuit Coeptis' translates as 'He favours our undertaking'. We are told the 'He' is the Christian god, but 'He' is not as I will be explaining.



Figure 3: The all-seeing eye of the Cult 'god' on the Freemason-designed Great Seal of the United States and also on the dollar bill.

Having you on

Two major Cult techniques of perceptual manipulation that relate to all this are what I have called since the 1990s Problem-Reaction-Solution (PRS) and the Totalitarian Tiptoe (TT). They can be uncovered by the inquiring mind with a simple question: Who benefits? The answer usually identifies the perpetrators of a given action or happening through the concept of 'he who most benefits from a crime is the one most likely to have committed it'. The Latin 'Cue bono?' – Who benefits? – is widely attributed to the Roman orator and statesman Marcus Tullius Cicero. No wonder it goes back so far when the concept has been relevant to human behaviour since

history was recorded. Problem-Reaction-Solution is the technique used to manipulate us every day by covertly creating a problem (or the illusion of one) and offering the solution to the problem (or the illusion of one). In the first phase you create the problem and blame someone or something else for why it has happened. This may relate to a financial collapse, terrorist attack, war, global warming or pandemic, anything in fact that will allow you to impose the 'solution' to change society in the way you desire at that time. The 'problem' doesn't have to be real. PRS is manipulation of perception and all you need is the population to believe the problem is real. Human-caused global warming and the 'Covid pandemic' only have to be *perceived* to be real for the population to accept the 'solutions' of authority. I refer to this technique as NO-Problem-Reaction-Solution. Billions did not meekly accept house arrest from early 2020 because there was a real deadly 'Covid pandemic' but because they perceived – believed – that to be the case. The antidote to Problem-Reaction-Solution is to ask who benefits from the proposed solution. Invariably it will be anyone who wants to justify more control through deletion of freedom and centralisation of power and decision-making.

The two world wars were Problem-Reaction-Solutions that transformed and realigned global society. Both were manipulated into being by the Cult as I have detailed in books since the mid-1990s. They dramatically centralised global power, especially World War Two, which led to the United Nations and other global bodies thanks to the overt and covert manipulations of the Rockefeller family and other Cult bloodlines like the Rothschilds. The UN is a stalking horse for full-blown world government that I will come to shortly. The land on which the UN building stands in New York was donated by the Rockefellers and the same Cult family was behind Big Pharma scalpel and drug 'medicine' and the creation of the World Health Organization as part of the UN. They have been stalwarts of the eugenics movement and funded Hitler's race-purity expert' Ernst Rudin. The human-caused global warming hoax has been orchestrated by the Club of Rome through the UN which is

manufacturing both the 'problem' through its Intergovernmental Panel on Climate Change and imposing the 'solution' through its Agenda 21 and Agenda 2030 which demand the total centralisation of global power to 'save the world' from a climate hoax the United Nations is itself perpetrating. What a small world the Cult can be seen to be particularly among the inner circles. The bedfellow of Problem-Reaction-Solution is the Totalitarian Tiptoe which became the Totalitarian Sprint in 2020. The technique is fashioned to hide the carefully-coordinated behind the cover of apparently random events. You start the sequence at 'A' and you know you are heading for 'Z'. You don't want people to know that and each step on the journey is presented as a random happening while all the steps strung together lead in the same direction. The speed may have quickened dramatically in recent times, but you can still see the incremental approach of the Tiptoe in the case of 'Covid' as each new imposition takes us deeper into fascism. Tell people they have to do this or that to get back to 'normal', then this and this and this. With each new demand adding to the ones that went before the population's freedom is deleted until it disappears. The spider wraps its web around the flies more comprehensively with each new diktat. I'll highlight this in more detail when I get to the 'Covid' hoax and how it has been pulled off. Another prime example of the Totalitarian Tiptoe is how the Cult-created European Union went from a 'free-trade zone' to a centralised bureaucratic dictatorship through the Tiptoe of incremental centralisation of power until nations became mere administrative units for Cult-owned dark suits in Brussels.

The antidote to ignorance is knowledge which the Cult seeks vehemently to deny us, but despite the systematic censorship to that end the Renegade Mind can overcome this by vociferously seeking out the facts no matter the impediments put in the way. There is also a method of thinking and perceiving – *knowing* – that doesn't even need names, dates, place-type facts to identify the patterns that reveal the story. I'll get to that in the final chapter. All you need to know about the manipulation of human society and to what end is still out there – *at the time of writing* – in the form of books, videos

and websites for those that really want to breach the walls of programmed perception. To access this knowledge requires the abandonment of the mainstream media as a source of information in the awareness that this is owned and controlled by the Cult and therefore promotes mass perceptions that suit the Cult. Mainstream media lies all day, every day. That is its function and very reason for being. Where it does tell the truth, here and there, is only because the truth and the Cult agenda very occasionally coincide. If you look for fact and insight to the BBC, CNN and virtually all the rest of them you are asking to be conned and perceptually programmed.

Know the outcome and you'll see the journey

Events seem random when you have no idea where the world is being taken. Once you do the random becomes the carefully planned. Know the outcome and you'll see the journey is a phrase I have been using for a long time to give context to daily happenings that appear unconnected. Does a problem, or illusion of a problem, trigger a proposed 'solution' that further drives society in the direction of the outcome? Invariably the answer will be yes and the random – *abracadabra* – becomes the clearly coordinated. So what is this outcome that unlocks the door to a massively expanded understanding of daily events? I will summarise its major aspects – the fine detail is in my other books – and those new to this information will see that the world they thought they were living in is a very different place. The foundation of the Cult agenda is the incessant centralisation of power and all such centralisation is ultimately in pursuit of Cult control on a global level. I have described for a long time the planned world structure of top-down dictatorship as the Hunger Games Society. The term obviously comes from the movie series which portrayed a world in which a few living in military-protected hi-tech luxury were the overlords of a population condemned to abject poverty in isolated 'sectors' that were not allowed to interact. 'Covid' lockdowns and travel bans anyone? The 'Hunger Games' pyramid of structural control has the inner circle of the Cult at the top with pretty much the entire

population at the bottom under their control through dependency for survival on the Cult. The whole structure is planned to be protected and enforced by a military-police state (Fig 4).

Here you have the reason for the global lockdowns of the fake pandemic to coldly destroy independent incomes and livelihoods and make everyone dependent on the 'state' (the Cult that controls the 'states'). I have warned in my books for many years about the plan to introduce a 'guaranteed income' – a barely survivable pittance – designed to impose dependency when employment was destroyed by AI technology and now even more comprehensively at great speed by the 'Covid' scam. Once the pandemic was played and lockdown consequences began to delete independent income the authorities began to talk right on cue about the need for a guaranteed income and a 'Great Reset'. Guaranteed income will be presented as benevolent governments seeking to help a desperate people – desperate as a direct result of actions of the same governments. The truth is that such payments are a trap. You will only get them if you do exactly what the authorities demand including mass vaccination (genetic manipulation). We have seen this theme already in Australia where those dependent on government benefits have them reduced if parents don't agree to have their children vaccinated according to an insane health-destroying government-dictated schedule. Calculated economic collapse applies to governments as well as people. The Cult wants rid of countries through the creation of a world state with countries broken up into regions ruled by a world government and super states like the European Union. Countries must be bankrupted, too, to this end and it's being achieved by the trillions in 'rescue packages' and furlough payments, trillions in lost taxation, and money-no-object spending on 'Covid' including constant all-medium advertising (programming) which has made the media dependent on government for much of its income. The day of reckoning is coming – as planned – for government spending and given that it has been made possible by printing money and not by production/taxation there is inflation on the way that has the

potential to wipe out monetary value. In that case there will be no need for the Cult to steal your money. It just won't be worth anything (see the German Weimar Republic before the Nazis took over). Many have been okay with lockdowns while getting a percentage of their income from so-called furlough payments without having to work. Those payments are dependent, however, on people having at least a theoretical job with a business considered non-essential and ordered to close. As these business go under because they are closed by lockdown after lockdown the furlough stops and it will for everyone eventually. Then what? The 'then what?' is precisely the idea.



Figure 4: The Hunger Games Society structure I have long warned was planned and now the 'Covid' hoax has made it possible. This is the real reason for lockdowns.

Hired hands

Between the Hunger Games Cult elite and the dependent population is planned to be a vicious military-police state (a fusion of the two into one force). This has been in the making for a long time with police looking ever more like the military and carrying weapons to match. The pandemic scam has seen this process accelerate so fast as

lockdown house arrest is brutally enforced by carefully recruited fascist minds and gormless system-servers. The police and military are planned to merge into a centrally-directed world army in a global structure headed by a world government which wouldn't be elected even by the election fixes now in place. The world army is not planned even to be human and instead wars would be fought, primarily against the population, using robot technology controlled by artificial intelligence. I have been warning about this for decades and now militaries around the world are being transformed by this very AI technology. The global regime that I describe is a particular form of fascism known as a technocracy in which decisions are not made by clueless and co-opted politicians but by unelected technocrats – scientists, engineers, technologists and bureaucrats. Cult-owned-and-controlled Silicon Valley giants are examples of technocracy and they already have far more power to direct world events than governments. They are with their censorship *selecting* governments. I know that some are calling the 'Great Reset' a Marxist communist takeover, but fascism and Marxism are different labels for the same tyranny. Tell those who lived in fascist Germany and Stalinist Russia that there was a difference in the way their freedom was deleted and their lives controlled. I could call it a fascist technocracy or a Marxist technocracy and they would be equally accurate. The Hunger Games society with its world government structure would oversee a world army, world central bank and single world cashless currency imposing its will on a microchipped population (Fig 5). Scan its different elements and see how the illusory pandemic is forcing society in this very direction at great speed. Leaders of 23 countries and the World Health Organization (WHO) backed the idea in March, 2021, of a global treaty for 'international cooperation' in 'health emergencies' and nations should 'come together as a global community for peaceful cooperation that extends beyond this crisis'. Cut the Orwellian bullshit and this means another step towards global government. The plan includes a cashless digital money system that I first warned about in 1993. Right at the start of 'Covid' the deeply corrupt Tedros

Adhanom Ghebreyesus, the crooked and merely gofer 'head' of the World Health Organization, said it was possible to catch the 'virus' by touching cash and it was better to use cashless means. The claim was ridiculous nonsense and like the whole 'Covid' mind-trick it was nothing to do with 'health' and everything to do with pushing every aspect of the Cult agenda. As a result of the Tedros lie the use of cash has plummeted. The Cult script involves a single world digital currency that would eventually be technologically embedded in the body. China is a massive global centre for the Cult and if you watch what is happening there you will know what is planned for everywhere. The Chinese government is developing a digital currency which would allow fines to be deducted immediately via AI for anyone caught on camera breaking its fantastic list of laws and the money is going to be programmable with an expiry date to ensure that no one can accrue wealth except the Cult and its operatives.



Figure 5: The structure of global control the Cult has been working towards for so long and this has been enormously advanced by the 'Covid' illusion.

Serfdom is so smart

The Cult plan is far wider, extreme, and more comprehensive than even most conspiracy researchers appreciate and I will come to the true depths of deceit and control in the chapters 'Who controls the

Cult?' and 'Escaping Wetiko'. Even the world that we know is crazy enough. We are being deluged with ever more sophisticated and controlling technology under the heading of 'smart'. We have smart televisions, smart meters, smart cards, smart cars, smart driving, smart roads, smart pills, smart patches, smart watches, smart skin, smart borders, smart pavements, smart streets, smart cities, smart communities, smart environments, smart growth, smart planet ... smart *everything* around us. Smart technologies and methods of operation are designed to interlock to create a global Smart Grid connecting the entirety of human society including human minds to create a centrally-dictated 'hive' mind. 'Smart cities' is code for densely-occupied megacities of total surveillance and control through AI. Ever more destructive frequency communication systems like 5G have been rolled out without any official testing for health and psychological effects (colossal). 5G/6G/7G systems are needed to run the Smart Grid and each one becomes more destructive of body and mind. Deleting independent income is crucial to forcing people into these AI-policed prisons by ending private property ownership (except for the Cult elite). The Cult's Great Reset now openly foresees a global society in which no one will own any possessions and everything will be rented while the Cult would own literally everything under the guise of government and corporations. The aim has been to use the lockdowns to destroy sources of income on a mass scale and when the people are destitute and in unrepayable amounts of debt (problem) Cult assets come forward with the pledge to write-off debt in return for handing over all property and possessions (solution). Everything – literally everything including people – would be connected to the Internet via AI. I was warning years ago about the coming Internet of Things (IoT) in which all devices and technology from your car to your fridge would be plugged into the Internet and controlled by AI. Now we are already there with much more to come. The next stage is the Internet of Everything (IoE) which is planned to include the connection of AI to the human brain and body to replace the human mind with a centrally-controlled AI mind. Instead of perceptions

being manipulated through control of information and censorship those perceptions would come direct from the Cult through AI. What do you think? You think whatever AI decides that you think. In human terms there would be no individual 'think' any longer. Too incredible? The ravings of a lunatic? Not at all. Cult-owned crazies in Silicon Valley have been telling us the plan for years without explaining the real motivation and calculated implications. These include Google executive and 'futurist' Ray Kurzweil who highlights the year 2030 for when this would be underway. He said:

Our thinking ... will be a hybrid of biological and non-biological thinking ... humans will be able to extend their limitations and 'think in the cloud' ... We're going to put gateways to the cloud in our brains ... We're going to gradually merge and enhance ourselves ... In my view, that's the nature of being human – we transcend our limitations.

As the technology becomes vastly superior to what we are then the small proportion that is still human gets smaller and smaller and smaller until it's just utterly negligible.

The sales-pitch of Kurzweil and Cult-owned Silicon Valley is that this would make us 'super-human' when the real aim is to make us post-human and no longer 'human' in the sense that we have come to know. The entire global population would be connected to AI and become the centrally-controlled 'hive-mind' of externally-delivered perceptions. The Smart Grid being installed to impose the Cult's will on the world is being constructed to allow particular locations – even one location – to control the whole global system. From these prime control centres, which absolutely include China and Israel, anything connected to the Internet would be switched on or off and manipulated at will. Energy systems could be cut, communication via the Internet taken down, computer-controlled driverless autonomous vehicles driven off the road, medical devices switched off, the potential is limitless given how much AI and Internet connections now run human society. We have seen nothing yet if we allow this to continue. Autonomous vehicle makers are working with law enforcement to produce cars designed to automatically pull over if they detect a police or emergency vehicle flashing from up to 100 feet away. At a police stop the car would be unlocked and the

window rolled down automatically. Vehicles would only take you where the computer (the state) allowed. The end of petrol vehicles and speed limiters on all new cars in the UK and EU from 2022 are steps leading to electric computerised transport over which ultimately you have no control. The picture is far bigger even than the Cult global network or web and that will become clear when I get to the nature of the 'spider'. There is a connection between all these happenings and the instigation of DNA-manipulating 'vaccines' (which aren't 'vaccines') justified by the 'Covid' hoax. That connection is the unfolding plan to transform the human body from a biological to a synthetic biological state and this is why synthetic biology is such a fast-emerging discipline of mainstream science. 'Covid vaccines' are infusing self-replicating synthetic genetic material into the cells to cumulatively take us on the Totalitarian Tiptoe from Human 1.0 to the synthetic biological Human 2.0 which will be physically and perceptually attached to the Smart Grid to one hundred percent control every thought, perception and deed. Humanity needs to wake up and *fast*.

This is the barest explanation of where the 'outcome' is planned to go but it's enough to see the journey happening all around us. Those new to this information will already see 'Covid' in a whole new context. I will add much more detail as we go along, but for the minutiae evidence see my mega-works, *The Answer, The Trigger* and *Everything You Need to Know But Have Never Been Told*.

Now – how does a Renegade Mind see the 'world'?

CHAPTER TWO

Renegade Perception

It is one thing to be clever and another to be wise
George R.R. Martin

A simple definition of the difference between a programmed mind and a Renegade Mind would be that one sees only dots while the other connects them to see the picture. Reading reality with accuracy requires the observer to (a) know the planned outcome and (b) realise that everything, but *everything*, is connected.

The entirety of infinite reality is connected – that's its very nature – and with human society an expression of infinite reality the same must apply. Simple cause and effect is a connection. The effect is triggered by the cause and the effect then becomes the cause of another effect. Nothing happens in isolation because it *can't*. Life in whatever reality is simple choice and consequence. We make choices and these lead to consequences. If we don't like the consequences we can make different choices and get different consequences which lead to other choices and consequences. The choice and the consequence are not only connected they are indivisible. You can't have one without the other as an old song goes. A few cannot control the world unless those being controlled allow that to happen – cause and effect, choice and consequence. Control – who has it and who doesn't – is a two-way process, a symbiotic relationship, involving the controller and controlled. 'They took my freedom away!!' Well, yes, but you also gave it to them. Humanity is

subjected to mass control because humanity has acquiesced to that control. This is all cause and effect and literally a case of give and take. In the same way world events of every kind are connected and the Cult works incessantly to sell the illusion of the random and coincidental to maintain the essential (to them) perception of dots that hide the picture. Renegade Minds know this and constantly scan the world for patterns of connection. This is absolutely pivotal in understanding the happenings in the world and without that perspective clarity is impossible. First you know the planned outcome and then you identify the steps on the journey – the day-by-day apparently random which, when connected in relation to the outcome, no longer appear as individual events, but as the proverbial *chain* of events leading in the same direction. I'll give you some examples:

Political puppet show

We are told to believe that politics is 'adversarial' in that different parties with different beliefs engage in an endless tussle for power. There may have been some truth in that up to a point – and only a point – but today divisions between 'different' parties are rhetorical not ideological. Even the rhetorical is fusing into one-speak as the parties eject any remaining free thinkers while others succumb to the ever-gathering intimidation of anyone with the 'wrong' opinion. The Cult is not a new phenomenon and can be traced back thousands of years as my books have documented. Its intergenerational initiates have been manipulating events with increasing effect the more that global power has been centralised. In ancient times the Cult secured control through the system of monarchy in which 'special' bloodlines (of which more later) demanded the right to rule as kings and queens simply by birthright and by vanquishing others who claimed the same birthright. There came a time, however, when people had matured enough to see the unfairness of such tyranny and demanded a say in who governed them. Note the word – *governed* them. Not served them – *governed* them, hence government defined as 'the political direction and control exercised over the

actions of the members, citizens, or inhabitants of communities, societies, and states; direction of the affairs of a state, community, etc.' Governments exercise control over rather than serve just like the monarchies before them. Bizarrely there are still countries like the United Kingdom which are ruled by a monarch *and* a government that officially answers to the monarch. The UK head of state and that of Commonwealth countries such as Canada, Australia and New Zealand is 'selected' by who in a *single family* had unprotected sex with whom and in what order. Pinch me it can't be true. Ouch! Shit, it is. The demise of monarchies in most countries offered a potential vacuum in which some form of free and fair society could arise and the Cult had that base covered. Monarchies had served its interests but they couldn't continue in the face of such widespread opposition and, anyway, replacing a 'royal' dictatorship that people could see with a dictatorship 'of the people' hiding behind the concept of 'democracy' presented far greater manipulative possibilities and ways of hiding coordinated tyranny behind the illusion of 'freedom'.

Democracy is quite wrongly defined as government selected by the population. This is not the case at all. It is government selected by *some* of the population (and then only in theory). This 'some' doesn't even have to be the majority as we have seen so often in first-past-the-post elections in which the so-called majority party wins fewer votes than the 'losing' parties combined. Democracy can give total power to a party in government from a minority of the votes cast. It's a sleight of hand to sell tyranny as freedom. Seventy-four million Trump-supporting Americans didn't vote for the 'Democratic' Party of Joe Biden in the distinctly dodgy election in 2020 and yet far from acknowledging the wishes and feelings of that great percentage of American society the Cult-owned Biden government set out from day one to destroy them and their right to a voice and opinion. Empty shell Biden and his Cult handlers said they were doing this to 'protect democracy'. Such is the level of lunacy and sickness to which politics has descended. Connect the dots and relate them to the desired outcome – a world government run by self-appointed technocrats and no longer even elected

politicians. While operating through its political agents in government the Cult is at the same time encouraging public distain for politicians by putting idiots and incompetents in theoretical power on the road to deleting them. The idea is to instil a public reaction that says of the technocrats: 'Well, they couldn't do any worse than the pathetic politicians.' It's all about controlling perception and Renegade Minds can see through that while programmed minds cannot when they are ignorant of both the planned outcome and the manipulation techniques employed to secure that end. This knowledge can be learned, however, and fast if people choose to get informed.

Politics may at first sight appear very difficult to control from a central point. I mean look at the 'different' parties and how would you be able to oversee them all and their constituent parts? In truth, it's very straightforward because of their structure. We are back to the pyramid of imposition and acquiescence. Organisations are structured in the same way as the system as a whole. Political parties are not open forums of free expression. They are hierarchies. I was a national spokesman for the British Green Party which claimed to be a different kind of politics in which influence and power was devolved; but I can tell you from direct experience – and it's far worse now – that Green parties are run as hierarchies like all the others however much they may try to hide that fact or kid themselves that it's not true. A very few at the top of all political parties are directing policy and personnel. They decide if you are elevated in the party or serve as a government minister and to do that you have to be a yes man or woman. Look at all the maverick political thinkers who never ascended the greasy pole. If you want to progress within the party or reach 'high-office' you need to fall into line and conform. Exceptions to this are rare indeed. Should you want to run for parliament or Congress you have to persuade the local or state level of the party to select you and for that you need to play the game as dictated by the hierarchy. If you secure election and wish to progress within the greater structure you need to go on conforming to what is acceptable to those running the hierarchy

from the peak of the pyramid. Political parties are perceptual gulags and the very fact that there are party 'Whips' appointed to 'whip' politicians into voting the way the hierarchy demands exposes the ridiculous idea that politicians are elected to serve the people they are supposed to represent. Cult operatives and manipulation has long seized control of major parties that have any chance of forming a government and at least most of those that haven't. A new party forms and the Cult goes to work to infiltrate and direct. This has reached such a level today that you see video compilations of 'leaders' of all parties whether Democrats, Republicans, Conservative, Labour and Green parroting the same Cult mantra of 'Build Back Better' and the 'Great Reset' which are straight off the Cult song-sheet to describe the transformation of global society in response to the Cult-instigated hoaxes of the 'Covid pandemic' and human-caused 'climate change'. To see Caroline Lucas, the Green Party MP that I knew when I was in the party in the 1980s, speaking in support of plans proposed by Cult operative Klaus Schwab representing the billionaire global elite is a real head-shaker.

Many parties – one master

The party system is another mind-trick and was instigated to change the nature of the dictatorship by swapping 'royalty' for dark suits that people believed – though now ever less so – represented their interests. Understanding this trick is to realise that a single force (the Cult) controls all parties either directly in terms of the major ones or through manipulation of perception and ideology with others. You don't need to manipulate Green parties to demand your transformation of society in the name of 'climate change' when they are obsessed with the lie that this is essential to 'save the planet'. You just give them a platform and away they go serving your interests while believing they are being environmentally virtuous. America's political structure is a perfect blueprint for how the two or multi-party system is really a one-party state. The Republican Party is controlled from one step back in the shadows by a group made up of billionaires and their gofers known as neoconservatives or Neocons.

I have exposed them in fine detail in my books and they were the driving force behind the policies of the imbecilic presidency of Boy George Bush which included 9/11 (see *The Trigger* for a comprehensive demolition of the official story), the subsequent 'war on terror' (war *of* terror) and the invasions of Afghanistan and Iraq. The latter was a No-Problem-Reaction-Solution based on claims by Cult operatives, including Bush and British Prime Minister Tony Blair, about Saddam Hussein's 'weapons of mass destruction' which did not exist as war criminals Bush and Blair well knew.

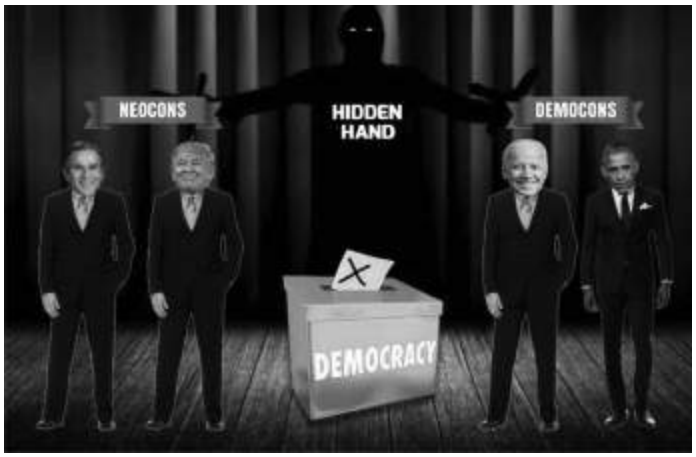


Figure 6: Different front people, different parties – same control system.

The Democratic Party has its own 'Neocon' group controlling from the background which I call the 'Democons' and here's the penny-drop – the Neocons and Democons answer to the same masters one step further back into the shadows (Fig 6). At that level of the Cult the Republican and Democrat parties are controlled by the same people and no matter which is in power the Cult is in power. This is how it works in almost every country and certainly in Britain with Conservative, Labour, Liberal Democrat and Green parties now all on the same page whatever the rhetoric may be in their feeble attempts to appear different. Neocons operated at the time of Bush through a think tank called The Project for the New American Century which in September, 2000, published a document entitled *Rebuilding America's Defenses: Strategies, Forces, and Resources*

For a New Century demanding that America fight 'multiple, simultaneous major theatre wars' as a 'core mission' to force regime-change in countries including Iraq, Libya and Syria. Neocons arranged for Bush ('Republican') and Blair ('Labour Party') to front-up the invasion of Iraq and when they departed the Democons orchestrated the targeting of Libya and Syria through Barack Obama ('Democrat') and British Prime Minister David Cameron ('Conservative Party'). We have 'different' parties and 'different' people, but the same unfolding script. The more the Cult has seized the reigns of parties and personnel the more their policies have transparently pursued the same agenda to the point where the fascist 'Covid' impositions of the Conservative junta of Jackboot Johnson in Britain were opposed by the Labour Party because they were not fascist enough. The Labour Party is likened to the US Democrats while the Conservative Party is akin to a British version of the Republicans and on both sides of the Atlantic they all speak the same language and support the direction demanded by the Cult although some more enthusiastically than others. It's a similar story in country after country because it's all centrally controlled. Oh, but what about Trump? I'll come to him shortly. Political 'choice' in the 'party' system goes like this: You vote for Party A and they get into government. You don't like what they do so next time you vote for Party B and they get into government. You don't like what they do when it's pretty much the same as Party A and why wouldn't that be with both controlled by the same force? Given that only two, sometimes three, parties have any chance of forming a government to get rid of Party B that you don't like you have to vote again for Party A which … you don't like. This, ladies and gentlemen, is what they call 'democracy' which we are told – wrongly – is a term interchangeable with 'freedom'.

The cult of cults

At this point I need to introduce a major expression of the Global Cult known as Sabbatian-Frankism. Sabbatian is also spelt as Sabbatean. I will summarise here. I have published major exposés

and detailed background in other works. Sabbatian-Frankism combines the names of two frauds posing as 'Jewish' men, Sabbatai Zevi (1626-1676), a rabbi, black magician and occultist who proclaimed he was the Jewish messiah; and Jacob Frank (1726-1791), the Polish 'Jew', black magician and occultist who said he was the reincarnation of 'messiah' Zevi and biblical patriarch Jacob. They worked across two centuries to establish the Sabbatian-Frankist cult that plays a major, indeed central, role in the manipulation of human society by the Global Cult which has its origins much further back in history than Sabbatai Zevi. I should emphasise two points here in response to the shrill voices that will scream 'anti-Semitism': (1) Sabbatian-Frankists are NOT Jewish and only pose as such to hide their cult behind a Jewish façade; and (2) my information about this cult has come from Jewish sources who have long realised that their society and community has been infiltrated and taken over by interloper Sabbatian-Frankists. Infiltration has been the foundation technique of Sabbatian-Frankism from its official origin in the 17th century. Zevi's Sabbatian sect attracted a massive following described as the biggest messianic movement in Jewish history, spreading as far as Africa and Asia, and he promised a return for the Jews to the 'Promised Land' of Israel. Sabbatianism was not Judaism but an inversion of everything that mainstream Judaism stood for. So much so that this sinister cult would have a feast day when Judaism had a fast day and whatever was forbidden in Judaism the Sabbatians were encouraged and even commanded to do. This included incest and what would be today called Satanism. Members were forbidden to marry outside the sect and there was a system of keeping their children ignorant of what they were part of until they were old enough to be trusted not to unknowingly reveal anything to outsiders. The same system is employed to this day by the Global Cult in general which Sabbatian-Frankism has enormously influenced and now largely controls.

Zevi and his Sabbatians suffered a setback with the intervention by the Sultan of the Islamic Ottoman Empire in the Middle East and what is now the Republic of Turkey where Zevi was located. The

Sultan gave him the choice of proving his 'divinity', converting to Islam or facing torture and death. Funnily enough Zevi chose to convert or at least appear to. Some of his supporters were disillusioned and drifted away, but many did not with 300 families also converting – only in theory – to Islam. They continued behind this Islamic smokescreen to follow the goals, rules and rituals of Sabbatianism and became known as 'crypto-Jews' or the 'Dönmeh' which means 'to turn'. This is rather ironic because they didn't 'turn' and instead hid behind a fake Islamic persona. The process of appearing to be one thing while being very much another would become the calling card of Sabbatianism especially after Zevi's death and the arrival of the Satanist Jacob Frank in the 18th century when the cult became Sabbatian-Frankism and plumbed still new depths of depravity and infiltration which included – still includes – human sacrifice and sex with children. Wherever Sabbatians go paedophilia and Satanism follow and is it really a surprise that Hollywood is so infested with child abuse and Satanism when it was established by Sabbatian-Frankists and is still controlled by them? Hollywood has been one of the prime vehicles for global perceptual programming and manipulation. How many believe the version of 'history' portrayed in movies when it is a travesty and inversion (again) of the truth? Rabbi Marvin Antelman describes Frankism in his book, *To Eliminate the Opiate*, as 'a movement of complete evil' while Jewish professor Gershom Scholem said of Frank in *The Messianic Idea in Judaism*: 'In all his actions [he was] a truly corrupt and degenerate individual ... one of the most frightening phenomena in the whole of Jewish history.' Frank was excommunicated by traditional rabbis, as was Zevi, but Frank was undeterred and enjoyed vital support from the House of Rothschild, the infamous banking dynasty whose inner-core are Sabbatian-Frankists and not Jews. Infiltration of the Roman Church and Vatican was instigated by Frank with many Dönmeh 'turning' again to convert to Roman Catholicism with a view to hijacking the reins of power. This was the ever-repeating modus operandi and continues to be so. Pose as an advocate of the religion, culture or country that you want to control and then

manipulate your people into the positions of authority and influence largely as advisers, administrators and Svengalis for those that appear to be in power. They did this with Judaism, Christianity (Christian Zionism is part of this), Islam and other religions and nations until Sabbatian-Frankism spanned the world as it does today.

Sabbatian Saudis and the terror network

One expression of the Sabbatian-Frankist Dönmeh within Islam is the ruling family of Saudi Arabia, the House of Saud, through which came the vile distortion of Islam known as Wahhabism. This is the violent creed followed by terrorist groups like Al-Qaeda and ISIS or Islamic State. Wahhabism is the hand-chopping, head-chopping 'religion' of Saudi Arabia which is used to keep the people in a constant state of fear so the interloper House of Saud can continue to rule. Al-Qaeda and Islamic State were lavishly funded by the House of Saud while being created and directed by the Sabbatian-Frankist network in the United States that operates through the Pentagon, CIA and the government in general of whichever 'party'. The front man for the establishment of Wahhabism in the middle of the 18th century was a Sabbatian-Frankist 'crypto-Jew' posing as Islamic called Muhammad ibn Abd al-Wahhab. His daughter would marry the son of Muhammad bin Saud who established the first Saudi state before his death in 1765 with support from the British Empire. Bin Saud's successors would establish modern Saudi Arabia in league with the British and Americans in 1932 which allowed them to seize control of Islam's major shrines in Mecca and Medina. They have dictated the direction of Sunni Islam ever since while Iran is the major centre of the Shiite version and here we have the source of at least the public conflict between them. The Sabbatian network has used its Wahhabi extremists to carry out Problem-Reaction-Solution terrorist attacks in the name of 'Al-Qaeda' and 'Islamic State' to justify a devastating 'war on terror', ever-increasing surveillance of the population and to terrify people into compliance. Another insight of the Renegade Mind is the streetwise understanding that

just because a country, location or people are attacked doesn't mean that those apparently representing that country, location or people are not behind the attackers. Often they are *orchestrating* the attacks because of the societal changes that can be then justified in the name of 'saving the population from terrorists'.

I show in great detail in *The Trigger* how Sabbatian-Frankists were the real perpetrators of 9/11 and not '19 Arab hijackers' who were blamed for what happened. Observe what was justified in the name of 9/11 alone in terms of Middle East invasions, mass surveillance and control that fulfilled the demands of the Project for the New American Century document published by the Sabbatian Neocons. What appear to be enemies are on the deep inside players on the same Sabbatian team. Israel and Arab 'royal' dictatorships are all ruled by Sabbatians and the recent peace agreements between Israel and Saudi Arabia, the United Arab Emirates (UAE) and others are only making formal what has always been the case behind the scenes. Palestinians who have been subjected to grotesque tyranny since Israel was bombed and terrorised into existence in 1948 have never stood a chance. Sabbatian-Frankists have controlled Israel (so the constant theme of violence and war which Sabbatians love) and they have controlled the Arab countries that Palestinians have looked to for real support that never comes. 'Royal families' of the Arab world in Saudi Arabia, Bahrain, UAE, etc., are all Sabbatians with allegiance to the aims of the cult and not what is best for their Arabic populations. They have stolen the oil and financial resources from their people by false claims to be 'royal dynasties' with a genetic right to rule and by employing vicious militaries to impose their will.

Satanic 'illumination'

The Satanist Jacob Frank formed an alliance in 1773 with two other Sabbatians, Mayer Amschel Rothschild (1744-1812), founder of the Rothschild banking dynasty, and Jesuit-educated fraudulent Jew, Adam Weishaupt, and this led to the formation of the Bavarian Illuminati, firstly under another name, in 1776. The Illuminati would

be the manipulating force behind the French Revolution (1789-1799) and was also involved in the American Revolution (1775-1783) before and after the Illuminati's official creation. Weishaupt would later become (in public) a Protestant Christian in archetypal Sabbatian style. I read that his name can be decoded as Adam-Weishaupt or 'the first man to lead those who know'. He wasn't a leader in the sense that he was a subordinate, but he did lead those below him in a crusade of transforming human society that still continues today. The theme was confirmed as early as 1785 when a horseman courier called Lanz was reported to be struck by lighting and extensive Illuminati documents were found in his saddlebags. They made the link to Weishaupt and detailed the plan for world takeover. Current events with 'Covid' fascism have been in the making for a very long time. Jacob Frank was jailed for 13 years by the Catholic Inquisition after his arrest in 1760 and on his release he headed for Frankfurt, Germany, home city and headquarters of the House of Rothschild where the alliance was struck with Mayer Amschel Rothschild and Weishaupt. Rothschild arranged for Frank to be given the title of Baron and he became a wealthy nobleman with a big following of Jews in Germany, the Austro-Hungarian Empire and other European countries. Most of them would have believed he was on their side.

The name 'Illuminati' came from the Zohar which is a body of works in the Jewish mystical 'bible' called the Kabbalah. 'Zohar' is the foundation of Sabbatian-Frankist belief and in Hebrew 'Zohar' means 'splendour', 'radiance', 'illuminated', and so we have 'Illuminati'. They claim to be the 'Illuminated Ones' from their knowledge systematically hidden from the human population and passed on through generations of carefully-chosen initiates in the global secret society network or Cult. Hidden knowledge includes an awareness of the Cult agenda for the world and the nature of our collective reality that I will explore later. Cult 'illumination' is symbolised by the torch held by the Statue of Liberty which was gifted to New York by French Freemasons in Paris who knew exactly what it represents. 'Liberty' symbolises the goddess worshipped in

Babylon as Queen Semiramis or Ishtar. The significance of this will become clear. Notice again the ubiquitous theme of inversion with the Statue of 'Liberty' really symbolising mass control (Fig 7). A mirror-image statute stands on an island in the River Seine in Paris from where New York Liberty originated (Fig 8). A large replica of the Liberty flame stands on top of the Pont de l'Alma tunnel in Paris where Princess Diana died in a Cult ritual described in *The Biggest Secret*. Lucifer 'the light bringer' is related to all this (and much more as we'll see) and 'Lucifer' is a central figure in Sabbatian-Frankism and its associated Satanism. Sabbatians reject the Jewish Torah, or Pentateuch, the 'five books of Moses' in the Old Testament known as Genesis, Exodus, Leviticus, Numbers, and Deuteronomy which are claimed by Judaism and Christianity to have been dictated by 'God' to Moses on Mount Sinai. Sabbatians say these do not apply to them and they seek to replace them with the Zohar to absorb Judaism and its followers into their inversion which is an expression of a much greater global inversion. They want to delete all religions and force humanity to worship a one-world religion – Sabbatian Satanism that also includes worship of the Earth goddess. Satanic themes are being more and more introduced into mainstream society and while Christianity is currently the foremost target for destruction the others are planned to follow.



Figure 7: The Cult goddess of Babylon disguised as the Statue of Liberty holding the flame of Lucifer the 'light bringer'.



Figure 8: Liberty's mirror image in Paris where the New York version originated.

Marx brothers

Rabbi Marvin Antelman connects the Illuminati to the Jacobins in *To Eliminate the Opiate* and Jacobins were the force behind the French Revolution. He links both to the Bund der Gerechten, or League of the Just, which was the network that inflicted communism/Marxism on the world. Antelman wrote:

> The original inner circle of the Bund der Gerechten consisted of born Catholics, Protestants and Jews [Sabbatian-Frankist infiltrators], and those representatives of respective subdivisions formulated schemes for the ultimate destruction of their faiths. The heretical Catholics laid plans which they felt would take a century or more for the ultimate destruction of the church; the apostate Jews for the ultimate destruction of the Jewish religion.

Sabbatian-created communism connects into this anti-religion agenda in that communism does not allow for the free practice of religion. The Sabbatian 'Bund' became the International Communist Party and Communist League and in 1848 'Marxism' was born with the Communist Manifesto of Sabbatian assets Karl Marx and Friedrich Engels. It is absolutely no coincidence that Marxism, just a different name for fascist and other centrally-controlled tyrannies, is being imposed worldwide as a result of the 'Covid' hoax and nor that Marxist/fascist China was the place where the hoax originated. The reason for this will become very clear in the chapter 'Covid: The calculated catastrophe'. The so-called 'Woke' mentality has hijacked

traditional beliefs of the political left and replaced them with far-right make-believe 'social justice' better known as Marxism. Woke will, however, be swallowed by its own perceived 'revolution' which is really the work of billionaires and billionaire corporations feigning being 'Woke'. Marxism is being touted by Wokers as a replacement for 'capitalism' when we don't have 'capitalism'. We have cartelism in which the market is stitched up by the very Cult billionaires and corporations bankrolling Woke. Billionaires love Marxism which keeps the people in servitude while they control from the top. Terminally naïve Wokers think they are 'changing the world' when it's the Cult that is doing the changing and when they have played their vital part and become surplus to requirements they, too, will be targeted. The Illuminati-Jacobins were behind the period known as 'The Terror' in the French Revolution in 1793 and 1794 when Jacobin Maximillian de Robespierre and his Orwellian 'Committee of Public Safety' killed 17,000 'enemies of the Revolution' who had once been 'friends of the Revolution'. Karl Marx (1818-1883), whose Sabbatian creed of Marxism has cost the lives of at least 100 million people, is a hero once again to Wokers who have been systematically kept ignorant of real history by their 'education' programming. As a result they now promote a Sabbatian 'Marxist' abomination destined at some point to consume them. Rabbi Antelman, who spent decades researching the Sabbatian plot, said of the League of the Just and Karl Marx:

> Contrary to popular opinion Karl Marx did not originate the Communist Manifesto. He was paid for his services by the League of the Just, which was known in its country of origin, Germany, as the Bund der Geaechteten.

Antelman said the text attributed to Marx was the work of other people and Marx 'was only repeating what others already said'. Marx was 'a hired hack – lackey of the wealthy Illuminists'. Marx famously said that religion was the 'opium of the people' (part of the Sabbatian plan to demonise religion) and Antelman called his books, *To Eliminate the Opiate*. Marx was born Jewish, but his family converted to Christianity (Sabbatian modus operandi) and he

attacked Jews, not least in his book, *A World Without Jews*. In doing so he supported the Sabbatian plan to destroy traditional Jewishness and Judaism which we are clearly seeing today with the vindictive targeting of orthodox Jews by the Sabbatian government of Israel over 'Covid' laws. I don't follow any religion and it has done much damage to the world over centuries and acted as a perceptual straightjacket. Renegade Minds, however, are always asking *why* something is being done. It doesn't matter if they agree or disagree with what is happening – *why* is it happening is the question. The 'why?' can be answered with regard to religion in that religions create interacting communities of believers when the Cult wants to dismantle all discourse, unity and interaction (see 'Covid' lockdowns) and the ultimate goal is to delete all religions for a one-world religion of Cult Satanism worshipping their 'god' of which more later. We see the same 'why?' with gun control in America. I don't have guns and don't want them, but why is the Cult seeking to disarm the population at the same time that law enforcement agencies are armed to their molars and why has every tyrant in history sought to disarm people before launching the final takeover? They include Hitler, Stalin, Pol Pot and Mao who followed confiscation with violent seizing of power. You know it's a Cult agenda by the people who immediately race to the microphones to exploit dead people in multiple shootings. Ultra-Zionist Cult lackey Senator Chuck Schumer was straight on the case after ten people were killed in Boulder, Colorado in March, 2121. Simple rule … if Schumer wants it the Cult wants it and the same with his ultra-Zionist mate the wild-eyed Senator Adam Schiff. At the same time they were calling for the disarmament of Americans, many of whom live a long way from a police response, Schumer, Schiff and the rest of these pampered clowns were sitting on Capitol Hill behind a razor-wired security fence protected by thousands of armed troops in addition to their own armed bodyguards. Mom and pop in an isolated home? They're just potential mass shooters.

Zion Mainframe

Sabbatian-Frankists and most importantly the Rothschilds were behind the creation of 'Zionism', a political movement that demanded a Jewish homeland in Israel as promised by Sabbatai Zevi. The very symbol of Israel comes from the German meaning of the name Rothschild. Dynasty founder Mayer Amschel Rothschild changed the family name from Bauer to Rothschild, or 'Red-Shield' in German, in deference to the six-pointed 'Star of David' hexagram displayed on the family's home in Frankfurt. The symbol later appeared on the flag of Israel after the Rothschilds were centrally involved in its creation. Hexagrams are not a uniquely Jewish symbol and are widely used in occult ('hidden') networks often as a symbol for Saturn (see my other books for why). Neither are Zionism and Jewishness interchangeable. Zionism is a political movement and philosophy and not a 'race' or a people. Many Jews oppose Zionism and many non-Jews, including US President Joe Biden, call themselves Zionists as does Israel-centric Donald Trump. America's support for the Israel government is pretty much a gimme with ultra-Zionist billionaires and corporations providing fantastic and dominant funding for both political parties. Former Congresswoman Cynthia McKinney has told how she was approached immediately she ran for office to 'sign the pledge' to Israel and confirm that she would always vote in that country's best interests. All American politicians are approached in this way. Anyone who refuses will get no support or funding from the enormous and all-powerful Zionist lobby that includes organisations like mega-lobby group AIPAC, the American Israel Public Affairs Committee. Trump's biggest funder was ultra-Zionist casino and media billionaire Sheldon Adelson while major funders of the Democratic Party include ultra-Zionist George Soros and ultra-Zionist financial and media mogul, Haim Saban. Some may reel back at the suggestion that Soros is an Israel-firster (Sabbatian-controlled Israel-firster), but Renegade Minds watch the actions not the words and everywhere Soros donates his billions the Sabbatian agenda benefits. In the spirit of Sabbatian inversion Soros pledged $1 billion for a new university network to promote 'liberal values and tackle intolerance'. He made the announcement during his annual speech

at the Cult-owned World Economic Forum in Davos, Switzerland, in January, 2020, after his 'harsh criticism' of 'authoritarian rulers' around the world. You can only laugh at such brazen mendacity. How *he* doesn't laugh is the mystery. Translated from the Orwellian 'liberal values and tackle intolerance' means teaching non-white people to hate white people and for white people to loathe themselves for being born white. The reason for that will become clear.

The 'Anti-Semitism' fraud

Zionists support the Jewish homeland in the land of Palestine which has been the Sabbatian-Rothschild goal for so long, but not for the benefit of Jews. Sabbatians and their global Anti-Semitism Industry have skewed public and political opinion to equate opposing the violent extremes of Zionism to be a blanket attack and condemnation of all Jewish people. Sabbatians and their global Anti-Semitism Industry have skewed public and political opinion to equate opposing the violent extremes of Zionism to be a blanket attack and condemnation of all Jewish people. This is nothing more than a Sabbatian protection racket to stop legitimate investigation and exposure of their agendas and activities. The official definition of 'anti-Semitism' has more recently been expanded to include criticism of Zionism – a *political movement* – and this was done to further stop exposure of Sabbatian infiltrators who created Zionism as we know it today in the 19th century. Renegade Minds will talk about these subjects when they know the shit that will come their way. People must decide if they want to know the truth or just cower in the corner in fear of what others will say. Sabbatians have been trying to label me as 'anti-Semitic' since the 1990s as I have uncovered more and more about their background and agendas. Useless, gutless, fraudulent 'journalists' then just repeat the smears without question and on the day I was writing this section a pair of unquestioning repeaters called Ben Quinn and Archie Bland (how appropriate) outright called me an 'anti-Semite' in the establishment propaganda sheet, the London *Guardian*, with no supporting evidence. The

Sabbatian Anti-Semitism Industry said so and who are they to question that? They wouldn't dare. Ironically 'Semitic' refers to a group of languages in the Middle East that are almost entirely Arabic. 'Anti-Semitism' becomes 'anti-Arab' which if the consequences of this misunderstanding were not so grave would be hilarious. Don't bother telling Quinn and Bland. I don't want to confuse them, bless 'em. One reason I am dubbed 'anti-Semitic' is that I wrote in the 1990s that Jewish operatives (Sabbatians) were heavily involved in the Russian Revolution when Sabbatians overthrew the Romanov dynasty. This apparently made me 'anti-Semitic'. Oh, really? Here is a section from *The Trigger*:

British journalist Robert Wilton confirmed these themes in his 1920 book *The Last Days of the Romanovs* when he studied official documents from the Russian government to identify the members of the Bolshevik ruling elite between 1917 and 1919. The Central Committee included 41 Jews among 62 members; the Council of the People's Commissars had 17 Jews out of 22 members; and 458 of the 556 most important Bolshevik positions between 1918 and 1919 were occupied by Jewish people. Only 17 were Russian. Then there were the 23 Jews among the 36 members of the vicious Cheka Soviet secret police established in 1917 who would soon appear all across the country.

Professor Robert Service of Oxford University, an expert on 20th century Russian history, found evidence that ['Jewish'] Leon Trotsky had sought to make sure that Jews were enrolled in the Red Army and were disproportionately represented in the Soviet civil bureaucracy that included the Cheka which performed mass arrests, imprisonment and executions of 'enemies of the people'. A US State Department Decimal File (861.00/5339) dated November 13th, 1918, names [Rothschild banking agent in America] Jacob Schiff and a list of ultra-Zionists as funders of the Russian Revolution leading to claims of a 'Jewish plot', but the key point missed by all is they were not 'Jews' – they were Sabbatian-Frankists.

Britain's Winston Churchill made the same error by mistake or otherwise. He wrote in a 1920 edition of the *Illustrated Sunday Herald* that those behind the Russian revolution were part of a 'worldwide conspiracy for the overthrow of civilisation and for the reconstitution of society on the basis of arrested development, of envious malevolence, and impossible equality' (see 'Woke' today because that has been created by the same network). Churchill said there was no need to exaggerate the part played in the creation of Bolshevism and in the actual bringing about of the Russian

Revolution 'by these international and for the most part atheistical Jews' ['atheistical Jews' = Sabbatians]. Churchill said it is certainly a very great one and probably outweighs all others: 'With the notable exception of Lenin, the majority of the leading figures are Jews.' He went on to describe, knowingly or not, the Sabbatian modus operandi of placing puppet leaders nominally in power while they control from the background:

> Moreover, the principal inspiration and driving power comes from the Jewish leaders. Thus Tchitcherin, a pure Russian, is eclipsed by his nominal subordinate, Litvinoff, and the influence of Russians like Bukharin or Lunacharski cannot be compared with the power of Trotsky, or of Zinovieff, the Dictator of the Red Citadel (Petrograd), or of Krassin or Radek – all Jews. In the Soviet institutions the predominance of Jews is even more astonishing. And the prominent, if not indeed the principal, part in the system of terrorism applied by the Extraordinary Commissions for Combatting Counter-Revolution has been taken by Jews, and in some notable cases by Jewesses.

What I said about seriously disproportionate involvement in the Russian Revolution by Jewish 'revolutionaries' (Sabbatians) is provable fact, but truth is no defence against the Sabbatian Anti-Semitism Industry, its repeater parrots like Quinn and Bland, and the now breathtaking network of so-called 'Woke' 'anti-hate' groups with interlocking leaderships and funding which have the role of discrediting and silencing anyone who gets too close to exposing the Sabbatians. We have seen 'truth is no defence' confirmed in legal judgements with the Saskatchewan Human Rights Commission in Canada decreeing this: 'Truthful statements can be presented in a manner that would meet the definition of hate speech, and not all truthful statements must be free from restriction.' Most 'anti-hate' activists, who are themselves consumed by hatred, are too stupid and ignorant of the world to know how they are being used. They are far too far up their own virtue-signalling arses and it's far too dark for them to see anything.

The 'revolution' game

The background and methods of the 'Russian' Revolution are straight from the Sabbatian playbook seen in the French Revolution

and endless others around the world that appear to start as a revolution of the people against tyrannical rule and end up with a regime change to more tyrannical rule overtly or covertly. Wars, terror attacks and regime overthrows follow the Sabbatian cult through history with its agents creating them as Problem-Reaction-Solutions to remove opposition on the road to world domination. Sabbatian dots connect the Rothschilds with the Illuminati, Jacobins of the French Revolution, the 'Bund' or League of the Just, the International Communist Party, Communist League and the Communist Manifesto of Karl Marx and Friedrich Engels that would lead to the Rothschild-funded Russian Revolution. The sequence comes under the heading of 'creative destruction' when you advance to your global goal by continually destroying the status quo to install a new status quo which you then also destroy. The two world wars come to mind. With each new status quo you move closer to your planned outcome. Wars and mass murder are to Sabbatians a collective blood sacrifice ritual. They are obsessed with death for many reasons and one is that death is an inversion of life. Satanists and Sabbatians are obsessed with death and often target churches and churchyards for their rituals. Inversion-obsessed Sabbatians explain the use of inverted symbolism including the *inverted* pentagram and *inverted* cross. The inversion of the cross has been related to targeting Christianity, but the cross was a religious symbol long before Christianity and its inversion is a statement about the Sabbatian mentality and goals more than any single religion.

Sabbatians operating in Germany were behind the rise of the occult-obsessed Nazis and the subsequent Jewish exodus from Germany and Europe to Palestine and the United States after World War Two. The Rothschild dynasty was at the forefront of this both as political manipulators and by funding the operation. Why would Sabbatians help to orchestrate the horrors inflicted on Jews by the Nazis and by Stalin after they organised the Russian Revolution? Sabbatians hate Jews and their religion, that's why. They pose as Jews and secure positions of control within Jewish society and play the 'anti-Semitism' card to protect themselves from exposure

through a global network of organisations answering to the Sabbatian-created-and-controlled globe-spanning intelligence network that involves a stunning web of military-intelligence operatives and operations for a tiny country of just nine million. Among them are Jewish assets who are not Sabbatians but have been convinced by them that what they are doing is for the good of Israel and the Jewish community to protect them from what they have been programmed since childhood to believe is a Jew-hating hostile world. The Jewish community is just a highly convenient cover to hide the true nature of Sabbatians. Anyone getting close to exposing their game is accused by Sabbatian place-people and gofers of 'anti-Semitism' and claiming that all Jews are part of a plot to take over the world. I am not saying that. I am saying that Sabbatians – the *real* Jew-haters – have infiltrated the Jewish community to use them both as a cover and an 'anti-Semitic' defence against exposure. Thus we have the Anti-Semitism Industry targeted researchers in this way and most Jewish people think this is justified and genuine. They don't know that their 'Jewish' leaders and institutions of state, intelligence and military are not controlled by Jews at all, but cultists and stooges of Sabbatian-Frankism. I once added my name to a pro-Jewish freedom petition online and the next time I looked my name was gone and text had been added to the petition blurb to attack me as an 'anti-Semite' such is the scale of perceptual programming.

Moving on America

I tell the story in *The Trigger* and a chapter called 'Atlantic Crossing' how particularly after Israel was established the Sabbatians moved in on the United States and eventually grasped control of government administration, the political system via both Democrats and Republicans, the intelligence community like the CIA and National Security Agency (NSA), the Pentagon and mass media. Through this seriously compartmentalised network Sabbatians and their operatives in Mossad, Israeli Defense Forces (IDF) and US agencies pulled off 9/11 and blamed it on 19 'Al-Qaeda hijackers' dominated by men from, or connected to, Sabbatian-ruled Saudi

Arabia. The '19' were not even on the planes let alone flew those big passenger jets into buildings while being largely incompetent at piloting one-engine light aircraft. 'Hijacker' Hani Hanjour who is said to have flown American Airlines Flight 77 into the Pentagon with a turn and manoeuvre most professional pilots said they would have struggled to do was banned from renting a small plane by instructors at the Freeway Airport in Bowie, Maryland, just *six weeks* earlier on the grounds that he was an incompetent pilot. The Jewish population of the world is just 0.2 percent with even that almost entirely concentrated in Israel (75 percent Jewish) and the United States (around two percent). This two percent and globally 0.2 percent refers to *Jewish* people and not Sabbatian interlopers who are a fraction of that fraction. What a sobering thought when you think of the fantastic influence on world affairs of tiny Israel and that the Project for the New America Century (PNAC) which laid out the blueprint in September, 2000, for America's war on terror and regime change wars in Iraq, Libya and Syria was founded and dominated by Sabbatians known as 'Neocons'. The document conceded that this plan would not be supported politically or publicly without a major attack on American soil and a Problem-Reaction-Solution excuse to send troops to war across the Middle East. Sabbatian Neocons said:

> ... [The] process of transformation ... [war and regime change] ... is likely to be a long one, absent some catastrophic and catalysing event – like a new Pearl Harbor.

Four months later many of those who produced that document came to power with their inane puppet George Bush from the long-time Sabbatian Bush family. They included Sabbatian Dick Cheney who was officially vice-president, but really de-facto president for the entirety of the 'Bush' government. Nine months after the 'Bush' inauguration came what Bush called at the time 'the Pearl Harbor of the 21st century' and with typical Sabbatian timing and symbolism 2001 was the 60th anniversary of the attack in 1941 by the Japanese Air Force on Pearl Harbor, Hawaii, which allowed President Franklin Delano Roosevelt to take the United States into a Sabbatian-

instigated Second World War that he said in his election campaign that he never would. The evidence is overwhelming that Roosevelt and his military and intelligence networks knew the attack was coming and did nothing to stop it, but they did make sure that America's most essential naval ships were not in Hawaii at the time. Three thousand Americans died in the Pearl Harbor attacks as they did on September 11th. By the 9/11 year of 2001 Sabbatians had widely infiltrated the US government, military and intelligence operations and used their compartmentalised assets to pull off the 'Al-Qaeda' attacks. If you read *The Trigger* it will blow your mind to see the utterly staggering concentration of 'Jewish' operatives (Sabbatian infiltrators) in essential positions of political, security, legal, law enforcement, financial and business power before, during, and after the attacks to make them happen, carry them out, and then cover their tracks – and I do mean *staggering* when you think of that 0.2 percent of the world population and two percent of Americans which are Jewish while Sabbatian infiltrators are a fraction of that. A central foundation of the 9/11 conspiracy was the hijacking of government, military, Air Force and intelligence computer systems in real time through 'back-door' access made possible by Israeli (Sabbatian) 'cyber security' software. Sabbatian-controlled Israel is on the way to rivalling Silicon Valley for domination of cyberspace and is becoming the dominant force in cyber-security which gives them access to entire computer systems and their passcodes across the world. Then add to this that Zionists head (officially) Silicon Valley giants like Google (Larry Page and Sergey Brin), Google-owned YouTube (Susan Wojcicki), Facebook (Mark Zuckerberg and Sheryl Sandberg), and Apple (Chairman Arthur D. Levinson), and that ultra-Zionist hedge fund billionaire Paul Singer has a $1 billion stake in Twitter which is only nominally headed by 'CEO' pothead Jack Dorsey. As cable news host Tucker Carlson said of Dorsey: 'There used to be debate in the medical community whether dropping a ton of acid had permanent effects and I think that debate has now ended.' Carlson made the comment after Dorsey told a hearing on Capitol Hill (if you cut through his bullshit) that he

believed in free speech so long as he got to decide what you can hear and see. These 'big names' of Silicon Valley are only front men and women for the Global Cult, not least the Sabbatians, who are the true controllers of these corporations. Does anyone still wonder why these same people and companies have been ferociously censoring and banning people (like me) for exposing any aspect of the Cult agenda and especially the truth about the 'Covid' hoax which Sabbatians have orchestrated?

The Jeffrey Epstein paedophile ring was a Sabbatian operation. He was officially 'Jewish' but he was a Sabbatian and women abused by the ring have told me about the high number of 'Jewish' people involved. The Epstein horror has Sabbatian written all over it and matches perfectly their modus operandi and obsession with sex and ritual. Epstein was running a Sabbatian blackmail ring in which famous people with political and other influence were provided with young girls for sex while everything was being filmed and recorded on hidden cameras and microphones at his New York house, Caribbean island and other properties. Epstein survivors have described this surveillance system to me and some have gone public. Once the famous politician or other figure knew he or she was on video they tended to do whatever they were told. Here we go again …when you've got them by the balls their hearts and minds will follow. Sabbatians use this blackmail technique on a wide scale across the world to entrap politicians and others they need to act as demanded. Epstein's private plane, the infamous 'Lolita Express', had many well-known passengers including Bill Clinton while Bill Gates has flown on an Epstein plane and met with him four years after Epstein had been jailed for paedophilia. They subsequently met many times at Epstein's home in New York according to a witness who was there. Epstein's infamous side-kick was Ghislaine Maxwell, daughter of Mossad agent and ultra-Zionist mega-crooked British businessman, Bob Maxwell, who at one time owned the *Daily Mirror* newspaper. Maxwell was murdered at sea on his boat in 1991 by Sabbatian-controlled Mossad when he became a liability with his

business empire collapsing as a former Mossad operative has confirmed (see *The Trigger*).

Money, money, money, funny money …

Before I come to the Sabbatian connection with the last three US presidents I will lay out the crucial importance to Sabbatians of controlling banking and finance. Sabbatian Mayer Amschel Rothschild set out to dominate this arena in his family's quest for total global control. What is freedom? It is, in effect, choice. The more choices you have the freer you are and the fewer your choices the more you are enslaved. In the global structure created over centuries by Sabbatians the biggest decider and restrictor of choice is … money. Across the world if you ask people what they would like to do with their lives and why they are not doing that they will reply 'I don't have the money'. This is the idea. A global elite of multi-billionaires are described as 'greedy' and that is true on one level; but control of money – who has it and who doesn't – is not primarily about greed. It's about control. Sabbatians have seized ever more control of finance and sucked the wealth of the world out of the hands of the population. We talk now, after all, about the 'One-percent' and even then the wealthiest are a lot fewer even than that. This has been made possible by a money scam so outrageous and so vast it could rightly be called the scam of scams founded on creating 'money' out of nothing and 'loaning' that with interest to the population. Money out of nothing is called 'credit'. Sabbatians have asserted control over governments and banking ever more completely through the centuries and secured financial laws that allow banks to lend hugely more than they have on deposit in a confidence trick known as fractional reserve lending. Imagine if you could lend money that doesn't exist and charge the recipient interest for doing so. You would end up in jail. Bankers by contrast end up in mansions, private jets, Malibu and Monaco.

Banks are only required to keep a fraction of their deposits and wealth in their vaults and they are allowed to lend 'money' they don't have called 'credit. Go into a bank for a loan and if you succeed

the banker will not move any real wealth into your account. They will type into your account the amount of the agreed 'loan' – say £100,000. This is not wealth that really exists; it is non-existent, fresh-air, created-out-of-nothing 'credit' which has never, does not, and will never exist except in theory. Credit is backed by nothing except wind and only has buying power because people think that it has buying power and accept it in return for property, goods and services. I have described this situation as like those cartoon characters you see chasing each other and when they run over the edge of a cliff they keep running forward on fresh air until one of them looks down, realises what's happened, and they all crash into the ravine. The whole foundation of the Sabbatian financial system is to stop people looking down except for periodic moments when they want to crash the system (as in 2008 and 2020 ongoing) and reap the rewards from all the property, businesses and wealth their borrowers had signed over as 'collateral' in return for a 'loan' of fresh air. Most people think that money is somehow created by governments when it comes into existence from the start as a debt through banks 'lending' illusory money called credit. Yes, the very currency of exchange is a *debt* from day one issued as an interest-bearing loan. Why don't governments create money interest-free and lend it to their people interest-free? Governments are controlled by Sabbatians and the financial system is controlled by Sabbatians for whom interest-free money would be a nightmare come true. Sabbatians underpin their financial domination through their global network of central banks, including the privately-owned US Federal Reserve and Britain's Bank of England, and this is orchestrated by a privately-owned central bank coordination body called the Bank for International Settlements in Basle, Switzerland, created by the usual suspects including the Rockefellers and Rothschilds. Central bank chiefs don't answer to governments or the people. They answer to the Bank for International Settlements or, in other words, the Global Cult which is dominated today by Sabbatians.

Built-in disaster

There are so many constituent scams within the overall banking scam. When you take out a loan of thin-air credit only the amount of that loan is theoretically brought into circulation to add to the amount in circulation; but you are paying back the principle plus interest. The additional interest is not created and this means that with every 'loan' there is a shortfall in the money in circulation between what is borrowed and what has to be paid back. There is never even close to enough money in circulation to repay all outstanding public and private debt including interest. Coldly weaved in the very fabric of the system is the certainty that some will lose their homes, businesses and possessions to the banking 'lender'. This is less obvious in times of 'boom' when the amount of money in circulation (and the debt) is expanding through more people wanting and getting loans. When a downturn comes and the money supply contracts it becomes painfully obvious that there is not enough money to service all debt and interest. This is less obvious in times of 'boom' when the amount of money in circulation (and the debt) is expanding through more people wanting and getting loans. When a downturn comes and the money supply contracts and it becomes painfully obvious – as in 2008 and currently – that there is not enough money to service all debt and interest. Sabbatian banksters have been leading the human population through a calculated series of booms (more debt incurred) and busts (when the debt can't be repaid and the banks get the debtor's tangible wealth in exchange for non-existent 'credit'). With each 'bust' Sabbatian bankers have absorbed more of the world's tangible wealth and we end up with the One-percent. Governments are in bankruptcy levels of debt to the same system and are therefore owned by a system they do not control. The Federal Reserve, 'America's central bank', is privately-owned and American presidents only nominally appoint its chairman or woman to maintain the illusion that it's an arm of government. It's not. The 'Fed' is a cartel of private banks which handed billions to its associates and friends after the crash of 2008 and has been Sabbatian-controlled since it was manipulated into being in 1913 through the covert trickery of Rothschild banking agents Jacob Schiff and Paul

Warburg, and the Sabbatian Rockefeller family. Somehow from a Jewish population of two-percent and globally 0.2 percent (Sabbatian interlopers remember are far smaller) ultra-Zionists headed the Federal Reserve for 31 years between 1987 and 2018 in the form of Alan Greenspan, Bernard Bernanke and Janet Yellen (now Biden's Treasury Secretary) with Yellen's deputy chairman a Israeli-American duel citizen and ultra-Zionist Stanley Fischer, a former governor of the Bank of Israel. Ultra-Zionist Fed chiefs spanned the presidencies of Ronald Reagan ('Republican'), Father George Bush ('Republican'), Bill Clinton ('Democrat'), Boy George Bush ('Republican') and Barack Obama ('Democrat'). We should really add the pre-Greenspan chairman, Paul Adolph Volcker, 'appointed' by Jimmy Carter ('Democrat') who ran the Fed between 1979 and 1987 during the Carter and Reagan administrations before Greenspan took over. Volcker was a long-time associate and business partner of the Rothschilds. No matter what the 'party' officially in power the United States economy was directed by the same force. Here are members of the Obama, Trump and Biden administrations and see if you can make out a common theme.

Barack Obama ('Democrat')

Ultra-Zionists Robert Rubin, Larry Summers, and Timothy Geithner ran the US Treasury in the Clinton administration and two of them reappeared with Obama. Ultra-Zionist Fed chairman Alan Greenspan had manipulated the crash of 2008 through deregulation and jumped ship just before the disaster to make way for ultra-Zionist Bernard Bernanke to hand out trillions to Sabbatian 'too big to fail' banks and businesses, including the ubiquitous ultra-Zionist Goldman Sachs which has an ongoing staff revolving door operation between itself and major financial positions in government worldwide. Obama inherited the fallout of the crash when he took office in January, 2009, and fortunately he had the support of his ultra-Zionist White House Chief of Staff Rahm Emmanuel, son of a terrorist who helped to bomb Israel into being in 1948, and his ultra-Zionist senior adviser David Axelrod, chief strategist in Obama's two

successful presidential campaigns. Emmanuel, later mayor of Chicago and former senior fundraiser and strategist for Bill Clinton, is an example of the Sabbatian policy after Israel was established of migrating insider families to America so their children would be born American citizens. 'Obama' chose this financial team throughout his administration to respond to the Sabbatian-instigated crisis:

Timothy Geithner (ultra-Zionist) Treasury Secretary; Jacob J. Lew, Treasury Secretary; Larry Summers (ultra-Zionist), director of the White House National Economic Council; Paul Adolph Volcker (Rothschild business partner), chairman of the Economic Recovery Advisory Board; Peter Orszag (ultra-Zionist), director of the Office of Management and Budget overseeing all government spending; Penny Pritzker (ultra-Zionist), Commerce Secretary; Jared Bernstein (ultra-Zionist), chief economist and economic policy adviser to Vice President Joe Biden; Mary Schapiro (ultra-Zionist), chair of the Securities and Exchange Commission (SEC); Gary Gensler (ultra-Zionist), chairman of the Commodity Futures Trading Commission (CFTC); Sheila Bair (ultra-Zionist), chair of the Federal Deposit Insurance Corporation (FDIC); Karen Mills (ultra-Zionist), head of the Small Business Administration (SBA); Kenneth Feinberg (ultra-Zionist), Special Master for Executive [bail-out] Compensation. Feinberg would be appointed to oversee compensation (with strings) to 9/11 victims and families in a campaign to stop them having their day in court to question the official story. At the same time ultra-Zionist Bernard Bernanke was chairman of the Federal Reserve and these are only some of the ultra-Zionists with allegiance to Sabbatian-controlled Israel in the Obama government. Obama's biggest corporate donor was ultra-Zionist Goldman Sachs which had employed many in his administration.

Donald Trump ('Republican')

Trump claimed to be an outsider (he wasn't) who had come to 'drain the swamp'. He embarked on this goal by immediately appointing ultra-Zionist Steve Mnuchin, a Goldman Sachs employee for 17

years, as his Treasury Secretary. Others included Gary Cohn (ultra-Zionist), chief operating officer of Goldman Sachs, his first Director of the National Economic Council and chief economic adviser, who was later replaced by Larry Kudlow (ultra-Zionist). Trump's senior adviser throughout his four years in the White House was his sinister son-in-law Jared Kushner, a life-long friend of Israel Prime Minister Benjamin Netanyahu. Kushner is the son of a convicted crook who was pardoned by Trump in his last days in office. Other ultra-Zionists in the Trump administration included: Stephen Miller, Senior Policy Adviser; Avrahm Berkowitz, Deputy Adviser to Trump and his Senior Adviser Jared Kushner; Ivanka Trump, Adviser to the President, who converted to Judaism when she married Jared Kushner; David Friedman, Trump lawyer and Ambassador to Israel; Jason Greenblatt, Trump Organization executive vice president and chief legal officer, who was made Special Representative for International Negotiations and the Israeli-Palestinian Conflict; Rod Rosenstein, Deputy Attorney General; Elliot Abrams, Special Representative for Venezuela, then Iran; John Eisenberg, National Security Council Legal Adviser and Deputy Council to the President for National Security Affairs; Anne Neuberger, Deputy National Manager, National Security Agency; Ezra Cohen-Watnick, Acting Under Secretary of Defense for Intelligence; Elan Carr, Special Envoy to monitor and combat anti-Semitism; Len Khodorkovsky, Deputy Special Envoy to monitor and combat anti-Semitism; Reed Cordish, Assistant to the President, Intragovernmental and Technology Initiatives. Trump Vice President Mike Pence and Secretary of State Mike Pompeo, both Christian Zionists, were also vehement supporters of Israel and its goals and ambitions.

Donald 'free-speech believer' Trump pardoned a number of financial and violent criminals while ignoring calls to pardon Julian Assange and Edward Snowden whose crimes are revealing highly relevant information about government manipulation and corruption and the widespread illegal surveillance of the American people by US 'security' agencies. It's so good to know that Trump is on the side of freedom and justice and not mega-criminals with

allegiance to Sabbatian-controlled Israel. These included a pardon for Israeli spy Jonathan Pollard who was jailed for life in 1987 under the Espionage Act. Aviem Sella, the Mossad agent who recruited Pollard, was also pardoned by Trump while Assange sat in jail and Snowden remained in exile in Russia. Sella had 'fled' (was helped to escape) to Israel in 1987 and was never extradited despite being charged under the Espionage Act. A Trump White House statement said that Sella's clemency had been 'supported by Benjamin Netanyahu, Ron Dermer, Israel's US Ambassador, David Friedman, US Ambassador to Israel and Miriam Adelson, wife of leading Trump donor Sheldon Adelson who died shortly before. Other friends of Jared Kushner were pardoned along with Sholom Weiss who was believed to be serving the longest-ever white-collar prison sentence of more than 800 years in 2000. The sentence was commuted of Ponzi-schemer Eliyahu Weinstein who defrauded Jews and others out of $200 million. I did mention that Assange and Snowden were ignored, right? Trump gave Sabbatians almost everything they asked for in military and political support, moving the US Embassy from Tel Aviv to Jerusalem with its critical symbolic and literal implications for Palestinian statehood, and the 'deal of the Century' designed by Jared Kushner and David Friedman which gave the Sabbatian Israeli government the green light to substantially expand its already widespread program of building illegal Jewish-only settlements in the occupied land of the West Bank. This made a two-state 'solution' impossible by seizing all the land of a potential Palestinian homeland and that had been the plan since 1948 and then 1967 when the Arab-controlled Gaza Strip, West Bank, Sinai Peninsula and Syrian Golan Heights were occupied by Israel. All the talks about talks and road maps and delays have been buying time until the West Bank was physically occupied by Israeli real estate. Trump would have to be a monumentally ill-informed idiot not to see that this was the plan he was helping to complete. The Trump administration was in so many ways the Kushner administration which means the Netanyahu administration which means the Sabbatian administration. I understand why many opposing Cult fascism in all its forms gravitated to Trump, but he

was a crucial part of the Sabbatian plan and I will deal with this in the next chapter.

Joe Biden ('Democrat')

A barely cognitive Joe Biden took over the presidency in January, 2021, along with his fellow empty shell, Vice-President Kamala Harris, as the latest Sabbatian gofers to enter the White House. Names on the door may have changed and the 'party' – the force behind them remained the same as Zionists were appointed to a stream of pivotal areas relating to Sabbatian plans and policy. They included: Janet Yellen, Treasury Secretary, former head of the Federal Reserve, and still another ultra-Zionist running the US Treasury after Mnuchin (Trump), Lew and Geithner (Obama), and Summers and Rubin (Clinton); Anthony Blinken, Secretary of State; Wendy Sherman, Deputy Secretary of State (so that's 'Biden's' Sabbatian foreign policy sorted); Jeff Zients, White House coronavirus coordinator; Rochelle Walensky, head of the Centers for Disease Control; Rachel Levine, transgender deputy health secretary (that's 'Covid' hoax policy under control); Merrick Garland, Attorney General; Alejandro Mayorkas, Secretary of Homeland Security; Cass Sunstein, Homeland Security with responsibility for new immigration laws; Avril Haines, Director of National Intelligence; Anne Neuberger, National Security Agency cybersecurity director (note, cybersecurity); David Cohen, CIA Deputy Director; Ronald Klain, Biden's Chief of Staff (see Rahm Emanuel); Eric Lander, a 'leading geneticist', Office of Science and Technology Policy director (see Smart Grid, synthetic biology agenda); Jessica Rosenworcel, acting head of the Federal Communications Commission (FCC) which controls Smart Grid technology policy and electromagnetic communication systems including 5G. How can it be that so many pivotal positions are held by two-percent of the American population and 0.2 percent of the world population administration after administration no matter who is the president and what is the party? It's a coincidence? Of course it's not and this is why Sabbatians have built their colossal global web of interlocking 'anti-

hate' hate groups to condemn anyone who asks these glaring questions as an 'anti-Semite'. The way that Jewish people horrifically abused in Sabbatian-backed Nazi Germany are exploited to this end is stomach-turning and disgusting beyond words.

Political fusion

Sabbatian manipulation has reversed the roles of Republicans and Democrats and the same has happened in Britain with the Conservative and Labour Parties. Republicans and Conservatives were always labelled the 'right' and Democrats and Labour the 'left', but look at the policy positions now and the Democrat-Labour 'left' has moved further to the 'right' than Republicans and Conservatives under the banner of 'Woke', the Cult-created far-right tyranny. Where once the Democrat-Labour 'left' defended free speech and human rights they now seek to delete them and as I said earlier despite the 'Covid' fascism of the Jackboot Johnson Conservative government in the UK the Labour Party of leader Keir Starmer demanded even more extreme measures. The Labour Party has been very publicly absorbed by Sabbatians after a political and media onslaught against the previous leader, the weak and inept Jeremy Corbyn, over made-up allegations of 'anti-Semitism' both by him and his party. The plan was clear with this 'anti-Semite' propaganda and what was required in response was a swift and decisive 'fuck off' from Corbyn and a statement to expose the Anti-Semitism Industry (Sabbatian) attempt to silence Labour criticism of the Israeli government (Sabbatians) and purge the party of all dissent against the extremes of ultra-Zionism (Sabbatians). Instead Corbyn and his party fell to their knees and appeased the abusers which, by definition, is impossible. Appeasing one demand leads only to a new demand to be appeased until takeover is complete. Like I say – 'fuck off' would have been a much more effective policy and I have used it myself with great effect over the years when Sabbatians are on my case which is most of the time. I consider that fact a great compliment, by the way. The outcome of the Labour Party capitulation is that we now have a Sabbatian-controlled

Conservative Party 'opposed' by a Sabbatian-controlled Labour Party in a one-party Sabbatian state that hurtles towards the extremes of tyranny (the Sabbatian cult agenda). In America the situation is the same. Labour's Keir Starmer spends his days on his knees with his tongue out pointing to Tel Aviv, or I guess now Jerusalem, while Boris Johnson has an 'anti-Semitism czar' in the form of former Labour MP John Mann who keeps Starmer company on his prayer mat.

Sabbatian influence can be seen in Jewish members of the Labour Party who have been ejected for criticism of Israel including those from families that suffered in Nazi Germany. Sabbatians despise real Jewish people and target them even more harshly because it is so much more difficult to dub them 'anti-Semitic' although in their desperation they do try.

CHAPTER THREE

The Pushbacker sting

Until you realize how easy it is for your mind to be manipulated, you remain the puppet of someone else's game
Evita Ochel

I will use the presidencies of Trump and Biden to show how the manipulation of the one-party state plays out behind the illusion of political choice across the world. No two presidencies could – on the face of it – be more different and apparently at odds in terms of direction and policy.

A Renegade Mind sees beyond the obvious and focuses on outcomes and consequences and not image, words and waffle. The Cult embarked on a campaign to divide America between those who blindly support its agenda (the mentality known as 'Woke') and those who are pushing back on where the Cult and its Sabbatians want to go. This presents infinite possibilities for dividing and ruling the population by setting them at war with each other and allows a perceptual ring fence of demonisation to encircle the Pushbackers in a modern version of the Little Big Horn in 1876 when American cavalry led by Lieutenant Colonel George Custer were drawn into a trap, surrounded and killed by Native American tribes defending their land of thousands of years from being seized by the government. In this modern version the roles are reversed and it's those defending themselves from the Sabbatian government who are surrounded and the government that's seeking to destroy them. This trap was set years ago and to explain how we must return to 2016

and the emergence of Donald Trump as a candidate to be President of the United States. He set out to overcome the best part of 20 other candidates in the Republican Party before and during the primaries and was not considered by many in those early stages to have a prayer of living in the White House. The Republican Party was said to have great reservations about Trump and yet somehow he won the nomination. When you know how American politics works – politics in general – there is no way that Trump could have become the party's candidate unless the Sabbatian-controlled 'Neocons' that run the Republican Party wanted that to happen. We saw the proof in emails and documents made public by WikiLeaks that the Democratic Party hierarchy, or Democons, systematically undermined the campaign of Bernie Sanders to make sure that Sabbatian gofer Hillary Clinton won the nomination to be their presidential candidate. If the Democons could do that then the Neocons in the Republican Party could have derailed Trump in the same way. But they didn't and at that stage I began to conclude that Trump could well be the one chosen to be president. If that was the case the 'why' was pretty clear to see – the goal of dividing America between Cult agenda-supporting Wokers and Pushbackers who gravitated to Trump because he was telling them what they wanted to hear. His constituency of support had been increasingly ignored and voiceless for decades and profoundly through the eight years of Sabbatian puppet Barack Obama. Now here was someone speaking their language of pulling back from the incessant globalisation of political and economic power, the exporting of American jobs to China and elsewhere by 'American' (Sabbatian) corporations, the deletion of free speech, and the mass immigration policies that had further devastated job opportunities for the urban working class of all races and the once American heartlands of the Midwest.

Beware the forked tongue

Those people collectively sighed with relief that at last a political leader was apparently on their side, but another trait of the Renegade Mind is that you look even harder at people telling you

what you want to hear than those who are telling you otherwise. Obviously as I said earlier people wish what they want to hear to be true and genuine and they are much more likely to believe that than someone saying what they don't want to here and don't want to be true. Sales people are taught to be skilled in eliciting by calculated questioning what their customers want to hear and repeating that back to them as their own opinion to get their targets to like and trust them. Assets of the Cult are also sales people in the sense of selling perception. To read Cult manipulation you have to play the long and expanded game and not fall for the Vaudeville show of party politics. Both American parties are vehicles for the Cult and they exploit them in different ways depending on what the agenda requires at that moment. Trump and the Republicans were used to be the focus of dividing America and isolating Pushbackers to open the way for a Biden presidency to become the most extreme in American history by advancing the full-blown Woke (Cult) agenda with the aim of destroying and silencing Pushbackers now labelled Nazi Trump supporters and white supremacists.

Sabbatians wanted Trump in office for the reasons described by ultra-Zionist Saul Alinsky (1909-1972) who was promoting the Woke philosophy through 'community organising' long before anyone had heard of it. In those days it still went by its traditional name of Marxism. The reason for the manipulated Trump phenomenon was laid out in Alinsky's 1971 book, *Rules for Radicals*, which was his blueprint for overthrowing democratic and other regimes and replacing them with Sabbatian Marxism. Not surprisingly his to-do list was evident in the Sabbatian French and Russian 'Revolutions' and that in China which will become very relevant in the next chapter about the 'Covid' hoax. Among Alinsky's followers have been the deeply corrupt Barack Obama, House Speaker Nancy Pelosi and Hillary Clinton who described him as a 'hero'. All three are Sabbatian stooges with Pelosi personifying the arrogant corrupt idiocy that so widely fronts up for the Cult inner core. Predictably as a Sabbatian advocate of the 'light-bringer' Alinsky features Lucifer on the dedication page of his book as the original radical who gained

his own kingdom ('Earth' as we shall see). One of Alinsky's golden radical rules was to pick an individual and focus all attention, hatred and blame on them and not to target faceless bureaucracies and corporations. *Rules for Radicals* is really a Sabbatian handbook with its contents repeatedly employed all over the world for centuries and why wouldn't Sabbatians bring to power their designer-villain to be used as the individual on which all attention, hatred and blame was bestowed? This is what they did and the only question for me is how much Trump knew that and how much he was manipulated. A bit of both, I suspect. This was Alinsky's Trump technique from a man who died in 1972. The technique has spanned history:

Pick the target, freeze it, personalize it, polarize it. Don't try to attack abstract corporations or bureaucracies. Identify a responsible individual. Ignore attempts to shift or spread the blame.

From the moment Trump came to illusory power everything was about him. It wasn't about Republican policy or opinion, but all about Trump. Everything he did was presented in negative, derogatory and abusive terms by the Sabbatian-dominated media led by Cult operations such as CNN, MSNBC, *The New York Times* and the Jeff Bezos-owned *Washington Post* – 'Pick the target, freeze it, personalize it, polarize it.' Trump was turned into a demon to be vilified by those who hated him and a demi-god loved by those who worshipped him. This, in turn, had his supporters, too, presented as equally demonic in preparation for the punchline later down the line when Biden was about to take office. It was here's a Trump, there's a Trump, everywhere a Trump, Trump. Virtually every news story or happening was filtered through the lens of 'The Donald'. You loved him or hated him and which one you chose was said to define you as Satan's spawn or a paragon of virtue. Even supporting some Trump policies or statements and not others was enough for an assault on your character. No shades of grey were or are allowed. Everything is black and white (literally and figuratively). A Californian I knew had her head utterly scrambled by her hatred for Trump while telling people they should love each other. She was so totally consumed by

Trump Derangement Syndrome as it became to be known that this glaring contradiction would never have occurred to her. By definition anyone who criticised Trump or praised his opponents was a hero and this lady described Joe Biden as 'a kind, honest gentleman' when he's a provable liar, mega-crook and vicious piece of work to boot. Sabbatians had indeed divided America using Trump as the fall-guy and all along the clock was ticking on the consequences for his supporters.

In hock to his masters

Trump gave Sabbatians via Israel almost everything they wanted in his four years. Ask and you shall receive was the dynamic between himself and Benjamin Netanyahu orchestrated by Trump's ultra-Zionist son-in-law Jared Kushner, his ultra-Zionist Ambassador to Israel, David Friedman, and ultra-Zionist 'Israel adviser', Jason Greenblatt. The last two were central to the running and protecting from collapse of his business empire, the Trump Organisation, and colossal business failures made him forever beholding to Sabbatian networks that bailed him out. By the start of the 1990s Trump owed $4 billion to banks that he couldn't pay and almost $1billion of that was down to him personally and not his companies. This mega-disaster was the result of building two new casinos in Atlantic City and buying the enormous Taj Mahal operation which led to crippling debt payments. He had borrowed fantastic sums from 72 banks with major Sabbatian connections and although the scale of debt should have had him living in a tent alongside the highway they never foreclosed. A plan was devised to lift Trump from the mire by BT Securities Corporation and Rothschild Inc. and the case was handled by Wilber Ross who had worked for the Rothschilds for 27 years. Ross would be named US Commerce Secretary after Trump's election. Another crucial figure in saving Trump was ultra-Zionist 'investor' Carl Icahn who bought the Taj Mahal casino. Icahn was made special economic adviser on financial regulation in the Trump administration. He didn't stay long but still managed to find time to make a tidy sum of a reported $31.3 million when he sold his

holdings affected by the price of steel three days before Trump imposed a 235 percent tariff on steel imports. What amazing bits of luck these people have. Trump and Sabbatian operatives have long had a close association and his mentor and legal adviser from the early 1970s until 1986 was the dark and genetically corrupt ultra-Zionist Roy Cohn who was chief counsel to Senator Joseph McCarthy's 'communist' witch-hunt in the 1950s. *Esquire* magazine published an article about Cohn with the headline 'Don't mess with Roy Cohn'. He was described as the most feared lawyer in New York and 'a ruthless master of dirty tricks ... [with] ... more than one Mafia Don on speed dial'. Cohn's influence, contacts, support and protection made Trump a front man for Sabbatians in New York with their connections to one of Cohn's many criminal employers, the 'Russian' Sabbatian Mafia. Israel-centric media mogul Rupert Murdoch was introduced to Trump by Cohn and they started a long friendship. Cohn died in 1986 weeks after being disbarred for unethical conduct by the Appellate Division of the New York State Supreme Court. The wheels of justice do indeed run slow given the length of Cohn's crooked career.

QAnon-sense

We are asked to believe that Donald Trump with his fundamental connections to Sabbatian networks and operatives has been leading the fight to stop the Sabbatian agenda for the fascistic control of America and the world. Sure he has. A man entrapped during his years in the White House by Sabbatian operatives and whose biggest financial donor was casino billionaire Sheldon Adelson who was Sabbatian to his DNA?? Oh, do come on. Trump has been used to divide America and isolate Pushbackers on the Cult agenda under the heading of 'Trump supporters', 'insurrectionists' and 'white supremacists'. The US Intelligence/Mossad Psyop or psychological operation known as QAnon emerged during the Trump years as a central pillar in the Sabbatian campaign to lead Pushbackers into the trap set by those that wished to destroy them. I knew from the start that QAnon was a scam because I had seen the same scenario many

times before over 30 years under different names and I had written about one in particular in the books. 'Not again' was my reaction when QAnon came to the fore. The same script is pulled out every few years and a new name added to the letterhead. The story always takes the same form: 'Insiders' or 'the good guys' in the government-intelligence-military 'Deep State' apparatus were going to instigate mass arrests of the 'bad guys' which would include the Rockefellers, Rothschilds, Barack Obama, Hillary Clinton, George Soros, etc., etc. Dates are given for when the 'good guys' are going to move in, but the dates pass without incident and new dates are given which pass without incident. The central message to Pushbackers in each case is that they don't have to do anything because there is 'a plan' and it is all going to be sorted by the 'good guys' on the inside. 'Trust the plan' was a QAnon mantra when the only plan was to misdirect Pushbackers into putting their trust in a Psyop they believed to be real. Beware, beware, those who tell you what you want to hear and always check it out. Right up to Biden's inauguration QAnon was still claiming that 'the Storm' was coming and Trump would stay on as president when Biden and his cronies were arrested and jailed. It was never going to happen and of course it didn't, but what did happen as a result provided that punchline to the Sabbatian Trump/QAnon Psyop.

On January 6th, 2021, a very big crowd of Trump supporters gathered in the National Mall in Washington DC down from the Capitol Building to protest at what they believed to be widespread corruption and vote fraud that stopped Trump being re-elected for a second term as president in November, 2020. I say as someone that does not support Trump or Biden that the evidence is clear that major vote-fixing went on to favour Biden, a man with cognitive problems so advanced he can often hardly string a sentence together without reading the words written for him on the Teleprompter. Glaring ballot discrepancies included serious questions about electronic voting machines that make vote rigging a comparative cinch and hundreds of thousands of paper votes that suddenly appeared during already advanced vote counts and virtually all of

them for Biden. Early Trump leads in crucial swing states suddenly began to close and disappear. The pandemic hoax was used as the excuse to issue almost limitless numbers of mail-in ballots with no checks to establish that the recipients were still alive or lived at that address. They were sent to streams of people who had not even asked for them. Private organisations were employed to gather these ballots and who knows what they did with them before they turned up at the counts. The American election system has been manipulated over decades to become a sick joke with more holes than a Swiss cheese for the express purpose of dictating the results. Then there was the criminal manipulation of information by Sabbatian tech giants like Facebook, Twitter and Google-owned YouTube which deleted pro-Trump, anti-Biden accounts and posts while everything in support of Biden was left alone. Sabbatians wanted Biden to win because after the dividing of America it was time for full-on Woke and every aspect of the Cult agenda to be unleashed.

Hunter gatherer

Extreme Silicon Valley bias included blocking information by the *New York Post* exposing a Biden scandal that should have ended his bid for president in the final weeks of the campaign. Hunter Biden, his monumentally corrupt son, is reported to have sent a laptop to be repaired at a local store and failed to return for it. Time passed until the laptop became the property of the store for non-payment of the bill. When the owner saw what was on the hard drive he gave a copy to the FBI who did nothing even though it confirmed widespread corruption in which the Joe Biden family were using his political position, especially when he was vice president to Obama, to make multiple millions in countries around the world and most notably Ukraine and China. Hunter Biden's one-time business partner Tony Bobulinski went public when the story broke in the *New York Post* to confirm the corruption he saw and that Joe Biden not only knew what was going on he also profited from the spoils. Millions were handed over by a Chinese company with close

connections – like all major businesses in China – to the Chinese communist party of President Xi Jinping. Joe Biden even boasted at a meeting of the Cult's World Economic Forum that as vice president he had ordered the government of Ukraine to fire a prosecutor. What he didn't mention was that the same man just happened to be investigating an energy company which was part of Hunter Biden's corrupt portfolio. The company was paying him big bucks for no other reason than the influence his father had. Overnight Biden's presidential campaign should have been over given that he had lied publicly about not knowing what his son was doing. Instead almost the entire Sabbatian-owned mainstream media and Sabbatian-owned Silicon Valley suppressed circulation of the story. This alone went a mighty way to rigging the election of 2020. Cult assets like Mark Zuckerberg at Facebook also spent hundreds of millions to be used in support of Biden and vote 'administration'.

The Cult had used Trump as the focus to divide America and was now desperate to bring in moronic, pliable, corrupt Biden to complete the double-whammy. No way were they going to let little things like the will of the people thwart their plan. Silicon Valley widely censored claims that the election was rigged because it *was* rigged. For the same reason anyone claiming it was rigged was denounced as a 'white supremacist' including the pathetically few Republican politicians willing to say so. Right across the media where the claim was mentioned it was described as a 'false claim' even though these excuses for 'journalists' would have done no research into the subject whatsoever. Trump won seven million more votes than any sitting president had ever achieved while somehow a cognitively-challenged soon to be 78-year-old who was hidden away from the public for most of the campaign managed to win more votes than any presidential candidate in history. It makes no sense. You only had to see election rallies for both candidates to witness the enthusiasm for Trump and the apathy for Biden. Tens of thousands would attend Trump events while Biden was speaking in empty car parks with often only television crews attending and framing their shots to hide the fact that no one was there. It was pathetic to see

footage come to light of Biden standing at a podium making speeches only to TV crews and party fixers while reading the words written for him on massive Teleprompter screens. So, yes, those protestors on January 6th had a point about election rigging, but some were about to walk into a trap laid for them in Washington by the Cult Deep State and its QAnon Psyop. This was the Capitol Hill riot ludicrously dubbed an 'insurrection'.

The spider and the fly

Renegade Minds know there are not two 'sides' in politics, only one side, the Cult, working through all 'sides'. It's a stage show, a puppet show, to direct the perceptions of the population into focusing on diversions like parties and candidates while missing the puppeteers with their hands holding all the strings. The Capitol Hill 'insurrection' brings us back to the Little Big Horn. Having created two distinct opposing groupings – Woke and Pushbackers – the trap was about to be sprung. Pushbackers were to be encircled and isolated by associating them all in the public mind with Trump and then labelling Trump as some sort of Confederate leader. I knew immediately that the Capitol riot was a set-up because of two things. One was how easy the rioters got into the building with virtually no credible resistance and secondly I could see – as with the 'Covid' hoax in the West at the start of 2020 – how the Cult could exploit the situation to move its agenda forward with great speed. My experience of Cult techniques and activities over more than 30 years has showed me that while they do exploit situations they haven't themselves created this never happens with events of fundamental agenda significance. Every time major events giving cultists the excuse to rapidly advance their plan you find they are manipulated into being for the specific reason of providing that excuse – Problem-Reaction-Solution. Only a tiny minority of the huge crowd of Washington protestors sought to gain entry to the Capitol by smashing windows and breaching doors. That didn't matter. The whole crowd and all Pushbackers, even if they did not support Trump, were going to be lumped together as dangerous

insurrectionists and conspiracy theorists. The latter term came into widespread use through a CIA memo in the 1960s aimed at discrediting those questioning the nonsensical official story of the Kennedy assassination and it subsequently became widely employed by the media. It's still being used by inept 'journalists' with no idea of its origin to discredit anyone questioning anything that authority claims to be true. When you are perpetrating a conspiracy you need to discredit the very word itself even though the dictionary definition of conspiracy is merely 'the activity of secretly planning with other people to do something bad or illegal' and 'a general agreement to keep silent about a subject for the purpose of keeping it secret'. On that basis there are conspiracies almost wherever you look. For obvious reasons the Cult and its lapdog media have to claim there are no conspiracies even though the word appears in state laws as with conspiracy to defraud, to murder, and to corrupt public morals.

Agent provocateurs are widely used by the Cult Deep State to manipulate genuine people into acting in ways that suit the desired outcome. By genuine in this case I mean protestors genuinely supporting Trump and claims that the election was stolen. In among them, however, were agents of the state wearing the garb of Trump supporters and QAnon to pump-prime the Capital riot which some genuine Trump supporters naively fell for. I described the situation as 'Come into my parlour said the spider to the fly'. Leaflets appeared through the Woke paramilitary arm Antifa, the anti-fascist fascists, calling on supporters to turn up in Washington looking like Trump supporters even though they hated him. Some of those arrested for breaching the Capitol Building were sourced to Antifa and its stable mate Black Lives Matter. Both organisations are funded by Cult billionaires and corporations. One man charged for the riot was according to his lawyer a former FBI agent who had held top secret security clearance for 40 years. Attorney Thomas Plofchan said of his client, 66-year-old Thomas Edward Caldwell:

He has held a Top Secret Security Clearance since 1979 and has undergone multiple Special Background Investigations in support of his clearances. After retiring from the Navy, he

worked as a section chief for the Federal Bureau of Investigation from 2009-2010 as a GS-12 [mid-level employee].

He also formed and operated a consulting firm performing work, often classified, for U.S government customers including the US. Drug Enforcement Agency, Department of Housing and Urban Development, the US Coast Guard, and the US Army Personnel Command.

A judge later released Caldwell pending trial in the absence of evidence about a conspiracy or that he tried to force his way into the building. *The New York Post* reported a 'law enforcement source' as saying that 'at least two known Antifa members were spotted' on camera among Trump supporters during the riot while one of the rioters arrested was John Earle Sullivan, a seriously extreme Black Lives Matter Trump-hater from Utah who was previously arrested and charged in July, 2020, over a BLM-Antifa riot in which drivers were threatened and one was shot. Sullivan is the founder of Utah-based Insurgence USA which is an affiliate of the Cult-created-and-funded Black Lives Matter movement. Footage appeared and was then deleted by Twitter of Trump supporters calling out Antifa infiltrators and a group was filmed changing into pro-Trump clothing before the riot. Security at the building was *pathetic* – as planned. Colonel Leroy Fletcher Prouty, a man with long experience in covert operations working with the US security apparatus, once described the tell-tale sign to identify who is involved in an assassination. He said:

No one has to direct an assassination – it happens. The active role is played secretly by permitting it to happen. This is the greatest single clue. Who has the power to call off or reduce the usual security precautions?

This principle applies to many other situations and certainly to the Capitol riot of January 6th, 2021.

The sting

With such a big and potentially angry crowd known to be gathering near the Capitol the security apparatus would have had a major police detail to defend the building with National Guard troops on

standby given the strength of feeling among people arriving from all over America encouraged by the QAnon Psyop and statements by Donald Trump. Instead Capitol Police 'security' was flimsy, weak, and easily breached. The same number of officers was deployed as on a regular day and that is a blatant red flag. They were not staffed or equipped for a possible riot that had been an obvious possibility in the circumstances. No protective and effective fencing worth the name was put in place and there were no contingency plans. The whole thing was basically a case of standing aside and waving people in. Once inside police mostly backed off apart from one Capitol police officer who ridiculously shot dead unarmed Air Force veteran protestor Ashli Babbitt without a warning as she climbed through a broken window. The 'investigation' refused to name or charge the officer after what must surely be considered a murder in the circumstances. They just lifted a carpet and swept. The story was endlessly repeated about five people dying in the 'armed insurrection' when there was no report of rioters using weapons. Apart from Babbitt the other four died from a heart attack, strokes and apparently a drug overdose. Capitol police officer Brian Sicknick was reported to have died after being bludgeoned with a fire extinguisher when he was alive after the riot was over and died later of what the Washington Medical Examiner's Office said was a stroke. Sicknick had no external injuries. The lies were delivered like rapid fire. There was a narrative to build with incessant repetition of the lie until the lie became the accepted 'everybody knows that' truth. The 'Big Lie' technique of Nazi Propaganda Minister Joseph Goebbels is constantly used by the Cult which was behind the Nazis and is today behind the 'Covid' and 'climate change' hoaxes. Goebbels said:

If you tell a lie big enough and keep repeating it, people will eventually come to believe it. The lie can be maintained only for such time as the State can shield the people from the political, economic and/or military consequences of the lie. It thus becomes vitally important for the State to use all of its powers to repress dissent, for the truth is the mortal enemy of the lie, and thus by extension, the truth is the greatest enemy of the State.

Most protestors had a free run of the Capitol Building. This allowed pictures to be taken of rioters in iconic parts of the building including the Senate chamber which could be used as propaganda images against all Pushbackers. One Congresswoman described the scene as 'the worst kind of non-security anybody could ever imagine'. Well, the first part was true, but someone obviously did imagine it and made sure it happened. Some photographs most widely circulated featured people wearing QAnon symbols and now the Psyop would be used to dub all QAnon followers with the ubiquitous fit-all label of 'white supremacist' and 'insurrectionists'. When a Muslim extremist called Noah Green drove his car at two police officers at the Capitol Building killing one in April, 2021, there was no such political and media hysteria. They were just disappointed he wasn't white.

The witch-hunt

Government prosecutor Michael Sherwin, an aggressive, dark-eyed, professional Rottweiler led the 'investigation' and to call it over the top would be to understate reality a thousand fold. Hundreds were tracked down and arrested for the crime of having the wrong political views and people were jailed who had done nothing more than walk in the building, committed no violence or damage to property, took a few pictures and left. They were labelled a 'threat to the Republic' while Biden sat in the White House signing executive orders written for him that were dismantling 'the Republic'. Even when judges ruled that a mother and son should not be in jail the government kept them there. Some of those arrested have been badly beaten by prison guards in Washington and lawyers for one man said he suffered a fractured skull and was made blind in one eye. Meanwhile a woman is shot dead for no reason by a Capitol Police officer and we are not allowed to know who he is never mind what has happened to him although that will be *nothing*. The Cult's QAnon/Trump sting to identify and isolate Pushbackers and then target them on the road to crushing and deleting them was a resounding success. You would have thought the Russians had

invaded the building at gunpoint and lined up senators for a firing squad to see the political and media reaction. Congresswoman Alexandria Ocasio-Cortez is a child in a woman's body, a terrible-twos, me, me, me, Woker narcissist of such proportions that words have no meaning. She said she thought she was going to die when 'insurrectionists' banged on her office door. It turned out she wasn't even in the Capitol Building when the riot was happening and the 'banging' was a Capitol Police officer. She referred to herself as a 'survivor' which is an insult to all those true survivors of violent and sexual abuse while she lives her pampered and privileged life talking drivel for a living. Her Woke colleague and fellow mega-narcissist Rashida Tlaib broke down describing the devastating effect on her, too, of *not being* in the building when the rioters were there. Ocasio-Cortez and Tlaib are members of a fully-Woke group of Congresswomen known as 'The Squad' along with Ilhan Omar and Ayanna Pressley. The Squad from what I can see can be identified by its vehement anti-white racism, anti-white men agenda, and, as always in these cases, the absence of brain cells on active duty.

The usual suspects were on the riot case immediately in the form of Democrat ultra-Zionist senators and operatives Chuck Schumer and Adam Schiff demanding that Trump be impeached for 'his part in the insurrection'. The same pair of prats had led the failed impeachment of Trump over the invented 'Russia collusion' nonsense which claimed Russia had helped Trump win the 2016 election. I didn't realise that Tel Aviv had been relocated just outside Moscow. I must find an up-to-date map. The Russia hoax was a Sabbatian operation to keep Trump occupied and impotent and to stop any rapport with Russia which the Cult wants to retain as a perceptual enemy to be pulled out at will. Puppet Biden began attacking Russia when he came to office as the Cult seeks more upheaval, division and war across the world. A two-year stage show 'Russia collusion inquiry' headed by the not-very-bright former 9/11 FBI chief Robert Mueller, with support from 19 lawyers, 40 FBI agents plus intelligence analysts, forensic accountants and other

staff, devoured tens of millions of dollars and found no evidence of Russia collusion which a ten-year-old could have told them on day one. Now the same moronic Schumer and Schiff wanted a second impeachment of Trump over the Capitol 'insurrection' (riot) which the arrested development of Schumer called another 'Pearl Harbor' while others compared it with 9/11 in which 3,000 died and, in the case of CNN, with the Rwandan genocide in the 1990s in which an estimated 500,000 to 600,000 were murdered, between 250, 000 and 500,000 women were raped, and populations of whole towns were hacked to death with machetes. To make those comparisons purely for Cult political reasons is beyond insulting to those that suffered and lost their lives and confirms yet again the callous inhumanity that we are dealing with. Schumer is a monumental idiot and so is Schiff, but they serve the Cult agenda and do whatever they're told so they get looked after. Talking of idiots – another inane man who spanned the Russia and Capitol impeachment attempts was Senator Eric Swalwell who had the nerve to accuse Trump of collusion with the Russians while sleeping with a Chinese spy called Christine Fang or 'Fang Fang' which is straight out of a Bond film no doubt starring Klaus Schwab as the bloke living on a secret island and controlling laser weapons positioned in space and pointing at world capitals. Fang Fang plays the part of Bond's infiltrator girlfriend which I'm sure she would enjoy rather more than sharing a bed with the brainless Swalwell, lying back and thinking of China. The FBI eventually warned Swalwell about Fang Fang which gave her time to escape back to the Chinese dictatorship. How very thoughtful of them. The second Trump impeachment also failed and hardly surprising when an impeachment is supposed to remove a sitting president and by the time it happened Trump was no longer president. These people are running your country America, well, officially anyway. Terrifying isn't it?

Outcomes tell the story - always

The outcome of all this – and it's the *outcome* on which Renegade Minds focus, not the words – was that a vicious, hysterical and

obviously pre-planned assault was launched on Pushbackers to censor, silence and discredit them and even targeted their right to earn a living. They have since been condemned as 'domestic terrorists' that need to be treated like Al-Qaeda and Islamic State. 'Domestic terrorists' is a label the Cult has been trying to make stick since the period of the Oklahoma bombing in 1995 which was blamed on 'far-right domestic terrorists'. If you read *The Trigger* you will see that the bombing was clearly a Problem-Reaction-Solution carried out by the Deep State during a Bill Clinton administration so corrupt that no dictionary definition of the term would even nearly suffice. Nearly 30, 000 troops were deployed from all over America to the empty streets of Washington for Biden's inauguration. Ten thousand of them stayed on with the pretext of protecting the capital from insurrectionists when it was more psychological programming to normalise the use of the military in domestic law enforcement in support of the Cult plan for a police-military state. Biden's fascist administration began a purge of 'wrong-thinkers' in the military which means anyone that is not on board with Woke. The Capitol Building was surrounded by a fence with razor wire and the Land of the Free was further symbolically and literally dismantled. The circle was completed with the installation of Biden and the exploitation of the QAnon Psyop.

America had never been so divided since the civil war of the 19th century, Pushbackers were isolated and dubbed terrorists and now, as was always going to happen, the Cult immediately set about deleting what little was left of freedom and transforming American society through a swish of the hand of the most controlled 'president' in American history leading (officially at least) the most extreme regime since the country was declared an independent state on July 4th, 1776. Biden issued undebated, dictatorial executive orders almost by the hour in his opening days in office across the whole spectrum of the Cult wish-list including diluting controls on the border with Mexico allowing thousands of migrants to illegally enter the United States to transform the demographics of America and import an election-changing number of perceived Democrat

voters. Then there were Biden deportation amnesties for the already illegally resident (estimated to be as high as 20 or even 30 million). A bill before Congress awarded American citizenship to anyone who could prove they had worked in agriculture for just 180 days in the previous two years as 'Big Ag' secured its slave labour long-term. There were the plans to add new states to the union such as Puerto Rico and making Washington DC a state. They are all parts of a plan to ensure that the Cult-owned Woke Democrats would be permanently in power.

Border – what border?

I have exposed in detail in other books how mass immigration into the United States and Europe is the work of Cult networks fuelled by the tens of billions spent to this and other ends by George Soros and his global Open Society (open borders) Foundations. The impact can be seen in America alone where the population has increased by *100 million* in little more than 30 years mostly through immigration. I wrote in *The Answer* that the plan was to have so many people crossing the southern border that the numbers become unstoppable and we are now there under Cult-owned Biden. El Salvador in Central America puts the scale of what is happening into context. A third of the population now lives in the United States, much of it illegally, and many more are on the way. The methodology is to crush Central and South American countries economically and spread violence through machete-wielding psychopathic gangs like MS-13 based in El Salvador and now operating in many American cities. Biden-imposed lax security at the southern border means that it is all but open. He said before his 'election' that he wanted to see a surge towards the border if he became president and that was the green light for people to do just that after election day to create the human disaster that followed for both America and the migrants. When that surge came the imbecilic Alexandria Ocasio-Cortez said it wasn't a 'surge' because they are 'children, not insurgents' and the term 'surge' (used by Biden) was a claim of 'white supremacists'.

This disingenuous lady may one day enter the realm of the most basic intelligence, but it won't be any time soon.

Sabbatians and the Cult are in the process of destroying America by importing violent people and gangs in among the genuine to terrorise American cities and by overwhelming services that cannot cope with the sheer volume of new arrivals. Something similar is happening in Europe as Western society in general is targeted for demographic and cultural transformation and upheaval. The plan demands violence and crime to create an environment of intimidation, fear and division and Soros has been funding the election of district attorneys across America who then stop prosecuting many crimes, reduce sentences for violent crimes and free as many violent criminals as they can. Sabbatians are creating the chaos from which order – their order – can respond in a classic Problem-Reaction-Solution. A Freemasonic moto says 'Ordo Ab Chao' (Order out of Chaos) and this is why the Cult is constantly creating chaos to impose a new 'order'. Here you have the reason the Cult is constantly creating chaos. The 'Covid' hoax can be seen with those entering the United States by plane being forced to take a 'Covid' test while migrants flooding through southern border processing facilities do not. Nothing is put in the way of mass migration and if that means ignoring the government's own 'Covid' rules then so be it. They know it's all bullshit anyway. Any pushback on this is denounced as 'racist' by Wokers and Sabbatian fronts like the ultra-Zionist Anti-Defamation League headed by the appalling Jonathan Greenblatt which at the same time argues that Israel should not give citizenship and voting rights to more Palestinian Arabs or the 'Jewish population' (in truth the Sabbatian network) will lose control of the country.

Society-changing numbers

Biden's masters have declared that countries like El Salvador are so dangerous that their people must be allowed into the United States for humanitarian reasons when there are fewer murders in large parts of many Central American countries than in US cities like

Baltimore. That is not to say Central America cannot be a dangerous place and Cult-controlled American governments have been making it so since way back, along with the dismantling of economies, in a long-term plan to drive people north into the United States. Parts of Central America are very dangerous, but in other areas the story is being greatly exaggerated to justify relaxing immigration criteria. Migrants are being offered free healthcare and education in the United States as another incentive to head for the border and there is no requirement to be financially independent before you can enter to prevent the resources of America being drained. You can't blame migrants for seeking what they believe will be a better life, but they are being played by the Cult for dark and nefarious ends. The numbers since Biden took office are huge. In February, 2021, more than 100,000 people were known to have tried to enter the US illegally through the southern border (it was 34,000 in the same month in 2020) and in March it was 170,000 – a 418 percent increase on March, 2020. These numbers are only known people, not the ones who get in unseen. The true figure for migrants illegally crossing the border in a single month was estimated by one congressman at 250,000 and that number will only rise under Biden's current policy. Gangs of murdering drug-running thugs that control the Mexican side of the border demand money – thousands of dollars – to let migrants cross the Rio Grande into America. At the same time gun battles are breaking out on the border several times a week between rival Mexican drug gangs (which now operate globally) who are equipped with sophisticated military-grade weapons, grenades and armoured vehicles. While the Capitol Building was being 'protected' from a non-existent 'threat' by thousands of troops, and others were still deployed at the time in the Cult Neocon war in Afghanistan, the southern border of America was left to its fate. This is not incompetence, it is cold calculation.

By March, 2021, there were 17,000 unaccompanied children held at border facilities and many of them are ensnared by people traffickers for paedophile rings and raped on their journey north to America. This is not conjecture – this is fact. Many of those designated

children are in reality teenage boys or older. Meanwhile Wokers posture their self-purity for encouraging poor and tragic people to come to America and face this nightmare both on the journey and at the border with the disgusting figure of House Speaker Nancy Pelosi giving disingenuous speeches about caring for migrants. The woman's evil. Wokers condemned Trump for having children in cages at the border (so did Obama, *Shhhh*), but now they are sleeping on the floor without access to a shower with one border facility 729 percent over capacity. The Biden insanity even proposed flying migrants from the southern border to the northern border with Canada for 'processing'. The whole shambles is being overseen by ultra-Zionist Secretary of Homeland Security, the moronic liar Alejandro Mayorkas, who banned news cameras at border facilities to stop Americans seeing what was happening. Mayorkas said there was not a ban on news crews; it was just that they were not allowed to film. Alongside him at Homeland Security is another ultra-Zionist Cass Sunstein appointed by Biden to oversee new immigration laws. Sunstein despises conspiracy researchers to the point where he suggests they should be banned or *taxed* for having such views. The man is not bonkers or anything. He's perfectly well-adjusted, but adjusted to what is the question. Criticise what is happening and you are a 'white supremacist' when earlier non-white immigrants also oppose the numbers which effect their lives and opportunities. Black people in poor areas are particularly damaged by uncontrolled immigration and the increased competition for work opportunities with those who will work for less. They are also losing voting power as Hispanics become more dominant in former black areas. It's a downward spiral for them while the billionaires behind the policy drone on about how much they care about black people and 'racism'. None of this is about compassion for migrants or black people – that's just wind and air. Migrants are instead being mercilessly exploited to transform America while the countries they leave are losing their future and the same is true in Europe. Mass immigration may now be the work of Woke Democrats, but it can be traced back to the 1986 Immigration Reform and Control Act (it

wasn't) signed into law by Republican hero President Ronald Reagan which gave amnesty to millions living in the United States illegally and other incentives for people to head for the southern border. Here we have the one-party state at work again.

Save me syndrome

Almost every aspect of what I have been exposing as the Cult agenda was on display in even the first days of 'Biden' with silencing of Pushbackers at the forefront of everything. A Renegade Mind will view the Trump years and QAnon in a very different light to their supporters and advocates as the dots are connected. The QAnon/Trump Psyop has given the Cult all it was looking for. We may not know how much, or little, that Trump realised he was being used, but that's a side issue. This pincer movement produced the desired outcome of dividing America and having Pushbackers isolated. To turn this around we have to look at new routes to empowerment which do not include handing our power to other people and groups through what I will call the 'Save Me Syndrome' – 'I want someone else to do it so that I don't have to'. We have seen this at work throughout human history and the QAnon/Trump Psyop is only the latest incarnation alongside all the others. Religion is an obvious expression of this when people look to a 'god' or priest to save them or tell them how to be saved and then there are 'save me' politicians like Trump. Politics is a diversion and not a 'saviour'. It is a means to block positive change, not make it possible.

Save Me Syndrome always comes with the same repeating theme of handing your power to whom or what you believe will save you while your real 'saviour' stares back from the mirror every morning. Renegade Minds are constantly vigilant in this regard and always asking the question 'What can I do?' rather than 'What can someone else do for me?' Gandhi was right when he said: 'You must be the change you want to see in the world.' We are indeed the people we have been waiting for. We are presented with a constant raft of reasons to concede that power to others and forget where the real power is. Humanity has the numbers and the Cult does not. It has to

use diversion and division to target the unstoppable power that comes from unity. Religions, governments, politicians, corporations, media, QAnon, are all different manifestations of this power-diversion and dilution. Refusing to give your power to governments and instead handing it to Trump and QAnon is not to take a new direction, but merely to recycle the old one with new names on the posters. I will explore this phenomenon as we proceed and how to break the cycles and recycles that got us here through the mists of repeating perception and so repeating history.

For now we shall turn to the most potent example in the entire human story of the consequences that follow when you give your power away. I am talking, of course, of the 'Covid' hoax.

CHAPTER FOUR

'Covid': Calculated catastrophe

Facts are threatening to those invested in fraud
DaShanne Stokes

We can easily unravel the real reason for the 'Covid pandemic' hoax by employing the Renegade Mind methodology that I have outlined this far. We'll start by comparing the long-planned Cult outcome with the 'Covid pandemic' outcome. Know the outcome and you'll see the journey.

I have highlighted the plan for the Hunger Games Society which has been in my books for so many years with the very few controlling the very many through ongoing dependency. To create this dependency it is essential to destroy independent livelihoods, businesses and employment to make the population reliant on the state (the Cult) for even the basics of life through a guaranteed pittance income. While independence of income remained these Cult ambitions would be thwarted. With this knowledge it was easy to see where the 'pandemic' hoax was going once talk of 'lockdowns' began and the closing of all but perceived 'essential' businesses to 'save' us from an alleged 'deadly virus'. Cult corporations like Amazon and Walmart were naturally considered 'essential' while mom and pop shops and stores had their doors closed by fascist decree. As a result with every new lockdown and new regulation more small and medium, even large businesses not owned by the Cult, went to the wall while Cult giants and their frontmen and women grew financially fatter by the second. Mom and pop were

denied an income and the right to earn a living and the wealth of people like Jeff Bezos (Amazon), Mark Zuckerberg (Facebook) and Sergei Brin and Larry Page (Google/Alphabet) have reached record levels. The Cult was increasing its own power through further dramatic concentrations of wealth while the competition was being destroyed and brought into a state of dependency. Lockdowns have been instigated to secure that very end and were never anything to do with health. My brother Paul spent 45 years building up a bus repair business, but lockdowns meant buses were running at a fraction of normal levels for months on end. Similar stories can told in their hundreds of millions worldwide. Efforts of a lifetime coldly destroyed by Cult multi-billionaires and their lackeys in government and law enforcement who continued to earn their living from the taxation of the people while denying the right of the same people to earn theirs. How different it would have been if those making and enforcing these decisions had to face the same financial hardships of those they affected, but they never do.

Gates of Hell

Behind it all in the full knowledge of what he is doing and why is the psychopathic figure of Cult operative Bill Gates. His puppet Tedros at the World Health Organization declared 'Covid' a pandemic in March, 2020. The WHO had changed the definition of a 'pandemic' in 2009 just a month before declaring the 'swine flu pandemic' which would not have been so under the previous definition. The same applies to 'Covid'. The definition had included… 'an infection by an infectious agent, occurring simultaneously in different countries, with a significant mortality rate relative to the proportion of the population infected'. The new definition removed the need for 'significant mortality'. The 'pandemic' has been fraudulent even down to the definition, but Gates demanded economy-destroying lockdowns, school closures, social distancing, mandatory masks, a 'vaccination' for every man, woman and child on the planet and severe consequences and restrictions for those that refused. Who gave him this power? The

Cult did which he serves like a little boy in short trousers doing what his daddy tells him. He and his psychopathic missus even smiled when they said that much worse was to come (what they knew was planned to come). Gates responded in the matter-of-fact way of all psychopaths to a question about the effect on the world economy of what he was doing:

> Well, it won't go to zero but it will shrink. Global GDP is probably going to take the biggest hit ever [Gates was smiling as he said this] … in my lifetime this will be the greatest economic hit. But you don't have a choice. People act as if you have a choice. People don't feel like going to the stadium when they might get infected … People are deeply affected by seeing these stats, by knowing they could be part of the transmission chain, old people, their parents and grandparents, could be affected by this, and so you don't get to say ignore what is going on here.
>
> There will be the ability to open up, particularly in rich countries, if things are done well over the next few months, but for the world at large normalcy only returns when we have largely vaccinated the entire population.

The man has no compassion or empathy. How could he when he's a psychopath like all Cult players? My own view is that even beyond that he is very seriously mentally ill. Look in his eyes and you can see this along with his crazy flailing arms. You don't do what he has done to the world population since the start of 2020 unless you are mentally ill and at the most extreme end of psychopathic. You especially don't do it when to you know, as we shall see, that cases and deaths from 'Covid' are fakery and a product of monumental figure massaging. 'These stats' that Gates referred to are based on a 'test' that's not testing for the 'virus' as he has known all along. He made his fortune with big Cult support as an infamously ruthless software salesman and now buys global control of 'health' (death) policy without the population he affects having any say. It's a breathtaking outrage. Gates talked about people being deeply affected by fear of 'Covid' when that was because of *him* and his global network lying to them minute-by-minute supported by a lying media that he seriously influences and funds to the tune of hundreds of millions. He's handed big sums to media operations including the BBC, NBC, Al Jazeera, Univision, *PBS NewsHour*,

ProPublica, National Journal, The Guardian, The Financial Times, The Atlantic, Texas Tribune, USA Today publisher Gannett, *Washington Monthly, Le Monde,* Center for Investigative Reporting, Pulitzer Center on Crisis Reporting, National Press Foundation, International Center for Journalists, Solutions Journalism Network, the Poynter Institute for Media Studies, and many more. Gates is everywhere in the 'Covid' hoax and the man must go to prison – or a mental facility – for the rest of his life and his money distributed to those he has taken such enormous psychopathic pleasure in crushing.

The Muscle

The Hunger Games global structure demands a police-military state – a fusion of the two into one force – which viciously imposes the will of the Cult on the population and protects the Cult from public rebellion. In that regard, too, the 'Covid' hoax just keeps on giving. Often unlawful, ridiculous and contradictory 'Covid' rules and regulations have been policed across the world by moronic automatons and psychopaths made faceless by face-nappy masks and acting like the Nazi SS and fascist blackshirts and brownshirts of Hitler and Mussolini. The smallest departure from the rules decreed by the psychos in government and their clueless gofers were jumped upon by the face-nappy fascists. Brutality against public protestors soon became commonplace even on girls, women and old people as the brave men with the batons – the Face-Nappies as I call them – broke up peaceful protests and handed out fines like confetti to people who couldn't earn a living let alone pay hundreds of pounds for what was once an accepted human right. Robot Face-Nappies of Nottingham police in the English East Midlands fined one group £11,000 for attending a child's birthday party. For decades I charted the transformation of law enforcement as genuine, decent officers were replaced with psychopaths and the brain dead who would happily and brutally do whatever their masters told them. Now they were let loose on the public and I would emphasise the point that none of this just happened. The step-by-step change in the dynamic between police and public was orchestrated from the shadows by

those who knew where this was all going and the same with the perceptual reframing of those in all levels of authority and official administration through 'training courses' by organisations such as Common Purpose which was created in the late 1980s and given a massive boost in Blair era Britain until it became a global phenomenon. Supposed public 'servants' began to view the population as the enemy and the same was true of the police. This was the start of the explosion of behaviour manipulation organisations and networks preparing for the all-war on the human psyche unleashed with the dawn of 2020. I will go into more detail about this later in the book because it is a core part of what is happening.

Police desecrated beauty spots to deter people gathering and arrested women for walking in the countryside alone 'too far' from their homes. We had arrogant, clueless sergeants in the Isle of Wight police where I live posting on Facebook what they insisted the population must do or else. A schoolmaster sergeant called Radford looked young enough for me to ask if his mother knew he was out, but he was posting what he *expected* people to do while a Sergeant Wilkinson boasted about fining lads for meeting in a McDonald's car park where they went to get a lockdown takeaway. Wilkinson added that he had even cancelled their order. What a pair of prats these people are and yet they have increasingly become the norm among Jackboot Johnson's Yellowshirts once known as the British police. This was the theme all over the world with police savagery common during lockdown protests in the United States, the Netherlands, and the fascist state of Victoria in Australia under its tyrannical and again moronic premier Daniel Andrews. Amazing how tyrannical and moronic tend to work as a team and the same combination could be seen across America as arrogant, narcissistic Woke governors and mayors such as Gavin Newsom (California), Andrew Cuomo (New York), Gretchen Whitmer (Michigan), Lori Lightfoot (Chicago) and Eric Garcetti (Los Angeles) did their Nazi and Stalin impressions with the full support of the compliant brutality of their enforcers in uniform as they arrested small business owners defying

fascist shutdown orders and took them to jail in ankle shackles and handcuffs. This happened to bistro owner Marlena Pavlos-Hackney in Gretchen Whitmer's fascist state of Michigan when police arrived to enforce an order by a state-owned judge for 'putting the community at risk' at a time when other states like Texas were dropping restrictions and migrants were pouring across the southern border without any 'Covid' questions at all. I'm sure there are many officers appalled by what they are ordered to do, but not nearly enough of them. If they were truly appalled they would not do it. As the months passed every opportunity was taken to have the military involved to make their presence on the streets ever more familiar and 'normal' for the longer-term goal of police-military fusion.

Another crucial element to the Hunger Games enforcement network has been encouraging the public to report neighbours and others for 'breaking the lockdown rules'. The group faced with £11,000 in fines at the child's birthday party would have been dobbed-in by a neighbour with a brain the size of a pea. The technique was most famously employed by the Stasi secret police in communist East Germany who had public informants placed throughout the population. A police chief in the UK says his force doesn't need to carry out 'Covid' patrols when they are flooded with so many calls from the public reporting other people for visiting the beach. Dorset police chief James Vaughan said people were so enthusiastic about snitching on their fellow humans they were now operating as an auxiliary arm of the police: 'We are still getting around 400 reports a week from the public, so we will respond to reports …We won't need to be doing hotspot patrols because people are very quick to pick the phone up and tell us.' Vaughan didn't say that this is a pillar of all tyrannies of whatever complexion and the means to hugely extend the reach of enforcement while spreading distrust among the people and making them wary of doing anything that might get them reported. Those narcissistic Isle of Wight sergeants Radford and Wilkinson never fail to add a link to their Facebook posts where the public can inform on their fellow slaves.

Neither would be self-aware enough to realise they were imitating the Stasi which they might well never have heard of. Government psychologists that I will expose later laid out a policy to turn communities against each other in the same way.

A coincidence? Yep, and I can knit fog

I knew from the start of the alleged pandemic that this was a Cult operation. It presented limitless potential to rapidly advance the Cult agenda and exploit manipulated fear to demand that every man, woman and child on the planet was 'vaccinated' in a process never used on humans before which infuses self-replicating *synthetic* material into human cells. Remember the plan to transform the human body from a biological to a synthetic biological state. I'll deal with the 'vaccine' (that's not actually a vaccine) when I focus on the genetic agenda. Enough to say here that mass global 'vaccination' justified by this 'new virus' set alarms ringing after 30 years of tracking these people and their methods. The 'Covid' hoax officially beginning in China was also a big red flag for reasons I will be explaining. The agenda potential was so enormous that I could dismiss any idea that the 'virus' appeared naturally. Major happenings with major agenda implications never occur without Cult involvement in making them happen. My questions were twofold in early 2020 as the media began its campaign to induce global fear and hysteria: Was this alleged infectious agent released on purpose by the Cult or did it even exist at all? I then did what I always do in these situations. I sat, observed and waited to see where the evidence and information would take me. By March and early April synchronicity was strongly – and ever more so since then – pointing me in the direction of *there is no 'virus'*. I went public on that with derision even from swathes of the alternative media that voiced a scenario that the Chinese government released the 'virus' in league with Deep State elements in the United States from a top-level bio-lab in Wuhan where the 'virus' is said to have first appeared. I looked at that possibility, but I didn't buy it for several reasons. Deaths from the 'virus' did not in any way match what they

would have been with a 'deadly bioweapon' and it is much more effective if you sell the *illusion* of an infectious agent rather than having a real one unless you can control through injection who has it and who doesn't. Otherwise you lose control of events. A made-up 'virus' gives you a blank sheet of paper on which you can make it do whatever you like and have any symptoms or mutant 'variants' you choose to add while a real infectious agent would limit you to what it actually does. A phantom disease allows you to have endless ludicrous 'studies' on the 'Covid' dollar to widen the perceived impact by inventing ever more 'at risk' groups including one study which said those who walk slowly may be almost four times more likely to die from the 'virus'. People are in psychiatric wards for less.

A real 'deadly bioweapon' can take out people in the hierarchy that are not part of the Cult, but essential to its operation. Obviously they don't want that. Releasing a real disease means you immediately lose control of it. Releasing an illusory one means you don't. Again it's vital that people are extra careful when dealing with what they want to hear. A bioweapon unleashed from a Chinese laboratory in collusion with the American Deep State may fit a conspiracy narrative, but is it true? Would it not be far more effective to use the excuse of a 'virus' to justify the real bioweapon – the 'vaccine'? That way your disease agent does not have to be transmitted and arrives directly through a syringe. I saw a French virologist Luc Montagnier quoted in the alternative media as saying he had discovered that the alleged 'new' severe acute respiratory syndrome coronavirus , or SARS-CoV-2, was made artificially and included elements of the human immunodeficiency 'virus' (HIV) and a parasite that causes malaria. SARS-CoV-2 is alleged to trigger an alleged illness called Covid-19. I remembered Montagnier's name from my research years before into claims that an HIV 'retrovirus' causes AIDs – claims that were demolished by Berkeley virologist Peter Duesberg who showed that no one had ever proved that HIV causes acquired immunodeficiency syndrome or AIDS. Claims that become accepted as fact, publicly and medically, with no proof whatsoever are an ever-recurring story that profoundly applies to

'Covid'. Nevertheless, despite the lack of proof, Montagnier's team at the Pasteur Institute in Paris had a long dispute with American researcher Robert Gallo over which of them discovered and isolated the HIV 'virus' and with *no evidence* found it to cause AIDS. You will see later that there is also no evidence that any 'virus' causes any disease or that there is even such a thing as a 'virus' in the way it is said to exist. The claim to have 'isolated' the HIV 'virus' will be presented in its real context as we come to the shocking story – and it is a story – of SARS-CoV-2 and so will Montagnier's assertion that he identified the full SARS-CoV-2 genome.

Hoax in the making

We can pick up the 'Covid' story in 2010 and the publication by the Rockefeller Foundation of a document called 'Scenarios for the Future of Technology and International Development'. The inner circle of the Rockefeller family has been serving the Cult since John D. Rockefeller (1839-1937) made his fortune with Standard Oil. It is less well known that the same Rockefeller – the Bill Gates of his day – was responsible for establishing what is now referred to as 'Big Pharma', the global network of pharmaceutical companies that make outrageous profits dispensing scalpel and drug 'medicine' and are obsessed with pumping vaccines in ever-increasing number into as many human arms and backsides as possible. John D. Rockefeller was the driving force behind the creation of the 'education' system in the United States and elsewhere specifically designed to program the perceptions of generations thereafter. The Rockefeller family donated exceptionally valuable land in New York for the United Nations building and were central in establishing the World Health Organization in 1948 as an agency of the UN which was created from the start as a Trojan horse and stalking horse for world government. Now enter Bill Gates. His family and the Rockefellers have long been extremely close and I have seen genealogy which claims that if you go back far enough the two families fuse into the same bloodline. Gates has said that the Bill and Melinda Gates Foundation was inspired by the Rockefeller Foundation and why not

when both are serving the same Cult? Major tax-exempt foundations are overwhelmingly criminal enterprises in which Cult assets fund the Cult agenda in the guise of 'philanthropy' while avoiding tax in the process. Cult operatives can become mega-rich in their role of front men and women for the psychopaths at the inner core and they, too, have to be psychopaths to knowingly serve such evil. Part of the deal is that a big percentage of the wealth gleaned from representing the Cult has to be spent advancing the ambitions of the Cult and hence you have the Rockefeller Foundation, Bill and Melinda Gates Foundation (and *so* many more) and people like George Soros with his global Open Society Foundations spending their billions in pursuit of global Cult control. Gates is a global public face of the Cult with his interventions in world affairs including Big Tech influence; a central role in the 'Covid' and 'vaccine' scam; promotion of the climate change shakedown; manipulation of education; geoengineering of the skies; and his food-control agenda as the biggest owner of farmland in America, his GMO promotion and through other means. As one writer said: 'Gates monopolizes or wields disproportionate influence over the tech industry, global health and vaccines, agriculture and food policy (including biopiracy and fake food), weather modification and other climate technologies, surveillance, education and media.' The almost limitless wealth secured through Microsoft and other not-allowed-to-fail ventures (including vaccines) has been ploughed into a long, long list of Cult projects designed to enslave the entire human race. Gates and the Rockefellers have been working as one unit with the Rockefeller-established World Health Organization leading global 'Covid' policy controlled by Gates through his mouth-piece Tedros. Gates became the WHO's biggest funder when Trump announced that the American government would cease its donations, but Biden immediately said he would restore the money when he took office in January, 2021. The Gates Foundation (the Cult) owns through limitless funding the world health system and the major players across the globe in the 'Covid' hoax.

Okay, with that background we return to that Rockefeller Foundation document of 2010 headed 'Scenarios for the Future of Technology and International Development' and its 'imaginary' epidemic of a virulent and deadly influenza strain which infected 20 percent of the global population and killed eight million in seven months. The Rockefeller scenario was that the epidemic destroyed economies, closed shops, offices and other businesses and led to governments imposing fierce rules and restrictions that included mandatory wearing of face masks and body-temperature checks to enter communal spaces like railway stations and supermarkets. The document predicted that even after the height of the Rockefeller-envisaged epidemic the authoritarian rule would continue to deal with further pandemics, transnational terrorism, environmental crises and rising poverty. Now you may think that the Rockefellers are our modern-day seers or alternatively, and rather more likely, that they well knew what was planned a few years further on. Fascism had to be imposed, you see, to 'protect citizens from risk and exposure'. The Rockefeller scenario document said:

During the pandemic, national leaders around the world flexed their authority and imposed airtight rules and restrictions, from the mandatory wearing of face masks to body-temperature checks at the entries to communal spaces like train stations and supermarkets. Even after the pandemic faded, this more authoritarian control and oversight of citizens and their activities stuck and even intensified. In order to protect themselves from the spread of increasingly global problems – from pandemics and transnational terrorism to environmental crises and rising poverty – leaders around the world took a firmer grip on power.

At first, the notion of a more controlled world gained wide acceptance and approval. Citizens willingly gave up some of their sovereignty – and their privacy – to more paternalistic states in exchange for greater safety and stability. Citizens were more tolerant, and even eager, for top-down direction and oversight, and national leaders had more latitude to impose order in the ways they saw fit.

In developed countries, this heightened oversight took many forms: biometric IDs for all citizens, for example, and tighter regulation of key industries whose stability was deemed vital to national interests. In many developed countries, enforced cooperation with a suite of new regulations and agreements slowly but steadily restored both order and, importantly, economic growth.

There we have the prophetic Rockefellers in 2010 and three years later came their paper for the Global Health Summit in Beijing, China, when government representatives, the private sector, international organisations and groups met to discuss the next 100 years of 'global health'. The Rockefeller Foundation-funded paper was called 'Dreaming the Future of Health for the Next 100 Years and more prophecy ensued as it described a dystopian future: 'The abundance of data, digitally tracking and linking people may mean the 'death of privacy' and may replace physical interaction with transient, virtual connection, generating isolation and raising questions of how values are shaped in virtual networks.' Next in the 'Covid' hoax preparation sequence came a 'table top' simulation in 2018 for another 'imaginary' pandemic of a disease called Clade X which was said to kill 900 million people. The exercise was organised by the Gates-funded Johns Hopkins University's Center for Health Security in the United States and this is the very same university that has been compiling the disgustingly and systematically erroneous global figures for 'Covid' cases and deaths. Similar Johns Hopkins health crisis scenarios have included the Dark Winter exercise in 2001 and Atlantic Storm in 2005.

Nostradamus 201

For sheer predictive genius look no further prophecy-watchers than the Bill Gates-funded Event 201 held only six weeks before the 'coronavirus pandemic' is supposed to have broken out in China and Event 201 was based on a scenario of a global 'coronavirus pandemic'. Melinda Gates, the great man's missus, told the BBC that he had 'prepared for years' for a coronavirus pandemic which told us what we already knew. Nostradamugates had predicted in a TED talk in 2015 that a pandemic was coming that would kill a lot of people and demolish the world economy. My god, the man is a machine – possibly even literally. Now here he was only weeks before the real thing funding just such a simulated scenario and involving his friends and associates at Johns Hopkins, the World Economic Forum Cult-front of Klaus Schwab, the United Nations,

Johnson & Johnson, major banks, and officials from China and the Centers for Disease Control in the United States. What synchronicity – Johns Hopkins would go on to compile the fraudulent 'Covid' figures, the World Economic Forum and Schwab would push the 'Great Reset' in response to 'Covid', the Centers for Disease Control would be at the forefront of 'Covid' policy in the United States, Johnson & Johnson would produce a 'Covid vaccine', and everything would officially start just weeks later in China. Spooky, eh? They were even accurate in creating a simulation of a 'virus' pandemic because the 'real thing' would also be a simulation. Event 201 was not an exercise preparing for something that might happen; it was a rehearsal for what those in control knew was *going* to happen and very shortly. Hours of this simulation were posted on the Internet and the various themes and responses mirrored what would soon be imposed to transform human society. News stories were inserted and what they said would be commonplace a few weeks later with still more prophecy perfection. Much discussion focused on the need to deal with misinformation and the 'anti-vax movement' which is exactly what happened when the 'virus' arrived – was said to have arrived – in the West.

Cult-owned social media banned criticism and exposure of the official 'virus' narrative and when I said there *was* no 'virus' in early April, 2020, I was banned by one platform after another including YouTube, Facebook and later Twitter. The mainstream broadcast media in Britain was in effect banned from interviewing me by the Tony-Blair-created government broadcasting censor Ofcom headed by career government bureaucrat Melanie Dawes who was appointed just as the 'virus' hoax was about to play out in January, 2020. At the same time the Ickonic media platform was using Vimeo, another ultra-Zionist-owned operation, while our own player was being created and they deleted in an instant hundreds of videos, documentaries, series and shows to confirm their unbelievable vindictiveness. We had copies, of course, and they had to be restored one by one when our player was ready. These people have no class. Sabbatian Facebook promised free advertisements for the Gates-

controlled World Health Organization narrative while deleting 'false claims and conspiracy theories' to stop 'misinformation' about the alleged coronavirus. All these responses could be seen just a short while earlier in the scenarios of Event 201. Extreme censorship was absolutely crucial for the Cult because the official story was so ridiculous and unsupportable by the evidence that it could never survive open debate and the free-flow of information and opinion. If you can't win a debate then don't have one is the Cult's approach throughout history. Facebook's little boy front man – front boy – Mark Zuckerberg equated 'credible and accurate information' with official sources and exposing their lies with 'misinformation'.

Silencing those that can see

The censorship dynamic of Event 201 is now the norm with an army of narrative-supporting 'fact-checker' organisations whose entire reason for being is to tell the public that official narratives are true and those exposing them are lying. One of the most appalling of these 'fact-checkers' is called NewsGuard founded by ultra-Zionist Americans Gordon Crovitz and Steven Brill. Crovitz is a former publisher of *The Wall Street Journal*, former Executive Vice President of Dow Jones, a member of the Council on Foreign Relations (CFR), and on the board of the American Association of Rhodes Scholars. The CFR and Rhodes Scholarships, named after Rothschild agent Cecil Rhodes who plundered the gold and diamonds of South Africa for his masters and the Cult, have featured widely in my books. NewsGuard don't seem to like me for some reason – I really can't think why – and they have done all they can to have me censored and discredited which is, to quote an old British politician, like being savaged by a dead sheep. They are, however, like all in the censorship network, very well connected and funded by organisations themselves funded by, or connected to, Bill Gates. As you would expect with anything associated with Gates NewsGuard has an offshoot called HealthGuard which 'fights online health care hoaxes'. How very kind. Somehow the NewsGuard European Managing Director Anna-Sophie Harling, a remarkably young-

looking woman with no broadcasting experience and little hands-on work in journalism, has somehow secured a position on the 'Content Board' of UK government broadcast censor Ofcom. An executive of an organisation seeking to discredit dissidents of the official narratives is making decisions for the government broadcast 'regulator' about content?? Another appalling 'fact-checker' is Full Fact funded by George Soros and global censors Google and Facebook.

It's amazing how many activists in the 'fact-checking', 'anti-hate', arena turn up in government-related positions – people like UK Labour Party activist Imran Ahmed who heads the Center for Countering Digital Hate founded by people like Morgan McSweeney, now chief of staff to the Labour Party's hapless and useless 'leader' Keir Starmer. Digital Hate – which is what it really is – uses the American spelling of Center to betray its connection to a transatlantic network of similar organisations which in 2020 shapeshifted from attacking people for 'hate' to attacking them for questioning the 'Covid' hoax and the dangers of the 'Covid vaccine'. It's just a coincidence, you understand. This is one of Imran Ahmed's hysterical statements: 'I would go beyond calling anti-vaxxers conspiracy theorists to say they are an extremist group that pose a national security risk.' No one could ever accuse this prat of understatement and he's including in that those parents who are now against vaccines after their children were damaged for life or killed by them. He's such a nice man. Ahmed does the rounds of the Woke media getting soft-ball questions from spineless 'journalists' who never ask what right he has to campaign to destroy the freedom of speech of others while he demands it for himself. There also seems to be an overrepresentation in Ofcom of people connected to the narrative-worshipping BBC. This incredible global network of narrative-support was super-vital when the 'Covid' hoax was played in the light of the mega-whopper lies that have to be defended from the spotlight cast by the most basic intelligence.

Setting the scene

The Cult plays the long game and proceeds step-by-step ensuring that everything is in place before major cards are played and they don't come any bigger than the 'Covid' hoax. The psychopaths can't handle events where the outcome isn't certain and as little as possible – preferably nothing – is left to chance. Politicians, government and medical officials who would follow direction were brought to illusory power in advance by the Cult web whether on the national stage or others like state governors and mayors of America. For decades the dynamic between officialdom, law enforcement and the public was changed from one of service to one of control and dictatorship. Behaviour manipulation networks established within government were waiting to impose the coming 'Covid' rules and regulations specifically designed to subdue and rewire the psyche of the people in the guise of protecting health. These included in the UK the Behavioural Insights Team part-owned by the British government Cabinet Office; the Scientific Pandemic Insights Group on Behaviours (SPI-B); and a whole web of intelligence and military groups seeking to direct the conversation on social media and control the narrative. Among them are the cyberwarfare (on the people) 77th Brigade of the British military which is also coordinated through the Cabinet Office as civilian and military leadership continues to combine in what they call the Fusion Doctrine. The 77th Brigade is a British equivalent of the infamous Israeli (Sabbatian) military cyberwarfare and Internet manipulation operation Unit 8200 which I expose at length in *The Trigger*. Also carefully in place were the medical and science advisers to government – many on the payroll past or present of Bill Gates – and a whole alternative structure of unelected government stood by to take control when elected parliaments were effectively closed down once the 'Covid' card was slammed on the table. The structure I have described here and so much more was installed in every major country through the Cult networks. The top-down control hierarchy looks like this: The Cult – Cult-owned Gates – the World Health Organization and Tedros – Gates-funded or controlled chief medical officers and science 'advisers' (dictators) in each country –

political 'leaders'– law enforcement – The People. Through this simple global communication and enforcement structure the policy of the Cult could be imposed on virtually the entire human population so long as they acquiesced to the fascism. With everything in place it was time for the button to be pressed in late 2019/early 2020.

These were the prime goals the Cult had to secure for its will to prevail:

1) Locking down economies, closing all but designated 'essential' businesses (Cult-owned corporations were 'essential'), and putting the population under house arrest was an imperative to destroy independent income and employment and ensure dependency on the Cult-controlled state in the Hunger Games Society. Lockdowns had to be established as the global blueprint from the start to respond to the 'virus' and followed by pretty much the entire world.

2) The global population had to be terrified into believing in a deadly 'virus' that didn't actually exist so they would unquestioningly obey authority in the belief that authority must know how best to protect them and their families. Software salesman Gates would suddenly morph into the world's health expert and be promoted as such by the Cult-owned media.

3) A method of testing that wasn't testing for the 'virus', but was only claimed to be, had to be in place to provide the illusion of 'cases' and subsequent 'deaths' that had a very different cause to the 'Covid-19' that would be scribbled on the death certificate.

4) Because there was no 'virus' and the great majority testing positive with a test not testing for the 'virus' would have no symptoms of anything the lie had to be sold that people without symptoms (without the 'virus') could still pass it on to others. This was crucial to justify for the first time quarantining – house arresting – healthy people. Without this the economy-destroying lockdown of *everybody* could not have been credibly sold.

5) The 'saviour' had to be seen as a vaccine which beyond evil drug companies were working like angels of mercy to develop as quickly as possible, with all corners cut, to save the day. The public must absolutely not know that the 'vaccine' had nothing to do with a 'virus' or that the contents were ready and waiting with a very different motive long before the 'Covid' card was even lifted from the pack.

I said in March, 2020, that the 'vaccine' would have been created way ahead of the 'Covid' hoax which justified its use and the following December an article in the New York *Intelligencer* magazine said the Moderna 'vaccine' had been 'designed' by

January, 2020. This was 'before China had even acknowledged that the disease could be transmitted from human to human, more than a week before the first confirmed coronavirus case in the United States'. The article said that by the time the first American death was announced a month later 'the vaccine had already been manufactured and shipped to the National Institutes of Health for the beginning of its Phase I clinical trial'. The 'vaccine' was actually 'designed' long before that although even with this timescale you would expect the article to ask how on earth it could have been done that quickly. Instead it asked why the 'vaccine' had not been rolled out then and not months later. Journalism in the mainstream is truly dead. I am going to detail in the next chapter why the 'virus' has never existed and how a hoax on that scale was possible, but first the foundation on which the Big Lie of 'Covid' was built.

The test that doesn't test

Fraudulent 'testing' is the bottom line of the whole 'Covid' hoax and was the means by which a 'virus' that did not exist *appeared* to exist. They could only achieve this magic trick by using a test not testing for the 'virus'. To use a test that *was* testing for the 'virus' would mean that every test would come back negative given there was no 'virus'. They chose to exploit something called the RT-PCR test invented by American biochemist Kary Mullis in the 1980s who said publicly that his PCR test … *cannot detect infectious disease*. Yes, the 'test' used worldwide to detect infectious 'Covid' to produce all the illusory 'cases' and 'deaths' compiled by Johns Hopkins and others *cannot detect infectious disease*. This fact came from the mouth of the man who invented PCR and was awarded the Nobel Prize in Chemistry in 1993 for doing so. Sadly, and incredibly conveniently for the Cult, Mullis died in August, 2019, at the age of 74 just before his test would be fraudulently used to unleash fascism on the world. He was said to have died from pneumonia which was an irony in itself. A few months later he would have had 'Covid-19' on his death certificate. I say the timing of his death was convenient because had he lived Mullis, a brilliant, honest and decent man, would have been

vociferously speaking out against the use of his test to detect 'Covid' when it was never designed, or able, to do that. I know that to be true given that Mullis made the same point when his test was used to 'detect' – not detect – HIV. He had been seriously critical of the Gallo/Montagnier claim to have isolated the HIV 'virus' and shown it to cause AIDS for which Mullis said there was no evidence. AIDS is actually not a disease but a series of diseases from which people die all the time. When they die from those *same diseases* after a positive 'test' for HIV then AIDS goes on their death certificate. I think I've heard that before somewhere. Countries instigated a policy with 'Covid' that anyone who tested positive with a test not testing for the 'virus' and died of any other cause within 28 days and even longer 'Covid-19' had to go on the death certificate. Cases have come from the test that can't test for infectious disease and the deaths are those who have died of *anything* after testing positive with a test not testing for the 'virus'. I'll have much more later about the death certificate scandal.

Mullis was deeply dismissive of the now US 'Covid' star Anthony Fauci who he said was a liar who didn't know anything about anything – 'and I would say that to his face – nothing.' He said of Fauci: 'The man thinks he can take a blood sample, put it in an electron microscope and if it's got a virus in there you'll know it – he doesn't understand electron microscopy and he doesn't understand medicine and shouldn't be in a position like he's in.' That position, terrifyingly, has made him the decider of 'Covid' fascism policy on behalf of the Cult in his role as director since 1984 of the National Institute of Allergy and Infectious Diseases (NIAID) while his record of being wrong is laughable; but being wrong, so long as it's the *right kind* of wrong, is why the Cult loves him. He'll say anything the Cult tells him to say. Fauci was made Chief Medical Adviser to the President immediately Biden took office. Biden was installed in the White House by Cult manipulation and one of his first decisions was to elevate Fauci to a position of even more control. This is a coincidence? Yes, and I identify as a flamenco dancer called Lola. How does such an incompetent criminal like Fauci remain in that

pivotal position in American health since *the 1980s*? When you serve the Cult it looks after you until you are surplus to requirements. Kary Mullis said prophetically of Fauci and his like: 'Those guys have an agenda and it's not an agenda we would like them to have … they make their own rules, they change them when they want to, and Tony Fauci does not mind going on television in front of the people who pay his salary and lie directly into the camera.' Fauci has done that almost daily since the 'Covid' hoax began. Lying is in Fauci's DNA. To make the situation crystal clear about the PCR test this is a direct quote from its inventor Kary Mullis:

> It [the PCR test] doesn't tell you that you're sick and doesn't tell you that the thing you ended up with was really going to hurt you ...'

Ask yourself why governments and medical systems the world over have been using this very test to decide who is 'infected' with the SARS-CoV-2 'virus' and the alleged disease it allegedly causes, 'Covid-19'. The answer to that question will tell you what has been going on. By the way, here's a little show-stopper – the 'new' SARS-CoV-2 'virus' was 'identified' as such right from the start using … *the PCR test not testing for the 'virus'*. If you are new to this and find that shocking then stick around. I have hardly started yet. Even worse, other 'tests', like the 'Lateral Flow Device' (LFD), are considered so useless that they have to be *confirmed* by the PCR test! Leaked emails written by Ben Dyson, adviser to UK 'Health' Secretary Matt Hancock, said they were 'dangerously unreliable'. Dyson, executive director of strategy at the Department of Health, wrote: 'As of today, someone who gets a positive LFD result in (say) London has at best a 25 per cent chance of it being a true positive, but if it is a self-reported test potentially as low as 10 per cent (on an optimistic assumption about specificity) or as low as 2 per cent (on a more pessimistic assumption).' These are the 'tests' that schoolchildren and the public are being urged to have twice a week or more and have to isolate if they get a positive. Each fake positive goes in the statistics as a 'case' no matter how ludicrously inaccurate and the

'cases' drive lockdown, masks and the pressure to 'vaccinate'. The government said in response to the email leak that the 'tests' were accurate which confirmed yet again what shocking bloody liars they are. The real false positive rate is *100 percent* as we'll see. In another 'you couldn't make it up' the UK government agreed to pay £2.8 billion to California's Innova Medical Group to supply the irrelevant lateral flow tests. The company's primary test-making centre is in China. Innova Medical Group, established in March, 2020, is owned by Pasaca Capital Inc, chaired by Chinese-American millionaire Charles Huang who was born in Wuhan.

How it works – and how it doesn't

The RT-PCR test, known by its full title of Polymerase chain reaction, is used across the world to make millions, even billions, of copies of a DNA/RNA genetic information sample. The process is called 'amplification' and means that a tiny sample of genetic material is amplified to bring out the detailed content. I stress that it is not testing for an infectious disease. It is simply amplifying a sample of genetic material. In the words of Kary Mullis: 'PCR is … just a process that's used to make a whole lot of something out of something.' To emphasise the point companies that make the PCR tests circulated around the world to 'test' for 'Covid' warn on the box that it can't be used to detect 'Covid' or infectious disease and is for research purposes only. It's okay, rest for a minute and you'll be fine. This is the test that produces the 'cases' and 'deaths' that have been used to destroy human society. All those global and national medical and scientific 'experts' demanding this destruction to 'save us' *KNOW* that the test is not testing for the 'virus' and the cases and deaths they claim to be real are an almost unimaginable fraud. Every one of them and so many others including politicians and psychopaths like Gates and Tedros must be brought before Nuremburg-type trials and jailed for the rest of their lives. The more the genetic sample is amplified by PCR the more elements of that material become sensitive to the test and by that I don't mean sensitive for a 'virus' but for elements of the genetic material which

is *naturally* in the body or relates to remnants of old conditions of various kinds lying dormant and causing no disease. Once the amplification of the PCR reaches a certain level *everyone* will test positive. So much of the material has been made sensitive to the test that everyone will have some part of it in their body. Even lying criminals like Fauci have said that once PCR amplifications pass 35 cycles everything will be a false positive that cannot be trusted for the reasons I have described. I say, like many proper doctors and scientists, that 100 percent of the 'positives' are false, but let's just go with Fauci for a moment.

He says that any amplification over 35 cycles will produce false positives and yet the US Centers for Disease Control (CDC) and Food and Drug Administration (FDA) have recommended up to *40 cycles* and the National Health Service (NHS) in Britain admitted in an internal document for staff that it was using *45 cycles* of amplification. A long list of other countries has been doing the same and at least one 'testing' laboratory has been using *50 cycles*. Have you ever heard a doctor, medical 'expert' or the media ask what level of amplification has been used to claim a 'positive'. The 'test' comes back 'positive' and so you have the 'virus', end of story. Now we can see how the government in Tanzania could send off samples from a goat and a pawpaw fruit under human names and both came back positive for 'Covid-19'. Tanzania president John Magufuli mocked the 'Covid' hysteria, the PCR test and masks and refused to import the DNA-manipulating 'vaccine'. The Cult hated him and an article sponsored by the Bill Gates Foundation appeared in the London *Guardian* in February, 2021, headed 'It's time for Africa to rein in Tanzania's anti-vaxxer president'. Well, 'reined in' he shortly was. Magufuli appeared in good health, but then, in March, 2021, he was dead at 61 from 'heart failure'. He was replaced by Samia Hassan Suhulu who is connected to Klaus Schwab's World Economic Forum and she immediately reversed Magufuli's 'Covid' policy. A sample of cola tested positive for 'Covid' with the PCR test in Germany while American actress and singer-songwriter Erykah Badu tested positive in one nostril and negative in the other. Footballer Ronaldo called

the PCR test 'bullshit' after testing positive three times and being forced to quarantine and miss matches when there was nothing wrong with him. The mantra from Tedros at the World Health Organization and national governments (same thing) has been test, test, test. They know that the more tests they can generate the more fake 'cases' they have which go on to become 'deaths' in ways I am coming to. The UK government has its Operation Moonshot planned to test multiple millions every day in workplaces and schools with free tests for everyone to use twice a week at home in line with the Cult plan from the start to make testing part of life. A government advertisement for an 'Interim Head of Asymptomatic Testing Communication' said the job included responsibility for delivering a 'communications strategy' (propaganda) 'to support the expansion of asymptomatic testing that *'normalises testing as part of everyday life*'. More tests means more fake 'cases', 'deaths' and fascism. I have heard of, and from, many people who booked a test, couldn't turn up, and yet got a positive result through the post for a test they'd never even had. The whole thing is crazy, but for the Cult there's method in the madness. Controlling and manipulating the level of amplification of the test means the authorities can control whenever they want the number of apparent 'cases' and 'deaths'. If they want to justify more fascist lockdown and destruction of livelihoods they keep the amplification high. If they want to give the illusion that lockdowns and the 'vaccine' are working then they lower the amplification and 'cases' and 'deaths' will appear to fall. In January, 2021, the Cult-owned World Health Organization suddenly warned laboratories about over-amplification of the test and to lower the threshold. Suddenly headlines began appearing such as: 'Why ARE "Covid" cases plummeting?' This was just when the vaccine rollout was underway and I had predicted months before they would make cases appear to fall through amplification tampering when the 'vaccine' came. These people are so predictable.

Cow vaccines?

The question must be asked of what is on the test swabs being poked far up the nose of the population to the base of the brain? A nasal swab punctured one woman's brain and caused it to leak fluid. Most of these procedures are being done by people with little training or medical knowledge. Dr Lorraine Day, former orthopaedic trauma surgeon and Chief of Orthopaedic Surgery at San Francisco General Hospital, says the tests are really a *'vaccine'*. Cows have long been vaccinated this way. She points out that masks have to cover the nose and the mouth where it is claimed the 'virus' exists in saliva. Why then don't they take saliva from the mouth as they do with a DNA test instead of pushing a long swab up the nose towards the brain? The ethmoid bone separates the nasal cavity from the brain and within that bone is the cribriform plate. Dr Day says that when the swab is pushed up against this plate and twisted the procedure is 'depositing things back there'. She claims that among these 'things' are nanoparticles that can enter the brain. Researchers have noted that a team at the Gates-funded Johns Hopkins have designed tiny, star-shaped micro-devices that can latch onto intestinal mucosa and release drugs into the body. Mucosa is the thin skin that covers the inside surface of parts of the body such as *the nose* and mouth and produces mucus to protect them. The Johns Hopkins micro-devices are called 'theragrippers' and were 'inspired' by a parasitic worm that digs its sharp teeth into a host's intestines. Nasal swabs are also coated in the sterilisation agent ethylene oxide. The US National Cancer Institute posts this explanation on its website:

> At room temperature, ethylene oxide is a flammable colorless gas with a sweet odor. It is used primarily to produce other chemicals, including antifreeze. In smaller amounts, ethylene oxide is used as a pesticide and a sterilizing agent. The ability of ethylene oxide to damage DNA makes it an effective sterilizing agent but also accounts for its cancer-causing activity.

The Institute mentions lymphoma and leukaemia as cancers most frequently reported to be associated with occupational exposure to ethylene oxide along with stomach and breast cancers. How does anyone think this is going to work out with the constant testing

regime being inflicted on adults and children at home and at school that will accumulate in the body anything that's on the swab?

Doctors know best

It is vital for people to realise that 'hero' doctors 'know' only what the Big Pharma-dominated medical authorities tell them to 'know' and if they refuse to 'know' what they are told to 'know' they are out the door. They are mostly not physicians or healers, but repeaters of the official narrative – or else. I have seen alleged professional doctors on British television make shocking statements that we are supposed to take seriously. One called 'Dr' Amir Khan, who is actually telling patients how to respond to illness, said that men could take the birth pill to 'help slow down the effects of Covid-19'. In March, 2021, another ridiculous 'Covid study' by an American doctor proposed injecting men with the female sex hormone progesterone as a 'Covid' treatment. British doctor Nighat Arif told the BBC that face coverings were now going to be part of ongoing normal. Yes, the vaccine protects you, she said (evidence?) … but the way to deal with viruses in the community was always going to come down to hand washing, face covering and keeping a physical distance. That's not what we were told before the 'vaccine' was circulating. Arif said she couldn't imagine ever again going on the underground or in a lift without a mask. I was just thanking my good luck that she was not my doctor when she said – in March, 2021 – that if 'we are *behaving* and we are doing all the right things' she thought we could 'have our nearest and dearest around us at home … around *Christmas* and N*ew Year!* Her patronising delivery was the usual school teacher talking to six-year-olds as she repeated every government talking point and probably believed them all. If we have learned anything from the 'Covid' experience surely it must be that humanity's perception of doctors needs a fundamental rethink. NHS 'doctor' Sara Kayat told her television audience that the 'Covid vaccine' would '100 percent prevent hospitalisation and death'. Not even Big Pharma claimed that. We have to stop taking 'experts' at their word without question when so many of them are

clueless and only repeating the party line on which their careers depend. That is not to say there are not brilliants doctors – there are and I have spoken to many of them since all this began – but you won't see them in the mainstream media or quoted by the psychopaths and yes-people in government.

Remember the name – Christian Drosten

German virologist Christian Drosten, Director of Charité Institute of Virology in Berlin, became a national star after the pandemic hoax began. He was feted on television and advised the German government on 'Covid' policy. Most importantly to the wider world Drosten led a group that produced the 'Covid' testing protocol for the PCR test. What a remarkable feat given the PCR cannot test for infectious disease and even more so when you think that Drosten said that his method of testing for SARS-CoV-2 was developed 'without having virus material available'. *He developed a test for a 'virus' that he didn't have and had never seen*. Let that sink in as you survey the global devastation that came from what he did. The whole catastrophe of Drosten's 'test' was based on the alleged genetic sequence published by Chinese scientists on the Internet. We will see in the next chapter that this alleged 'genetic sequence' has never been produced by China or anyone and cannot be when there *is no* SARS-CoV-2. Drosten, however, doesn't seem to let little details like that get in the way. He was the lead author with Victor Corman from the same Charité Hospital of the paper 'Detection of 2019 novel coronavirus (2019-nCoV) by real-time PCR' published in a magazine called *Eurosurveillance*. This became known as the Corman-Drosten paper. In November, 2020, with human society devastated by the effects of the Corman-Drosten test baloney, the protocol was publicly challenged by 22 international scientists and independent researchers from Europe, the United States, and Japan. Among them were senior molecular geneticists, biochemists, immunologists, and microbiologists. They produced a document headed 'External peer review of the RTPCR test to detect SARS-Cov-2 Reveals 10 Major Flaws At The Molecular and Methodological Level: Consequences

For False-Positive Results'. The flaws in the Corman-Drosten test included the following:

- The test is non-specific because of erroneous design
- Results are enormously variable
- The test is unable to discriminate between the whole 'virus' and viral fragments
- It doesn't have positive or negative controls
- The test lacks a standard operating procedure
- It is unsupported by proper peer view

The scientists said the PCR 'Covid' testing protocol was not founded on science and they demanded the Corman-Drosten paper be retracted by *Eurosurveillance*. They said all present and previous Covid deaths, cases, and 'infection rates' should be subject to a massive retroactive inquiry. Lockdowns and travel restrictions should be reviewed and relaxed and those diagnosed through PCR to have 'Covid-19' should not be forced to isolate. Dr Kevin Corbett, a health researcher and nurse educator with a long academic career producing a stream of peer-reviewed publications at many UK universities, made the same point about the PCR test debacle. He said of the scientists' conclusions: 'Every scientific rationale for the development of that test has been totally destroyed by this paper. It's like Hiroshima/Nagasaki to the Covid test.' He said that China hadn't given them an isolated 'virus' when Drosten developed the test. Instead they had developed the test from *a sequence in a gene bank*.' Put another way … *they made it up*! The scientists were supported in this contention by a Portuguese appeals court which ruled in November, 2020, that PCR tests are unreliable and it is unlawful to quarantine people based solely on a PCR test. The point about China not providing an isolated virus must be true when the 'virus' has never been isolated to this day and the consequences of that will become clear. Drosten and company produced this useless 'protocol' right on cue in January, 2020, just as the 'virus' was said to

be moving westward and it somehow managed to successfully pass a peer-review in 24 hours. In other words there was no peer-review for a test that would be used to decide who had 'Covid' and who didn't across the world. The Cult-created, Gates-controlled World Health Organization immediately recommended all its nearly 200 member countries to use the Drosten PCR protocol to detect 'cases' and 'deaths'. The sting was underway and it continues to this day.

So who is this Christian Drosten that produced the means through which death, destruction and economic catastrophe would be justified? His education background, including his doctoral thesis, would appear to be somewhat shrouded in mystery and his track record is dire as with another essential player in the 'Covid' hoax, the Gates-funded Professor Neil Ferguson at the Gates-funded Imperial College in London of whom more shortly. Drosten predicted in 2003 that the alleged original SARS 'virus' (SARS-1') was an epidemic that could have serious effects on economies and an effective vaccine would take at least two years to produce. Drosten's answer to every alleged 'outbreak' is a vaccine which you won't be shocked to know. What followed were just 774 official deaths worldwide and none in Germany where there were only nine cases. That is even if you believe there ever was a SARS 'virus' when the evidence is zilch and I will expand on this in the next chapter. Drosten claims to be co-discoverer of 'SARS-1' and developed a test for it in 2003. He was screaming warnings about 'swine flu' in 2009 and how it was a widespread infection far more severe than any dangers from a vaccine could be and people should get vaccinated. It would be helpful for Drosten's vocal chords if he simply recorded the words 'the virus is deadly and you need to get vaccinated' and copies could be handed out whenever the latest made-up threat comes along. Drosten's swine flu epidemic never happened, but Big Pharma didn't mind with governments spending hundreds of millions on vaccines that hardly anyone bothered to use and many who did wished they hadn't. A study in 2010 revealed that the risk of dying from swine flu, or H1N1, was no higher than that of the annual seasonal flu which is what at least most of 'it' really was as in

the case of 'Covid-19'. A media investigation into Drosten asked how with such a record of inaccuracy he could be *the* government adviser on these issues. The answer to that question is the same with Drosten, Ferguson and Fauci – they keep on giving the authorities the 'conclusions' and 'advice' they want to hear. Drosten certainly produced the goods for them in January, 2020, with his PCR protocol garbage and provided the foundation of what German internal medicine specialist Dr Claus Köhnlein, co-author of *Virus Mania*, called the 'test pandemic'. The 22 scientists in the *Eurosurveillance* challenge called out conflicts of interest within the Drosten 'protocol' group and with good reason. Olfert Landt, a regular co-author of Drosten 'studies', owns the biotech company TIB Molbiol Syntheselabor GmbH in Berlin which manufactures and sells the tests that Drosten and his mates come up with. They have done this with SARS, Enterotoxigenic E. coli (ETEC), MERS, Zika 'virus', yellow fever, and now 'Covid'. Landt told the *Berliner Zeitung* newspaper:

> The testing, design and development came from the Charité [Drosten and Corman]. We simply implemented it immediately in the form of a kit. And if we don't have the virus, which originally only existed in Wuhan, we can make a synthetic gene to simulate the genome of the virus. That's what we did very quickly.

This is more confirmation that the Drosten test was designed without access to the 'virus' and only a synthetic simulation which is what SARS-CoV-2 really is – a computer-generated synthetic fiction. It's quite an enterprise they have going here. A Drosten team decides what the test for something should be and Landt's biotech company flogs it to governments and medical systems across the world. His company must have made an absolute fortune since the 'Covid' hoax began. Dr Reiner Fuellmich, a prominent German consumer protection trial lawyer in Germany and California, is on Drosten's case and that of Tedros at the World Health Organization for crimes against humanity with a class-action lawsuit being prepared in the United States and other legal action in Germany.

Why China?

Scamming the world with a 'virus' that doesn't exist would seem impossible on the face of it, but not if you have control of the relatively few people that make policy decisions and the great majority of the global media. Remember it's not about changing 'real' reality it's about controlling *perception* of reality. You don't have to make something happen you only have make people *believe* that it's happening. Renegade Minds understand this and are therefore much harder to swindle. 'Covid-19' is not a 'real' 'virus'. It's a mind virus, like a computer virus, which has infected the minds, not the bodies, of billions. It all started, publically at least, in China and that alone is of central significance. The Cult was behind the revolution led by its asset Mao Zedong, or Chairman Mao, which established the People's Republic of China on October 1st, 1949. It should have been called The Cult's Republic of China, but the name had to reflect the recurring illusion that vicious dictatorships are run by and for the people (see all the 'Democratic Republics' controlled by tyrants). In the same way we have the 'Biden' Democratic Republic of America officially ruled by a puppet tyrant (at least temporarily) on behalf of Cult tyrants. The creation of Mao's merciless communist/fascist dictatorship was part of a frenzy of activity by the Cult at the conclusion of World War Two which, like the First World War, it had instigated through its assets in Germany, Britain, France, the United States and elsewhere. Israel was formed in 1948; the Soviet Union expanded its 'Iron Curtain' control, influence and military power with the Warsaw Pact communist alliance in 1955; the United Nations was formed in 1945 as a Cult precursor to world government; and a long list of world bodies would be established including the World Health Organization (1948), World Trade Organization (1948 under another name until 1995), International Monetary Fund (1945) and World Bank (1944). Human society was redrawn and hugely centralised in the global Problem-Reaction-Solution that was World War Two. All these changes were significant. Israel would become the headquarters of the Sabbatians

and the revolution in China would prepare the ground and control system for the events of 2019/2020.

Renegade Minds know there are no borders except for public consumption. The Cult is a seamless, borderless global entity and to understand the game we need to put aside labels like borders, nations, countries, communism, fascism and democracy. These delude the population into believing that countries are ruled within their borders by a government of whatever shade when these are mere agencies of a global power. America's illusion of democracy and China's communism/fascism are subsidiaries – vehicles – for the same agenda. We may hear about conflict and competition between America and China and on the lower levels that will be true; but at the Cult level they are branches of the same company in the way of the McDonald's example I gave earlier. I have tracked in the books over the years support by US governments of both parties for Chinese Communist Party infiltration of American society through allowing the sale of land, even military facilities, and the acquisition of American business and university influence. All this is underpinned by the infamous stealing of intellectual property and technological know-how. Cult-owned Silicon Valley corporations waive their fraudulent 'morality' to do business with human-rights-free China; Cult-controlled Disney has become China's PR department; and China in effect owns 'American' sports such as basketball which depends for much of its income on Chinese audiences. As a result any sports player, coach or official speaking out against China's horrific human rights record is immediately condemned or fired by the China-worshipping National Basketball Association. One of the first acts of China-controlled Biden was to issue an executive order telling federal agencies to stop making references to the 'virus' by the 'geographic location of its origin'. Long-time Congressman Jerry Nadler warned that criticising China, America's biggest rival, leads to hate crimes against Asian people in the United States. So shut up you bigot. China is fast closing in on Israel as a country that must not be criticised which is apt, really, given that Sabbatians control them both. The two countries have

developed close economic, military, technological and strategic ties which include involvement in China's 'Silk Road' transport and economic initiative to connect China with Europe. Israel was the first country in the Middle East to recognise the establishment of Mao's tyranny in 1950 months after it was established.

Project Wuhan – the 'Covid' Psyop

I emphasise again that the Cult plays the long game and what is happening to the world today is the result of centuries of calculated manipulation following a script to take control step-by-step of every aspect of human society. I will discuss later the common force behind all this that has spanned those centuries and thousands of years if the truth be told. Instigating the Mao revolution in China in 1949 with a 2020 'pandemic' in mind is not only how they work – the 71 years between them is really quite short by the Cult's standards of manipulation preparation. The reason for the Cult's Chinese revolution was to create a fiercely-controlled environment within which an extreme structure for human control could be incubated to eventually be unleashed across the world. We have seen this happen since the 'pandemic' emerged from China with the Chinese control-structure founded on AI technology and tyrannical enforcement sweep across the West. Until the moment when the Cult went for broke in the West and put its fascism on public display Western governments had to pay some lip-service to freedom and democracy to not alert too many people to the tyranny-in-the-making. Freedoms were more subtly eroded and power centralised with covert government structures put in place waiting for the arrival of 2020 when that smokescreen of 'freedom' could be dispensed with. The West was not able to move towards tyranny before 2020 anything like as fast as China which was created as a tyranny and had no limits on how fast it could construct the Cult's blueprint for global control. When the time came to impose that structure on the world it was the same Cult-owned Chinese communist/fascist government that provided the excuse – the 'Covid pandemic'. It was absolutely crucial to the Cult plan for the Chinese response to the 'pandemic' –

draconian lockdowns of the entire population – to become the blueprint that Western countries would follow to destroy the livelihoods and freedom of their people. This is why the Cult-owned, Gates-owned, WHO Director-General Tedros said early on:

The Chinese government is to be congratulated for the extraordinary measures it has taken to contain the outbreak. China is actually setting a new standard for outbreak response and it is not an exaggeration.

Forbes magazine said of China: '… those measures protected untold millions from getting the disease'. The Rockefeller Foundation 'epidemic scenario' document in 2010 said 'prophetically':

However, a few countries did fare better – China in particular. The Chinese government's quick imposition and enforcement of mandatory quarantine for all citizens, as well as its instant and near-hermetic sealing off of all borders, saved millions of lives, stopping the spread of the virus far earlier than in other countries and enabling a swifter post-pandemic recovery.

Once again – *spooky*.

The first official story was the 'bat theory' or rather the bat diversion. The source of the 'virus outbreak' we were told was a ''wet market' in Wuhan where bats and other animals are bought and eaten in horrifically unhygienic conditions. Then another story emerged through the alternative media that the 'virus' had been released on purpose or by accident from a BSL-4 (biosafety level 4) laboratory in Wuhan not far from the wet market. The lab was reported to create and work with lethal concoctions and bioweapons. Biosafety level 4 is the highest in the World Health Organization system of safety and containment. Renegade Minds are aware of what I call designer manipulation. The ideal for the Cult is for people to buy its prime narrative which in the opening salvoes of the 'pandemic' was the wet market story. It knows, however, that there is now a considerable worldwide alternative media of researchers sceptical of anything governments say and they are often given a version of events in a form they can perceive as credible while misdirecting them from the real truth. In this case let them

think that the conspiracy involved is a 'bioweapon virus' released from the Wuhan lab to keep them from the real conspiracy – *there is no 'virus'*. The WHO's current position on the source of the outbreak at the time of writing appears to be: 'We haven't got a clue, mate.' This is a good position to maintain mystery and bewilderment. The inner circle will know where the 'virus' came from – *nowhere*. The bottom line was to ensure the public believed there *was* a 'virus' and it didn't much matter if they thought it was natural or had been released from a lab. The belief that there was a 'deadly virus' was all that was needed to trigger global panic and fear. The population was terrified into handing their power to authority and doing what they were told. They had to or they were 'all gonna die'.

In March, 2020, information began to come my way from real doctors and scientists and my own additional research which had my intuition screaming: 'Yes, that's it! *There is no virus*.' The 'bioweapon' was not the 'virus'; it was the *'vaccine'* already being talked about that would be the bioweapon. My conclusion was further enhanced by happenings in Wuhan. The 'virus' was said to be sweeping the city and news footage circulated of people collapsing in the street (which they've never done in the West with the same 'virus'). The Chinese government was building 'new hospitals' in a matter of ten days to 'cope with demand' such was the virulent nature of the 'virus'. Yet in what seemed like no time the 'new hospitals' closed – even if they even opened – and China declared itself 'virus-free'. It was back to business as usual. This was more propaganda to promote the Chinese draconian lockdowns in the West as the way to 'beat the virus'. Trouble was that we subsequently had lockdown after lockdown, but never business as usual. As the people of the West and most of the rest of the world were caught in an ever-worsening spiral of lockdown, social distancing, masks, isolated old people, families forced apart, and livelihood destruction, it was party-time in Wuhan. Pictures emerged of thousands of people enjoying pool parties and concerts. It made no sense until you realised there never was a 'virus' and the

whole thing was a Cult set-up to transform human society out of one its major global strongholds – China.

How is it possible to deceive virtually the entire world population into believing there is a deadly virus when there is not even a 'virus' let alone a deadly one? It's nothing like as difficult as you would think and that's clearly true because it happened.

Postscript: See end of book Postscript for more on the 'Wuhan lab virus release' story which the authorities and media were pushing heavily in the summer of 2021 to divert attention from the truth that the 'Covid virus' is pure invention.

CHAPTER FIVE

There *is no* 'virus'

You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time
Abraham Lincoln

The greatest form of mind control is repetition. The more you repeat the same mantra of alleged 'facts' the more will accept them to be true. It becomes an 'everyone knows that, mate'. If you can also censor any other version or alternative to your alleged 'facts' you are pretty much home and cooking.

By the start of 2020 the Cult owned the global mainstream media almost in its entirety to spew out its 'Covid' propaganda and ignore or discredit any other information and view. Cult-owned social media platforms in Cult-owned Silicon Valley were poised and ready to unleash a campaign of ferocious censorship to obliterate all but the official narrative. To complete the circle many demands for censorship by Silicon Valley were led by the mainstream media as 'journalists' became full-out enforcers for the Cult both as propagandists and censors. Part of this has been the influx of young people straight out of university who have become 'journalists' in significant positions. They have no experience and a headful of programmed perceptions from their years at school and university at a time when today's young are the most perceptually-targeted generations in known human history given the insidious impact of technology. They enter the media perceptually prepared and ready to repeat the narratives of the system that programmed them to

repeat its narratives. The BBC has a truly pathetic 'specialist disinformation reporter' called Marianna Spring who fits this bill perfectly. She is clueless about the world, how it works and what is really going on. Her role is to discredit anyone doing the job that a proper journalist would do and system-serving hacks like Spring wouldn't dare to do or even see the need to do. They are too busy licking the arse of authority which can never be wrong and, in the case of the BBC propaganda programme, *Panorama*, contacting payments systems such as PayPal to have a donations page taken down for a film company making documentaries questioning vaccines. Even the BBC soap opera *EastEnders* included a disgracefully biased scene in which an inarticulate white working class woman was made to look foolish for questioning the 'vaccine' while a well-spoken black man and Asian woman promoted the government narrative. It ticked every BBC box and the fact that the black and minority community was resisting the 'vaccine' had nothing to do with the way the scene was written. The BBC has become a disgusting tyrannical propaganda and censorship operation that should be defunded and disbanded and a free media take its place with a brief to stop censorship instead of demanding it. A BBC 'interview' with Gates goes something like: 'Mr Gates, sir, if I can call you sir, would you like to tell our audience why you are such a great man, a wonderful humanitarian philanthropist, and why you should absolutely be allowed as a software salesman to decide health policy for approaching eight billion people? Thank you, sir, please sir.' Propaganda programming has been incessant and merciless and when all you hear is the same story from the media, repeated by those around you who have only heard the same story, is it any wonder that people on a grand scale believe absolute mendacious garbage to be true? You are about to see, too, why this level of information control is necessary when the official 'Covid' narrative is so nonsensical and unsupportable by the evidence.

Structure of Deceit

The pyramid structure through which the 'Covid' hoax has been manifested is very simple and has to be to work. As few people as possible have to be involved with full knowledge of what they are doing – and why – or the real story would get out. At the top of the pyramid are the inner core of the Cult which controls Bill Gates who, in turn, controls the World Health Organization through his pivotal funding and his puppet Director-General mouthpiece, Tedros. Before he was appointed Tedros was chair of the Gates-founded Global Fund to 'fight against AIDS, tuberculosis and malaria', a board member of the Gates-funded 'vaccine alliance' GAVI, and on the board of another Gates-funded organisation. Gates owns him and picked him for a specific reason – Tedros is a crook and worse. 'Dr' Tedros (he's not a medical doctor, the first WHO chief not to be) was a member of the tyrannical Marxist government of Ethiopia for decades with all its human rights abuses. He has faced allegations of corruption and misappropriation of funds and was exposed three times for covering up cholera epidemics while Ethiopia's health minister. Tedros appointed the mass-murdering genocidal Zimbabwe dictator Robert Mugabe as a WHO goodwill ambassador for public health which, as with Tedros, is like appointing a psychopath to run a peace and love campaign. The move was so ridiculous that he had to drop Mugabe in the face of widespread condemnation. American economist David Steinman, a Nobel peace prize nominee, lodged a complaint with the International Criminal Court in The Hague over alleged genocide by Tedros when he was Ethiopia's foreign minister. Steinman says Tedros was a 'crucial decision maker' who directed the actions of Ethiopia's security forces from 2013 to 2015 and one of three officials in charge when those security services embarked on the 'killing' and 'torturing' of Ethiopians. You can see where Tedros is coming from and it's sobering to think that he has been the vehicle for Gates and the Cult to direct the global response to 'Covid'. Think about that. A psychopathic Cult dictates to psychopath Gates who dictates to psychopath Tedros who dictates how countries of the world must respond to a 'Covid virus' never scientifically shown to exist. At the same time psychopathic Cult-owned Silicon Valley information

giants like Google, YouTube, Facebook and Twitter announced very early on that they would give the Cult/Gates/Tedros/WHO version of the narrative free advertising and censor those who challenged their intelligence-insulting, mendacious story.

The next layer in the global 'medical' structure below the Cult, Gates and Tedros are the chief medical officers and science 'advisers' in each of the WHO member countries which means virtually all of them. Medical officers and arbiters of science (they're not) then take the WHO policy and recommended responses and impose them on their country's population while the political 'leaders' say they are deciding policy (they're clearly not) by 'following the science' on the advice of the 'experts' – the same medical officers and science 'advisers' (dictators). In this way with the rarest of exceptions the entire world followed the same policy of lockdown, people distancing, masks and 'vaccines' dictated by the psychopathic Cult, psychopathic Gates and psychopathic Tedros who we are supposed to believe give a damn about the health of the world population they are seeking to enslave. That, amazingly, is all there is to it in terms of crucial decision-making. Medical staff in each country then follow like sheep the dictates of the shepherds at the top of the national medical hierarchies – chief medical officers and science 'advisers' who themselves follow like sheep the shepherds of the World Health Organization and the Cult. Shepherds at the national level often have major funding and other connections to Gates and his Bill and Melinda Gates Foundation which carefully hands out money like confetti at a wedding to control the entire global medical system from the WHO down.

Follow the money

Christopher Whitty, Chief Medical Adviser to the UK Government at the centre of 'virus' policy, a senior adviser to the government's Scientific Advisory Group for Emergencies (SAGE), and Executive Board member of the World Health Organization, was gifted a grant of $40 million by the Bill and Melinda Gates Foundation for malaria research in Africa. The BBC described the unelected Whitty as 'the

official who will probably have the greatest impact on our everyday lives of any individual policymaker in modern times' and so it turned out. What Gates and Tedros have said Whitty has done like his equivalents around the world. Patrick Vallance, co-chair of SAGE and the government's Chief Scientific Adviser, is a former executive of Big Pharma giant GlaxoSmithKline with its fundamental financial and business connections to Bill Gates. In September, 2020, it was revealed that Vallance owned a deferred bonus of shares in GlaxoSmithKline worth £600,000 while the company was 'developing' a 'Covid vaccine'. Move along now – nothing to see here – what could possibly be wrong with that? Imperial College in London, a major player in 'Covid' policy in Britain and elsewhere with its 'Covid-19' Response Team, is funded by Gates and has big connections to China while the now infamous Professor Neil Ferguson, the useless 'computer modeller' at Imperial College is also funded by Gates. Ferguson delivered the dramatically inaccurate excuse for the first lockdowns (much more in the next chapter). The Institute for Health Metrics and Evaluation (IHME) in the United States, another source of outrageously false 'Covid' computer models to justify lockdowns, is bankrolled by Gates who is a vehement promotor of lockdowns. America's version of Whitty and Vallance, the again now infamous Anthony Fauci, has connections to 'Covid vaccine' maker Moderna as does Bill Gates through funding from the Bill and Melinda Gates Foundation. Fauci is director of the National Institute of Allergy and Infectious Diseases (NIAID), a major recipient of Gates money, and they are very close. Deborah Birx who was appointed White House Coronavirus Response Coordinator in February, 2020, is yet another with ties to Gates. Everywhere you look at the different elements around the world behind the coordination and decision making of the 'Covid' hoax there is Bill Gates and his money. They include the World Health Organization; Centers for Disease Control (CDC) in the United States; National Institutes of Health (NIH) of Anthony Fauci; Imperial College and Neil Ferguson; the London School of Hygiene where Chris Whitty worked; Regulatory agencies like the UK Medicines & Healthcare products Regulatory Agency (MHRA)

which gave emergency approval for 'Covid vaccines'; Wellcome Trust; GAVI, the Vaccine Alliance; the Coalition for Epidemic Preparedness Innovations (CEPI); Johns Hopkins University which has compiled the false 'Covid' figures; and the World Economic Forum. A Nationalfile.com article said:

> Gates has a lot of pull in the medical world, he has a multi-million dollar relationship with Dr. Fauci, and Fauci originally took the Gates line supporting vaccines and casting doubt on [the drug hydroxychloroquine]. Coronavirus response team member Dr. Deborah Birx, appointed by former president Obama to serve as United States Global AIDS Coordinator, also sits on the board of a group that has received billions from Gates' foundation, and Birx reportedly used a disputed Bill Gates-funded model for the White House's Coronavirus effort. Gates is a big proponent for a population lockdown scenario for the Coronavirus outbreak.

Another funder of Moderna is the Defense Advanced Research Projects Agency (DARPA), the technology-development arm of the Pentagon and one of the most sinister organisations on earth. DARPA had a major role with the CIA covert technology-funding operation In-Q-Tel in the development of Google and social media which is now at the centre of global censorship. Fauci and Gates are extremely close and openly admit to talking regularly about 'Covid' policy, but then why wouldn't Gates have a seat at every national 'Covid' table after his Foundation committed $1.75 billion to the 'fight against Covid-19'. When passed through our Orwellian Translation Unit this means that he has bought and paid for the Cult-driven 'Covid' response worldwide. Research the major 'Covid' response personnel in your own country and you will find the same Gates funding and other connections again and again. Medical and science chiefs following World Health Organization 'policy' sit atop a medical hierarchy in their country of administrators, doctors and nursing staff. These 'subordinates' are told they must work and behave in accordance with the policy delivered from the 'top' of the national 'health' pyramid which is largely the policy delivered by the WHO which is the policy delivered by Gates and the Cult. The whole 'Covid' narrative has been imposed on medical staff by a climate of fear although great numbers don't even need that to comply. They do so through breathtaking levels of ignorance and

include doctors who go through life simply repeating what Big Pharma and their hierarchical masters tell them to say and believe. No wonder Big Pharma 'medicine' is one of the biggest killers on Planet Earth.

The same top-down system of intimidation operates with regard to the Cult Big Pharma cartel which also dictates policy through national and global medical systems in this way. The Cult and Big Pharma agendas are the same because the former controls and owns the latter. 'Health' administrators, doctors, and nursing staff are told to support and parrot the dictated policy or they will face consequences which can include being fired. How sad it's been to see medical staff meekly repeating and imposing Cult policy without question and most of those who can see through the deceit are only willing to speak anonymously off the record. They know what will happen if their identity is known. This has left the courageous few to expose the lies about the 'virus', face masks, overwhelmed hospitals that aren't, and the dangers of the 'vaccine' that isn't a vaccine. When these medical professionals and scientists, some renowned in their field, have taken to the Internet to expose the truth their articles, comments and videos have been deleted by Cult-owned Facebook, Twitter and YouTube. What a real head-shaker to see YouTube videos with leading world scientists and highly qualified medical specialists with an added link underneath to the notorious Cult propaganda website *Wikipedia* to find the 'facts' about the same subject.

HIV – the 'Covid' trial-run

I'll give you an example of the consequences for health and truth that come from censorship and unquestioning belief in official narratives. The story was told by PCR inventor Kary Mullis in his book *Dancing Naked in the Mind Field*. He said that in 1984 he accepted as just another scientific fact that Luc Montagnier of France's Pasteur Institute and Robert Gallo of America's National Institutes of Health had independently discovered that a 'retrovirus' dubbed HIV (human immunodeficiency virus) caused AIDS. They

were, after all, Mullis writes, specialists in retroviruses. This is how the medical and science pyramids work. Something is announced or *assumed* and then becomes an everybody-knows-that purely through repetition of the assumption as if it is fact. Complete crap becomes accepted truth with no supporting evidence and only repetition of the crap. This is how a 'virus' that doesn't exist became the 'virus' that changed the world. The HIV-AIDS fairy story became a multi-billion pound industry and the media poured out propaganda terrifying the world about the deadly HIV 'virus' that caused the lethal AIDS. By then Mullis was working at a lab in Santa Monica, California, to detect retroviruses with his PCR test in blood donations received by the Red Cross. In doing so he asked a virologist where he could find a reference for HIV being the cause of AIDS. 'You don't need a reference,' the virologist said … *'Everybody knows it.'* Mullis said he wanted to quote a reference in the report he was doing and he said he felt a little funny about not knowing the source of such an important discovery when everyone else seemed to. The virologist suggested he cite a report by the Centers for Disease Control and Prevention (CDC) on morbidity and mortality. Mullis read the report, but it only said that an organism had been identified and did not say how. The report did not identify the original scientific work. Physicians, however, *assumed* (key recurring theme) that if the CDC was convinced that HIV caused AIDS then proof must exist. Mullis continues:

I did computer searches. Neither Montagnier, Gallo, nor anyone else had published papers describing experiments which led to the conclusion that HIV probably caused AIDS. I read the papers in Science for which they had become well known as AIDS doctors, but all they had said there was that they had found evidence of a past infection by something which was probably HIV in some AIDS patients.

They found antibodies. Antibodies to viruses had always been considered evidence of past disease, not present disease. Antibodies signaled that the virus had been defeated. The patient had saved himself. There was no indication in these papers that this virus caused a disease. They didn't show that everybody with the antibodies had the disease. In fact they found some healthy people with antibodies.

Mullis asked why their work had been published if Montagnier and Gallo hadn't really found this evidence, and why had they been fighting so hard to get credit for the discovery? He says he was hesitant to write 'HIV is the probable cause of AIDS' until he found published evidence to support that. 'Tens of thousands of scientists and researchers were spending billions of dollars a year doing research based on this idea,' Mullis writes. 'The reason had to be there somewhere; otherwise these people would not have allowed their research to settle into one narrow channel of investigation.' He said he lectured about PCR at numerous meetings where people were always talking about HIV and he asked them how they knew that HIV was the cause of AIDS:

> Everyone said something. Everyone had the answer at home, in the office, in some drawer. They all knew, and they would send me the papers as soon as they got back. But I never got any papers. Nobody ever sent me the news about how AIDS was caused by HIV.

Eventually Mullis was able to ask Montagnier himself about the reference proof when he lectured in San Diego at the grand opening of the University of California AIDS Research Center. Mullis says this was the last time he would ask his question without showing anger. Montagnier said he should reference the CDC report. 'I read it', Mullis said, and it didn't answer the question. 'If Montagnier didn't know the answer who the hell did?' Then one night Mullis was driving when an interview came on National Public Radio with Peter Duesberg, a prominent virologist at Berkeley and a California Scientist of the Year. Mullis says he finally understood why he could not find references that connected HIV to AIDS – *there weren't any!* No one had ever proved that HIV causes AIDS even though it had spawned a multi-billion pound global industry and the media was repeating this as fact every day in their articles and broadcasts terrifying the shit out of people about AIDS and giving the impression that a positive test for HIV (see 'Covid') was a death sentence. Duesberg was a threat to the AIDS gravy train and the agenda that underpinned it. He was therefore abused and castigated after he told the Proceedings of the National Academy of Sciences

there was no good evidence implicating the new 'virus'. Editors rejected his manuscripts and his research funds were deleted. Mullis points out that the CDC has defined AIDS as one of more than 30 diseases *if accompanied* by a positive result on a test that detects antibodies to HIV; but those same diseases are not defined as AIDS cases when antibodies are not detected:

> If an HIV-positive woman develops uterine cancer, for example, she is considered to have AIDS. If she is not HIV positive, she simply has uterine cancer. An HIV-positive man with tuberculosis has AIDS; if he tests negative he simply has tuberculosis. If he lives in Kenya or Colombia, where the test for HIV antibodies is too expensive, he is simply presumed to have the antibodies and therefore AIDS, and therefore he can be treated in the World Health Organization's clinic. It's the only medical help available in some places. And it's free, because the countries that support WHO are worried about AIDS.

Mullis accuses the CDC of continually adding new diseases (see ever more 'Covid symptoms') to the grand AIDS definition and of virtually doctoring the books to make it appear as if the disease continued to spread. He cites how in 1993 the CDC enormously broadened its AIDS definition and county health authorities were delighted because they received $2,500 per year from the Federal government for every reported AIDS case. Ladies and gentlemen, I have just described, via Kary Mullis, the 'Covid pandemic' of 2020 and beyond. Every element is the same and it's been pulled off in the same way by the same networks.

The 'Covid virus' exists? Okay – prove it. Er … still waiting

What Kary Mullis described with regard to 'HIV' has been repeated with 'Covid'. A claim is made that a new, or 'novel', infection has been found and the entire medical system of the world repeats that as fact exactly as they did with HIV and AIDS. No one in the mainstream asks rather relevant questions such as 'How do you know?' and 'Where is your proof?' The SARS-Cov-2 'virus' and the 'Covid-19 disease' became an overnight 'everybody-knows-that'. The origin could be debated and mulled over, but what you could not suggest was that 'SARS-Cov-2' didn't exist. That would be

ridiculous. 'Everybody knows' the 'virus' exists. Well, I didn't for one along with American proper doctors like Andrew Kaufman and Tom Cowan and long-time American proper journalist Jon Rappaport. We dared to pursue the obvious and simple question: 'Where's the evidence?' The overwhelming majority in medicine, journalism and the general public did not think to ask that. After all, *everyone knew* there was a new 'virus'. Everyone was saying so and I heard it on the BBC. Some would eventually argue that the 'deadly virus' was nothing like as deadly as claimed, but few would venture into the realms of its very existence. Had they done so they would have found that the evidence for that claim had gone AWOL as with HIV causes AIDS. In fact, not even that. For something to go AWOL it has to exist in the first place and scientific proof for a 'SARS-Cov-2' can be filed under nothing, nowhere and zilch.

Dr Andrew Kaufman is a board-certified forensic psychiatrist in New York State, a Doctor of Medicine and former Assistant Professor and Medical Director of Psychiatry at SUNY Upstate Medical University, and Medical Instructor of Hematology and Oncology at the Medical School of South Carolina. He also studied biology at the Massachusetts Institute of Technology (MIT) and trained in Psychiatry at Duke University. Kaufman is retired from allopathic medicine, but remains a consultant and educator on natural healing, I saw a video of his very early on in the 'Covid' hoax in which he questioned claims about the 'virus' in the absence of any supporting evidence and with plenty pointing the other way. I did everything I could to circulate his work which I felt was asking the pivotal questions that needed an answer. I can recommend an excellent pull-together interview he did with the website The Last Vagabond entitled *Dr Andrew Kaufman: Virus Isolation, Terrain Theory and Covid-19* and his website is andrewkaufmanmd.com. Kaufman is not only a forensic psychiatrist; he is forensic in all that he does. He always reads original scientific papers, experiments and studies instead of second-third-fourth-hand reports about the 'virus' in the media which are repeating the repeated repetition of the narrative. When he did so with the original Chinese 'virus' papers Kaufman

realised that there was no evidence of a 'SARS-Cov-2'. They had never – from the start – shown it to exist and every repeat of this claim worldwide was based on the accepted existence of proof that was nowhere to be found – see Kary Mullis and HIV. Here we go again.

Let's postulate

Kaufman discovered that the Chinese authorities immediately concluded that the cause of an illness that broke out among about 200 initial patients in Wuhan was a 'new virus' when there were no grounds to make that conclusion. The alleged 'virus' was not isolated from other genetic material in their samples and then shown through a system known as Koch's postulates to be the causative agent of the illness. The world was told that the SARS-Cov-2 'virus' caused a disease they called 'Covid-19' which had 'flu-like' symptoms and could lead to respiratory problems and pneumonia. If it wasn't so tragic it would almost be funny. *'Flu-like' symptoms'? Pneumonia? Respiratory disease?* What in *CHINA* and particularly in *Wuhan*, one of the most polluted cities in the world with a resulting epidemic of respiratory disease?? Three hundred thousand people get pneumonia in China every year and there are nearly a billion cases worldwide of 'flu-like symptoms'. These have a whole range of causes – including pollution in Wuhan – but no other possibility was credibly considered in late 2019 when the world was told there was a new and deadly 'virus'. The global prevalence of pneumonia and 'flu-like systems' gave the Cult networks unlimited potential to re-diagnose these other causes as the mythical 'Covid-19' and that is what they did from the very start. Kaufman revealed how Chinese medical and science authorities (all subordinates to the Cult-owned communist government) took genetic material from the lungs of only a few of the first patients. The material contained their own cells, bacteria, fungi and other microorganisms living in their bodies. The only way you could prove the existence of the 'virus' and its responsibility for the alleged 'Covid-19' was to isolate the virus from all the other material – a process also known as 'purification' – and

then follow the postulates sequence developed in the late 19th century by German physician and bacteriologist Robert Koch which became the 'gold standard' for connecting an alleged causation agent to a disease:

1. The microorganism (bacteria, fungus, virus, etc.) must be present in every case of the disease and all patients must have the same symptoms. It must also *not be present in healthy individuals*.

2. The microorganism must be isolated from the host with the disease. If the microorganism is a bacteria or fungus it must be grown in a pure culture. If it is a virus, it must be purified (i.e. containing no other material except the virus particles) from a clinical sample.

3. The specific disease, with all of its characteristics, must be reproduced when the infectious agent (the purified virus or a pure culture of bacteria or fungi) is inoculated into a healthy, susceptible host.

4. The microorganism must be recoverable from the experimentally infected host as in step 2.

Not one of these criteria has been met in the case of 'SARS-Cov-2' and 'Covid-19'. Not ONE. *EVER*. Robert Koch refers to bacteria and not viruses. What are called 'viral particles' are so minute (hence masks are useless by any definition) that they could only be seen after the invention of the electron microscope in the 1930s and can still only be observed through that means. American bacteriologist and virologist Thomas Milton Rivers, the so-called 'Father of Modern Virology' who was very significantly director of the Rockefeller Institute for Medical Research in the 1930s, developed a less stringent version of Koch's postulates to identify 'virus' causation known as 'Rivers criteria'. 'Covid' did not pass that process either. Some even doubt whether any 'virus' can be isolated from other particles containing genetic material in the Koch method. Freedom of Information requests in many countries asking for scientific proof that the 'Covid virus' has been purified and isolated and shown to exist have all come back with a 'we don't have that' and when this happened with a request to the UK Department of Health they added this comment:

However, outside of the scope of the [Freedom of Information Act] and on a discretionary basis, the following information has been advised to us, which may be of interest. Most infectious diseases are caused by viruses, bacteria or fungi. Some bacteria or fungi have the capacity to grow on their own in isolation, for example in colonies on a petri dish. Viruses are different in that they are what we call 'obligate pathogens' – that is, they cannot survive or reproduce without infecting a host ...

… For some diseases, it is possible to establish causation between a microorganism and a disease by isolating the pathogen from a patient, growing it in pure culture and reintroducing it to a healthy organism. These are known as 'Koch's postulates' and were developed in 1882. However, as our understanding of disease and different disease-causing agents has advanced, these are no longer the method for determining causation [Andrew Kaufman asks why in that case are there two published articles falsely claiming to satisfy Koch's postulates].

It has long been known that viral diseases cannot be identified in this way as viruses cannot be grown in 'pure culture'. When a patient is tested for a viral illness, this is normally done by looking for the presence of antigens, or viral genetic code in a host with molecular biology techniques [Kaufman asks how you could know the origin of these chemicals without having a pure culture for comparison].

For the record 'antigens' are defined so:

Invading microorganisms have antigens on their surface that the human body can recognise as being foreign – meaning not belonging to it. When the body recognises a foreign antigen, lymphocytes (white blood cells) produce antibodies, which are complementary in shape to the antigen.

Notwithstanding that this is open to question in relation to 'SARS-Cov-2' the presence of 'antibodies' can have many causes and they are found in people that are perfectly well. Kary Mullis said: 'Antibodies … had always been considered evidence of past disease, not present disease.'

'Covid' really is a *computer* 'virus'

Where the UK Department of Health statement says 'viruses' are now 'diagnosed' through a 'viral genetic code in a host with molecular biology techniques', they mean … *the PCR test* which its inventor said cannot test for infectious disease. They have no credible method of connecting a 'virus' to a disease and we will see that there is no scientific proof that any 'virus' causes any disease or there is any such thing as a 'virus' in the way that it is described. Tenacious Canadian researcher Christine Massey and her team made

some 40 Freedom of Information requests to national public health agencies in different countries asking for proof that SARS-CoV-2 has been isolated and not one of them could supply that information. Massey said of her request in Canada: 'Freedom of Information reveals Public Health Agency of Canada has no record of 'SARS-COV-2' isolation performed by anyone, anywhere, ever.' If you accept the comment from the UK Department of Health it's because they can't isolate a 'virus'. Even so many 'science' papers claimed to have isolated the 'Covid virus' until they were questioned and had to admit they hadn't. A reply from the Robert Koch Institute in Germany was typical: 'I am not aware of a paper which purified isolated SARS-CoV-2.' So what the hell was Christian Drosten and his gang using to design the 'Covid' testing protocol that has produced all the illusory Covid' cases and 'Covid' deaths when the head of the Chinese version of the CDC admitted there was a problem right from the start in that the 'virus' had never been isolated/purified? Breathe deeply: What they are calling 'Covid' is actually created by a *computer program* i.e. *they made it up* – er, that's it. They took lung fluid, with many sources of genetic material, from one single person alleged to be infected with Covid-19 by a PCR test which they *claimed*, without clear evidence, contained a 'virus'. They used several computer programs to create a model of a theoretical virus genome sequence from more than fifty-six million small sequences of RNA, each of an unknown source, assembling them like a puzzle with no known solution. The computer filled in the gaps with sequences from bits in the gene bank to make it look like a bat SARS-like coronavirus! A wave of the magic wand and poof, an in silico (computer-generated) genome, a scientific fantasy, was created. UK health researcher Dr Kevin Corbett made the same point with this analogy:

> … It's like giving you a few bones and saying that's your fish. It could be any fish. Not even a skeleton. Here's a few fragments of bones. That's your fish … It's all from gene bank and the bits of the virus sequence that weren't there they made up.
>
> They synthetically created them to fill in the blanks. That's what genetics is; it's a code. So it's ABBBCCDDD and you're missing some what you think is EEE so you put it in. It's all

synthetic. You just manufacture the bits that are missing. This is the end result of the geneticization of virology. This is basically a computer virus.

Further confirmation came in an email exchange between British citizen journalist Frances Leader and the government's Medicines & Healthcare Products Regulatory Agency (the Gates-funded MHRA) which gave emergency permission for untested 'Covid vaccines' to be used. The agency admitted that the 'vaccine' is not based on an isolated 'virus', but comes from a *computer-generated model*. Frances Leader was naturally banned from Cult-owned fascist Twitter for making this exchange public. The process of creating computer-generated alleged 'viruses' is called 'in silico' or 'in silicon' – computer chips – and the term 'in silico' is believed to originate with biological experiments using only a computer in 1989. 'Vaccines' involved with 'Covid' are also produced 'in silico' or by computer not a natural process. If the original 'virus' is nothing more than a made-up computer model how can there be 'new variants' of something that never existed in the first place? They are not new 'variants'; they are new *computer models* only minutely different to the original program and designed to further terrify the population into having the 'vaccine' and submitting to fascism. You want a 'new variant'? Click, click, enter – there you go. Tell the medical profession that you have discovered a 'South African variant', 'UK variants' or a 'Brazilian variant' and in the usual HIV-causes-AIDS manner they will unquestioningly repeat it with no evidence whatsoever to support these claims. They will go on television and warn about the dangers of 'new variants' while doing nothing more than repeating what they have been told to be true and knowing that any deviation from that would be career suicide. Big-time insiders will know it's a hoax, but much of the medical community is clueless about the way they are being played and themselves play the public without even being aware they are doing so. What an interesting 'coincidence' that AstraZeneca and Oxford University were conducting 'Covid vaccine trials' in the three countries – the UK, South Africa and Brazil – where the first three 'variants' were claimed to have 'broken out'.

Here's your 'virus' – it's a unicorn

Dr Andrew Kaufman presented a brilliant analysis describing how the 'virus' was imagined into fake existence when he dissected an article published by *Nature* and written by 19 authors detailing *alleged* 'sequencing of a complete viral genome' of the 'new SARS-CoV-2 virus'. This computer-modelled *in silico* genome was used as a template for all subsequent genome sequencing experiments that resulted in the so-called variants which he said now number more than *6,000*. The fake genome was constructed from more than 56 million individual short strands of RNA. Those little pieces were assembled into longer pieces by finding areas of overlapping sequences. The computer programs created over two million possible combinations from which the authors simply chose the longest one. They then compared this to a 'bat virus' and the computer 'alignment' rearranged the sequence and filled in the gaps! They called this computer-generated abomination the 'complete genome'. Dr Tom Cowan, a fellow medical author and collaborator with Kaufman, said such computer-generation constitutes scientific fraud and he makes this superb analogy:

> Here is an equivalency: A group of researchers claim to have found a unicorn because they found a piece of a hoof, a hair from a tail, and a snippet of a horn. They then add that information into a computer and program it to re-create the unicorn, and they then claim this computer re-creation is the real unicorn. Of course, they had never actually seen a unicorn so could not possibly have examined its genetic makeup to compare their samples with the actual unicorn's hair, hooves and horn.
>
> The researchers claim they decided which is the real genome of SARS-CoV-2 by 'consensus', sort of like a vote. Again, different computer programs will come up with different versions of the imaginary 'unicorn', so they come together as a group and decide which is the real imaginary unicorn.

This is how the 'virus' that has transformed the world was brought into fraudulent 'existence'. Extraordinary, yes, but as the Nazis said the bigger the lie the more will believe it. Cowan, however, wasn't finished and he went on to identify what he called the real blockbuster in the paper. He quotes this section from a paper written

by virologists and published by the CDC and then explains what it means:

Therefore, we examined the capacity of SARS-CoV-2 to infect and replicate in several common primate and human cell lines, including human adenocarcinoma cells (A549), human liver cells (HUH 7.0), and human embryonic kidney cells (HEK-293T). In addition to Vero E6 and Vero CCL81 cells. ... Each cell line was inoculated at high multiplicity of infection and examined 24h post-infection.

No CPE was observed in any of the cell lines except in Vero cells, which grew to greater than 10 to the 7th power at 24 h post-infection. In contrast, HUH 7.0 and 293T showed only modest viral replication, and A549 cells were incompatible with SARS CoV-2 infection.

Cowan explains that when virologists attempt to prove infection they have three possible 'hosts' or models on which they can test. The first was humans. Exposure to humans was generally not done for ethical reasons and has never been done with SARS-CoV-2 or any coronavirus. The second possible host was animals. Cowan said that forgetting for a moment that they never actually use purified virus when exposing animals they do use solutions that they *claim* contain the virus. Exposure to animals has been done with SARS-CoV-2 in an experiment involving mice and this is what they found: *None of the wild (normal) mice got sick*. In a group of genetically-modified mice, a statistically insignificant number lost weight and had slightly bristled fur, but they experienced nothing like the illness called 'Covid-19'. Cowan said the third method – the one they mostly rely on – is to inoculate solutions they *say* contain the virus onto a variety of tissue cultures. This process had never been shown to kill tissue *unless* the sample material was starved of nutrients and poisoned as *part of the process*. Yes, incredibly, in tissue experiments designed to show the 'virus' is responsible for killing the tissue they starve the tissue of nutrients and add toxic drugs including antibiotics and they do not have control studies to see if it's the starvation and poisoning that is degrading the tissue rather than the 'virus' they allege to be in there somewhere. You want me to pinch you? Yep, I understand. Tom Cowan said this about the whole nonsensical farce as he explains what that quote from the CDC paper really means:

The shocking thing about the above quote is that using their own methods, the virologists found that solutions containing SARS-CoV-2 – even in high amounts – were NOT, I repeat NOT, infective to any of the three human tissue cultures they tested. In plain English, this means they proved, on their terms, that this 'new coronavirus' is not infectious to human beings. It is ONLY infective to monkey kidney cells, and only then when you add two potent drugs (gentamicin and amphotericin), known to be toxic to kidneys, to the mix.

My friends, read this again and again. These virologists, published by the CDC, performed a clear proof, on their terms, showing that the SARS-CoV-2 virus is harmless to human beings. That is the only possible conclusion, but, unfortunately, this result is not even mentioned in their conclusion. They simply say they can provide virus stocks cultured only on monkey Vero cells, thanks for coming.

Cowan concluded: 'If people really understood how this "science" was done, I would hope they would storm the gates and demand honesty, transparency and truth.' Dr Michael Yeadon, former Vice President and Chief Scientific Adviser at drug giant Pfizer has been a vocal critic of the 'Covid vaccine' and its potential for multiple harm. He said in an interview in April, 2021, that 'not one [vaccine] has the virus. He was asked why vaccines normally using a 'dead' version of a disease to activate the immune system were not used for 'Covid' and instead we had the synthetic methods of the 'mRNA Covid vaccine'. Yeadon said that to do the former 'you'd have to have some of [the virus] wouldn't you?' He added: 'No-one's got any – seriously.' Yeadon said that surely they couldn't have fooled the whole world for a year without having a virus, 'but oddly enough ask around – no one's got it'. He didn't know why with all the 'great labs' around the world that the virus had not been isolated – 'Maybe they've been too busy running bad PCR tests and vaccines that people don't need.' What is today called 'science' is not 'science' at all. Science is no longer what is, but whatever people can be manipulated to *believe* that it is. Real science has been hijacked by the Cult to dispense and produce the 'expert scientists' and contentions that suit the agenda of the Cult. How big-time this has happened with the 'Covid' hoax which is entirely based on fake science delivered by fake 'scientists' and fake 'doctors'. The human-caused climate change hoax is also entirely based on fake science delivered by fake 'scientists' and fake 'climate experts'. In both cases real

scientists, climate experts and doctors have their views suppressed and deleted by the Cult-owned science establishment, media and Silicon Valley. This is the 'science' that politicians claim to be 'following' and a common denominator of 'Covid' and climate are Cult psychopaths Bill Gates and his mate Klaus Schwab at the Gates-funded World Economic Forum. But, don't worry, it's all just a coincidence and absolutely nothing to worry about. *Zzzzzzzz*.

What is a 'virus' REALLY?

Dr Tom Cowan is one of many contesting the very existence of viruses let alone that they cause disease. This is understandable when there is no scientific evidence for a disease-causing 'virus'. German virologist Dr Stefan Lanka won a landmark case in 2017 in the German Supreme Court over his contention that there is no such thing as a measles virus. He had offered a big prize for anyone who could prove there is and Lanka won his case when someone sought to claim the money. There is currently a prize of more than 225,000 euros on offer from an Isolate Truth Fund for anyone who can prove the isolation of SARS-CoV-2 and its genetic substance. Lanka wrote in an article headed 'The Misconception Called Virus' that scientists think a 'virus' is causing tissue to become diseased and degraded when in fact it is the *processes they are using* which do that – not a 'virus'. Lanka has done an important job in making this point clear as Cowan did in his analysis of the CDC paper. Lanka says that all claims about viruses as disease-causing pathogens are wrong and based on 'easily recognisable, understandable and verifiable misinterpretations.' Scientists believed they were working with 'viruses' in their laboratories when they were really working with 'typical particles of specific dying tissues or cells …' Lanka said that the tissue decaying process claimed to be caused by a 'virus' still happens when no alleged 'virus' is involved. It's the *process* that does the damage and not a 'virus'. The genetic sample is deprived of nutrients, removed from its energy supply through removal from the body and then doused in toxic antibiotics to remove any bacteria. He confirms again that establishment scientists do not (pinch me)

conduct control experiments to see if this is the case and if they did they would see the claims that 'viruses' are doing the damage is nonsense. He adds that during the measles 'virus' court case he commissioned an independent laboratory to perform just such a control experiment and the result was that the tissues and cells died in the exact same way as with alleged 'infected' material. This is supported by a gathering number of scientists, doctors and researchers who reject what is called 'germ theory' or the belief in the body being infected by contagious sources emitted by other people. Researchers Dawn Lester and David Parker take the same stance in their highly-detailed and sourced book *What Really Makes You Ill – Why everything you thought you knew about disease is wrong* which was recommended to me by a number of medical professionals genuinely seeking the truth. Lester and Parker say there is no provable scientific evidence to show that a 'virus' can be transmitted between people or people and animals or animals and people:

> The definition also claims that viruses are the cause of many diseases, as if this has been definitively proven. But this is not the case; there is no original scientific evidence that definitively demonstrates that any virus is the cause of any disease. The burden of proof for any theory lies with those who proposed it; but none of the existing documents provides 'proof' that supports the claim that 'viruses' are pathogens.

Dr Tom Cowan employs one of his clever analogies to describe the process by which a 'virus' is named as the culprit for a disease when what is called a 'virus' is only material released by cells detoxing themselves from infiltration by chemical or radiation poisoning. The tidal wave of technologically-generated radiation in the 'smart' modern world plus all the toxic food and drink are causing this to happen more than ever. Deluded 'scientists' misread this as a gathering impact of what they wrongly label 'viruses'.

Paper can infect houses

Cowan said in an article for davidicke.com – with his tongue only mildly in his cheek – that he believed he had made a tremendous

discovery that may revolutionise science. He had discovered that small bits of paper are alive, 'well alive-ish', can 'infect' houses, and then reproduce themselves inside the house. The result was that this explosion of growth in the paper inside the house causes the house to explode, blowing it to smithereens. His evidence for this new theory is that in the past months he had carefully examined many of the houses in his neighbourhood and found almost no scraps of paper on the lawns and surrounds of the house. There was an occasional stray label, but nothing more. Then he would return to these same houses a week or so later and with a few, not all of them, particularly the old and decrepit ones, he found to his shock and surprise they were littered with stray bits of paper. He knew then that the paper had infected these houses, made copies of itself, and blew up the house. A young boy on a bicycle at one of the sites told him he had seen a demolition crew using dynamite to explode the house the previous week, but Cowan dismissed this as the idle thoughts of silly boys because 'I was on to something big'. He was on to how 'scientists' mistake genetic material in the detoxifying process for something they call a 'virus'. Cowan said of his house and paper story:

> If this sounds crazy to you, it's because it should. This scenario is obviously nuts. But consider this admittedly embellished, for effect, current viral theory that all scientists, medical doctors and virologists currently believe.

He takes the example of the 'novel SARS-Cov2' virus to prove the point. First they take someone with an undefined illness called 'Covid-19' and don't even attempt to find any virus in their sputum. Never mind the scientists still describe how this 'virus', which they have not located attaches to a cell receptor, injects its genetic material, in 'Covid's' case, RNA, into the cell. The RNA once inserted exploits the cell to reproduce itself and makes 'thousands, nay millions, of copies of itself … Then it emerges victorious to claim its next victim':

If you were to look in the scientific literature for proof, actual scientific proof, that uniform SARS-CoV2 viruses have been properly isolated from the sputum of a sick person, that actual spike proteins could be seen protruding from the virus (which has not been found), you would find that such evidence doesn't exist.

If you go looking in the published scientific literature for actual pictures, proof, that these spike proteins or any viral proteins are ever attached to any receptor embedded in any cell membrane, you would also find that no such evidence exists. If you were to look for a video or documented evidence of the intact virus injecting its genetic material into the body of the cell, reproducing itself and then emerging victorious by budding off the cell membrane, you would find that no such evidence exists.

The closest thing you would find is electron micrograph pictures of cellular particles, possibly attached to cell debris, both of which to be seen were stained by heavy metals, a process that completely distorts their architecture within the living organism. This is like finding bits of paper stuck to the blown-up bricks, thereby proving the paper emerged by taking pieces of the bricks on its way out.

The Enders baloney

Cowan describes the 'Covid' story as being just as make-believe as his paper story and he charts back this fantasy to a Nobel Prize winner called John Enders (1897-1985), an American biomedical scientist who has been dubbed 'The Father of Modern Vaccines'. Enders is claimed to have 'discovered' the process of the viral culture which 'proved' that a 'virus' caused measles. Cowan explains how Enders did this 'by using the EXACT same procedure that has been followed by every virologist to find and characterize every new virus since 1954'. Enders took throat swabs from children with measles and immersed them in 2ml of milk. Penicillin (100u/ml) and the antibiotic streptomycin (50,g/ml) were added and the whole mix was centrifuged – rotated at high speed to separate large cellular debris from small particles and molecules as with milk and cream, for example. Cowan says that if the aim is to find little particles of genetic material ('viruses') in the snot from children with measles it would seem that the last thing you would do is mix the snot with other material – milk –that also has genetic material. 'How are you ever going to know whether whatever you found came from the snot or the milk?' He points out that streptomycin is a 'nephrotoxic' or poisonous-to-the-kidney drug. You will see the relevance of that

shortly. Cowan says that it gets worse, much worse, when Enders describes the culture medium upon which the virus 'grows': 'The culture medium consisted of bovine amniotic fluid (90%), beef embryo extract (5%), horse serum (5%), antibiotics and phenol red as an indicator of cell metabolism.' Cowan asks incredulously: 'Did he just say that the culture medium also contained fluids and tissues that are themselves rich sources of genetic material?' The genetic cocktail, or 'medium', is inoculated onto tissue and cells from rhesus monkey *kidney* tissue. This is where the importance of streptomycin comes in and currently-used antimicrobials and other drugs that are *poisonous to kidneys* and used in ALL modern viral cultures (e.g. gentamicin, streptomycin, and amphotericin). Cowan asks: 'How are you ever going to know from this witch's brew where any genetic material comes from as we now have five different sources of rich genetic material in our mix?' Remember, he says, that all genetic material, whether from monkey kidney tissues, bovine serum, milk, etc., is made from the exact same components. The same central question returns: 'How are you possibly going to know that it was the virus that killed the kidney tissue and not the toxic antibiotic and starvation rations on which you are growing the tissue?' John Enders answered the question himself – *you can't*:

A second agent was obtained from an uninoculated culture of monkey kidney cells. The cytopathic changes [death of the cells] it induced in the unstained preparations could not be distinguished with confidence from the viruses isolated from measles.

The death of the cells ('cytopathic changes') happened in exactly the same manner, whether they inoculated the kidney tissue with the measles snot or not, Cowan says. 'This is evidence that the destruction of the tissue, the very proof of viral causation of illness, was not caused by anything in the snot because they saw the same destructive effect when the snot was not even used … the cytopathic, i.e., cell-killing, changes come from the process of the culture itself, not from any virus in any snot, period.' Enders quotes in his 1957 paper a virologist called Ruckle as reporting similar findings 'and in addition has isolated an agent from monkey kidney tissue that is so

far indistinguishable from human measles virus'. In other words, Cowan says, these particles called 'measles viruses' are simply and clearly breakdown products of the starved and poisoned tissue. For measles 'virus' see all 'viruses' including the so-called 'Covid virus'. Enders, the 'Father of Modern Vaccines', also said:

There is a potential risk in employing cultures of primate cells for the production of vaccines composed of attenuated virus, since the presence of other agents possibly latent in primate tissues cannot be definitely excluded by any known method.

Cowan further quotes from a paper published in the journal *Viruses* in May, 2020, while the 'Covid pandemic' was well underway in the media if not in reality. 'EVs' here refers to particles of genetic debris from our own tissues, such as exosomes of which more in a moment: 'The remarkable resemblance between EVs and viruses has caused quite a few problems in the studies focused on the analysis of EVs released during viral infections.' Later the paper adds that to date a reliable method that can actually guarantee a complete separation (of EVs from viruses) DOES NOT EXIST. This was published at a time when a fairy tale 'virus' was claimed in total certainty to be causing a fairy tale 'viral disease' called 'Covid-19' – a fairy tale that was already well on the way to transforming human society in the image that the Cult has worked to achieve for so long. Cowan concludes his article:

To summarize, there is no scientific evidence that pathogenic viruses exist. What we think of as 'viruses' are simply the normal breakdown products of dead and dying tissues and cells. When we are well, we make fewer of these particles; when we are starved, poisoned, suffocated by wearing masks, or afraid, we make more.

There is no engineered virus circulating and making people sick. People in laboratories all over the world are making genetically modified products to make people sick. These are called vaccines. There is no virome, no 'ecosystem' of viruses, viruses are not 8%, 50% or 100 % of our genetic material. These are all simply erroneous ideas based on the misconception called a virus.

What is 'Covid'? Load of bollocks

The background described here by Cowan and Lanka was emphasised in the first video presentation that I saw by Dr Andrew Kaufman when he asked whether the 'Covid virus' was in truth a natural defence mechanism of the body called 'exosomes'. These are released by cells when in states of toxicity – see the same themes returning over and over. They are released ever more profusely as chemical and radiation toxicity increases and think of the potential effect therefore of 5G alone as its destructive frequencies infest the human energetic information field with a gathering pace (5G went online in Wuhan in 2019 as the 'virus' emerged). I'll have more about this later. Exosomes transmit a warning to the rest of the body that 'Houston, we have a problem'. Kaufman presented images of exosomes and compared them with 'Covid' under an electron microscope and the similarity was remarkable. They both attach to the same cell receptors (*claimed* in the case of 'Covid'), contain the same genetic material in the form of RNA or ribonucleic acid, and both are found in 'viral cell cultures' with damaged or dying cells. James Hildreth MD, President and Chief Executive Officer of the Meharry Medical College at Johns Hopkins, said: 'The virus is fully an exosome in every sense of the word.' Kaufman's conclusion was that there is no 'virus': 'This entire pandemic is a completely manufactured crisis … there is no evidence of anyone dying from [this] illness.' Dr Tom Cowan and Sally Fallon Morell, authors of *The Contagion Myth*, published a statement with Dr Kaufman in February, 2021, explaining why the 'virus' does not exist and you can read it that in full in the Appendix.

'Virus' theory can be traced to the 'cell theory' in 1858 of German physician Rudolf Virchow (1821-1920) who contended that disease originates from a single cell infiltrated by a 'virus'. Dr Stefan Lanka said that findings and insights with respect to the structure, function and central importance of tissues in the creation of life, which were already known in 1858, comprehensively refute the cell theory. Virchow ignored them. We have seen the part later played by John Enders in the 1950s and Lanka notes that infection theories were only established as a global dogma through the policies and

eugenics of the Third Reich in Nazi Germany (creation of the same Sabbatian cult behind the 'Covid' hoax). Lanka said: 'Before 1933, scientists dared to contradict this theory; after 1933, these critical scientists were silenced'. Dr Tom Cowan's view is that ill-heath is caused by too much of something, too little of something, or toxification from chemicals and radiation – not contagion. We must also highlight as a major source of the 'virus' theology a man still called the 'Father of Modern Virology' – Thomas Milton Rivers (1888-1962). There is no way given the Cult's long game policy that it was a coincidence for the 'Father of Modern Virology' to be director of the Rockefeller Institute for Medical Research from 1937 to 1956 when he is credited with making the Rockefeller Institute a leader in 'viral research'. Cult Rockefellers were the force behind the creation of Big Pharma 'medicine', established the World Health Organisation in 1948, and have long and close associations with the Gates family that now runs the WHO during the pandemic hoax through mega-rich Cult gofer and psychopath Bill Gates.

Only a Renegade Mind can see through all this bullshit by asking the questions that need to be answered, not taking 'no' or prevarication for an answer, and certainly not hiding from the truth in fear of speaking it. Renegade Minds have always changed the world for the better and they will change this one no matter how bleak it may currently appear to be.

CHAPTER SIX

Sequence of deceit

If you tell the truth, you don't have to remember anything
Mark Twain

Against the background that I have laid out this far the sequence that took us from an invented 'virus' in Cult-owned China in late 2019 to the fascist transformation of human society can be seen and understood in a whole new context.

We were told that a deadly disease had broken out in Wuhan and the world media began its campaign (coordinated by behavioural psychologists as we shall see) to terrify the population into unquestioning compliance. We were shown images of Chinese people collapsing in the street which never happened in the West with what was supposed to be the same condition. In the earliest days when alleged cases and deaths were few the fear register was hysterical in many areas of the media and this would expand into the common media narrative across the world. The real story was rather different, but we were never told that. The Chinese government, one of the Cult's biggest centres of global operation, said they had discovered a new illness with flu-like and pneumonia-type symptoms in a city with such toxic air that it is overwhelmed with flu-like symptoms, pneumonia and respiratory disease. Chinese scientists said it was a new – 'novel' – coronavirus which they called Sars-Cov-2 and that it caused a disease they labelled 'Covid-19'. There was no evidence for this and the 'virus' has never to this day been isolated, purified and its genetic code established from that. It

was from the beginning a computer-generated fiction. Stories of Chinese whistleblowers saying the number of deaths was being supressed or that the 'new disease' was related to the Wuhan bio-lab misdirected mainstream and alternative media into cul-de-sacs to obscure the real truth – there was no 'virus'.

Chinese scientists took genetic material from the lung fluid of just a few people and said they had found a 'new' disease when this material had a wide range of content. There was no evidence for a 'virus' for the very reasons explained in the last two chapters. The 'virus' has never been shown to (a) exist and (b) cause any disease. People were diagnosed on symptoms that are so widespread in Wuhan and polluted China and with a PCR test that can't detect infectious disease. On this farce the whole global scam was sold to the rest of the world which would also diagnose respiratory disease as 'Covid-19' from symptoms alone or with a PCR test not testing for a 'virus'. Flu miraculously disappeared *worldwide* in 2020 and into 2021 as it was redesignated 'Covid-19'. It was really the same old flu with its 'flu-like' symptoms attributed to 'flu-like' 'Covid-19'. At the same time with very few exceptions the Chinese response of draconian lockdown and fascism was the chosen weapon to respond across the West as recommended by the Cult-owned Tedros at the Cult-owned World Health Organization run by the Cult-owned Gates. All was going according to plan. Chinese scientists – everything in China is controlled by the Cult-owned government – compared their contaminated RNA lung-fluid material with other RNA sequences and said it appeared to be just under 80 percent identical to the SARS-CoV-1 'virus' claimed to be the cause of the SARS (severe acute respiratory syndrome) 'outbreak' in 2003. They decreed that because of this the 'new virus' had to be related and they called it SARS-CoV-2. There are some serious problems with this assumption and *assumption* was all it was. Most 'factual' science turns out to be assumptions repeated into everyone-knows-that. A match of under 80-percent is meaningless. Dr Kaufman makes the point that there's a *96 percent* genetic correlation between humans and chimpanzees, but 'no one would say our genetic material is part

of the chimpanzee family'. Yet the Chinese authorities were claiming that a much lower percentage, less than 80 percent, proved the existence of a new 'coronavirus'. For goodness sake human DNA is 60 percent similar to a *banana*.

You are feeling sleepy

The entire 'Covid' hoax is a global Psyop, a psychological operation to program the human mind into believing and fearing a complete fantasy. A crucial aspect of this was what *appeared* to happen in Italy. It was all very well streaming out daily images of an alleged catastrophe in Wuhan, but to the Western mind it was still on the other side of the world in a very different culture and setting. A reaction of 'this could happen to me and my family' was still nothing like as intense enough for the mind-doctors. The Cult needed a Western example to push people over that edge and it chose Italy, one of its major global locations going back to the Roman Empire. An Italian 'Covid' crisis was manufactured in a particular area called Lombardy which just happens to be notorious for its toxic air and therefore respiratory disease. Wuhan, China, déjà vu. An hysterical media told horror stories of Italians dying from 'Covid' in their droves and how Lombardy hospitals were being overrun by a tidal wave of desperately ill people needing treatment after being struck down by the 'deadly virus'. Here was the psychological turning point the Cult had planned. Wow, if this is happening in Italy, the Western mind concluded, this indeed could happen to me and my family. Another point is that Italian authorities responded by following the Chinese blueprint so vehemently recommended by the Cult-owned World Health Organization. They imposed fascistic lockdowns on the whole country viciously policed with the help of surveillance drones sweeping through the streets seeking out anyone who escaped from mass house arrest. Livelihoods were destroyed and psychology unravelled in the way we have witnessed since in all lockdown countries. Crucial to the plan was that Italy responded in this way to set the precedent of suspending freedom and imposing fascism in a 'Western liberal democracy'. I emphasised in an

animated video explanation on davidicke.com posted in the summer of 2020 how important it was to the Cult to expand the Chinese lockdown model across the West. Without this, and the bare-faced lie that non-symptomatic people could still transmit a 'disease' they didn't have, there was no way locking down the whole population, sick and not sick, could be pulled off. At just the right time and with no evidence Cult operatives and gofers claimed that people without symptoms could pass on the 'disease'. In the name of protecting the 'vulnerable' like elderly people, who lockdowns would kill by the tens of thousands, we had for the first time healthy people told to isolate as well as the sick. The great majority of people who tested positive had no symptoms because there was nothing wrong with them. It was just a trick made possible by a test not testing for the 'virus'.

Months after my animated video the Gates-funded Professor Neil Ferguson at the Gates-funded Imperial College confirmed that I was right. He didn't say it in those terms, naturally, but he did say it. Ferguson will enter the story shortly for his outrageously crazy 'computer models' that led to Britain, the United States and many other countries following the Chinese and now Italian methods of response. Put another way, following the Cult script. Ferguson said that SAGE, the UK government's scientific advisory group which has controlled 'Covid' policy from the start, wanted to follow the Chinese lockdown model (while they all continued to work and be paid), but they wondered if they could possibly, in Ferguson's words, 'get away with it in Europe'. 'Get away with it'? Who the hell do these moronic, arrogant people think they are? This appalling man Ferguson said that once Italy went into national lockdown they realised they, too, could mimic China:

> It's a communist one-party state, we said. We couldn't get away with it in Europe, we thought … and then Italy did it. And we realised we could. Behind this garbage from Ferguson is a simple fact: Doing the same as China in every country was the plan from the start and Ferguson's 'models' would play a central role in achieving that. It's just a coincidence, of course, and absolutely nothing to worry your little head about.

Oops, sorry, our mistake

Once the Italian segment of the Psyop had done the job it was designed to do a very different story emerged. Italian authorities revealed that 99 percent of those who had 'died from Covid-19' in Italy had one, two, three, or more 'co-morbidities' or illnesses and health problems that could have ended their life. The US Centers for Disease Control and Prevention (CDC) published a figure of 94 percent for Americans dying of 'Covid' while having other serious medical conditions – on average two to three (some five or six) other potential causes of death. In terms of death from an unproven 'virus' I say it is 100 percent. The other one percent in Italy and six percent in the US would presumably have died from 'Covid's' flu-like symptoms with a range of other possible causes in conjunction with a test not testing for the 'virus'. Fox News reported that even more startling figures had emerged in one US county in which 410 of 422 deaths attributed to 'Covid-19' had other potentially deadly health conditions. The Italian National Health Institute said later that the average age of people dying with a 'Covid-19' diagnosis in Italy was about 81. Ninety percent were over 70 with ten percent over 90. In terms of other reasons to die some 80 percent had two or more chronic diseases with half having three or more including cardiovascular problems, diabetes, respiratory problems and cancer. Why is the phantom 'Covid-19' said to kill overwhelmingly old people and hardly affect the young? Old people continually die of many causes and especially respiratory disease which you can re-diagnose 'Covid-19' while young people die in tiny numbers by comparison and rarely of respiratory disease. Old people 'die of Covid' because they die of other things that can be redesignated 'Covid' and it really is that simple.

Flu has flown

The blueprint was in place. Get your illusory 'cases' from a test not testing for the 'virus' and redesignate other causes of death as 'Covid-19'. You have an instant 'pandemic' from something that is nothing more than a computer-generated fiction. With near-on a

billion people having 'flu-like' symptoms every year the potential was limitless and we can see why flu quickly and apparently miraculously disappeared *worldwide* by being diagnosed 'Covid-19'. The painfully bloody obvious was explained away by the childlike media in headlines like this in the UK '*Independent*': 'Not a single case of flu detected by Public Health England this year as Covid restrictions suppress virus'. I kid you not. The masking, social distancing and house arrest that did not make the 'Covid virus' disappear somehow did so with the 'flu virus'. Even worse the article, by a bloke called Samuel Lovett, suggested that maybe the masking, sanitising and other 'Covid' measures should continue to keep the flu away. With a ridiculousness that disturbs your breathing (it's 'Covid-19') the said Lovett wrote: 'With widespread social distancing and mask-wearing measures in place throughout the UK, the usual routes of transmission for influenza have been blocked.' He had absolutely no evidence to support that statement, but look at the consequences of him acknowledging the obvious. With flu not disappearing at all and only being relabelled 'Covid-19' he would have to contemplate that 'Covid' was a hoax on a scale that is hard to imagine. You need guts and commitment to truth to even go there and that's clearly something Samuel Lovett does not have in abundance. He would never have got it through the editors anyway.

Tens of thousands die in the United States alone every winter from flu including many with pneumonia complications. CDC figures record *45 million* Americans diagnosed with flu in 2017-2018 of which 61,000 died and some reports claim 80,000. Where was the same hysteria then that we have seen with 'Covid-19'? Some 250,000 Americans are admitted to hospital with pneumonia every year with about 50,000 cases proving fatal. About 65 million suffer respiratory disease every year and three million deaths makes this the third biggest cause of death worldwide. You only have to redesignate a portion of all these people 'Covid-19' and you have an instant global pandemic or the *appearance* of one. Why would doctors do this? They are told to do this and all but a few dare not refuse those who must be obeyed. Doctors in general are not researching their own

knowledge and instead take it direct and unquestioned from the authorities that own them and their careers. The authorities say they must now diagnose these symptoms 'Covid-19' and not flu, or whatever, and they do it. Dark suits say put 'Covid-19' on death certificates no matter what the cause of death and the doctors do it. Renegade Minds don't fall for the illusion that doctors and medical staff are all highly-intelligent, highly-principled, seekers of medical truth. *Some are*, but not the majority. They are repeaters, gofers, and yes sir, no sir, purveyors of what the system demands they purvey. The 'Covid' con is not merely confined to diseases of the lungs. Instructions to doctors to put 'Covid-19' on death certificates for anyone dying of *anything* within 28 days (or much more) of a positive test not testing for the 'virus' opened the floodgates. The term dying *with* 'Covid' and not *of* 'Covid' was coined to cover the truth. Whether it was a *with* or an *of* they were all added to the death numbers attributed to the 'deadly virus' compiled by national governments and globally by the Gates-funded Johns Hopkins operation in the United States that was so involved in those 'pandemic' simulations. Fraudulent deaths were added to the ever-growing list of fraudulent 'cases' from false positives from a false test. No wonder Professor Walter Ricciardi, scientific advisor to the Italian minister of health, said after the Lombardy hysteria had done its job that 'Covid' death rates were due to Italy having the second oldest population in the world and to *how hospitals record deaths*:

The way in which we code deaths in our country is very generous in the sense that all the people who die in hospitals with the coronavirus are deemed to be dying of the coronavirus. On re-evaluation by the National Institute of Health, only 12 per cent of death certificates have shown a direct causality from coronavirus, while 88 per cent of patients who have died have at least one pre-morbidity – many had two or three.

This is extraordinary enough when you consider the propaganda campaign to use Italy to terrify the world, but how can they even say twelve percent were genuine when the 'virus' has not been shown to exist, its 'code' is a computer program, and diagnosis comes from a test not testing for it? As in China, and soon the world, 'Covid-19' in

Italy was a redesignation of diagnosis. Lies and corruption were to become the real 'pandemic' fuelled by a pathetically-compliant medical system taking its orders from the tiny few at the top of their national hierarchy who answered to the World Health Organization which answers to Gates and the Cult. Doctors were told – ordered – to diagnose a particular set of symptoms 'Covid-19' and put that on the death certificate for any cause of death if the patient had tested positive with a test not testing for the virus or had 'Covid' symptoms like the flu. The United States even introduced big financial incentives to manipulate the figures with hospitals receiving £4,600 from the Medicare system for diagnosing someone with regular pneumonia, $13,000 if they made the diagnosis from the same symptoms 'Covid-19' pneumonia, and $39, 000 if they put a 'Covid' diagnosed patient on a ventilator that would almost certainly kill them. A few – painfully and pathetically few – medical whistleblowers revealed (before Cult-owned YouTube deleted their videos) that they had been instructed to 'let the patient crash' and put them straight on a ventilator instead of going through a series of far less intrusive and dangerous methods as they would have done before the pandemic hoax began and the financial incentives kicked in. We are talking cold-blooded murder given that ventilators are so damaging to respiratory systems they are usually the last step before heaven awaits. Renegade Minds never fall for the belief that people in white coats are all angels of mercy and cannot be full-on psychopaths. I have explained in detail in *The Answer* how what I am describing here played out across the world coordinated by the World Health Organization through the medical hierarchies in almost every country.

Medical scientist calls it

Information about the non-existence of the 'virus' began to emerge for me in late March, 2020, and mushroomed after that. I was sent an email by Sir Julian Rose, a writer, researcher, and organic farming promotor, from a medical scientist friend of his in the United States. Even at that early stage in March the scientist was able to explain

how the 'Covid' hoax was being manipulated. He said there were no reliable tests for a specific 'Covid-19 virus' and nor were there any reliable agencies or media outlets for reporting numbers of actual 'Covid-19' cases. We have seen in the long period since then that he was absolutely right. 'Every action and reaction to Covid-19 is based on totally flawed data and we simply cannot make accurate assessments,' he said. Most people diagnosed with 'Covid-19' were showing nothing more than cold and flu-like symptoms 'because most coronavirus strains *are* nothing more than cold/flu-like symptoms'. We had farcical situations like an 84-year-old German man testing positive for 'Covid-19' and his nursing home ordered to quarantine only for him to be found to have a common cold. The scientist described back then why PCR tests and what he called the 'Mickey Mouse test kits' were useless for what they were claimed to be identifying. 'The idea these kits can isolate a specific virus like Covid-19 is nonsense,' he said. Significantly, he pointed out that 'if you want to create a totally false panic about a totally false pandemic – pick a coronavirus'. This is exactly what the Cult-owned Gates, World Economic Forum and Johns Hopkins University did with their Event 201 'simulation' followed by their real-life simulation called the 'pandemic'. The scientist said that all you had to do was select the sickest of people with respiratory-type diseases in a single location – 'say Wuhan' – and administer PCR tests to them. You can then claim that anyone showing 'viral sequences' similar to a coronavirus 'which will inevitably be quite a few' is suffering from a 'new' disease:

> Since you already selected the sickest flu cases a fairly high proportion of your sample will go on to die. You can then say this 'new' virus has a CFR [case fatality rate] higher than the flu and use this to infuse more concern and do more tests which will of course produce more 'cases', which expands the testing, which produces yet more 'cases' and so on and so on. Before long you have your 'pandemic', and all you have done is use a simple test kit trick to convert the worst flu and pneumonia cases into something new that doesn't ACTUALLY EXIST [my emphasis].

He said that you then 'just run the same scam in other countries' and make sure to keep the fear message running high 'so that people

will feel panicky and less able to think critically'. The only problem to overcome was the fact *there is no* actual new deadly pathogen and only regular sick people. This meant that deaths from the 'new deadly pathogen' were going to be way too low for a real new deadly virus pandemic, but he said this could be overcome in the following ways – all of which would go on to happen:

1. You can claim this is just the beginning and more deaths are imminent [you underpin this with fantasy 'computer projections']. Use this as an excuse to quarantine everyone and then claim the quarantine prevented the expected millions of dead.

2. You can [say that people] 'minimizing' the dangers are irresponsible and bully them into not talking about numbers.

3. You can talk crap about made up numbers hoping to blind people with pseudoscience.

4. You can start testing well people (who, of course, will also likely have shreds of coronavirus [RNA] in them) and thus inflate your 'case figures' with 'asymptomatic carriers' (you will of course have to spin that to sound deadly even though any virologist knows the more symptom-less cases you have the less deadly is your pathogen).

The scientist said that if you take these simple steps 'you can have your own entirely manufactured pandemic up and running in weeks'. His analysis made so early in the hoax was brilliantly prophetic of what would actually unfold. Pulling all the information together in these recent chapters we have this is simple 1, 2, 3, of how you can delude virtually the entire human population into believing in a 'virus' that doesn't exist:

- A 'Covid case' is someone who tests positive with a test not testing for the 'virus'.

- A 'Covid death' is someone who dies of *any cause* within 28 days (or much longer) of testing positive with a test not testing for the 'virus.

- Asymptomatic means there is nothing wrong with you, but they claim you can pass on what you don't have to justify locking

down (quarantining) healthy people in totality.

The foundations of the hoax are that simple. A study involving ten million people in Wuhan, published in November, 2020, demolished the whole lie about those without symptoms passing on the 'virus'. They found '300 asymptomatic cases' and traced their contacts to find that not one of them was detected with the 'virus'. 'Asymptomatic' patients and their contacts were isolated for no less than two weeks and nothing changed. I know it's all crap, but if you are going to claim that those without symptoms can transmit 'the virus' then you must produce evidence for that and they never have. Even World Health Organization official Dr Maria Van Kerkhove, head of the emerging diseases and zoonosis unit, said as early as June, 2020, that she doubted the validity of asymptomatic transmission. She said that 'from the data we have, it still seems to be rare that an asymptomatic person actually transmits onward to a secondary individual' and by 'rare' she meant that she couldn't cite any case of asymptomatic transmission.

The Ferguson factor

The problem for the Cult as it headed into March, 2020, when the script had lockdown due to start, was that despite all the manipulation of the case and death figures they still did not have enough people alleged to have died from 'Covid' to justify mass house arrest. This was overcome in the way the scientist described: 'You can claim this is just the beginning and more deaths are imminent … Use this as an excuse to quarantine everyone and then claim the quarantine prevented the expected millions of dead.' Enter one Professor Neil Ferguson, the Gates-funded 'epidemiologist' at the Gates-funded Imperial College in London. Ferguson is Britain's Christian Drosten in that he has a dire record of predicting health outcomes, but is still called upon to advise government on the next health outcome when another 'crisis' comes along. This may seem to be a strange and ridiculous thing to do. Why would you keep turning for policy guidance to people who have a history of being

monumentally wrong? Ah, but it makes sense from the Cult point of view. These 'experts' keep on producing predictions that suit the Cult agenda for societal transformation and so it was with Neil Ferguson as he revealed his horrific (and clearly insane) computer model predictions that allowed lockdowns to be imposed in Britain, the United States and many other countries. Ferguson does not have even an A-level in biology and would appear to have no formal training in computer modelling, medicine or epidemiology, according to Derek Winton, an MSc in Computational Intelligence. He wrote an article somewhat aghast at what Ferguson did which included taking no account of respiratory disease 'seasonality' which means it is far worse in the winter months. Who would have thought that respiratory disease could be worse in the winter? Well, certainly not Ferguson.

The massively China-connected Imperial College and its bizarre professor provided the excuse for the long-incubated Chinese model of human control to travel westward at lightning speed. Imperial College confirms on its website that it collaborates with the Chinese Research Institute; publishes more than 600 research papers every year with Chinese research institutions; has 225 Chinese staff; 2,600 Chinese students – the biggest international group; 7,000 former students living in China which is the largest group outside the UK; and was selected for a tour by China's President Xi Jinping during his state visit to the UK in 2015. The college takes major donations from China and describes itself as the UK's number one university collaborator with Chinese research institutions. The China communist/fascist government did not appear phased by the woeful predictions of Ferguson and Imperial when during the lockdown that Ferguson induced the college signed a five-year collaboration deal with China tech giant Huawei that will have Huawei's indoor 5G network equipment installed at the college's West London tech campus along with an 'AI cloud platform'. The deal includes Chinese sponsorship of Imperial's Venture Catalyst entrepreneurship competition. Imperial is an example of the enormous influence the Chinese government has within British and North American

universities and research centres – and further afield. Up to 200 academics from more than a dozen UK universities are being investigated on suspicion of 'unintentionally' helping the Chinese government build weapons of mass destruction by 'transferring world-leading research in advanced military technology such as aircraft, missile designs and cyberweapons'. Similar scandals have broken in the United States, but it's all a coincidence. Imperial College serves the agenda in many other ways including the promotion of every aspect of the United Nations Agenda 21/2030 (the Great Reset) and produced computer models to show that human-caused 'climate change' is happening when in the real world it isn't. Imperial College is driving the climate agenda as it drives the 'Covid' agenda (both Cult hoaxes) while Patrick Vallance, the UK government's Chief Scientific Adviser on 'Covid', was named Chief Scientific Adviser to the UN 'climate change' conference known as COP26 hosted by the government in Glasgow, Scotland. 'Covid' and 'climate' are fundamentally connected.

Professor Woeful

From Imperial's bosom came Neil Ferguson still advising government despite his previous disasters and it was announced early on that he and other key people like UK Chief Medical Adviser Chris Whitty had caught the 'virus' as the propaganda story was being sold. Somehow they managed to survive and we had Prime Minister Boris Johnson admitted to hospital with what was said to be a severe version of the 'virus' in this same period. His whole policy and demeanour changed when he returned to Downing Street. It's a small world with these government advisors – especially in their communal connections to Gates – and Ferguson had partnered with Whitty to write a paper called 'Infectious disease: Tough choices to reduce Ebola transmission' which involved another scare-story that didn't happen. Ferguson's 'models' predicted that up to150, 000 could die from 'mad cow disease', or BSE, and its version in sheep if it was transmitted to humans. BSE was not transmitted and instead triggered by an organophosphate pesticide used to treat a pest on

cows. Fewer than 200 deaths followed from the human form. Models by Ferguson and his fellow incompetents led to the unnecessary culling of millions of pigs, cattle and sheep in the foot and mouth outbreak in 2001 which destroyed the lives and livelihoods of farmers and their families who had often spent decades building their herds and flocks. Vast numbers of these animals did not have foot and mouth and had no contact with the infection. Another 'expert' behind the cull was Professor Roy Anderson, a computer modeller at Imperial College specialising in the epidemiology of *human*, not animal, disease. Anderson has served on the Bill and Melinda Gates Grand Challenges in Global Health advisory board and chairs another Gates-funded organisation. Gates is everywhere.

In a precursor to the 'Covid' script Ferguson backed closing schools 'for prolonged periods' over the swine flu 'pandemic' in 2009 and said it would affect a third of the world population if it continued to spread at the speed he claimed to be happening. His mates at Imperial College said much the same and a news report said: 'One of the authors, the epidemiologist and disease modeller Neil Ferguson, who sits on the World Health Organisation's emergency committee for the outbreak, said the virus had "full pandemic potential".' Professor Liam Donaldson, the Chris Whitty of his day as Chief Medical Officer, said the worst case could see 30 percent of the British people infected by swine flu with 65,000 dying. Ferguson and Donaldson were indeed proved correct when at the end of the year the number of deaths attributed to swine flu was 392. The term 'expert' is rather liberally applied unfortunately, not least to complete idiots. Swine flu 'projections' were great for GlaxoSmithKline (GSK) as millions rolled in for its Pandemrix influenza vaccine which led to brain damage with children most affected. The British government (taxpayers) paid out more than £60 million in compensation after GSK was given immunity from prosecution. Yet another 'Covid' déjà vu. Swine flu was supposed to have broken out in Mexico, but Dr Wolfgang Wodarg, a German doctor, former member of parliament and critic of the 'Covid' hoax, observed 'the spread of swine flu' in Mexico City at the time. He

said: 'What we experienced in Mexico City was a very mild flu which did not kill more than usual – which killed even fewer people than usual.' Hyping the fear against all the facts is not unique to 'Covid' and has happened many times before. Ferguson is reported to have over-estimated the projected death toll of bird flu (H5N1) by some three million-fold, but bird flu vaccine makers again made a killing from the scare. This is some of the background to the Neil Ferguson who produced the perfectly-timed computer models in early 2020 predicting that half a million people would die in Britain without draconian lockdown and 2.2 million in the United States. Politicians panicked, people panicked, and lockdowns of alleged short duration were instigated to 'flatten the curve' of cases gleaned from a test not testing for the 'virus'. I said at the time that the public could forget the 'short duration' bit. This was an agenda to destroy the livelihoods of the population and force them into mass control through dependency and there was going to be nothing 'short' about it. American researcher Daniel Horowitz described the consequences of the 'models' spewed out by Gates-funded Ferguson and Imperial College:

> What led our government and the governments of many other countries into panic was a single Imperial College of UK study, funded by global warming activists, that predicted 2.2 million deaths if we didn't lock down the country. In addition, the reported 8-9% death rate in Italy scared us into thinking there was some other mutation of this virus that they got, which might have come here.
>
> Together with the fact that we were finally testing and had the ability to actually report new cases, we thought we were headed for a death spiral. But again … we can't flatten a curve if we don't know when the curve started.

How about it *never* started?

Giving them what they want

An investigation by German news outlet *Welt Am Sonntag (World on Sunday)* revealed how in March, 2020, the German government gathered together 'leading scientists from several research institutes and universities' and 'together, they were to produce a [modelling]

paper that would serve as legitimization for further tough political measures'. The Cult agenda was justified by computer modelling not based on evidence or reality; it was specifically constructed to justify the Cult demand for lockdowns all over the world to destroy the independent livelihoods of the global population. All these modellers and everyone responsible for the 'Covid' hoax have a date with a trial like those in Nuremberg after World War Two when Nazis faced the consequences of their war crimes. These corrupt-beyond-belief 'modellers' wrote the paper according to government instructions and it said that that if lockdown measures were lifted then up to one million Germans would die from 'Covid-19' adding that some would die 'agonizingly at home, gasping for breath' unable to be treated by hospitals that couldn't cope. All lies. No matter – it gave the Cult all that it wanted. What did long-time government 'modeller' Neil Ferguson say? If the UK and the United States didn't lockdown half a million would die in Britain and 2.2 million Americans. Anyone see a theme here? 'Modellers' are such a crucial part of the lockdown strategy that we should look into their background and follow the money. Researcher Rosemary Frei produced an excellent article headlined 'The Modelling-paper Mafiosi'. She highlights a guy called John Edmunds, a British epidemiologist, and professor in the Faculty of Epidemiology and Population Health at the London School of Hygiene & Tropical Medicine. He studied at Imperial College. Edmunds is a member of government 'Covid' advisory bodies which have been dictating policy, the New and Emerging Respiratory Virus Threats Advisory Group (NERVTAG) and the Scientific Advisory Group for Emergencies (SAGE).

Ferguson, another member of NERVTAG and SAGE, led the way with the original 'virus' and Edmunds has followed in the 'variant' stage and especially the so-called UK or Kent variant known as the 'Variant of Concern' (VOC) B.1.1.7. He said in a co-written report for the Centre for Mathematical modelling of Infectious Diseases at the London School of Hygiene and Tropical Medicine, with input from the Centre's 'Covid-19' Working Group, that there was 'a realistic

possibility that VOC B.1.1.7 is associated with an increased risk of death compared to non-VOC viruses'. Fear, fear, fear, get the vaccine, fear, fear, fear, get the vaccine. Rosemary Frei reveals that almost all the paper's authors and members of the modelling centre's 'Covid-19' Working Group receive funding from the Bill and Melinda Gates Foundation and/or the associated Gates-funded Wellcome Trust. The paper was published by e-journal *Medr χiv* which only publishes papers not peer-reviewed and the journal was established by an organisation headed by Facebook's Mark Zuckerberg and his missus. What a small world it is. Frei discovered that Edmunds is on the Scientific Advisory Board of the Coalition for Epidemic Preparedness Innovations (CEPI) which was established by the Bill and Melinda Gates Foundation, Klaus Schwab's Davos World Economic Forum and Big Pharma giant Wellcome. CEPI was 'launched in Davos [in 2017] to develop vaccines to stop future epidemics', according to its website. 'Our mission is to accelerate the development of vaccines against emerging infectious diseases and enable equitable access to these vaccines for people during outbreaks.' What kind people they are. Rosemary Frei reveals that Public Health England (PHE) director Susan Hopkins is an author of her organisation's non-peer-reviewed reports on 'new variants'. Hopkins is a professor of infectious diseases at London's Imperial College which is gifted tens of millions of dollars a year by the Bill and Melinda Gates Foundation. Gates-funded modelling disaster Neil Ferguson also co-authors Public Health England reports and he spoke in December, 2020, about the potential danger of the B.1.1.7. 'UK variant' promoted by Gates-funded modeller John Edmunds. When I come to the 'Covid vaccines' the 'new variants' will be shown for what they are – bollocks.

Connections, connections

All these people and modellers are lockdown-obsessed or, put another way, they demand what the Cult demands. Edmunds said in January, 2021, that to ease lockdowns too soon would be a disaster and they had to 'vaccinate much, much, much more widely than the

elderly'. Rosemary Frei highlights that Edmunds is married to Jeanne Pimenta who is described in a LinkedIn profile as director of epidemiology at GlaxoSmithKline (GSK) and she held shares in the company. Patrick Vallance, co-chair of SAGE and the government's Chief Scientific Adviser, is a former executive of GSK and has a deferred bonus of shares in the company worth £600,000. GSK has serious business connections with Bill Gates and is collaborating with mRNA-'vaccine' company CureVac to make 'vaccines' for the new variants that Edmunds is talking about. GSK is planning a 'Covid vaccine' with drug giant Sanofi. Puppet Prime Minister Boris Johnson announced in the spring of 2021 that up to 60 million vaccine doses were to be made at the GSK facility at Barnard Castle in the English North East. Barnard Castle, with a population of just 6,000, was famously visited in breach of lockdown rules in April, 2020, by Johnson aide Dominic Cummings who said that he drove there 'to test his eyesight' before driving back to London. Cummings would be better advised to test his integrity – not that it would take long. The GSK facility had nothing to do with his visit then although I'm sure Patrick Vallance would have been happy to arrange an introduction and some tea and biscuits. Ruthless psychopath Gates has made yet another fortune from vaccines in collaboration with Big Pharma companies and gushes at the phenomenal profits to be made from vaccines – more than a 20-to-1 return as he told one interviewer. Gates also tweeted in December, 2019, with the foreknowledge of what was coming: 'What's next for our foundation? I'm particularly excited about what the next year could mean for one of the best buys in global health: vaccines.'

Modeller John Edmunds is a big promotor of vaccines as all these people appear to be. He's the dean of the London School of Hygiene & Tropical Medicine's Faculty of Epidemiology and Population Health which is primarily funded by the Bill and Melinda Gates Foundation and the Gates-established and funded GAVI vaccine alliance which is the Gates vehicle to vaccinate the world. The organisation Doctors Without Borders has described GAVI as being 'aimed more at supporting drug-industry desires to promote new

products than at finding the most efficient and sustainable means for fighting the diseases of poverty'. But then that's why the psychopath Gates created it. John Edmunds said in a video that the London School of Hygiene & Tropical Medicine is involved in every aspect of vaccine development including large-scale clinical trials. He contends that mathematical modelling can show that vaccines protect individuals and society. That's on the basis of shit in and shit out, I take it. Edmunds serves on the UK Vaccine Network as does Ferguson and the government's foremost 'Covid' adviser, the grim-faced, dark-eyed Chris Whitty. The Vaccine Network says it works 'to support the government to identify and shortlist targeted investment opportunities for the most promising vaccines and vaccine technologies that will help combat infectious diseases with epidemic potential, and to address structural issues related to the UK's broader vaccine infrastructure'. Ferguson is acting Director of the Imperial College Vaccine Impact Modelling Consortium which has funding from the Bill and Melina Gates Foundation and the Gates-created GAVI 'vaccine alliance'. Anyone wonder why these characters see vaccines as the answer to every problem? Ferguson is wildly enthusiastic in his support for GAVI's campaign to vaccine children en masse in poor countries. You would expect someone like Gates who has constantly talked about the need to reduce the population to want to fund vaccines to keep more people alive. I'm sure that's why he does it. The John Edmunds London School of Hygiene & Tropical Medicine (LSHTM) has a Vaccines Manufacturing Innovation Centre which develops, tests and commercialises vaccines. Rosemary Frei writes:

> The vaccines centre also performs affiliated activities like combating 'vaccine hesitancy'. The latter includes the Vaccine Confidence Project. The project's stated purpose is, among other things, 'to provide analysis and guidance for early response and engagement with the public to ensure sustained confidence in vaccines and immunisation'. The Vaccine Confidence Project's director is LSHTM professor Heidi Larson. For more than a decade she's been researching how to combat vaccine hesitancy.

How the bloody hell can blokes like John Edmunds and Neil Ferguson with those connections and financial ties model 'virus' case

and death projections for the government and especially in a way that gives their paymasters like Gates exactly what they want? It's insane, but this is what you find throughout the world.

'Covid' is not dangerous, oops, wait, yes it is

Only days before Ferguson's nightmare scenario made Jackboot Johnson take Britain into a China-style lockdown to save us from a deadly 'virus' the UK government website gov.uk was reporting something very different to Ferguson on a page of official government guidance for 'high consequence infectious diseases (HCID)'. It said this about 'Covid-19':

As of 19 March 2020, COVID-19 *is no longer considered to be a high consequence infectious diseases (HCID) in the UK* [my emphasis]. The 4 nations public health HCID group made an interim recommendation in January 2020 to classify COVID-19 as an HCID. This was based on consideration of the UK HCID criteria about the virus and the disease with information available during the early stages of the outbreak.

Now that more is known about COVID-19, the public health bodies in the UK have reviewed the most up to date information about COVID-19 against the UK HCID criteria. They have determined that several features have now changed; in particular, more information is available about mortality rates (low overall), and there is now greater clinical awareness and a specific and sensitive laboratory test, the availability of which continues to increase. The Advisory Committee on Dangerous Pathogens (ACDP) is also of the opinion that COVID-19 should no longer be classified as an HCID.

Soon after the government had been exposed for downgrading the risk they upgraded it again and everyone was back to singing from the same Cult hymn book. Ferguson and his fellow Gates clones indicated that lockdowns and restrictions would have to continue until a Gates-funded vaccine was developed. Gates said the same because Ferguson and his like were repeating the Gates script which is the Cult script. 'Flatten the curve' became an ongoing nightmare of continuing lockdowns with periods in between of severe restrictions in pursuit of destroying independent incomes and had nothing to do with protecting health about which the Cult gives not a shit. Why wouldn't Ferguson be pushing a vaccine 'solution' when he's owned by vaccine-obsessive Gates who makes a fortune from them and

when Ferguson heads the Vaccine Impact Modelling Consortium at Imperial College funded by the Gates Foundation and GAVI, the 'vaccine alliance', created by Gates as his personal vaccine promotion operation? To compound the human catastrophe that Ferguson's 'models' did so much to create he was later exposed for breaking his own lockdown rules by having sexual liaisons with his married girlfriend Antonia Staats at his home while she was living at another location with her husband and children. Staats was a 'climate' activist and senior campaigner at the Soros-funded Avaaz which I wouldn't trust to tell me that grass is green. Ferguson had to resign as a government advisor over this hypocrisy in May, 2020, but after a period of quiet he was back being quoted by the ridiculous media on the need for more lockdowns and a vaccine rollout. Other government-advising 'scientists' from Imperial College' held the fort in his absence and said lockdown could be indefinite until a vaccine was found. The Cult script was being sung by the payrolled choir. I said there was no intention of going back to 'normal' when the 'vaccine' came because the 'vaccine' is part of a very different agenda that I will discuss in Human 2.0. Why would the Cult want to let the world go back to normal when destroying that normal forever was the whole point of what was happening? House arrest, closing businesses and schools through lockdown, (un)social distancing and masks all followed the Ferguson fantasy models. Again as I predicted (these people are so predictable) when the 'vaccine' arrived we were told that house arrest, lockdown, (un)social distancing and masks would still have to continue. I will deal with the masks in the next chapter because they are of fundamental importance.

Where's the 'pandemic'?

Any mildly in-depth assessment of the figures revealed what was really going on. Cult-funded and controlled organisations still have genuine people working within them such is the number involved. So it is with Genevieve Briand, assistant program director of the Applied Economics master's degree program at Johns Hopkins

University. She analysed the impact that 'Covid-19' had on deaths from *all* causes in the United States using official data from the CDC for the period from early February to early September, 2020. She found that allegedly 'Covid' *related*-deaths exceeded those from heart disease which she found strange with heart disease always the biggest cause of fatalities. Her research became even more significant when she noted the sudden decline in 2020 of *all* non-'Covid' deaths: 'This trend is completely contrary to the pattern observed in all previous years … the total decrease in deaths by other causes almost exactly equals the increase in deaths by Covid-19.' This was such a game, set and match in terms of what was happening that Johns Hopkins University deleted the article on the grounds that it 'was being used to support false and dangerous inaccuracies about the impact of the pandemic'. No – because it exposed the scam from official CDC figures and this was confirmed when those figures were published in January, 2021. Here we can see the effect of people dying from heart attacks, cancer, road accidents and gunshot wounds – *anything* – having 'Covid-19' on the death certificate along with those diagnosed from 'symptoms' who had even not tested positive with a test not testing for the 'virus'. I am not kidding with the gunshot wounds, by the way. Brenda Bock, coroner in Grand County, Colorado, revealed that two gunshot victims tested positive for the 'virus' within the previous 30 days and were therefore classified as 'Covid deaths'. Bock said: 'These two people had tested positive for Covid, but that's not what killed them. A gunshot wound is what killed them.' She said she had not even finished her investigation when the state listed the gunshot victims as deaths due to the 'virus'. The death and case figures for 'Covid-19' are an absolute joke and yet they are repeated like parrots by the media, politicians and alleged medical 'experts'. The official Cult narrative is the only show in town.

Genevieve Briand found that deaths from all causes were not exceptional in 2020 compared with previous years and a Spanish magazine published figures that said the same about Spain which was a 'Covid' propaganda hotspot at one point. *Discovery Salud*, a

health and medicine magazine, quoted government figures which showed how 17,000 *fewer* people died in Spain in 2020 than in 2019 and more than 26,000 fewer than in 2018. The age-standardised mortality rate for England and Wales when age distribution is taken into account was significantly lower in 2020 than the 1970s, 80s and 90s, and was only the ninth highest since 2000. Where is the 'pandemic'?

Post mortems and autopsies virtually disappeared for 'Covid' deaths amid claims that 'virus-infected' bodily fluids posed a risk to those carrying out the autopsy. This was rejected by renowned German pathologist and forensic doctor Klaus Püschel who said that he and his staff had by then done 150 autopsies on 'Covid' patients with no problems at all. He said they were needed to know why some 'Covid' patients suffered blood clots and not severe respiratory infections. The 'virus' is, after all, called SARS or 'severe acute respiratory syndrome'. I highlighted in the spring of 2020 this phenomenon and quoted New York intensive care doctor Cameron Kyle-Sidell who posted a soon deleted YouTube video to say that they had been told to prepare to treat an infectious disease called 'Covid-19', but that was not what they were dealing with. Instead he likened the lung condition of the most severely ill patients to what you would expect with cabin depressurisation in a plane at 30,000 feet or someone dropped on the top of Everest without oxygen or acclimatisation. I have never said this is not happening to a small minority of alleged 'Covid' patients – I am saying this is not caused by a phantom 'contagious virus'. Indeed Kyle-Sidell said that 'Covid-19' was not the disease they were told was coming their way. 'We are operating under a medical paradigm that is untrue,' he said, and he believed they were treating the wrong disease: 'These people are being slowly starved of oxygen.' Patients would take off their oxygen masks in a state of fear and stress and while they were blue in the face on the brink of death. They did not look like patients dying of pneumonia. You can see why they don't want autopsies when their virus doesn't exist and there is another condition in some people that they don't wish to be uncovered. I should add here that

the 5G system of millimetre waves was being rapidly introduced around the world in 2020 and even more so now as they fire 5G at the Earth from satellites. At 60 gigahertz within the 5G range that frequency interacts with the oxygen molecule and stops people breathing in sufficient oxygen to be absorbed into the bloodstream. They are installing 5G in schools and hospitals. The world is not mad or anything. 5G can cause major changes to the lungs and blood as I detail in *The Answer* and these consequences are labelled 'Covid-19', the alleged symptoms of which can be caused by 5G and other electromagnetic frequencies as cells respond to radiation poisoning.

The 'Covid death' scam

Dr Scott Jensen, a Minnesota state senator and medical doctor, exposed 'Covid' Medicare payment incentives to hospitals and death certificate manipulation. He said he was sent a seven-page document by the US Department of Health 'coaching' him on how to fill out death certificates which had never happened before. The document said that he didn't need to have a laboratory test for 'Covid-19' to put that on the death certificate and that shocked him when death certificates are supposed to be about facts. Jensen described how doctors had been 'encouraged, if not pressured' to make a diagnosis of 'Covid-19' if they thought it was probable or *'presumed*'. No positive test was necessary – not that this would have mattered anyway. He said doctors were told to diagnose 'Covid' by symptoms when these were the same as colds, allergies, other respiratory problems, and certainly with influenza which 'disappeared' in the 'Covid' era. A common sniffle was enough to get the dreaded verdict. Ontario authorities decreed that a single care home resident with *one* symptom from a long list must lead to the isolation of the entire home. Other courageous doctors like Jensen made the same point about death figure manipulation and how deaths by other causes were falling while 'Covid-19 deaths' were rising at the same rate due to re-diagnosis. Their videos rarely survive long on YouTube with its Cult-supporting algorithms courtesy of CEO Susan Wojcicki and her bosses at Google. Figure-tampering was so glaring

and ubiquitous that even officials were letting it slip or outright saying it. UK chief scientific adviser Patrick Vallance said on one occasion that 'Covid' on the death certificate doesn't mean 'Covid' was the cause of death (so why the hell is it there?) and we had the rare sight of a BBC reporter telling the truth when she said: 'Someone could be successfully treated for Covid, in say April, discharged, and then in June, get run over by a bus and die … That person would still be counted as a Covid death in England.' Yet the BBC and the rest of the world media went on repeating the case and death figures as if they were real. Illinois Public Health Director Dr Ngozi Ezike revealed the deceit while her bosses must have been clenching their buttocks:

> If you were in a hospice and given a few weeks to live and you were then found to have Covid that would be counted as a Covid death. [There might be] a clear alternate cause, but it is still listed as a Covid death. So everyone listed as a Covid death doesn't mean that was the cause of the death, but that they had Covid at the time of death.

Yes, a 'Covid virus' never shown to exist and tested for with a test not testing for the 'virus'. In the first period of the pandemic hoax through the spring of 2020 the process began of designating almost everything a 'Covid' death and this has continued ever since. I sat in a restaurant one night listening to a loud conversation on the next table where a family was discussing in bewilderment how a relative who had no symptoms of 'Covid', and had died of a long-term problem, could have been diagnosed a death by the 'virus'. I could understand their bewilderment. If they read this book they will know why this medical fraud has been perpetrated the world over.

Some media truth shock

The media ignored the evidence of death certificate fraud until eventually one columnist did speak out when she saw it first-hand. Bel Mooney is a long-time national newspaper journalist in Britain currently working for the *Daily Mail*. Her article on February 19th, 2021, carried this headline: 'My dad Ted passed three Covid tests

and died of a chronic illness yet he's officially one of Britain's 120,000 victims of the virus and is far from alone ... so how many more are there?' She told how her 99-year-old father was in a care home with a long-standing chronic obstructive pulmonary disease and vascular dementia. Maybe, but he was still aware enough to tell her from the start that there was no 'virus' and he refused the 'vaccine' for that reason. His death was not unexpected given his chronic health problems and Mooney said she was shocked to find that 'Covid-19' was declared the cause of death on his death certificate. She said this was a 'bizarre and unacceptable untruth' for a man with long-time health problems who had tested negative twice at the home for the 'virus'. I was also shocked by this story although not by what she said. I had been highlighting the death certificate manipulation for ten months. It was the confirmation that a professional full-time journalist only realised this was going on when it affected her directly and neither did she know that whether her dad tested positive or negative was irrelevant with the test not testing for the 'virus'. Where had she been? She said she did not believe in 'conspiracy theories' without knowing I'm sure that this and 'conspiracy theorists' were terms put into widespread circulation by the CIA in the 1960s to discredit those who did not accept the ridiculous official story of the Kennedy assassination. A blanket statement of 'I don't believe in conspiracy theories' is always bizarre. The dictionary definition of the term alone means the world is drowning in conspiracies. What she said was even more daft when her dad had just been affected by the 'Covid' conspiracy. Why else does she think that 'Covid-19' was going on the death certificates of people who died of something else?

To be fair once she saw from personal experience what was happening she didn't mince words. Mooney was called by the care home on the morning of February 9th to be told her father had died in his sleep. When she asked for the official cause of death what came back was 'Covid-19'. Mooney challenged this and was told there had been deaths from Covid on the dementia floor (confirmed by a test not testing for the 'virus') so they considered it 'reasonable

to assume'. 'But doctor,' Mooney rightly protested, 'an assumption isn't a diagnosis.' She said she didn't blame the perfectly decent and sympathetic doctor – 'he was just doing his job'. Sorry, but that's *bullshit*. He wasn't doing his job at all. He was putting a false cause of death on the death certificate and that is a criminal offence for which he should be brought to account and the same with the millions of doctors worldwide who have done the same. They were not doing their job they were following orders and that must not wash at new Nuremberg trials any more than it did at the first ones. Mooney's doctor was 'assuming' (presuming) as he was told to, but 'just following orders' makes no difference to his actions. A doctor's job is to serve the patient and the truth, not follow orders, but that's what they have done all over the world and played a central part in making the 'Covid' hoax possible with all its catastrophic consequences for humanity. Shame on them and they must answer for their actions. Mooney said her disquiet worsened when she registered her father's death by telephone and was told by the registrar there had been very many other cases like hers where 'the deceased' had not tested positive for 'Covid' yet it was recorded as the cause of death. The test may not matter, but those involved at their level *think* it matters and it shows a callous disregard for accurate diagnosis. The pressure to do this is coming from the top of the national 'health' pyramids which in turn obey the World Health Organization which obeys Gates and the Cult. Mooney said the registrar agreed that this must distort the national figures adding that 'the strangest thing is that every winter we record countless deaths from flu, and this winter there have been none. Not one!' She asked if the registrar thought deaths from flu were being misdiagnosed and lumped together with 'Covid' deaths. The answer was a 'puzzled yes'. Mooney said that the funeral director said the same about 'Covid' deaths which had nothing to do with 'Covid'. They had lost count of the number of families upset by this and other funeral companies in different countries have had the same experience. Mooney wrote:

The nightly shroud-waving and shocking close-ups of pain imposed on us by the TV news bewildered and terrified the population into eager compliance with lockdowns. We were invited to 'save the NHS' and to grieve for strangers – the real-life loved ones behind those shocking death counts. Why would the public imagine what I now fear, namely that the way Covid-19 death statistics are compiled might make the numbers seem greater than they are?

Oh, just a little bit – like 100 percent.

Do the maths

Mooney asked why a country would wish to skew its mortality figures by wrongly certifying deaths? What had been going on? Well, if you don't believe in conspiracies you will never find the answer which is that *it's a conspiracy*. She did, however, describe what she had discovered as a 'national scandal'. In reality it's a global scandal and happening everywhere. Pillars of this conspiracy were all put into place before the button was pressed with the Drosten PCR protocol and high amplifications to produce the cases and death certificate changes to secure illusory 'Covid' deaths. Mooney notes that normally two doctors were needed to certify a death, with one having to know the patient, and how the rules were changed in the spring of 2020 to allow one doctor to do this. In the same period 'Covid deaths' were decreed to be all cases where Covid-19 was put on the death certificate even without a positive test or any symptoms. Mooney asked: 'How many of the 30,851 (as of January 15) care home resident deaths with Covid-19 on the certificate (32.4 per cent of all deaths so far) were based on an assumption, like that of my father? And what has that done to our national psyche?'All of them is the answer to the first question and it has devastated and dismantled the national psyche, actually the global psyche, on a colossal scale. In the UK case and death data is compiled by organisations like Public Health England (PHE) and the Office for National Statistics (ONS). Mooney highlights the insane policy of counting a death from any cause as 'Covid-19' if this happens within 28 days of a positive test (with a test not testing for the 'virus') and she points out that ONS statistics reflect deaths 'involving Covid' 'or due to Covid' which meant in practice any

death where 'Covid-19' was mentioned on the death certificate. She described the consequences of this fraud:

Most people will accept the narrative they are fed, so panicky governments here and in Europe witnessed the harsh measures enacted in totalitarian China and jumped into lockdown. Headlines about Covid deaths tolled like the knell that would bring doomsday to us all. Fear stalked our empty streets. Politicians parroted the frankly ridiculous aim of 'zero Covid' and shut down the economy, while most British people agreed that lockdown was essential and (astonishingly to me, as a patriotic Brit) even wanted more restrictions.

For what? Lies on death certificates? Never mind the grim toll of lives ruined, suicides, schools closed, rising inequality, depression, cancelled hospital treatments, cancer patients in a torture of waiting, poverty, economic devastation, loneliness, families kept apart, and so on. How many lives have been lost as a direct result of lockdown?

She said that we could join in a national chorus of shock and horror at reaching the 120,000 death toll which was surely certain to have been totally skewed all along, but what about the human cost of lockdown justified by these 'death figures'? *The British Medical Journal* had reported a 1,493 percent increase in cases of children taken to Great Ormond Street Hospital with abusive head injuries alone and then there was the effect on families:

Perhaps the most shocking thing about all this is that families have been kept apart – and obeyed the most irrational, changing rules at the whim of government – because they believed in the statistics. They succumbed to fear, which his generation rejected in that war fought for freedom. Dad (God rest his soul) would be angry. And so am I.

Another theme to watch is that in the winter months when there are more deaths from all causes they focus on 'Covid' deaths and in the summer when the British Lung Foundation says respiratory disease plummets by 80 percent they rage on about 'cases'. Either way fascism on population is always the answer.

Nazi eugenics in the 21st century

Elderly people in care homes have been isolated from their families month after lonely month with no contact with relatives and grandchildren who were banned from seeing them. We were told

that lockdown fascism was to 'protect the vulnerable' like elderly people. At the same time Do Not Resuscitate (DNR) orders were placed on their medical files so that if they needed resuscitation it wasn't done and 'Covid-19' went on their death certificates. Old people were not being 'protected' they were being culled – murdered in truth. DNR orders were being decreed for disabled and young people with learning difficulties or psychological problems. The UK Care Quality Commission, a non-departmental body of the Department of Health and Social Care, found that 34 percent of those working in health and social care were pressured into placing 'do not attempt cardiopulmonary resuscitation' orders on 'Covid' patients who suffered from disabilities and learning difficulties without involving the patient or their families in the decision. UK judges ruled that an elderly woman with dementia should have the DNA-manipulating 'Covid vaccine' against her son's wishes and that a man with severe learning difficulties should have the jab despite his family's objections. Never mind that many had already died. The judiciary always supports doctors and government in fascist dictatorships. They wouldn't dare do otherwise. A horrific video was posted showing fascist officers from Los Angeles police forcibly giving the 'Covid' shot to women with special needs who were screaming that they didn't want it. The same fascists are seen giving the jab to a sleeping elderly woman in a care home. This is straight out of the Nazi playbook. Hitler's Nazis committed mass murder of the mentally ill and physically disabled throughout Germany and occupied territories in the programme that became known as Aktion T4, or just T4. Sabbatian-controlled Hitler and his grotesque crazies set out to kill those they considered useless and unnecessary. The Reich Committee for the Scientific Registering of Hereditary and Congenital Illnesses registered the births of babies identified by physicians to have 'defects'. By 1941 alone more than 5,000 children were murdered by the state and it is estimated that in total the number of innocent people killed in Aktion T4 was between 275,000 and 300,000. Parents were told their children had been sent away for 'special treatment' never to return. It is rather pathetic to see claims about plans for new extermination camps being dismissed today

when the same force behind current events did precisely that 80 years ago. Margaret Sanger was a Cult operative who used 'birth control' to sanitise her programme of eugenics. Organisations she founded became what is now Planned Parenthood. Sanger proposed that 'the whole dysgenic population would have its choice of segregation or sterilization'. These included epileptics, 'feeble-minded', and prostitutes. Sanger opposed charity because it perpetuated 'human waste'. She reveals the Cult mentality and if anyone thinks that extermination camps are a 'conspiracy theory' their naivety is touching if breathtakingly stupid.

If you don't believe that doctors can act with callous disregard for their patients it is worth considering that doctors and medical staff agreed to put government-decreed DNR orders on medical files and do nothing when resuscitation is called for. I don't know what you call such people in your house. In mine they are Nazis from the Josef Mengele School of Medicine. Phenomenal numbers of old people have died worldwide from the effects of lockdown, depression, lack of treatment, the 'vaccine' (more later) and losing the will to live. A common response at the start of the manufactured pandemic was to remove old people from hospital beds and transfer them to nursing homes. The decision would result in a mass cull of elderly people in those homes through lack of treatment – *not* 'Covid'. Care home whistleblowers have told how once the 'Covid' era began doctors would not come to their homes to treat patients and they were begging for drugs like antibiotics that often never came. The most infamous example was ordered by New York governor Andrew Cuomo, brother of a moronic CNN host, who amazingly was given an Emmy Award for his handling of the 'Covid crisis' by the ridiculous Wokers that hand them out. Just how ridiculous could be seen in February, 2021, when a Department of Justice and FBI investigation began into how thousands of old people in New York died in nursing homes after being discharged from hospital to make way for 'Covid' patients on Cuomo's say-so – and how he and his staff covered up these facts. This couldn't have happened to a nicer psychopath. Even then there was a 'Covid' spin. Reports said that

thousands of old people who tested positive for 'Covid' in hospital were transferred to nursing homes to both die of 'Covid' and transmit it to others. No – they were in hospital because they were ill and the fact that they tested positive with a test not testing for the 'virus' is irrelevant. They were ill often with respiratory diseases ubiquitous in old people near the end of their lives. Their transfer out of hospital meant that their treatment stopped and many would go on to die.

They're old. Who gives a damn?

I have exposed in the books for decades the Cult plan to cull the world's old people and even to introduce at some point what they call a 'demise pill' which at a certain age everyone would take and be out of here by law. In March, 2021, Spain legalised euthanasia and assisted suicide following the Netherlands, Belgium, Luxembourg and Canada on the Tiptoe to the demise pill. Treatment of old people by many 'care' homes has been a disgrace in the 'Covid' era. There are many, many, caring staff – I know some. There have, however, been legions of stories about callous treatment of old people and their families. Police were called when families came to take their loved ones home in the light of isolation that was killing them. They became prisoners of the state. Care home residents in insane, fascist Ontario, Canada, were not allowed to leave their *room* once the 'Covid' hoax began. UK staff have even wheeled elderly people away from windows where family members were talking with them. Oriana Criscuolo from Stockport in the English North West dropped off some things for her 80-year-old father who has Parkinson's disease and dementia and she wanted to wave to him through a ground-floor window. She was told that was 'illegal'. When she went anyway they closed the curtains in the middle of the day. Oriana said:

> It's just unbelievable. I cannot understand how care home staff – people who are being paid to care – have become so uncaring. Their behaviour is inhumane and cruel. It's beyond belief.

She was right and this was not a one-off. What a way to end your life in such loveless circumstances. UK registered nurse Nicky Millen, a proper old school nurse for 40 years, said that when she started her career care was based on dignity, choice, compassion and empathy. Now she said 'the things that are important to me have gone out of the window.' She was appalled that people were dying without their loved ones and saying goodbye on iPads. Nicky described how a distressed 89-year-old lady stroked her face and asked her 'how many paracetamol would it take to finish me off'. Life was no longer worth living while not seeing her family. Nicky said she was humiliated in front of the ward staff and patients for letting the lady stroke her face and giving her a cuddle. Such is the dehumanisation that the 'Covid' hoax has brought to the surface. Nicky worked in care homes where patients told her they were being held prisoner. 'I want to live until I die', one said to her. 'I had a lady in tears because she hadn't seen her great-grandson.' Nicky was compassionate old school meeting psychopathic New Normal. She also said she had worked on a 'Covid' ward with no 'Covid' patients. Jewish writer Shai Held wrote an article in March, 2020, which was headlined 'The Staggering, Heartless Cruelty Toward the Elderly'. What he described was happening from the earliest days of lockdown. He said 'the elderly' were considered a group and not unique individuals (the way of the Woke). Shai Held said:

Notice how the all-too-familiar rhetoric of dehumanization works: 'The elderly' are bunched together as a faceless mass, all of them considered culprits and thus effectively deserving of the suffering the pandemic will inflict upon them. Lost entirely is the fact that the elderly are individual human beings, each with a distinctive face and voice, each with hopes and dreams, memories and regrets, friendships and marriages, loves lost and loves sustained.

'The elderly' have become another dehumanised group for which anything goes and for many that has resulted in cold disregard for their rights and their life. The distinctive face that Held talks about is designed to be deleted by masks until everyone is part of a faceless mass.

‘War-zone’ hospitals myth

Again and again medical professionals have told me what was really going on and how hospitals ‘overrun like war zones’ according to the media were virtually empty. The mantra from medical whistleblowers was please don’t use my name or my career is over. Citizen journalists around the world sneaked into hospitals to film evidence exposing the ‘war-zone’ lie. They really *were* largely empty with closed wards and operating theatres. I met a hospital worker in my town on the Isle of Wight during the first lockdown in 2020 who said the only island hospital had never been so quiet. Lockdown was justified by the psychopaths to stop hospitals being overrun. At the same time that the island hospital was near-empty the military arrived here to provide *extra beds*. It was all propaganda to ramp up the fear to ensure compliance with fascism as were never-used temporary hospitals with thousands of beds known as Nightingales and never-used make-shift mortuaries opened by the criminal UK government. A man who helped to install those extra island beds attributed to the army said they were never used and the hospital was empty. Doctors and nurses ‘stood around talking or on their phones, wandering down to us to see what we were doing’. There were no masks or social distancing. He accused the useless local island paper, the *County Press*, of ‘pumping the fear as if our hospital was overrun and we only have one so it should have been’. He described ambulances parked up with crews outside in deck chairs. When his brother called an ambulance he was told there was a two-hour backlog which he called ‘bullshit’. An old lady on the island fell ‘and was in a bad way’, but a caller who rang for an ambulance was told the situation wasn’t urgent enough. Ambulance stations were working under capacity while people would hear ambulances with sirens blaring driving through the streets. When those living near the stations realised what was going on they would follow them as they left, circulated around an urban area with the sirens going, and then came back without stopping. All this was to increase levels of fear and the same goes for the ‘ventilator shortage crisis’ that cost tens of millions for hastily produced ventilators never to be used.

Ambulance crews that agreed to be exploited in this way for fear propaganda might find themselves a mirror. I wish them well with that. Empty hospitals were the obvious consequence of treatment and diagnoses of non-'Covid' conditions cancelled and those involved handed a death sentence. People have been dying at home from undiagnosed and untreated cancer, heart disease and other life-threatening conditions to allow empty hospitals to deal with a 'pandemic' that wasn't happening.

Death of the innocent

'War-zones' have been laying off nursing staff, even doctors where they can. There was no work for them. Lockdown was justified by saving lives and protecting the vulnerable they were actually killing with DNR orders and preventing empty hospitals being 'overrun'. In Britain the mantra of stay at home to 'save the NHS' was everywhere and across the world the same story was being sold when it was all lies. Two California doctors, Dan Erickson and Artin Massihi at Accelerated Urgent Care in Bakersfield, held a news conference in April, 2020, to say that intensive care units in California were 'empty, essentially', with hospitals shutting floors, not treating patients and laying off doctors. The California health system was working at minimum capacity 'getting rid of doctors because we just don't have the volume'. They said that people with conditions such as heart disease and cancer were not coming to hospital out of fear of 'Covid-19'. Their video was deleted by Susan Wojcicki's Cult-owned YouTube after reaching five million views. Florida governor Ron Desantis, who rejected the severe lockdowns of other states and is being targeted for doing so, said that in March, 2020, every US governor was given models claiming they would run out of hospital beds in days. That was never going to happen and the 'modellers' knew it. Deceit can be found at every level of the system. Urgent children's operations were cancelled including fracture repairs and biopsies to spot cancer. Eric Nicholls, a consultant paediatrician, said 'this is obviously concerning and we need to return to normal operating and to increase capacity as soon as possible'. Psychopaths

in power were rather less concerned *because* they are psychopaths. Deletion of urgent care and diagnosis has been happening all over the world and how many kids and others have died as a result of the actions of these cold and heartless lunatics dictating 'health' policy? The number must be stratospheric. Richard Sullivan, professor of cancer and global health at King's College London, said people feared 'Covid' more than cancer such was the campaign of fear. 'Years of lost life will be quite dramatic', Sullivan said, with 'a huge amount of avoidable mortality'. Sarah Woolnough, executive director for policy at Cancer Research UK, said there had been a 75 percent drop in urgent referrals to hospitals by family doctors of people with suspected cancer. Sullivan said that 'a lot of services have had to scale back – we've seen a dramatic decrease in the amount of elective cancer surgery'. Lockdown deaths worldwide has been absolutely fantastic with the *New York Post* reporting how data confirmed that 'lockdowns end more lives than they save':

> There was a sharp decline in visits to emergency rooms and an increase in fatal heart attacks because patients didn't receive prompt treatment. Many fewer people were screened for cancer. Social isolation contributed to excess deaths from dementia and Alzheimer's.
>
> Researchers predicted that the social and economic upheaval would lead to tens of thousands of "deaths of despair" from drug overdoses, alcoholism and suicide. As unemployment surged and mental-health and substance-abuse treatment programs were interrupted, the reported levels of anxiety, depression and suicidal thoughts increased dramatically, as did alcohol sales and fatal drug overdoses.

This has been happening while nurses and other staff had so much time on their hands in the 'war-zones' that Tic-Tok dancing videos began appearing across the Internet with medical staff dancing around in empty wards and corridors as people died at home from causes that would normally have been treated in hospital.

Mentions in dispatches

One brave and truth-committed whistleblower was Louise Hampton, a call handler with the UK NHS who made a viral Internet video saying she had done 'fuck all' during the 'pandemic'

which was 'a load of bollocks'. She said that 'Covid-19' was rebranded flu and of course she lost her job. This is what happens in the medical and endless other professions now when you tell the truth. Louise filmed inside 'war-zone' accident and emergency departments to show they were empty and I mean *empty* as in no one there. The mainstream media could have done the same and blown the gaff on the whole conspiracy. They haven't to their eternal shame. Not that most 'journalists' seem capable of manifesting shame as with the psychopaths they slavishly repeat without question. The relative few who were admitted with serious health problems were left to die alone with no loved ones allowed to see them because of 'Covid' rules and they included kids dying without the comfort of mum and dad at their bedside while the evil behind this couldn't give a damn. It was all good fun to them. A Scottish NHS staff nurse publicly quit in the spring of 2021 saying: 'I can no longer be part of the lies and the corruption by the government.' She said hospitals 'aren't full, the beds aren't full, beds have been shut, wards have been shut'. Hospitals were never busy throughout 'Covid'. The staff nurse said that Nicola Sturgeon, tragically the leader of the Scottish government, was on television saying save the hospitals and the NHS – 'but the beds are empty' and 'we've not seen flu, we always see flu every year'. She wrote to government and spoke with her union Unison (the unions are Cult-compromised and *useless*, but nothing changed. Many of her colleagues were scared of losing their jobs if they spoke out as they wanted to. She said nursing staff were being affected by wearing masks all day and 'my head is splitting every shift from wearing a mask'. The NHS is part of the fascist tyranny and must be dismantled so we can start again with human beings in charge. (Ironically, hospitals were reported to be busier again when official 'Covid' cases *fell* in spring/summer of 2021 and many other conditions required treatment at the same time as *the fake vaccine rollout*.)

I will cover the 'Covid vaccine' scam in detail later, but it is another indicator of the sickening disregard for human life that I am highlighting here. The DNA-manipulating concoctions do not fulfil

the definition of a 'vaccine', have never been used on humans before and were given only emergency approval because trials were not completed and they continued using the unknowing public. The result was what a NHS senior nurse with responsibility for 'vaccine' procedure said was 'genocide'. She said the 'vaccines' were not 'vaccines'. They had not been shown to be safe and claims about their effectiveness by drug companies were 'poetic licence'. She described what was happening as a 'horrid act of human annihilation'. The nurse said that management had instigated a policy of not providing a Patient Information Leaflet (PIL) before people were 'vaccinated' even though health care professionals are supposed to do this according to protocol. Patients should also be told that they are taking part in an ongoing clinical trial. Her challenges to what is happening had seen her excluded from meetings and ridiculed in others. She said she was told to 'watch my step … or I would find myself surplus to requirements'. The nurse, who spoke anonymously in fear of her career, said she asked her NHS manager why he/she was content with taking part in genocide against those having the 'vaccines'. The reply was that everyone had to play their part and to 'put up, shut up, and get it done'. Government was 'leaning heavily' on NHS management which was clearly leaning heavily on staff. This is how the global 'medical' hierarchy operates and it starts with the Cult and its World Health Organization.

She told the story of a doctor who had the Pfizer jab and when questioned had no idea what was in it. The doctor had never read the literature. We have to stop treating doctors as intellectual giants when so many are moral and medical pygmies. The doctor did not even know that the 'vaccines' were not fully approved or that their trials were ongoing. They were, however, asking their patients if they minded taking part in follow-ups for research purposes – yes, the *ongoing clinical trial*. The nurse said the doctor's ignorance was not rare and she had spoken to a hospital consultant who had the jab without any idea of the background or that the 'trials' had not been completed. Nurses and pharmacists had shown the same ignorance.

'My NHS colleagues have forsaken their duty of care, broken their code of conduct – Hippocratic Oath – and have been brainwashed just the same as the majority of the UK public through propaganda …' She said she had not been able to recruit a single NHS colleague, doctor, nurse or pharmacist to stand with her and speak out. Her union had refused to help. She said that if the genocide came to light she would not hesitate to give evidence at a Nuremberg-type trial against those in power who could have affected the outcomes but didn't.

And all for what?

To put the nonsense into perspective let's say the 'virus' does exist and let's go completely crazy and accept that the official manipulated figures for cases and deaths are accurate. *Even then* a study by Stanford University epidemiologist Dr John Ioannidis published on the World Health Organization website produced an average infection to fatality rate of … *0.23 percent!* Ioannidis said: 'If one could sample equally from all locations globally, the median infection fatality rate might even be substantially lower than the 0.23% observed in my analysis.' For healthy people under 70 it was … *0.05 percent*! This compares with the 3.4 percent claimed by the Cult-owned World Health Organization when the hoax was first played and maximum fear needed to be generated. An updated Stanford study in April, 2021, put the 'infection' to 'fatality' rate at just 0.15 percent. Another team of scientists led by Megan O'Driscoll and Henrik Salje studied data from 45 countries and published their findings on the Nature website. For children and young people the figure is so small it virtually does not register although authorities will be hyping dangers to the young when they introduce DNA-manipulating 'vaccines' for children. The O'Driscoll study produced an average infection-fatality figure of 0.003 for children from birth to four; 0.001 for 5 to 14; 0.003 for 15 to 19; and it was still only 0.456 up to 64. To claim that children must be 'vaccinated' to protect them from 'Covid' is an obvious lie and so there must be another reason and there is. What's more the average age of a 'Covid' death is akin

to the average age that people die in general. The average age of death in England is about 80 for men and 83 for women. The average age of death from alleged 'Covid' is between 82 and 83. California doctors, Dan Erickson and Artin Massihi, said at their April media conference that projection models of millions of deaths had been 'woefully inaccurate'. They produced detailed figures showing that Californians had a 0.03 chance of dying from 'Covid' based on the number of people who tested positive (with a test not testing for the 'virus'). Erickson said there was a 0.1 percent chance of dying from 'Covid' in the *state* of New York, not just the city, and a 0.05 percent chance in Spain, a centre of 'Covid-19' hysteria at one stage. The Stanford studies supported the doctors' data with fatality rate estimates of 0.23 and 0.15 percent. How close are these figures to my estimate of *zero*? Death-rate figures claimed by the World Health Organization at the start of the hoax were some 15 times higher. The California doctors said there was no justification for lockdowns and the economic devastation they caused. Everything they had ever learned about quarantine was that you quarantine the *sick* and not the healthy. They had never seen this before and it made no medical sense.

Why in the in the light of all this would governments and medical systems the world over say that billions must go under house arrest; lose their livelihood; in many cases lose their mind, their health and their life; force people to wear masks dangerous to health and psychology; make human interaction and even family interaction a criminal offence; ban travel; close restaurants, bars, watching live sport, concerts, theatre, and any activity involving human togetherness and discourse; and closing schools to isolate children from their friends and cause many to commit suicide in acts of hopelessness and despair? The California doctors said lockdown consequences included increased child abuse, partner abuse, alcoholism, depression, and other impacts they were seeing every day. Who would do that to the entire human race if not mentally-ill psychopaths of almost unimaginable extremes like Bill Gates? We must face the reality of what we are dealing with and come out of

denial. Fascism and tyranny are made possible only by the target population submitting and acquiescing to fascism and tyranny. The whole of human history shows that to be true. Most people naively and unquestioning believed what they were told about a 'deadly virus' and meekly and weakly submitted to house arrest. Those who didn't believe it – at least in total – still submitted in fear of the consequences of not doing so. For the rest who wouldn't submit draconian fines have been imposed, brutal policing by psychopaths *for* psychopaths, and condemnation from the meek and weak who condemn the Pushbackers on behalf of the very force that has them, too, in its gunsights. 'Pathetic' does not even begin to suffice. Britain's brainless 'Health' Secretary Matt Hancock warned anyone lying to border officials about returning from a list of 'hotspot' countries could face a jail sentence of up to ten years which is more than for racially-aggravated assault, incest and attempting to have sex with a child under 13. Hancock is a lunatic, but he has the state apparatus behind him in a Cult-led chain reaction and the same with UK 'Vaccine Minister' Nadhim Zahawi, a prominent member of the mega-Cult secret society, Le Cercle, which featured in my earlier books. The Cult enforces its will on governments and medical systems; government and medical systems enforce their will on business and police; business enforces its will on staff who enforce it on customers; police enforce the will of the Cult on the population and play their essential part in creating a world of fascist control that their own children and grandchildren will have to live in their entire lives. It is a hierarchical pyramid of imposition and acquiescence and, yes indeedy, of clinical insanity.

Does anyone bright enough to read this book have to ask what the answer is? I think not, but I will reveal it anyway in the fewest of syllables: Tell the psychos and their moronic lackeys to fuck off and let's get on with our lives. We are many – They are few.

CHAPTER SEVEN

War on your mind

One believes things because one has been conditioned to believe them
Aldous Huxley, *Brave New World*

I have described the 'Covid' hoax as a 'Psyop' and that is true in every sense and on every level in accordance with the definition of that term which is psychological warfare. Break down the 'Covid pandemic' to the foundation themes and it is psychological warfare on the human individual and collective mind.

The same can be said for the entire human belief system involving every subject you can imagine. Huxley was right in his contention that people believe what they are conditioned to believe and this comes from the repetition throughout their lives of the same falsehoods. They spew from government, corporations, media and endless streams of 'experts' telling you what the Cult wants you to believe and often believing it themselves (although *far* from always). 'Experts' are rewarded with 'prestigious' jobs and titles and as agents of perceptual programming with regular access to the media. The Cult has to control the narrative – control *information* – or they lose control of the vital, crucial, without-which-they-cannot-prevail public perception of reality. The foundation of that control today is the Internet made possible by the Defense Advanced Research Projects Agency (DARPA), the incredibly sinister technological arm of the Pentagon. The Internet is the result of military technology.

DARPA openly brags about establishing the Internet which has been a long-term project to lasso the minds of the global population. I have said for decades the plan is to control information to such an extreme that eventually no one would see or hear anything that the Cult does not approve. We are closing in on that end with ferocious censorship since the 'Covid' hoax began and in my case it started back in the 1990s in terms of books and speaking venues. I had to create my own publishing company in 1995 precisely because no one else would publish my books even then. I think they're all still running.

Cult Internet

To secure total control of information they needed the Internet in which pre-programmed algorithms can seek out 'unclean' content for deletion and even stop it being posted in the first place. The Cult had to dismantle print and non-Internet broadcast media to ensure the transfer of information to the appropriate-named 'Web' – a critical expression of the *Cult* web. We've seen the ever-quickening demise of traditional media and control of what is left by a tiny number of corporations operating worldwide. Independent journalism in the mainstream is already dead and never was that more obvious than since the turn of 2020. The Cult wants all information communicated via the Internet to globally censor and allow the plug to be pulled any time. Lockdowns and forced isolation has meant that communication between people has been through electronic means and no longer through face-to-face discourse and discussion. Cult psychopaths have targeted the bars, restaurants, sport, venues and meeting places in general for this reason. None of this is by chance and it's to stop people gathering in any kind of privacy or number while being able to track and monitor all Internet communications and block them as necessary. Even private messages between individuals have been censored by these fascists that control Cult fronts like Facebook, Twitter, Google and YouTube which are all officially run by Sabbatian place-people and from the background by higher-level Sabbatian place people.

Facebook, Google, Amazon and their like were seed-funded and supported into existence with money-no-object infusions of funds either directly or indirectly from DARPA and CIA technology arm In-Q-Tel. The Cult plays the long game and prepares very carefully for big plays like 'Covid'. Amazon is another front in the psychological war and pretty much controls the global market in book sales and increasingly publishing. Amazon's limitless funds have deleted fantastic numbers of independent publishers to seize global domination on the way to deciding which books can be sold and circulated and which cannot. Moves in that direction are already happening. Amazon's leading light Jeff Bezos is the grandson of Lawrence Preston Gise who worked with DARPA predecessor ARPA. Amazon has big connections to the CIA and the Pentagon. The plan I have long described went like this:

1. Employ military technology to establish the Internet.

2. Sell the Internet as a place where people can freely communicate without censorship and allow that to happen until the Net becomes the central and irreversible pillar of human society. If the Internet had been highly censored from the start many would have rejected it.

3. Fund and manipulate major corporations into being to control the circulation of information on your Internet using cover stories about geeks in garages to explain how they came about. Give them unlimited funds to expand rapidly with no need to make a profit for years while non-Cult companies who need to balance the books cannot compete. You know that in these circumstances your Googles, YouTubes, Facebooks and Amazons are going to secure near monopolies by either crushing or buying up the opposition.

4. Allow freedom of expression on both the Internet and communication platforms to draw people in until the Internet is the central and irreversible pillar of human society and your communication corporations have reached a stage of near monopoly domination.

5. Then unleash your always-planned frenzy of censorship on the basis of 'where else are you going to go?' and continue to expand that until nothing remains that the Cult does not want its human targets to see.

The process was timed to hit the 'Covid' hoax to ensure the best chance possible of controlling the narrative which they knew they had to do at all costs. They were, after all, about to unleash a 'deadly virus' that didn't really exist. If you do that in an environment of free-flowing information and opinion you would be dead in the

water before you could say Gates is a psychopath. The network was in place through which the Cult-created-and-owned World Health Organization could dictate the 'Covid' narrative and response policy slavishly supported by Cult-owned Internet communication giants and mainstream media while those telling a different story were censored. Google, YouTube, Facebook and Twitter openly announced that they would do this. What else would we expect from Cult-owned operations like Facebook which former executives have confirmed set out to make the platform more addictive than cigarettes and coldly manipulates emotions of its users to sow division between people and groups and scramble the minds of the young? If Zuckerberg lives out the rest of his life without going to jail for crimes against humanity, and most emphatically against the young, it will be a travesty of justice. Still, no matter, cause and effect will catch up with him eventually and the same with Sergey Brin and Larry Page at Google with its CEO Sundar Pichai who fix the Google search results to promote Cult narratives and hide the opposition. Put the same key words into Google and other search engines like DuckDuckGo and you will see how different results can be. Wikipedia is another intensely biased 'encyclopaedia' which skews its content to the Cult agenda. YouTube links to Wikipedia's version of 'Covid' and 'climate change' on video pages in which experts in their field offer a different opinion (even that is increasingly rare with Wojcicki censorship). Into this 'Covid' silence-them network must be added government media censors, sorry 'regulators', such as Ofcom in the UK which imposed tyrannical restrictions on British broadcasters that had the effect of banning me from ever appearing. Just to debate with me about my evidence and views on 'Covid' would mean breaking the fascistic impositions of Ofcom and its CEO career government bureaucrat Melanie Dawes. Gutless British broadcasters tremble at the very thought of fascist Ofcom.

Psychos behind 'Covid'

The reason for the 'Covid' catastrophe in all its facets and forms can be seen by whom and what is driving the policies worldwide in such a coordinated way. Decisions are not being made to protect health, but to target psychology. The dominant group guiding and 'advising' government policy are not medical professionals. They are psychologists and behavioural scientists. Every major country has its own version of this phenomenon and I'll use the British example to show how it works. In many ways the British version has been affecting the wider world in the form of the huge behaviour manipulation network in the UK which operates in other countries. The network involves private companies, government, intelligence and military. The Cabinet Office is at the centre of the government 'Covid' Psyop and part-owns, with 'innovation charity' Nesta, the Behavioural Insights Team (BIT) which claims to be independent of government but patently isn't. The BIT was established in 2010 and its job is to manipulate the psyche of the population to acquiesce to government demands and so much more. It is also known as the 'Nudge Unit', a name inspired by the 2009 book by two ultra-Zionists, Cass Sunstein and Richard Thaler, called *Nudge: Improving Decisions About Health, Wealth, and Happiness*. The book, as with the Behavioural Insights Team, seeks to 'nudge' behaviour (manipulate it) to make the public follow patterns of action and perception that suit those in authority (the Cult). Sunstein is so skilled at this that he advises the World Health Organization and the UK Behavioural Insights Team and was Administrator of the White House Office of Information and Regulatory Affairs in the Obama administration. Biden appointed him to the Department of Homeland Security – another ultra-Zionist in the fold to oversee new immigration laws which is another policy the Cult wants to control. Sunstein is desperate to silence anyone exposing conspiracies and co-authored a 2008 report on the subject in which suggestions were offered to ban 'conspiracy theorizing' or impose 'some kind of tax, financial or otherwise, on those who disseminate such theories'. I guess a psychiatrist's chair is out of the question?

Sunstein's mate Richard Thaler, an 'academic affiliate' of the UK Behavioural Insights Team, is a proponent of 'behavioural economics' which is defined as the study of 'the effects of psychological, cognitive, emotional, cultural and social factors on the decisions of individuals and institutions'. Study the effects so they can be manipulated to be what you want them to be. Other leading names in the development of behavioural economics are ultra-Zionists Daniel Kahneman and Robert J. Shiller and they, with Thaler, won the Nobel Memorial Prize in Economic Sciences for their work in this field. The Behavioural Insights Team is operating at the heart of the UK government and has expanded globally through partnerships with several universities including Harvard, Oxford, Cambridge, University College London (UCL) and Pennsylvania. They claim to have 'trained' (reframed) 20,000 civil servants and run more than 750 projects involving 400 randomised controlled trials in dozens of countries' as another version of mind reframers Common Purpose. BIT works from its office in New York with cities and their agencies, as well as other partners, across the United States and Canada – this is a company part-owned by the British government Cabinet Office. An executive order by President Cult-servant Obama established a US Social and Behavioral Sciences Team in 2015. They all have the same reason for being and that's to brainwash the population directly and by brainwashing those in positions of authority.

'Covid' mind game

Another prime aspect of the UK mind-control network is the 'independent' [joke] Scientific Pandemic Insights Group on Behaviours (SPI-B) which 'provides behavioural science advice aimed at anticipating and helping people adhere to interventions that are recommended by medical or epidemiological experts'. That means manipulating public perception and behaviour to do whatever government tells them to do. It's disgusting and if they really want the public to be 'safe' this lot should all be under lock and key. According to the government website SPI-B consists of

'behavioural scientists, health and social psychologists, anthropologists and historians' and advises the Whitty-Vallance-led Scientific Advisory Group for Emergencies (SAGE) which in turn advises the government on 'the science' (it doesn't) and 'Covid' policy. When politicians say they are being guided by 'the science' this is the rabble in each country they are talking about and that 'science' is dominated by behaviour manipulators to enforce government fascism through public compliance. The Behaviour Insight Team is headed by psychologist David Solomon Halpern, a visiting professor at King's College London, and connects with a national and global web of other civilian and military organisations as the Cult moves towards its goal of fusing them into one fascistic whole in every country through its 'Fusion Doctrine'. The behaviour manipulation network involves, but is not confined to, the Foreign Office; National Security Council; government communications headquarters (GCHQ); MI5; MI6; the Cabinet Office-based Media Monitoring Unit; and the Rapid Response Unit which 'monitors digital trends to spot emerging issues; including misinformation and disinformation; and identifies the best way to respond'.

There is also the 77th Brigade of the UK military which operates like the notorious Israeli military's Unit 8200 in manipulating information and discussion on the Internet by posing as members of the public to promote the narrative and discredit those who challenge it. Here we have the military seeking to manipulate *domestic* public opinion while the Nazis in government are fine with that. Conservative Member of Parliament Tobias Ellwood, an advocate of lockdown and control through 'vaccine passports', is a Lieutenant Colonel reservist in the 77th Brigade which connects with the military operation jHub, the 'innovation centre' for the Ministry of Defence and Strategic Command. jHub has also been involved with the civilian National Health Service (NHS) in 'symptom tracing' the population. The NHS is a key part of this mind control network and produced a document in December, 2020, explaining to staff how to use psychological manipulation with different groups and ages to get them to have the DNA-manipulating 'Covid vaccine'

that's designed to cumulatively rewrite human genetics. The document, called 'Optimising Vaccination Roll Out – Do's and Dont's for all messaging, documents and "communications" in the widest sense', was published by NHS England and the NHS Improvement *Behaviour Change Unit* in partnership with Public Health England and Warwick Business School. I hear the mantra about 'save the NHS' and 'protect the NHS' when we need to scrap the NHS and start again. The current version is far too corrupt, far too anti-human and totally compromised by Cult operatives and their assets. UK government broadcast media censor Ofcom will connect into this web – as will the BBC with its tremendous Ofcom influence – to control what the public see and hear and dictate mass perception. Nuremberg trials must include personnel from all these organisations.

The fear factor

The 'Covid' hoax has led to the creation of the UK Cabinet Office-connected Joint Biosecurity Centre (JBC) which is officially described as providing 'expert advice on pandemics' using its independent [all Cult operations are 'independent'] analytical function to provide real-time analysis about infection outbreaks to identify and respond to outbreaks of Covid-19'. Another role is to advise the government on a response to spikes in infections – 'for example by closing schools or workplaces in local areas where infection levels have risen'. Put another way, promoting the Cult agenda. The Joint Biosecurity Centre is modelled on the Joint Terrorism Analysis Centre which analyses intelligence to set 'terrorism threat levels' and here again you see the fusion of civilian and military operations and intelligence that has led to military intelligence producing documents about 'vaccine hesitancy' and how it can be combated. Domestic civilian matters and opinions should not be the business of the military. The Joint Biosecurity Centre is headed by Tom Hurd, director general of the Office for Security and Counter-Terrorism from the establishment-to-its-fingertips Hurd family. His father is former Foreign Secretary Douglas Hurd. How coincidental that Tom

Hurd went to the elite Eton College and Oxford University with Boris Johnson. Imperial College with its ridiculous computer modeller Neil Ferguson will connect with this gigantic web that will itself interconnect with similar set-ups in other major and not so major countries. Compared with this Cult network the politicians, be they Boris Johnson, Donald Trump or Joe Biden, are bit-part players 'following the science'. The network of psychologists was on the 'Covid' case from the start with the aim of generating maximum fear of the 'virus' to ensure compliance by the population. A government behavioural science group known as SPI-B produced a paper in March, 2020, for discussion by the main government science advisory group known as SAGE. It was headed 'Options for increasing adherence to social distancing measures' and it said the following in a section headed 'Persuasion':

- A substantial number of people still do not feel sufficiently personally threatened; it could be that they are reassured by the low death rate in their demographic group, although levels of concern may be rising. Having a good understanding of the risk has been found to be positively associated with adoption of COVID-19 social distancing measures in Hong Kong.

- The perceived level of personal threat needs to be increased among those who are complacent, using hard-hitting evaluation of options for increasing social distancing emotional messaging. To be effective this must also empower people by making clear the actions they can take to reduce the threat.

- Responsibility to others: There seems to be insufficient understanding of, or feelings of responsibility about, people's role in transmitting the infection to others … Messaging about actions need to be framed positively in terms of protecting oneself and the community, and increase confidence that they will be effective.

- Some people will be more persuaded by appeals to play by the rules, some by duty to the community, and some to personal risk.

> All these different approaches are needed. The messaging also needs to take account of the realities of different people's lives. Messaging needs to take account of the different motivational levers and circumstances of different people.

All this could be achieved the SPI-B psychologists said by *using the media to increase the sense of personal threat* which translates as terrify the shit out of the population, including children, so they all do what we want. That's not happened has it? Those excuses for 'journalists' who wouldn't know journalism if it bit them on the arse (the great majority) have played their crucial part in serving this Cult-government Psyop to enslave their own kids and grandkids. How they live with themselves I have no idea. The psychological war has been underpinned by constant government 'Covid' propaganda in almost every television and radio ad break, plus the Internet and print media, which has pounded out the fear with taxpayers footing the bill for their own programming. The result has been people terrified of a 'virus' that doesn't exist or one with a tiny fatality rate even if you believe it does. People walk down the street and around the shops wearing face-nappies damaging their health and psychology while others report those who refuse to be that naïve to the police who turn up in their own face-nappies. I had a cameraman come to my flat and he was so frightened of 'Covid' he came in wearing a mask and refused to shake my hand in case he caught something. He had – naïveitis – and the thought that he worked in the mainstream media was both depressing and made his behaviour perfectly explainable. The fear which has gripped the minds of so many and frozen them into compliance has been carefully cultivated by these psychologists who are really psychopaths. If lives get destroyed and a lot of young people commit suicide it shows our plan is working. SPI-B then turned to compulsion on the public to comply. 'With adequate preparation, rapid change can be achieved', it said. Some countries had introduced mandatory self-isolation on a wide scale without evidence of major public unrest and a large majority of the UK's population appeared to be supportive of more coercive measures with 64 percent of adults saying they would

support putting London under a lockdown (watch the 'polls' which are designed to make people believe that public opinion is in favour or against whatever the subject in hand).

For 'aggressive protective measures' to be effective, the SPI-B paper said, special attention should be devoted to those population groups that are more at risk. Translated from the Orwellian this means making the rest of population feel guilty for not protecting the 'vulnerable' such as old people which the Cult and its agencies were about to kill on an industrial scale with lockdown, lack of treatment and the Gates 'vaccine'. Psychopath psychologists sold their guilt-trip so comprehensively that Los Angeles County Supervisor Hilda Solis reported that children were apologising (from a distance) to their parents and grandparents for bringing 'Covid' into their homes and getting them sick. '… These apologies are just some of the last words that loved ones will ever hear as they die alone,' she said. Gut-wrenchingly Solis then used this childhood tragedy to tell children to stay at home and 'keep your loved ones alive'. Imagine heaping such potentially life-long guilt on a kid when it has absolutely nothing to do with them. These people are deeply disturbed and the psychologists behind this even more so.

Uncivil war – divide and rule

Professional mind-controllers at SPI-B wanted the media to increase a sense of responsibility to others (do as you're told) and promote 'positive messaging' for those actions while in contrast to invoke 'social disapproval' by the unquestioning, obedient, community of anyone with a mind of their own. Again the compliant Goebbels-like media obliged. This is an old, old, trick employed by tyrannies the world over throughout human history. You get the target population to keep the target population in line – *your* line. SPI-B said this could 'play an important role in preventing anti-social behaviour or discouraging failure to enact pro-social behaviour'. For 'anti-social' in the Orwellian parlance of SPI-B see any behaviour that government doesn't approve. SPI-B recommendations said that 'social disapproval' should be accompanied by clear messaging and

promotion of strong collective identity – hence the government and celebrity mantra of 'we're all in this together'. Sure we are. The mind doctors have such contempt for their targets that they think some clueless comedian, actor or singer telling them to do what the government wants will be enough to win them over. We have had UK comedian Lenny Henry, actor Michael Caine and singer Elton John wheeled out to serve the propagandists by urging people to have the DNA-manipulating 'Covid' non-'vaccine'. The role of Henry and fellow black celebrities in seeking to coax a 'vaccine' reluctant black community into doing the government's will was especially stomach-turning. An emotion-manipulating script and carefully edited video featuring these black 'celebs' was such an insult to the intelligence of black people and where's the self-respect of those involved selling their souls to a fascist government agenda? Henry said he heard black people's 'legitimate worries and concerns', but people must 'trust the facts' when they were doing exactly that by not having the 'vaccine'. They had to include the obligatory reference to Black Lives Matter with the line … 'Don't let coronavirus cost even more black lives – because we matter'. My god, it was pathetic. 'I know the vaccine is safe and what it does.' How? 'I'm a comedian and it says so in my script.'

SPI-B said social disapproval needed to be carefully managed to avoid victimisation, scapegoating and misdirected criticism, but they knew that their 'recommendations' would lead to exactly that and the media were specifically used to stir-up the divide-and-conquer hostility. Those who conform like good little baa, baas, are praised while those who have seen through the tidal wave of lies are 'Covidiots'. The awake have been abused by the fast asleep for not conforming to fascism and impositions that the awake know are designed to endanger their health, dehumanise them, and tear asunder the very fabric of human society. We have had the curtain-twitchers and morons reporting neighbours and others to the face-nappied police for breaking 'Covid rules' with fascist police delighting in posting links and phone numbers where this could be done. The Cult cannot impose its will without a compliant police

and military or a compliant population willing to play their part in enslaving themselves and their kids. The words of a pastor in Nazi Germany are so appropriate today:

First they came for the socialists and I did not speak out because I was not a socialist.

Then they came for the trade unionists and I did not speak out because I was not a trade unionist.

Then they came for the Jews and I did not speak out because I was not a Jew.

Then they came for me and there was no one left to speak for me.

Those who don't learn from history are destined to repeat it and so many are.

'Covid' rules: Rewiring the mind

With the background laid out to this gigantic national and global web of psychological manipulation we can put 'Covid' rules into a clear and sinister perspective. Forget the claims about protecting health. 'Covid' rules are about dismantling the human mind, breaking the human spirit, destroying self-respect, and then putting Humpty Dumpty together again as a servile, submissive slave. Social isolation through lockdown and distancing have devastating effects on the human psyche as the psychological psychopaths well know and that's the real reason for them. Humans need contact with each other, discourse, closeness and touch, or they eventually, and literarily, go crazy. Masks, which I will address at some length, fundamentally add to the effects of isolation and the Cult agenda to dehumanise and de-individualise the population. To do this while knowing – in fact *seeking* – this outcome is the very epitome of evil and psychologists involved in this *are* the epitome of evil. They must like all the rest of the Cult demons and their assets stand trial for crimes against humanity on a scale that defies the imagination. Psychopaths in uniform use isolation to break enemy troops and agents and make them subservient and submissive to tell what they know. The technique is rightly considered a form of torture and

torture is most certainly what has been imposed on the human population.

Clinically-insane American psychologist Harry Harlow became famous for his isolation experiments in the 1950s in which he separated baby monkeys from their mothers and imprisoned them for months on end in a metal container or 'pit of despair'. They soon began to show mental distress and depression as any idiot could have predicted. Harlow put other monkeys in steel chambers for three, six or twelve months while denying them any contact with animals or humans. He said that the effects of total social isolation for six months were 'so devastating and debilitating that we had assumed initially that twelve months of isolation would not produce any additional decrement'; but twelve months of isolation 'almost obliterated the animals socially'. This is what the Cult and its psychopaths are doing to you and your children. Even monkeys in partial isolation in which they were not allowed to form relationships with other monkeys became 'aggressive and hostile, not only to others, but also towards their own bodies'. We have seen this in the young as a consequence of lockdown. UK government psychopaths launched a public relations campaign telling people not to hug each other even after they received the 'Covid-19 vaccine' which we were told with more lies would allow a return to 'normal life'. A government source told *The Telegraph*: 'It will be along the lines that it is great that you have been vaccinated, but if you are going to visit your family and hug your grandchildren there is a chance you are going to infect people you love.' The source was apparently speaking from a secure psychiatric facility. Janet Lord, director of Birmingham University's Institute of Inflammation and Ageing, said that parents and grandparents should avoid hugging their children. Well, how can I put it, Ms Lord? Fuck off. Yep, that'll do.

Destroying the kids – where are the parents?

Observe what has happened to people enslaved and isolated by lockdown as suicide and self-harm has soared worldwide,

particularly among the young denied the freedom to associate with their friends. A study of 49,000 people in English-speaking countries concluded that almost half of young adults are at clinical risk of mental health disorders. A national survey in America of 1,000 currently enrolled high school and college students found that 5 percent reported attempting suicide during the pandemic. Data from the US CDC's National Syndromic Surveillance Program from January 1st to October 17th, 2020, revealed a *31 percent* increase in mental health issues among adolescents aged 12 to 17 compared with 2019. The CDC reported that America in general suffered the biggest drop in life expectancy since World War Two as it fell by a year in the first half of 2020 as a result of 'deaths of despair' – overdoses and suicides. Deaths of despair have leapt by more than 20 percent during lockdown and include the highest number of fatal overdoses ever recorded in a single year – 81,000. Internet addiction is another consequence of being isolated at home which lowers interest in physical activities as kids fall into inertia and what's the point? Children and young people are losing hope and giving up on life, sometimes literally. A 14-year-old boy killed himself in Maryland because he had 'given up' when his school district didn't reopen; an 11-year-old boy shot himself during a zoom class; a teenager in Maine succumbed to the isolation of the 'pandemic' when he ended his life after experiencing a disrupted senior year at school. Children as young as nine have taken their life and all these stories can be repeated around the world. Careers are being destroyed before they start and that includes those in sport in which promising youngsters have not been able to take part. The plan of the psycho-psychologists is working all right. Researchers at Cambridge University found that lockdowns cause significant harm to children's mental health. Their study was published in the *Archives of Disease in Childhood*, and followed 168 children aged between 7 and 11. The researchers concluded:

> During the UK lockdown, children's depression symptoms have increased substantially, relative to before lockdown. The scale of this effect has direct relevance for the continuation of different elements of lockdown policy, such as complete or partial school closures …

… Specifically, we observed a statistically significant increase in ratings of depression, with a medium-to-large effect size. Our findings emphasise the need to incorporate the potential impact of lockdown on child mental health in planning the ongoing response to the global pandemic and the recovery from it.

Not a chance when the Cult's psycho-psychologists were getting exactly what they wanted. The UK's Royal College of Paediatrics and Child Health has urged parents to look for signs of eating disorders in children and young people after a three to four fold increase. Specialists say the 'pandemic' is a major reason behind the rise. You don't say. The College said isolation from friends during school closures, exam cancellations, loss of extra-curricular activities like sport, and an increased use of social media were all contributory factors along with fears about the virus (psycho-psychologists again), family finances, and students being forced to quarantine. Doctors said young people were becoming severely ill by the time they were seen with 'Covid' regulations reducing face-to-face consultations. Nor is it only the young that have been devastated by the psychopaths. Like all bullies and cowards the Cult is targeting the young, elderly, weak and infirm. A typical story was told by a British lady called Lynn Parker who was not allowed to visit her husband in 2020 for the last ten and half months of his life 'when he needed me most' between March 20th and when he died on December 19th. This vacates the criminal and enters the territory of evil. The emotional impact on the immune system alone is immense as are the number of people of all ages worldwide who have died as a result of Cult-demanded, Gates-demanded, lockdowns.

Isolation is torture

The experience of imposing solitary confinement on millions of prisoners around the world has shown how a large percentage become 'actively psychotic and/or acutely suicidal'. Social isolation has been found to trigger 'a specific psychiatric syndrome, characterized by hallucinations; panic attacks; overt paranoia; diminished impulse control; hypersensitivity to external stimuli; and difficulties with thinking, concentration and memory'. Juan Mendez,

a United Nations rapporteur (investigator), said that isolation is a form of torture. Research has shown that even after isolation prisoners find it far more difficult to make social connections and I remember chatting to a shop assistant after one lockdown who told me that when her young son met another child again he had no idea how to act or what to do. Hannah Flanagan, Director of Emergency Services at Journey Mental Health Center in Dane County, Wisconsin, said: 'The specificity about Covid social distancing and isolation that we've come across as contributing factors to the suicides are really new to us this year.' But they are not new to those that devised them. They are getting the effect they want as the population is psychologically dismantled to be rebuilt in a totally different way. Children and the young are particularly targeted. They will be the adults when the full-on fascist AI-controlled technocracy is planned to be imposed and they are being prepared to meekly submit. At the same time older people who still have a memory of what life was like before – and how fascist the new normal really is – are being deleted. You are going to see efforts to turn the young against the old to support this geriatric genocide. Hannah Flanagan said the big increase in suicide in her county proved that social isolation is not only harmful, but deadly. Studies have shown that isolation from others is one of the main risk factors in suicide and even more so with women. Warnings that lockdown could create a 'perfect storm' for suicide were ignored. After all this was one of the *reasons* for lockdown. Suicide, however, is only the most extreme of isolation consequences. There are many others. Dr Dhruv Khullar, assistant professor of healthcare policy at Weill Cornell Medical College, said in a *New York Times* article in 2016 long before the fake 'pandemic':

> A wave of new research suggests social separation is bad for us. Individuals with less social connection have disrupted sleep patterns, altered immune systems, more inflammation and higher levels of stress hormones. One recent study found that isolation increases the risk of heart disease by 29 percent and stroke by 32 percent. Another analysis that pooled data from 70 studies and 3.4 million people found that socially isolated individuals had a 30 percent higher risk of dying in the next seven years, and that this effect was largest in middle age.

Loneliness can accelerate cognitive decline in older adults, and isolated individuals are twice as likely to die prematurely as those with more robust social interactions. These effects start early: Socially isolated children have significantly poorer health 20 years later, even after controlling for other factors. All told, loneliness is as important a risk factor for early death as obesity and smoking.

There you have proof from that one article alone four years before 2020 that those who have enforced lockdown, social distancing and isolation knew what the effect would be and that is even more so with professional psychologists that have been driving the policy across the globe. We can go back even further to the years 2000 and 2003 and the start of a major study on the effects of isolation on health by Dr Janine Gronewold and Professor Dirk M. Hermann at the University Hospital in Essen, Germany, who analysed data on 4,316 people with an average age of 59 who were recruited for the long-term research project. They found that socially isolated people are more than 40 percent more likely to have a heart attack, stroke, or other major cardiovascular event and nearly 50 percent more likely to die from any cause. Given the financial Armageddon unleashed by lockdown we should note that the study found a relationship between increased cardiovascular risk and lack of financial support. After excluding other factors social isolation was still connected to a 44 percent increased risk of cardiovascular problems and a 47 percent increased risk of death by any cause. Lack of financial support was associated with a 30 percent increase in the risk of cardiovascular health events. Dr Gronewold said it had been known for some time that feeling lonely or lacking contact with close friends and family can have an impact on physical health and the study had shown that having strong social relationships is of high importance for heart health. Gronewold said they didn't understand yet why people who are socially isolated have such poor health outcomes, but this was obviously a worrying finding, particularly during these times of prolonged social distancing. Well, it can be explained on many levels. You only have to identify the point in the body where people feel loneliness and missing people they are parted from – it's in the centre of the chest where they feel the ache of loneliness and the ache of missing people. 'My heart aches for

you' … 'My heart aches for some company.' I will explain this more in the chapter Escaping Wetiko, but when you realise that the body is the mind – they are expressions of each other – the reason why state of the mind dictates state of the body becomes clear.

American psychologist Ranjit Powar was highlighting the effects of lockdown isolation as early as April, 2020. She said humans have evolved to be social creatures and are wired to live in interactive groups. Being isolated from family, friends and colleagues could be unbalancing and traumatic for most people and could result in short or even long-term psychological and physical health problems. An increase in levels of anxiety, aggression, depression, forgetfulness and hallucinations were possible psychological effects of isolation. 'Mental conditions may be precipitated for those with underlying pre-existing susceptibilities and show up in many others without any pre-condition.' Powar said personal relationships helped us cope with stress and if we lost this outlet for letting off steam the result can be a big emotional void which, for an average person, was difficult to deal with. 'Just a few days of isolation can cause increased levels of anxiety and depression' – so what the hell has been the effect on the global population of *18 months* of this at the time of writing? Powar said: 'Add to it the looming threat of a dreadful disease being repeatedly hammered in through the media and you have a recipe for many shades of mental and physical distress.' For those with a house and a garden it is easy to forget that billions have had to endure lockdown isolation in tiny overcrowded flats and apartments with nowhere to go outside. The psychological and physical consequences of this are unimaginable and with lunatic and abusive partners and parents the consequences have led to tremendous increases in domestic and child abuse and alcoholism as people seek to shut out the horror. Ranjit Powar said:

Staying in a confined space with family is not all a rosy picture for everyone. It can be extremely oppressive and claustrophobic for large low-income families huddled together in small single-room houses. Children here are not lucky enough to have many board/electronic games or books to keep them occupied.

Add to it the deep insecurity of running out of funds for food and basic necessities. On the other hand, there are people with dysfunctional family dynamics, such as domineering, abusive or alcoholic partners, siblings or parents which makes staying home a period of trial. Incidence of suicide and physical abuse against women has shown a worldwide increase. Heightened anxiety and depression also affect a person's immune system, making them more susceptible to illness.

To think that Powar's article was published on April 11th, 2020.

Six-feet fantasy

Social (unsocial) distancing demanded that people stay six feet or two metres apart. UK government advisor Robert Dingwall from the New and Emerging Respiratory Virus Threats Advisory Group said in a radio interview that the two-metre rule was 'conjured up out of nowhere' and was not based on science. No, it was not based on *medical* science, but it didn't come out of nowhere. The distance related to *psychological* science. Six feet/two metres was adopted in many countries and we were told by people like the criminal Anthony Fauci and his ilk that it was founded on science. Many schools could not reopen because they did not have the space for six-feet distancing. Then in March, 2021, after a year of six-feet 'science', a study published in the *Journal of Infectious Diseases* involving more than 500,000 students and almost 100,000 staff over 16 weeks revealed no significant difference in 'Covid' cases between six feet and three feet and Fauci changed his tune. Now three feet was okay. There is no difference between six feet and three *inches* when there is no 'virus' and they got away with six feet for psychological reasons for as long as they could. I hear journalists and others talk about 'unintended consequences' of lockdown. They are not *unintended* at all; they have been coldly-calculated for a specific outcome of human control and that's why super-psychopaths like Gates have called for them so vehemently. Super-psychopath psychologists have demanded them and psychopathic or clueless, spineless, politicians have gone along with them by 'following the science'. But it's not science at all. 'Science' is not what is; it's only what people can be manipulated to believe it is. The whole 'Covid' catastrophe is

founded on mind control. Three word or three statement mantras issued by the UK government are a well-known mind control technique and so we've had 'Stay home/protect the NHS/save lives', 'Stay alert/control the virus/save lives' and 'hands/face/space'. One of the most vocal proponents of extreme 'Covid' rules in the UK has been Professor Susan Michie, a member of the British Communist Party, who is not a medical professional. Michie is the director of the Centre for Behaviour Change at University College London. She is a *behavioural psychologist* and another filthy rich 'Marxist' who praised China's draconian lockdown. She was known by fellow students at Oxford University as 'Stalin's nanny' for her extreme Marxism. Michie is an influential member of the UK government's Scientific Advisory Group for Emergencies (SAGE) and behavioural manipulation groups which have dominated 'Covid' policy. She is a consultant adviser to the World Health Organization on 'Covid-19' and behaviour. Why the hell are lockdowns anything to do with her when they are claimed to be about health? Why does a behavioural psychologist from a group charged with changing the behaviour of the public want lockdown, human isolation and mandatory masks? Does that question really need an answer? Michie *absolutely* has to explain herself before a Nuremberg court when humanity takes back its world again and even more so when you see the consequences of masks that she demands are compulsory. This is a Michie classic:

> The benefits of getting primary school children to wear masks is that regardless of what little degree of transmission is occurring in those age groups it could help normalise the practice. Young children wearing masks may be more likely to get their families to accept masks.

Those words alone should carry a prison sentence when you ponder on the callous disregard for children involved and what a statement it makes about the mind and motivations of Susan Michie. What a lovely lady and what she said there encapsulates the mentality of the psychopaths behind the 'Covid' horror. Let us compare what Michie said with a countrywide study in Germany published at researchsquare.com involving 25,000 school children and 17,854 health complaints submitted by parents. Researchers

found that masks are harming children physically, psychologically, and behaviourally with 24 health issues associated with mask wearing. They include: shortness of breath (29.7%); dizziness (26.4%); increased headaches (53%); difficulty concentrating (50%); drowsiness or fatigue (37%); and malaise (42%). Nearly a third of children experienced more sleep issues than before and a quarter developed new fears. Researchers found health issues and other impairments in 68 percent of masked children covering their faces for an average of 4.5 hours a day. Hundreds of those taking part experienced accelerated respiration, tightness in the chest, weakness, and short-term impairment of consciousness. A reminder of what Michie said again:

The benefits of getting primary school children to wear masks is that regardless of what little degree of transmission is occurring in those age groups it could help normalise the practice. Young children wearing masks may be more likely to get their families to accept masks.

Psychopaths in government and psychology now have children and young people – plus all the adults – wearing masks for hours on end while clueless teachers impose the will of the psychopaths on the young they should be protecting. What the hell are parents doing?

Cult lab rats

We have some schools already imposing on students microchipped buzzers that activate when they get 'too close' to their pals in the way they do with lab rats. How apt. To the Cult and its brain-dead servants our children *are* lab rats being conditioned to be unquestioning, dehumanised slaves for the rest of their lives. Children and young people are being weaned and frightened away from the most natural human instincts including closeness and touch. I have tracked in the books over the years how schools were banning pupils from greeting each other with a hug and the whole Cult-induced Me Too movement has terrified men and boys from a relaxed and natural interaction with female friends and work colleagues to the point where many men try never to be in a room

alone with a woman that's not their partner. Airhead celebrities have as always played their virtue-signalling part in making this happen with their gross exaggeration. For every monster like Harvey Weinstein there are at least tens of thousands of men that don't treat women like that; but everyone must be branded the same and policy changed for them as well as the monster. I am going to be using the word 'dehumanise' many times in this chapter because that is what the Cult is seeking to do and it goes very deep as we shall see. Don't let them kid you that social distancing is planned to end one day. That's not the idea. We are seeing more governments and companies funding and producing wearable gadgets to keep people apart and they would not be doing that if this was meant to be short-term. A tech start-up company backed by GCHQ, the British Intelligence and military surveillance headquarters, has created a social distancing wrist sensor that alerts people when they get too close to others. The CIA has also supported tech companies developing similar devices. The wearable sensor was developed by Tended, one of a number of start-up companies supported by GCHQ (see the CIA and DARPA). The device can be worn on the wrist or as a tag on the waistband and will vibrate whenever someone wearing the device breaches social distancing and gets anywhere near natural human contact. The company had a lucky break in that it was developing a distancing sensor when the 'Covid' hoax arrived which immediately provided a potentially enormous market. How fortunate. The government in big-time Cult-controlled Ontario in Canada is investing $2.5 million in wearable contact tracing technology that 'will alert users if they may have been exposed to the Covid-19 in the workplace and will beep or vibrate if they are within six feet of another person'. Facedrive Inc., the technology company behind this, was founded in 2016 with funding from the Ontario Together Fund and obviously they, too, had a prophet on the board of directors. The human surveillance and control technology is called TraceSCAN and would be worn by the human cyborgs in places such as airports, workplaces, construction sites, care homes and … *schools*.

I emphasise schools with children and young people the prime targets. You know what is planned for society as a whole if you keep your eyes on the schools. They have always been places where the state program the next generation of slaves to be its compliant worker-ants – or Woker-ants these days; but in the mist of the 'Covid' madness they have been transformed into mind laboratories on a scale never seen before. Teachers and head teachers are just as programmed as the kids – often more so. Children are kept apart from human interaction by walk lanes, classroom distancing, staggered meal times, masks, and the rolling-out of buzzer systems. Schools are now physically laid out as a laboratory maze for lab-rats. Lunatics at a school in Anchorage, Alaska, who should be prosecuted for child abuse, took away desks and forced children to kneel (know your place) on a mat for five hours a day while wearing a mask and using their chairs as a desk. How this was supposed to impact on a 'virus' only these clinically insane people can tell you and even then it would be clap-trap. The school banned recess (interaction), art classes (creativity), and physical exercise (getting body and mind moving out of inertia). Everyone behind this outrage should be in jail or better still a mental institution. The behavioural manipulators are all for this dystopian approach to schools. Professor Susan Michie, the mind-doctor and British Communist Party member, said it was wrong to say that schools were safe. They had to be made so by 'distancing', masks and ventilation (sitting all day in the cold). I must ask this lady round for dinner on a night I know I am going to be out and not back for weeks. She probably wouldn't be able to make it, anyway, with all the visits to her own psychologist she must have block-booked.

Masking identity

I know how shocking it must be for you that a behaviour manipulator like Michie wants everyone to wear masks which have long been a feature of mind-control programs like the infamous MKUltra in the United States, but, there we are. We live and learn. I spent many years from 1996 to right across the millennium

researching mind control in detail on both sides of the Atlantic and elsewhere. I met a large number of mind-control survivors and many had been held captive in body and mind by MKUltra. MK stands for mind-control, but employs the German spelling in deference to the Nazis spirited out of Germany at the end of World War Two by Operation Paperclip in which the US authorities, with help from the Vatican, transported Nazi mind-controllers and engineers to America to continue their work. Many of them were behind the creation of NASA and they included Nazi scientist and SS officer Wernher von Braun who swapped designing V-2 rockets to bombard London with designing the Saturn V rockets that powered the NASA moon programme's Apollo craft. I think I may have mentioned that the Cult has no borders. Among Paperclip escapees was Josef Mengele, the Angel of Death in the Nazi concentration camps where he conducted mind and genetic experiments on children often using twins to provide a control twin to measure the impact of his 'work' on the other. If you want to observe the Cult mentality in all its extremes of evil then look into the life of Mengele. I have met many people who suffered mercilessly under Mengele in the United States where he operated under the name Dr Greene and became a stalwart of MKUltra programming and torture. Among his locations was the underground facility in the Mojave Desert in California called the China Lake Naval Weapons Station which is almost entirely below the surface. My books *The Biggest Secret*, *Children of the Matrix* and *The Perception Deception* have the detailed background to MKUltra.

The best-known MKUltra survivor is American Cathy O'Brien. I first met her and her late partner Mark Phillips at a conference in Colorado in 1996. Mark helped her escape and deprogram from decades of captivity in an offshoot of MKUltra known as Project Monarch in which 'sex slaves' were provided for the rich and famous including Father George Bush, Dick Cheney and the Clintons. Read Cathy and Mark's book *Trance-Formation of America* and if you are new to this you will be shocked to the core. I read it in 1996 shortly before, with the usual synchronicity of my life, I found

myself given a book table at the conference right next to hers. MKUltra never ended despite being very publicly exposed (only a small part of it) in the 1970s and continues in other guises. I am still in touch with Cathy. She contacted me during 2020 after masks became compulsory in many countries to tell me how they were used as part of MKUltra programming. I had been observing 'Covid regulations' and the relationship between authority and public for months. I saw techniques that I knew were employed on individuals in MKUltra being used on the global population. I had read many books and manuals on mind control including one called *Silent Weapons for Quiet Wars* which came to light in the 1980s and was a guide on how to perceptually program on a mass scale. 'Silent Weapons' refers to mind-control. I remembered a line from the manual as governments, medical authorities and law enforcement agencies have so obviously talked to – or rather at – the adult population since the 'Covid' hoax began as if they are children. The document said:

> If a person is spoken to by a T.V. advertiser as if he were a twelve-year-old, then, due to suggestibility, he will, with a certain probability, respond or react to that suggestion with the uncritical response of a twelve-year-old and will reach in to his economic reservoir and deliver its energy to buy that product on impulse when he passes it in the store.

That's why authority has spoken to adults like children since all this began.

Why did Michael Jackson wear masks?

Every aspect of the 'Covid' narrative has mind-control as its central theme. Cathy O'Brien wrote an article for davidicke.com about the connection between masks and mind control. Her daughter Kelly who I first met in the 1990s was born while Cathy was still held captive in MKUltra. Kelly was forced to wear a mask as part of her programming from the age of *two* to dehumanise her, target her sense of individuality and reduce the amount of oxygen her brain and body received. *Bingo*. This is the real reason for compulsory

masks, why they have been enforced en masse, and why they seek to increase the number they demand you wear. First one, then two, with one disgraceful alleged 'doctor' recommending four which is nothing less than a death sentence. Where and how often they must be worn is being expanded for the purpose of mass mind control and damaging respiratory health which they can call 'Covid-19'. Canada's government headed by the man-child Justin Trudeau, says it's fine for children of two and older to wear masks. An insane 'study' in Italy involving just 47 children concluded there was no problem for babies as young as *four months* wearing them. Even after people were 'vaccinated' they were still told to wear masks by the criminal that is Anthony Fauci. Cathy wrote that mandating masks is allowing the authorities literally to control the air we breathe which is what was done in MKUltra. You might recall how the singer Michael Jackson wore masks and there is a reason for that. He was subjected to MKUltra mind control through Project Monarch and his psyche was scrambled by these simpletons. Cathy wrote:

In MKUltra Project Monarch mind control, Michael Jackson had to wear a mask to silence his voice so he could not reach out for help. Remember how he developed that whisper voice when he wasn't singing? Masks control the mind from the outside in, like the redefining of words is doing. By controlling what we can and cannot say for fear of being labeled racist or beaten, for example, it ultimately controls thought that drives our words and ultimately actions (or lack thereof).

Likewise, a mask muffles our speech so that we are not heard, which controls voice … words … mind. This is Mind Control. Masks are an obvious mind control device, and I am disturbed so many people are complying on a global scale. Masks depersonalize while making a person feel as though they have no voice. It is a barrier to others. People who would never choose to comply but are forced to wear a mask in order to keep their job, and ultimately their family fed, are compromised. They often feel shame and are subdued. People have stopped talking with each other while media controls the narrative.

The 'no voice' theme has often become literal with train passengers told not to speak to each other in case they pass on the 'virus', singing banned for the same reason and bonkers California officials telling people riding roller coasters that they cannot shout and scream. Cathy said she heard every day from healed MKUltra survivors who cannot wear a mask without flashing back on ways

their breathing was controlled – 'from ball gags and penises to water boarding'. She said that through the years when she saw images of people in China wearing masks 'due to pollution' that it was really to control their oxygen levels. 'I knew it was as much of a population control mechanism of depersonalisation as are burkas', she said. Masks are another Chinese communist/fascist method of control that has been swept across the West as the West becomes China at lightning speed since we entered 2020.

Mask-19

There are other reasons for mandatory masks and these include destroying respiratory health to call it 'Covid-19' and stunting brain development of children and the young. Dr Margarite Griesz-Brisson MD, PhD, is a Consultant Neurologist and Neurophysiologist and the Founder and Medical Director of the London Neurology and Pain Clinic. Her CV goes down the street and round the corner. She is clearly someone who cares about people and won't parrot the propaganda. Griesz-Brisson has a PhD in pharmacology, with special interest in neurotoxicology, environmental medicine, neuroregeneration and neuroplasticity (the way the brain can change in the light of information received). She went public in October, 2020, with a passionate warning about the effects of mask-wearing laws:

> The reinhalation of our exhaled air will without a doubt create oxygen deficiency and a flooding of carbon dioxide. We know that the human brain is very sensitive to oxygen deprivation. There are nerve cells for example in the hippocampus that can't be longer than 3 minutes without oxygen – they cannot survive. The acute warning symptoms are headaches, drowsiness, dizziness, issues in concentration, slowing down of reaction time – reactions of the cognitive system.

Oh, I know, let's tell bus, truck and taxi drivers to wear them and people working machinery. How about pilots, doctors and police? Griesz-Brisson makes the important point that while the symptoms she mentions may fade as the body readjusts this does not alter the fact that people continue to operate in oxygen deficit with long list of

potential consequences. She said it was well known that neurodegenerative diseases take years or decades to develop. 'If today you forget your phone number, the breakdown in your brain would have already started 20 or 30 years ago.' She said degenerative processes in your brain are getting amplified as your oxygen deprivation continues through wearing a mask. Nerve cells in the brain are unable to divide themselves normally in these circumstances and lost nerve cells will no longer be regenerated. 'What is gone is gone.' Now consider that people like shop workers and *schoolchildren* are wearing masks for hours every day. What in the name of sanity is going to be happening to them? 'I do not wear a mask, I need my brain to think', Griesz-Brisson said, 'I want to have a clear head when I deal with my patients and not be in a carbon dioxide-induced anaesthesia'. If you are told to wear a mask anywhere ask the organisation, police, store, whatever, for their risk assessment on the dangers and negative effects on mind and body of enforcing mask-wearing. They won't have one because it has never been done not even by government. All of them must be subject to class-action lawsuits as the consequences come to light. They don't do mask risk assessments for an obvious reason. They know what the conclusions would be and independent scientific studies that *have* been done tell a horror story of consequences.

'Masks are criminal'

Dr Griesz-Brisson said that for children and adolescents, masks are an absolute no-no. They had an extremely active and adaptive immune system and their brain was incredibly active with so much to learn. 'The child's brain, or the youth's brain, is thirsting for oxygen.' The more metabolically active an organ was, the more oxygen it required; and in children and adolescents every organ was metabolically active. Griesz-Brisson said that to deprive a child's or adolescent's brain of oxygen, or to restrict it in any way, was not only dangerous to their health, it was absolutely criminal. 'Oxygen deficiency inhibits the development of the brain, and the damage that has taken place as a result CANNOT be reversed.' Mind

manipulators of MKUltra put masks on two-year-olds they wanted to neurologically rewire and you can see why. Griesz-Brisson said a child needs the brain to learn and the brain needs oxygen to function. 'We don't need a clinical study for that. This is simple, indisputable physiology.' Consciously and purposely induced oxygen deficiency was an absolutely deliberate health hazard, and an absolute medical contraindication which means that 'this drug, this therapy, this method or measure should not be used, and is not allowed to be used'. To coerce an entire population to use an absolute medical contraindication by force, she said, there had to be definite and serious reasons and the reasons must be presented to competent interdisciplinary and independent bodies to be verified and authorised. She had this warning of the consequences that were coming if mask wearing continued:

When, in ten years, dementia is going to increase exponentially, and the younger generations couldn't reach their god-given potential, it won't help to say 'we didn't need the masks'. I know how damaging oxygen deprivation is for the brain, cardiologists know how damaging it is for the heart, pulmonologists know how damaging it is for the lungs. Oxygen deprivation damages every single organ. Where are our health departments, our health insurance, our medical associations? It would have been their duty to be vehemently against the lockdown and to stop it and stop it from the very beginning.

Why do the medical boards issue punishments to doctors who give people exemptions? Does the person or the doctor seriously have to prove that oxygen deprivation harms people? What kind of medicine are our doctors and medical associations representing? Who is responsible for this crime? The ones who want to enforce it? The ones who let it happen and play along, or the ones who don't prevent it?

All of the organisations and people she mentions there either answer directly to the Cult or do whatever hierarchical levels above them tell them to do. The outcome of both is the same. 'It's not about masks, it's not about viruses, it's certainly not about your health', Griesz-Brisson said. 'It is about much, much more. I am not participating. I am not afraid.' They were taking our air to breathe and there was no unfounded medical exemption from face masks. Oxygen deprivation was dangerous for every single brain. It had to be the free decision of every human being whether they want to

wear a mask that was absolutely ineffective to protect themselves from a virus. She ended by rightly identifying where the responsibility lies for all this:

> The imperative of the hour is personal responsibility. We are responsible for what we think, not the media. We are responsible for what we do, not our superiors. We are responsible for our health, not the World Health Organization. And we are responsible for what happens in our country, not the government.

Halle-bloody-lujah.

But surgeons wear masks, right?

Independent studies of mask-wearing have produced a long list of reports detailing mental, emotional and physical dangers. What a definition of insanity to see police officers imposing mask-wearing on the public which will cumulatively damage their health while the police themselves wear masks that will cumulatively damage *their* health. It's utter madness and both public and police do this because 'the government says so' – yes a government of brain-donor idiots like UK Health Secretary Matt Hancock reading the 'follow the science' scripts of psychopathic, lunatic psychologists. The response you get from Stockholm syndrome sufferers defending the very authorities that are destroying them and their families is that 'surgeons wear masks'. This is considered the game, set and match that they must work and don't cause oxygen deficit. Well, actually, scientific studies have shown that they *do* and oxygen levels are monitored in operating theatres to compensate. Surgeons wear masks to stop spittle and such like dropping into open wounds – not to stop 'viral particles' which are so miniscule they can only be seen through an electron microscope. Holes in the masks are significantly bigger than 'viral particles' and if you sneeze or cough they will breach the mask. I watched an incredibly disingenuous 'experiment' that claimed to prove that masks work in catching 'virus' material from the mouth and nose. They did this with a slow motion camera and the mask did block big stuff which stayed inside the mask and

against the face to be breathed in or cause infections on the face as we have seen with many children. 'Viral particles', however, would never have been picked up by the camera as they came through the mask when they are far too small to be seen. The 'experiment' was therefore disingenuous *and* useless.

Studies have concluded that wearing masks in operating theatres (and thus elsewhere) make no difference to preventing infection while the opposite is true with toxic shite building up in the mask and this had led to an explosion in tooth decay and gum disease dubbed by dentists 'mask mouth'. You might have seen the Internet video of a furious American doctor urging people to take off their masks after a four-year-old patient had been rushed to hospital the night before and nearly died with a lung infection that doctors sourced to mask wearing. A study in the journal *Cancer Discovery* found that inhalation of harmful microbes can contribute to advanced stage lung cancer in adults and long-term use of masks can help breed dangerous pathogens. Microbiologists have said frequent mask wearing creates a moist environment in which microbes can grow and proliferate before entering the lungs. The Canadian Agency for Drugs and Technologies in Health, or CADTH, a Canadian national organisation that provides research and analysis to healthcare decision-makers, said this as long ago as 2013 in a report entitled 'Use of Surgical Masks in the Operating Room: A Review of the Clinical Effectiveness and Guidelines'. It said:

- No evidence was found to support the use of surgical face masks to reduce the frequency of surgical site infections
- No evidence was found on the effectiveness of wearing surgical face masks to protect staff from infectious material in the operating room.
- Guidelines recommend the use of surgical face masks by staff in the operating room to protect both operating room staff and patients (despite the lack of evidence).

We were told that the world could go back to 'normal' with the arrival of the 'vaccines'. When they came, fraudulent as they are, the story changed as I knew that it would. We are in the midst of transforming 'normal', not going back to it. Mary Ramsay, head of immunisation at Public Health England, echoed the words of US criminal Anthony Fauci who said masks and other regulations must stay no matter if people are vaccinated. The Fauci idiot continued to wear two masks – different colours so both could be clearly seen – after he *claimed* to have been vaccinated. Senator Rand Paul told Fauci in one exchange that his double-masks were 'theatre' and he was right. It's all theatre. Mary Ramsay back-tracked on the vaccine-return-to-normal theme when she said the public may need to wear masks and social-distance for years despite the jabs. 'People have got used to those lower-level restrictions now, and [they] can live with them', she said telling us what the idea has been all along. 'The vaccine does not give you a pass, even if you have had it, you must continue to follow all the guidelines' said a Public Health England statement which reneged on what we had been told before and made having the 'vaccine' irrelevant to 'normality' even by the official story. Spain's fascist government trumped everyone by passing a law mandating the wearing of masks on the beach and even when swimming in the sea. The move would have devastated what's left of the Spanish tourist industry, posed potential breathing dangers to swimmers and had Northern European sunbathers walking around with their forehead brown and the rest of their face white as a sheet. The ruling was so crazy that it had to be retracted after pressure from public and tourist industry, but it confirmed where the Cult wants to go with masks and how clinically insane authority has become. The determination to make masks permanent and hide the serious dangers to body and mind can be seen in the censorship of scientist Professor Denis Rancourt by Bill Gates-funded academic publishing website ResearchGate over his papers exposing the dangers and uselessness of masks. Rancourt said:

ResearchGate today has permanently locked my account, which I have had since 2015. Their reasons graphically show the nature of their attack against democracy, and their corruption of

science … By their obscene non-logic, a scientific review of science articles reporting on harms caused by face masks has a 'potential to cause harm'. No criticism of the psychological device (face masks) is tolerated, if the said criticism shows potential to influence public policy.

This is what happens in a fascist world.

Where are the 'greens' (again)?

Other dangers of wearing masks especially regularly relate to the inhalation of minute plastic fibres into the lungs and the deluge of discarded masks in the environment and oceans. Estimates predicted that more than 1.5 billion disposable masks will end up in the world's oceans every year polluting the water with tons of plastic and endangering marine wildlife. Studies project that humans are using 129 billion face masks each month worldwide – about three million a minute. Most are disposable and made from plastic, non-biodegradable microfibers that break down into smaller plastic particles that become widespread in ecosystems. They are littering cities, clogging sewage channels and turning up in bodies of water. I have written in other books about the immense amounts of microplastics from endless sources now being absorbed into the body. Rolf Halden, director of the Arizona State University (ASU) Biodesign Center for Environmental Health Engineering, was the senior researcher in a 2020 study that analysed 47 human tissue samples and found microplastics in all of them. 'We have detected these chemicals of plastics in every single organ that we have investigated', he said. I wrote in *The Answer* about the world being deluged with microplastics. A study by the Worldwide Fund for Nature (WWF) found that people are consuming on average every week some 2,000 tiny pieces of plastic mostly through water and also through marine life and the air. Every year humans are ingesting enough microplastics to fill a heaped dinner plate and in a life-time of 79 years it is enough to fill two large waste bins. Marco Lambertini, WWF International director general said: 'Not only are plastics polluting our oceans and waterways and killing marine life – it's in all of us and we can't escape consuming plastics,' American

geologists found tiny plastic fibres, beads and shards in rainwater samples collected from the remote slopes of the Rocky Mountain National Park near Denver, Colorado. Their report was headed: 'It is raining plastic.' Rachel Adams, senior lecturer in Biomedical Science at Cardiff Metropolitan University, said that among health consequences are internal inflammation and immune responses to a 'foreign body'. She further pointed out that microplastics become carriers of toxins including mercury, pesticides and dioxins (a known cause of cancer and reproductive and developmental problems). These toxins accumulate in the fatty tissues once they enter the body through microplastics. Now this is being compounded massively by people putting plastic on their face and throwing it away.

Workers exposed to polypropylene plastic fibres known as 'flock' have developed 'flock worker's lung' from inhaling small pieces of the flock fibres which can damage lung tissue, reduce breathing capacity and exacerbate other respiratory problems. *Now* … commonly used surgical masks have three layers of melt-blown textiles made of … polypropylene. We have billions of people putting these microplastics against their mouth, nose and face for hours at a time day after day in the form of masks. How does anyone think that will work out? I mean – what could possibly go wrong? We posted a number of scientific studies on this at davidicke.com, but when I went back to them as I was writing this book the links to the science research website where they were hosted were dead. Anything that challenges the official narrative in any way is either censored or vilified. The official narrative is so unsupportable by the evidence that only deleting the truth can protect it. A study by Chinese scientists still survived – with the usual twist which it why it was still active, I guess. Yes, they found that virtually all the masks they tested increased the daily intake of microplastic fibres, but people should still wear them because the danger from the 'virus' was worse said the crazy 'team' from the Institute of Hydrobiology in Wuhan. Scientists first discovered microplastics in lung tissue of some patients who died of lung cancer

in the 1990s. Subsequent studies have confirmed the potential health damage with the plastic degrading slowly and remaining in the lungs to accumulate in volume. Wuhan researchers used a machine simulating human breathing to establish that masks shed up to nearly 4,000 microplastic fibres in a month with reused masks producing more. Scientists said some masks are laced with toxic chemicals and a variety of compounds seriously restricted for both health and environmental reasons. They include cobalt (used in blue dye) and formaldehyde known to cause watery eyes, burning sensations in the eyes, nose, and throat, plus coughing, wheezing and nausea. No – that must be 'Covid-19'.

Mask 'worms'

There is another and potentially even more sinister content of masks. Mostly new masks of different makes filmed under a microscope around the world have been found to contain strange black fibres or 'worms' that appear to move or 'crawl' by themselves and react to heat and water. The nearest I have seen to them are the self-replicating fibres that are pulled out through the skin of those suffering from Morgellons disease which has been connected to the phenomena of 'chemtrails' which I will bring into the story later on. Morgellons fibres continue to grow outside the body and have a form of artificial intelligence. Black 'worm' fibres in masks have that kind of feel to them and there is a nanotechnology technique called 'worm micelles' which carry and release drugs or anything else you want to deliver to the body. For sure the suppression of humanity by mind altering drugs is the Cult agenda big time and the more excuses they can find to gain access to the body the more opportunities there are to make that happen whether through 'vaccines' or masks pushed against the mouth and nose for hours on end.

So let us summarise the pros and cons of masks:

Against masks: Breathing in your own carbon dioxide; depriving the body and brain of sufficient oxygen; build-up of toxins in the mask that can be breathed into the lungs and cause rashes on the face and 'mask-mouth'; breathing microplastic fibres and toxic chemicals into the lungs; dehumanisation and deleting individualisation by literally making people faceless; destroying human emotional interaction through facial expression and deleting parental connection with their babies which look for guidance to their facial expression.

For masks: They don't protect you from a 'virus' that doesn't exist and even if it did 'viral' particles are so minute they are smaller than the holes in the mask.

Governments, police, supermarkets, businesses, transport companies, and all the rest who seek to impose masks have done no risk assessment on their consequences for health and psychology and are now open to group lawsuits when the impact becomes clear with a cumulative epidemic of respiratory and other disease. Authorities will try to exploit these effects and hide the real cause by dubbing them 'Covid-19'. Can you imagine setting out to force the population to wear health-destroying masks without doing any assessment of the risks? It is criminal and it is evil, but then how many people targeted in this way, who see their children told to wear them all day at school, have asked for a risk assessment? Billions can't be imposed upon by the few unless the billions allow it. Oh, yes, with just a tinge of irony, 85 percent of all masks made worldwide come from *China*.

Wash your hands in toxic shite

'Covid' rules include the use of toxic sanitisers and again the health consequences of constantly applying toxins to be absorbed through the skin is obvious to any level of Renegade Mind. America's Food and Drug Administration (FDA) said that sanitisers are drugs and issued a warning about 75 dangerous brands which contain

methanol used in antifreeze and can cause death, kidney damage and blindness. The FDA circulated the following warning even for those brands that it claims to be safe:

Store hand sanitizer out of the reach of pets and children, and children should use it only with adult supervision. Do not drink hand sanitizer. This is particularly important for young children, especially toddlers, who may be attracted by the pleasant smell or brightly colored bottles of hand sanitizer.

Drinking even a small amount of hand sanitizer can cause alcohol poisoning in children. (However, there is no need to be concerned if your children eat with or lick their hands after using hand sanitizer.) During this coronavirus pandemic, poison control centers have had an increase in calls about accidental ingestion of hand sanitizer, so it is important that adults monitor young children's use.

Do not allow pets to swallow hand sanitizer. If you think your pet has eaten something potentially dangerous, call your veterinarian or a pet poison control center right away. Hand sanitizer is flammable and should be stored away from heat and flames. When using hand sanitizer, rub your hands until they feel completely dry before performing activities that may involve heat, sparks, static electricity, or open flames.

There you go, perfectly safe, then, and that's without even a mention of the toxins absorbed through the skin. Come on kids – sanitise your hands everywhere you go. It will save you from the 'virus'. Put all these elements together of the 'Covid' normal and see how much health and psychology is being cumulatively damaged, even devastated, to 'protect your health'. Makes sense, right? They are only imposing these things because they care, right? *Right?*

Submitting to insanity

Psychological reframing of the population goes very deep and is done in many less obvious ways. I hear people say how contradictory and crazy 'Covid' rules are and how they are ever changing. This is explained away by dismissing those involved as idiots. It is a big mistake. The Cult is delighted if its cold calculation is perceived as incompetence and idiocy when it is anything but. Oh, yes, there are idiots within the system – lots of them – but they are *administering* the Cult agenda, mostly unknowingly. They are not deciding and dictating it. The bulwark against tyranny is self-

respect, always has been, always will be. It is self-respect that has broken every tyranny in history. By its very nature self-respect will not bow to oppression and its perpetrators. There is so little self-respect that it's always the few that overturn dictators. Many may eventually follow, but the few with the iron spines (self-respect) kick it off and generate the momentum. The Cult targets self-respect in the knowledge that once this has gone only submission remains. Crazy, contradictory, ever-changing 'Covid' rules are systematically applied by psychologists to delete self-respect. They *want* you to see that the rules make no sense. It is one thing to decide to do something when *you* have made the choice based on evidence and logic. You still retain your self-respect. It is quite another when you can see what you are being told to do is insane, ridiculous and makes no sense, and *yet you still do it*. Your self-respect is extinguished and this has been happening as ever more obviously stupid and nonsensical things have been demanded and the great majority have complied even when they can see they are stupid and nonsensical.

People walk around in face-nappies knowing they are damaging their health and make no difference to a 'virus'. They do it in fear of not doing it. I know it's daft, but I'll do it anyway. When that happens something dies inside of you and submissive reframing has begun. Next there's a need to hide from yourself that you have conceded your self-respect and you convince yourself that you have not really submitted to fear and intimidation. You begin to believe that you are complying with craziness because it's the right thing to do. When first you concede your self-respect of 2+2 = 4 to 2+2 = 5 you *know* you are compromising your self-respect. Gradually to avoid facing that fact you begin to *believe* that 2+2=5. You have been reframed and I have been watching this process happening in the human psyche on an industrial scale. The Cult is working to break your spirit and one of its major tools in that war is humiliation. I read how former American soldier Bradley Manning (later Chelsea Manning after a sex-change) was treated after being jailed for supplying WikiLeaks with documents exposing the enormity of

government and elite mendacity. Manning was isolated in solitary confinement for eight months, put under 24-hour surveillance, forced to hand over clothing before going to bed, and stand naked for every roll call. This is systematic humiliation. The introduction of anal swab 'Covid' tests in China has been done for the same reason to delete self-respect and induce compliant submission. Anal swabs are mandatory for incoming passengers in parts of China and American diplomats have said they were forced to undergo the indignity which would have been calculated humiliation by the Cult-owned Chinese government that has America in its sights.

Government-people: An abusive relationship

Spirit-breaking psychological techniques include giving people hope and apparent respite from tyranny only to take it away again. This happened in the UK during Christmas, 2020, when the psycho-psychologists and their political lackeys announced an easing of restrictions over the holiday only to reimpose them almost immediately on the basis of yet another lie. There is a big psychological difference between getting used to oppression and being given hope of relief only to have that dashed. Psychologists know this and we have seen the technique used repeatedly. Then there is traumatising people before you introduce more extreme regulations that require compliance. A perfect case was the announcement by the dark and sinister Whitty and Vallance in the UK that 'new data' predicted that 4,000 could die every day over the winter of 2020/2021 if we did not lockdown again. I think they call it lying and after traumatising people with that claim out came Jackboot Johnson the next day with new curbs on human freedom. Psychologists know that a frightened and traumatised mind becomes suggestable to submission and behaviour reframing. Underpinning all this has been to make people fearful and suspicious of each other and see themselves as a potential danger to others. In league with deleted self-respect you have the perfect psychological recipe for self-loathing. The relationship between authority and public is now demonstrably the same as that of

subservience to an abusive partner. These are signs of an abusive relationship explained by psychologist Leslie Becker-Phelps:

Psychological and emotional abuse: Undermining a partner's self-worth with verbal attacks, name-calling, and belittling. Humiliating the partner in public, unjustly accusing them of having an affair, or interrogating them about their every behavior. Keeping partner confused or off balance by saying they were just kidding or blaming the partner for 'making' them act this way … Feigning in public that they care while turning against them in private. This leads to victims frequently feeling confused, incompetent, unworthy, hopeless, and chronically self-doubting. [Apply these techniques to how governments have treated the population since New Year, 2020, and the parallels are obvious.]

Physical abuse: The abuser might physically harm their partner in a range of ways, such as grabbing, hitting, punching, or shoving them. They might throw objects at them or harm them with a weapon. [Observe the physical harm imposed by masks, lockdown, and so on.]

Threats and intimidation: One way abusers keep their partners in line is by instilling fear. They might be verbally threatening, or give threatening looks or gestures. Abusers often make it known that they are tracking their partner's every move. They might destroy their partner's possessions, threaten to harm them, or threaten to harm their family members. Not surprisingly, victims of this abuse often feel anxiety, fear, and panic. [No words necessary.]

Isolation: Abusers often limit their partner's activities, forbidding them to talk or interact with friends or family. They might limit access to a car or even turn off their phone. All of this might be done by physically holding them against their will, but is often accomplished through psychological abuse and intimidation. The more isolated a person feels, the fewer resources they have to help gain perspective on their situation and to escape from it. [No words necessary.]

Economic abuse: Abusers often make their partners beholden to them for money by controlling access to funds of any kind. They might prevent their partner from getting a job or withhold access to money they earn from a job. This creates financial dependency that makes leaving the relationship very difficult. [See destruction of livelihoods and the proposed meagre 'guaranteed income' so long as you do whatever you are told.]

Using children: An abuser might disparage their partner's parenting skills, tell their children lies about their partner, threaten to take custody of their children, or threaten to harm their children. These tactics instil fear and often elicit compliance. [See reframed social service mafia and how children are being mercilessly abused by the state over 'Covid' while their parents look on too frightened to do anything.]

A further recurring trait in an abusive relationship is the abused blaming themselves for their abuse and making excuses for the abuser. We have the public blaming each other for lockdown abuse by government and many making excuses for the government while attacking those who challenge the government. How often we have heard authorities say that rules are being imposed or reimposed only because people have refused to 'behave' and follow the rules. We don't want to do it – it's *you*.

Renegade Minds are an antidote to all of these things. They will never concede their self-respect no matter what the circumstances. Even when apparent humiliation is heaped upon them they laugh in its face and reflect back the humiliation on the abuser where it belongs. Renegade Minds will never wear masks they know are only imposed to humiliate, suppress and damage both physically and psychologically. Consequences will take care of themselves and they will never break their spirit or cause them to concede to tyranny. UK newspaper columnist Peter Hitchens was one of the few in the mainstream media to speak out against lockdowns and forced vaccinations. He then announced he had taken the jab. He wanted to see family members abroad and he believed vaccine passports were inevitable even though they had not yet been introduced. Hitchens

has a questioning and critical mind, but not a Renegade one. If he had no amount of pressure would have made him concede. Hitchens excused his action by saying that the battle has been lost. Renegade Minds never accept defeat when freedom is at stake and even if they are the last one standing the self-respect of not submitting to tyranny is more important than any outcome or any consequence.

That's why Renegade Minds are the only minds that ever changed anything worth changing.

CHAPTER EIGHT

'Reframing' insanity

Insanity is relative. It depends on who has who locked in what cage
Ray Bradbury

'Reframing' a mind means simply to change its perception and behaviour. This can be done subconsciously to such an extent that subjects have no idea they have been 'reframed' while to any observer changes in behaviour and attitudes are obvious.

Human society is being reframed on a ginormous scale since the start of 2020 and here we have the reason why psychologists rather than doctors have been calling the shots. Ask most people who have succumbed to 'Covid' reframing if they have changed and most will say 'no'; but they *have* and fundamentally. The Cult's long-game has been preparing for these times since way back and crucial to that has been to prepare both population and officialdom mentally and emotionally. To use the mind-control parlance they had to reframe the population with a mentality that would submit to fascism and reframe those in government and law enforcement to impose fascism or at least go along with it. The result has been the fact-deleted mindlessness of 'Wokeness' and officialdom that has either enthusiastically or unquestioningly imposed global tyranny demanded by reframed politicians on behalf of psychopathic and deeply evil cultists. 'Cognitive reframing' identifies and challenges the way someone sees the world in the form of situations, experiences and emotions and then restructures those perceptions to view the same set of circumstances in a different way. This can have

benefits if the attitudes are personally destructive while on the other side it has the potential for individual and collective mind control which the subject has no idea has even happened.

Cognitive therapy was developed in the 1960s by Aaron T. Beck who was born in Rhode Island in 1921 as the son of Jewish immigrants from the Ukraine. He became interested in the techniques as a treatment for depression. Beck's daughter Judith S. Beck is prominent in the same field and they founded the Beck Institute for Cognitive Behavior Therapy in Philadelphia in 1994. Cognitive reframing, however, began to be used worldwide by those with a very dark agenda. The Cult reframes politicians to change their attitudes and actions until they are completely at odds with what they once appeared to stand for. The same has been happening to government administrators at all levels, law enforcement, military and the human population. Cultists love mind control for two main reasons: It allows them to control what people think, do and say to secure agenda advancement and, by definition, it calms their legendary insecurity and fear of the unexpected. I have studied mind control since the time I travelled America in 1996. I may have been talking to next to no one in terms of an audience in those years, but my goodness did I gather a phenomenal amount of information and knowledge about so many things including the techniques of mind control. I have described this in detail in other books going back to *The Biggest Secret* in 1998. I met a very large number of people recovering from MKUltra and its offshoots and successors and I began to see how these same techniques were being used on the population in general. This was never more obvious than since the 'Covid' hoax began.

Reframing the enforcers

I have observed over the last two decades and more the very clear transformation in the dynamic between the police, officialdom and the public. I tracked this in the books as the relationship mutated from one of serving the public to seeing them as almost the enemy and certainly a lower caste. There has always been a class divide

based on income and always been some psychopathic, corrupt, and big-I-am police officers. This was different. Wholesale change was unfolding in the collective dynamic; it was less about money and far more about position and perceived power. An us-and-them was emerging. Noses were lifted skyward by government administration and law enforcement and their attitude to the public they were *supposed* to be serving changed to one of increasing contempt, superiority and control. The transformation was so clear and widespread that it had to be planned. Collective attitudes and dynamics do not change naturally and organically that quickly on that scale. I then came across an organisation in Britain called Common Purpose created in the late 1980s by Julia Middleton who would work in the office of Deputy Prime Minister John Prescott during the long and disastrous premiership of war criminal Tony Blair. When Blair speaks the Cult is speaking and the man should have been in jail a long time ago. Common Purpose proclaims itself to be one of the biggest 'leadership development' organisations in the world while functioning as a *charity* with all the financial benefits which come from that. It hosts 'leadership development' courses and programmes all over the world and claims to have 'brought together' what it calls 'leaders' from more than 100 countries on six continents. The modus operandi of Common Purpose can be compared with the work of the UK government's reframing network that includes the Behavioural Insights Team 'nudge unit' and 'Covid' reframing specialists at SPI-B. WikiLeaks described Common Purpose long ago as 'a hidden virus in our government and schools' which is unknown to the general public: 'It recruits and trains "leaders" to be loyal to the directives of Common Purpose and the EU, instead of to their own departments, which they then undermine or subvert, the NHS [National Health Service] being an example.' This is a vital point to understand the 'Covid' hoax. The NHS, and its equivalent around the world, has been utterly reframed in terms of administrators and much of the medical personnel with the transformation underpinned by recruitment policies. The outcome has been the criminal and psychopathic behaviour of the

NHS over 'Covid' and we have seen the same in every other major country. WikiLeaks said Common Purpose trainees are 'learning to rule without regard to democracy' and to usher in a police state (current events explained). Common Purpose operated like a 'glue' and had members in the NHS, BBC, police, legal profession, church, many of Britain's 7,000 quangos, local councils, the Civil Service, government ministries and Parliament, and controlled many RDA's (Regional Development Agencies). Here we have one answer for how and why British institutions and their like in other countries have changed so negatively in relation to the public. This further explains how and why the beyond-disgraceful reframed BBC has become a propaganda arm of 'Covid' fascism. They are all part of a network pursuing the same goal.

By 2019 Common Purpose was quoting a figure of 85,000 'leaders' that had attended its programmes. These 'students' of all ages are known as Common Purpose 'graduates' and they consist of government, state and local government officials and administrators, police chiefs and officers, and a whole range of others operating within the national, local and global establishment. Cressida Dick, Commissioner of the London Metropolitan Police, is the Common Purpose graduate who was the 'Gold Commander' that oversaw what can only be described as the murder of Brazilian electrician Jean Charles de Menezes in 2005. He was held down by psychopathic police and shot seven times in the head by a psychopathic lunatic after being mistaken for a terrorist when he was just a bloke going about his day. Dick authorised officers to pursue and keep surveillance on de Menezes and ordered that he be stopped from entering the underground train system. Police psychopaths took her at her word clearly. She was 'disciplined' for this outrage by being *promoted* – eventually to the top of the 'Met' police where she has been a disaster. Many Chief Constables controlling the police in different parts of the UK are and have been Common Purpose graduates. I have heard the 'graduate' network described as a sort of Mafia or secret society operating within the fabric of government at all levels pursuing a collective policy

ingrained at Common Purpose training events. Founder Julia Middleton herself has said:

> Locally and internationally, Common Purpose graduates will be 'lighting small fires' to create change in their organisations and communities … The Common Purpose effect is best illustrated by the many stories of small changes brought about by leaders, who themselves have changed.

A Common Purpose mission statement declared:

> Common Purpose aims to improve the way society works by expanding the vision, decision-making ability and influence of all kinds of leaders. The organisation runs a variety of educational programmes for leaders of all ages, backgrounds and sectors, in order to provide them with the inspirational, information and opportunities they need to change the world.

Yes, but into what? Since 2020 the answer has become clear.

NLP and the Delphi technique

Common Purpose would seem to be a perfect name or would common programming be better? One of the foundation methods of reaching 'consensus' (group think) is by setting the agenda theme and then encouraging, cajoling or pressuring everyone to agree a 'consensus' in line with the core theme promoted by Common Purpose. The methodology involves the 'Delphi technique', or an adaption of it, in which opinions are expressed that are summarised by a 'facilitator or change agent' at each stage. Participants are 'encouraged' to modify their views in the light of what others have said. Stage by stage the former individual opinions are merged into group consensus which just happens to be what Common Purpose wants them to believe. A key part of this is to marginalise anyone refusing to concede to group think and turn the group against them to apply pressure to conform. We are seeing this very technique used on the general population to make 'Covid' group-thinkers hostile to those who have seen through the bullshit. People can be reframed by using perception manipulation methods such as Neuro-Linguistic Programming (NLP) in which you change perception with the use of

carefully constructed language. An NLP website described the technique this way:

… A method of influencing brain behaviour (the 'neuro' part of the phrase) through the use of language (the 'linguistic' part) and other types of communication to enable a person to 'recode' the way the brain responds to stimuli (that's the 'programming') and manifest new and better behaviours. Neuro-Linguistic Programming often incorporates hypnosis and self-hypnosis to help achieve the change (or 'programming') that is wanted.

British alternative media operation UKColumn has done very detailed research into Common Purpose over a long period. I quoted co-founder and former naval officer Brian Gerrish in my book *Remember Who You Are*, published in 2011, as saying the following years before current times:

It is interesting that many of the mothers who have had children taken by the State speak of the Social Services people being icily cool, emotionless and, as two ladies said in slightly different words, '… like little robots'. We know that NLP is cumulative, so people can be given small imperceptible doses of NLP in a course here, another in a few months, next year etc. In this way, major changes are accrued in their personality, but the day by day change is almost unnoticeable.

In these and other ways 'graduates' have had their perceptions uniformly reframed and they return to their roles in the institutions of government, law enforcement, legal profession, military, 'education', the UK National Health Service and the whole swathe of the establishment structure to pursue a common agenda preparing for the 'post-industrial', 'post-democratic' society. I say 'preparing' but we are now there. 'Post-industrial' is code for the Great Reset and 'post-democratic' is 'Covid' fascism. UKColumn has spoken to partners of those who have attended Common Purpose 'training'. They have described how personalities and attitudes of 'graduates' changed very noticeably for the worse by the time they had completed the course. They had been 'reframed' and told they are the 'leaders' – the special ones – who know better than the population. There has also been the very demonstrable recruitment of psychopaths and narcissists into government administration at all

levels and law enforcement. If you want psychopathy hire psychopaths and you get a simple cause and effect. If you want administrators, police officers and 'leaders' to perceive the public as lesser beings who don't matter then employ narcissists. These personalities are identified using 'psychometrics' that identifies knowledge, abilities, attitudes and personality traits, mostly through carefully-designed questionnaires and tests. As this policy has passed through the decades we have had power-crazy, power-trippers appointed into law enforcement, security and government administration in preparation for current times and the dynamic between public and law enforcement/officialdom has been transformed. UKColumn's Brian Gerrish said of the narcissistic personality:

> Their love of themselves and power automatically means that they will crush others who get in their way. I received a major piece of the puzzle when a friend pointed out that when they made public officials re-apply for their own jobs several years ago they were also required to do psychometric tests. This was undoubtedly the start of the screening process to get 'their' sort of people in post.

How obvious that has been since 2020 although it was clear what was happening long before if people paid attention to the changing public-establishment dynamic.

Change agents

At the centre of events in 'Covid' Britain is the National Health Service (NHS) which has behaved disgracefully in slavishly following the Cult agenda. The NHS management structure is awash with Common Purpose graduates or 'change agents' working to a common cause. Helen Bevan, a Chief of Service Transformation at the NHS Institute for Innovation and Improvement, co-authored a document called 'Towards a million change agents, a review of the social movements literature: implications for large scale change in the NHS'. The document compared a project management approach to that of change and social movements where 'people change

themselves and each other – peer to peer'. Two definitions given for a 'social movement' were:

A group of people who consciously attempt to build a radically new social order; involves people of a broad range of social backgrounds; and deploys politically confrontational and socially disruptive tactics – Cyrus Zirakzadeh 1997

Collective challenges, based on common purposes and social solidarities, in sustained interaction with elites, opponents, and authorities – Sidney Tarrow 1994

Helen Bevan wrote another NHS document in which she defined 'framing' as 'the process by which leaders construct, articulate and put across their message in a powerful and compelling way in order to win people to their cause and call them to action'. I think I could come up with another definition that would be rather more accurate. The National Health Service and institutions of Britain and the wider world have been taken over by reframed 'change agents' and that includes everything from the United Nations to national governments, local councils and social services which have been kidnapping children from loving parents on an extraordinary and gathering scale on the road to the end of parenthood altogether. Children from loving homes are stolen and kidnapped by the state and put into the 'care' (inversion) of the local authority through council homes, foster parents and forced adoption. At the same time children are allowed to be abused without response while many are under council 'care'. UKColumn highlighted the Common Purpose connection between South Yorkshire Police and Rotherham council officers in the case of the scandal in that area of the sexual exploitation of children to which the authorities turned not one blind eye, but both:

We were alarmed to discover that the Chief Executive, the Strategic Director of Children and Young People's Services, the Manager for the Local Strategic Partnership, the Community Cohesion Manager, the Cabinet Member for Cohesion, the Chief Constable and his predecessor had all attended Leadership training courses provided by the pseudo-charity Common Purpose.

Once 'change agents' have secured positions of hire and fire within any organisation things start to move very quickly. Personnel are then hired and fired on the basis of whether they will work towards the agenda the change agent represents. If they do they are rapidly promoted even though they may be incompetent. Those more qualified and skilled who are pre-Common Purpose 'old school' see their careers stall and even disappear. This has been happening for decades in every institution of state, police, 'health' and social services and all of them have been transformed as a result in their attitudes to their jobs and the public. Medical professions, including nursing, which were once vocations for the caring now employ many cold, callous and couldn't give a shit personality types. The UKColumn investigation concluded:

By blurring the boundaries between people, professions, public and private sectors, responsibility and accountability, Common Purpose encourages 'graduates' to believe that as new selected leaders, they can work together, outside of the established political and social structures, to achieve a paradigm shift or CHANGE – so called 'Leading Beyond Authority'. In doing so, the allegiance of the individual becomes 'reframed' on CP colleagues and their NETWORK.

Reframing the Face-Nappies

Nowhere has this process been more obvious than in the police where recruitment of psychopaths and development of unquestioning mind-controlled group-thinkers have transformed law enforcement into a politically-correct 'Woke' joke and a travesty of what should be public service. Today they wear their face-nappies like good little gofers and enforce 'Covid' rules which are fascism under another name. Alongside the specifically-recruited psychopaths we have software minds incapable of free thought. Brian Gerrish again:

An example is the policeman who would not get on a bike for a press photo because he had not done the cycling proficiency course. Normal people say this is political correctness gone mad. Nothing could be further from the truth. The policeman has been reframed, and in his reality it is perfect common sense not to get on the bike 'because he hasn't done the cycling course'.

Another example of this is where the police would not rescue a boy from a pond until they had taken advice from above on the 'risk assessment'. A normal person would have arrived, perhaps thought of the risk for a moment, and dived in. To the police now 'reframed', they followed 'normal' procedure.

There are shocking cases of reframed ambulance crews doing the same. Sheer unthinking stupidity of London Face-Nappies headed by Common Purpose graduate Cressida Dick can be seen in their behaviour at a vigil in March, 2021, for a murdered woman, Sarah Everard. A police officer had been charged with the crime. Anyone with a brain would have left the vigil alone in the circumstances. Instead they 'manhandled' women to stop them breaking 'Covid rules' to betray classic reframing. Minds in the thrall of perception control have no capacity for seeing a situation on its merits and acting accordingly. 'Rules is rules' is their only mind-set. My father used to say that rules and regulations are for the guidance of the intelligent and the blind obedience of the idiot. Most of the intelligent, decent, coppers have gone leaving only the other kind and a few old school for whom the job must be a daily nightmare. The combination of psychopaths and rule-book software minds has been clearly on public display in the 'Covid' era with automaton robots in uniform imposing fascistic 'Covid' regulations on the population without any personal initiative or judging situations on their merits. There are thousands of examples around the world, but I'll make my point with the infamous Derbyshire police in the English East Midlands – the ones who think pouring dye into beauty spots and using drones to track people walking in the countryside away from anyone is called 'policing'. To them there are rules decreed by the government which they have to enforce and in their bewildered state a group gathering in a closed space and someone walking alone in the countryside are the same thing. It is beyond idiocy and enters the realm of clinical insanity.

Police officers in Derbyshire said they were 'horrified' – *horrified* – to find 15 to 20 'irresponsible' kids playing a football match at a closed leisure centre 'in breach of coronavirus restrictions'. When they saw the police the kids ran away leaving their belongings behind and the reframed men and women of Derbyshire police were seeking to establish their identities with a view to fining their parents. The most natural thing for youngsters to do – kicking a ball about – is turned into a criminal activity and enforced by the moronic software programs of Derbyshire police. You find the same mentality in every country. These barely conscious 'horrified' officers said they had to take action because 'we need to ensure these rules are being followed' and 'it is of the utmost importance that you ensure your children are following the rules and regulations for Covid-19'. Had any of them done ten seconds of research to see if this parroting of their masters' script could be supported by any evidence? Nope. Reframed people don't think – others think for them and that's the whole idea of reframing. I have seen police officers one after the other repeating without question word for word what officialdom tells them just as I have seen great swathes of the public doing the same. Ask either for 'their' opinion and out spews what they have been told to think by the official narrative. Police and public may seem to be in different groups, but their mentality is the same. Most people do whatever they are told in fear not doing so or because they believe what officialdom tells them; almost the entirety of the police do what they are told for the same reason. Ultimately it's the tiny inner core of the global Cult that's telling both what to do.

So Derbyshire police were 'horrified'. Oh, really? Why did they think those kids were playing football? It was to relieve the psychological consequences of lockdown and being denied human contact with their friends and interaction, touch and discourse vital to human psychological health. Being denied this month after month has dismantled the psyche of many children and young people as depression and suicide have exploded. Were Derbyshire police *horrified by that*? Are you kidding? Reframed people don't have those

mental and emotional processes that can see how the impact on the psychological health of youngsters is far more dangerous than any 'virus' even if you take the mendacious official figures to be true. The reframed are told (programmed) how to act and so they do. The Derbyshire Chief Constable in the first period of lockdown when the black dye and drones nonsense was going on was Peter Goodman. He was the man who severed the connection between his force and the Derbyshire Constabulary *Male Voice* Choir when he decided that it was not inclusive enough to allow women to join. The fact it was a male voice choir making a particular sound produced by male voices seemed to elude a guy who terrifyingly ran policing in Derbyshire. He retired weeks after his force was condemned as disgraceful by former Supreme Court Justice Jonathan Sumption for their behaviour over extreme lockdown impositions. Goodman was replaced by his deputy Rachel Swann who was in charge when her officers were 'horrified'. The police statement over the boys committing the hanging-offence of playing football included the line about the youngsters being 'irresponsible in the times we are all living through' missing the point that the real relevance of the 'times we are all living through' is the imposition of fascism enforced by psychopaths and reframed minds of police officers playing such a vital part in establishing the fascist tyranny that their own children and grandchildren will have to live in their entire lives. As a definition of insanity that is hard to beat although it might be run close by imposing masks on people that can have a serious effect on their health while wearing a face nappy all day themselves. Once again public and police do it for the same reason – the authorities tell them to and who are they to have the self-respect to say no?

Wokers in uniform

How reframed do you have to be to arrest a *six-year-old* and take him to court for *picking a flower* while waiting for a bus? Brain dead police and officialdom did just that in North Carolina where criminal proceedings happen regularly for children under nine. Attorney Julie Boyer gave the six-year-old crayons and a colouring book

during the 'flower' hearing while the 'adults' decided his fate. County Chief District Court Judge Jay Corpening asked: 'Should a child that believes in Santa Claus, the Easter Bunny and the tooth fairy be making life-altering decisions?' Well, of course not, but common sense has no meaning when you have a common purpose and a reframed mind. Treating children in this way, and police operating in American schools, is all part of the psychological preparation for children to accept a police state as normal all their adult lives. The same goes for all the cameras and biometric tracking technology in schools. Police training is focused on reframing them as snowflake Wokers and this is happening in the military. Pentagon top brass said that 'training sessions on extremism' were needed for troops who asked why they were so focused on the Capitol Building riot when Black Lives Matter riots were ignored. What's the difference between them some apparently and rightly asked. Actually, there is a difference. Five people died in the Capitol riot, only one through violence, and that was a police officer shooting an unarmed protestor. BLM riots killed at least 25 people and cost billions. Asking the question prompted the psychopaths and reframed minds that run the Pentagon to say that more 'education' (programming) was needed. Troop training is all based on psychological programming to make them fodder for the Cult – 'Military men are just dumb, stupid animals to be used as pawns in foreign policy' as Cult-to-his-DNA former Secretary of State Henry Kissinger famously said. Governments see the police in similar terms and it's time for those among them who can see this to defend the people and stop being enforcers of the Cult agenda upon the people.

The US military, like the country itself, is being targeted for destruction through a long list of Woke impositions. Cult-owned gaga 'President' Biden signed an executive order when he took office to allow taxpayer money to pay for transgender surgery for active military personnel and veterans. Are you a man soldier? No, I'm a LGBTQIA+ with a hint of Skoliosexual and Spectrasexual. Oh, good man. Bad choice of words you bigot. The Pentagon announced in March, 2021, the appointment of the first 'diversity and inclusion

officer' for US Special Forces. Richard Torres-Estrada arrived with the publication of a 'D&I Strategic Plan which will guide the enterprise-wide effort to institutionalize and sustain D&I'. If you think a Special Forces 'Strategic Plan' should have something to do with defending America you haven't been paying attention. Defending Woke is now the military's new role. Torres-Estrada has posted images comparing Donald Trump with Adolf Hitler and we can expect no bias from him as a representative of the supposedly non-political Pentagon. Cable news host Tucker Carlson said: 'The Pentagon is now the Yale faculty lounge but with cruise missiles.' Meanwhile Secretary of Defense Lloyd Austin, a board member of weapons-maker Raytheon with stock and compensation interests in October, 2020, worth $1.4 million, said he was purging the military of the 'enemy within' – anyone who isn't Woke and supports Donald Trump. Austin refers to his targets as 'racist extremists' while in true Woke fashion being himself a racist extremist. Pentagon documents pledge to 'eradicate, eliminate and conquer all forms of racism, sexism and homophobia'. The definitions of these are decided by 'diversity and inclusion committees' peopled by those who see racism, sexism and homophobia in every situation and opinion. Woke (the Cult) is dismantling the US military and purging testosterone as China expands its military and gives its troops 'masculinity training'. How do we think that is going to end when this is all Cult coordinated? The US military, like the British military, is controlled by Woke and spineless top brass who just go along with it out of personal career interests.

'Woke' means fast asleep

Mind control and perception manipulation techniques used on individuals to create group-think have been unleashed on the global population in general. As a result many have no capacity to see the obvious fascist agenda being installed all around them or what 'Covid' is really all about. Their brains are firewalled like a computer system not to process certain concepts, thoughts and realisations that are bad for the Cult. The young are most targeted as the adults they

will be when the whole fascist global state is planned to be fully implemented. They need to be prepared for total compliance to eliminate all pushback from entire generations. The Cult has been pouring billions into taking complete control of 'education' from schools to universities via its operatives and corporations and not least Bill Gates as always. The plan has been to transform 'education' institutions into programming centres for the mentality of 'Woke'. James McConnell, professor of psychology at the University of Michigan, wrote in *Psychology Today* in 1970:

> The day has come when we can combine sensory deprivation with drugs, hypnosis, and astute manipulation of reward and punishment, to gain almost absolute control over an individual's behaviour. It should then be possible to achieve a very rapid and highly effective type of brainwashing that would allow us to make dramatic changes in a person's behaviour and personality ...
>
> … We should reshape society so that we all would be trained from birth to want to do what society wants us to do. We have the techniques to do it... no-one owns his own personality you acquired, and there's no reason to believe you should have the right to refuse to acquire a new personality if your old one is anti-social.

This was the potential for mass brainwashing in 1970 and the mentality there displayed captures the arrogant psychopathy that drives it forward. I emphasise that not all young people have succumbed to Woke programming and those that haven't are incredibly impressive people given that today's young are the most perceptually-targeted generations in history with all the technology now involved. Vast swathes of the young generations, however, have fallen into the spell – and that's what it is – of Woke. The Woke mentality and perceptual program is founded on *inversion* and you will appreciate later why that is so significant. Everything with Woke is inverted and the opposite of what it is claimed to be. Woke was a term used in African-American culture from the 1900s and referred to an awareness of social and racial justice. This is not the meaning of the modern version or 'New Woke' as I call it in *The Answer*. Oh, no, Woke today means something very different no matter how much Wokers may seek to hide that and insist Old Woke and New

Woke are the same. See if you find any 'awareness of social justice' here in the modern variety:

- Woke demands 'inclusivity' while excluding anyone with a different opinion and calls for mass censorship to silence other views.

- Woke claims to stand against oppression when imposing oppression is the foundation of all that it does. It is the driver of political correctness which is nothing more than a Cult invention to manipulate the population to silence itself.

- Woke believes itself to be 'liberal' while pursuing a global society that can only be described as fascist (see 'anti-fascist' fascist Antifa).

- Woke calls for 'social justice' while spreading injustice wherever it goes against the common 'enemy' which can be easily identified as a differing view.

- Woke is supposed to be a metaphor for 'awake' when it is solid-gold asleep and deep in a Cult-induced coma that meets the criteria for 'off with the fairies'.

I state these points as obvious facts if people only care to look. I don't do this with a sense of condemnation. We need to appreciate that the onslaught of perceptual programming on the young has been incessant and merciless. I can understand why so many have been reframed, or, given their youth, framed from the start to see the world as the Cult demands. The Cult has had access to their minds day after day in its 'education' system for their entire formative years. Perception is formed from information received and the Cult-created system is a life-long download of information delivered to elicit a particular perception, thus behaviour. The more this has expanded into still new extremes in recent decades and ever-increasing censorship has deleted other opinions and information why wouldn't that lead to a perceptual reframing on a mass scale? I

have described already cradle-to-grave programming and in more recent times the targeting of young minds from birth to adulthood has entered the stratosphere. This has taken the form of skewing what is 'taught' to fit the Cult agenda and the omnipresent techniques of group-think to isolate non-believers and pressure them into line. There has always been a tendency to follow the herd, but we really are in a new world now in relation to that. We have parents who can see the 'Covid' hoax told by their children not to stop them wearing masks at school, being 'Covid' tested or having the 'vaccine' in fear of the peer-pressure consequences of being different. What is 'peer-pressure' if not pressure to conform to group-think? Renegade Minds never group-think and always retain a set of perceptions that are unique to them. Group-think is always underpinned by consequences for not group-thinking. Abuse now aimed at those refusing DNA-manipulating 'Covid vaccines' are a potent example of this. The biggest pressure to conform comes from the very group which is itself being manipulated. 'I am programmed to be part of a hive mind and so you must be.'

Woke control structures in 'education' now apply to every mainstream organisation. Those at the top of the 'education' hierarchy (the Cult) decide the policy. This is imposed on governments through the Cult network; governments impose it on schools, colleges and universities; their leadership impose the policy on teachers and academics and they impose it on children and students. At any level where there is resistance, perhaps from a teacher or university lecturer, they are targeted by the authorities and often fired. Students themselves regularly demand the dismissal of academics (increasingly few) at odds with the narrative that the students have been programmed to believe in. It is quite a thought that students who are being targeted by the Cult become so consumed by programmed group-think that they launch protests and demand the removal of those who are trying to push back against those targeting the students. Such is the scale of perceptual inversion. We see this with 'Covid' programming as the Cult imposes the rules via psycho-psychologists and governments on

shops, transport companies and businesses which impose them on their staff who impose them on their customers who pressure Pushbackers to conform to the will of the Cult which is in the process of destroying them and their families. Scan all aspects of society and you will see the same sequence every time.

Fact free Woke and hijacking the 'left'

There is no more potent example of this than 'Woke', a mentality only made possible by the deletion of factual evidence by an 'education' system seeking to produce an ever more uniform society. Why would you bother with facts when you don't know any? Deletion of credible history both in volume and type is highly relevant. Orwell said: 'Who controls the past controls the future: who controls the present controls the past.' They who control the perception of the past control the perception of the future and they who control the present control the perception of the past through the writing and deleting of history. Why would you oppose the imposition of Marxism in the name of Wokeism when you don't know that Marxism cost at least 100 million lives in the 20th century alone? Watch videos and read reports in which Woker generations are asked basic historical questions – it's mind-blowing. A survey of 2,000 people found that six percent of millennials (born approximately early1980s to early 2000s) believed the Second World War (1939-1945) broke out with the assassination of President Kennedy (in 1963) and one in ten thought Margaret Thatcher was British Prime Minister at the time. She was in office between 1979 and 1990. We are in a post-fact society. Provable facts are no defence against the fascism of political correctness or Silicon Valley censorship. Facts don't matter anymore as we have witnessed with the 'Covid' hoax. Sacrificing uniqueness to the Woke group-think religion is all you are required to do and that means thinking for yourself is the biggest Woke no, no. All religions are an expression of group-think and censorship and Woke is just another religion with an orthodoxy defended by group-think and censorship. Burned at

the stake becomes burned on Twitter which leads back eventually to burned at the stake as Woke humanity regresses to ages past.

The biggest Woke inversion of all is its creators and funders. I grew up in a traditional left of centre political household on a council estate in Leicester in the 1950s and 60s – you know, the left that challenged the power of wealth-hoarding elites and threats to freedom of speech and opinion. In those days students went on marches defending freedom of speech while today's Wokers march for its deletion. What on earth could have happened? Those very elites (collectively the Cult) that we opposed in my youth and early life have funded into existence the antithesis of that former left and hijacked the 'brand' while inverting everything it ever stood for. We have a mentality that calls itself 'liberal' and 'progressive' while acting like fascists. Cult billionaires and their corporations have funded themselves into control of 'education' to ensure that Woke programming is unceasing throughout the formative years of children and young people and that non-Wokers are isolated (that word again) whether they be students, teachers or college professors. The Cult has funded into existence the now colossal global network of Woke organisations that have spawned and promoted all the 'causes' on the Cult wish-list for global transformation and turned Wokers into demanders of them. Does anyone really think it's a coincidence that the Cult agenda for humanity is a carbon (sorry) copy of the societal transformations desired by Woke?? These are only some of them:

Political correctness: The means by which the Cult deletes all public debates that it knows it cannot win if we had the free-flow of information and evidence.

Human-caused 'climate change': The means by which the Cult seeks to transform society into a globally-controlled dictatorship imposing its will over the fine detail of everyone's lives 'to save the planet' which doesn't actually need saving.

Transgender obsession: Preparing collective perception to accept the 'new human' which would not have genders because it would be created technologically and not through procreation. I'll have much more on this in Human 2.0.

Race obsession: The means by which the Cult seeks to divide and rule the population by triggering racial division through the perception that society is more racist than ever when the opposite is the case. Is it perfect in that regard? No. But to compare today with the racism of apartheid and segregation brought to an end by the civil rights movement in the 1960s is to insult the memory of that movement and inspirations like Martin Luther King. Why is the 'anti-racism' industry (which it is) so dominated by privileged white people?

White supremacy: This is a label used by privileged white people to demonise poor and deprived white people pushing back on tyranny to marginalise and destroy them. White people are being especially targeted as the dominant race by number within Western society which the Cult seeks to transform in its image. If you want to change a society you must weaken and undermine its biggest group and once you have done that by using the other groups you next turn on them to do the same … 'Then they came for the Jews and I was not a Jew so I did nothing.'

Mass migration: The mass movement of people from the Middle East, Africa and Asia into Europe, from the south into the United States and from Asia into Australia are another way the Cult seeks to dilute the racial, cultural and political influence of white people on Western society. White people ask why their governments appear to be working against them while being politically and culturally biased towards incoming cultures. Well, here's your answer. In the same way sexually 'straight' people, men and women, ask why the

authorities are biased against them in favour of other sexualities. The answer is the same – that's the way the Cult wants it to be for very sinister motives.

These are all central parts of the Cult agenda and central parts of the Woke agenda and Woke was created and continues to be funded to an immense degree by Cult billionaires and corporations. If anyone begins to say 'coincidence' the syllables should stick in their throat.

Billionaire 'social justice warriors'

Joe Biden is a 100 percent-owned asset of the Cult and the Wokers' man in the White House whenever he can remember his name and for however long he lasts with his rapidly diminishing cognitive function. Even walking up the steps of an aircraft without falling on his arse would appear to be a challenge. He's not an empty-shell puppet or anything. From the minute Biden took office (or the Cult did) he began his executive orders promoting the Woke wish-list. You will see the Woke agenda imposed ever more severely because it's really the *Cult* agenda. Woke organisations and activist networks spawned by the Cult are funded to the extreme so long as they promote what the Cult wants to happen. Woke is funded to promote 'social justice' by billionaires who become billionaires by destroying social justice. The social justice mantra is only a cover for dismantling social justice and funded by billionaires that couldn't give a damn about social justice. Everything makes sense when you see that. One of Woke's premier funders is Cult billionaire financier George Soros who said: 'I am basically there to make money, I cannot and do not look at the social consequences of what I do.' This is the same Soros who has given more than $32 billion to his Open Society Foundations global Woke network and funded Black Lives Matter, mass immigration into Europe and the United States, transgender activism, climate change activism, political correctness and groups targeting 'white supremacy' in the form of privileged white thugs that dominate Antifa. What a scam it all is and when

you are dealing with the unquestioning fact-free zone of Woke scamming them is child's play. All you need to pull it off in all these organisations are a few in-the-know agents of the Cult and an army of naïve, reframed, uninformed, narcissistic, know-nothings convinced of their own self-righteousness, self-purity and virtue.

Soros and fellow billionaires and billionaire corporations have poured hundreds of millions into Black Lives Matter and connected groups and promoted them to a global audience. None of this is motivated by caring about black people. These are the billionaires that have controlled and exploited a system that leaves millions of black people in abject poverty and deprivation which they do absolutely nothing to address. The same Cult networks funding BLM were behind the *slave trade!* Black Lives Matter hijacked a phrase that few would challenge and they have turned this laudable concept into a political weapon to divide society. You know that BLM is a fraud when it claims that *All* Lives Matter, the most inclusive statement of all, is 'racist'. BLM and its Cult masters don't want to end racism. To them it's a means to an end to control all of humanity never mind the colour, creed, culture or background. What has destroying the nuclear family got to do with ending racism? Nothing – but that is one of the goals of BLM and also happens to be a goal of the Cult as I have been exposing in my books for decades. Stealing children from loving parents and giving schools ever more power to override parents is part of that same agenda. BLM is a Marxist organisation and why would that not be the case when the Cult created Marxism *and* BLM? Patrisse Cullors, a BLM co-founder, said in a 2015 video that she and her fellow organisers, including co-founder Alicia Garza, are 'trained Marxists'. The lady known after marriage as Patrisse Khan-Cullors bought a $1.4 million home in 2021 in one of the whitest areas of California with a black population of just 1.6 per cent and has so far bought *four* high-end homes for a total of $3.2 million. How very Marxist. There must be a bit of spare in the BLM coffers, however, when Cult corporations and billionaires have handed over the best part of $100 million. Many black people can see that Black Lives Matter is not

working for them, but against them, and this is still more confirmation. Black journalist Jason Whitlock, who had his account suspended by Twitter for simply linking to the story about the 'Marxist's' home buying spree, said that BLM leaders are 'making millions of dollars off the backs of these dead black men who they wouldn't spit on if they were on fire and alive'.

Black Lies Matter

Cult assets and agencies came together to promote BLM in the wake of the death of career criminal George Floyd who had been jailed a number of times including for forcing his way into the home of a black woman with others in a raid in which a gun was pointed at her stomach. Floyd was filmed being held in a Minneapolis street in 2020 with the knee of a police officer on his neck and he subsequently died. It was an appalling thing for the officer to do, but the same technique has been used by police on peaceful protestors of lockdown without any outcry from the Woke brigade. As unquestioning supporters of the Cult agenda Wokers have supported lockdown and all the 'Covid' claptrap while attacking anyone standing up to the tyranny imposed in its name. Court documents would later include details of an autopsy on Floyd by County Medical Examiner Dr Andrew Baker who concluded that Floyd had taken a fatal level of the drug fentanyl. None of this mattered to fact-free, question-free, Woke. Floyd's death was followed by worldwide protests against police brutality amid calls to defund the police. Throwing babies out with the bathwater is a Woke speciality. In the wake of the murder of British woman Sarah Everard a Green Party member of the House of Lords, Baroness Jones of Moulescoomb (Nincompoopia would have been better), called for a 6pm curfew for all men. This would be in breach of the Geneva Conventions on war crimes which ban collective punishment, but that would never have crossed the black and white Woke mind of Baroness Nincompoopia who would have been far too convinced of her own self-righteousness to compute such details. Many American cities did defund the police in the face of Floyd riots

and after $15 million was deleted from the police budget in Washington DC under useless Woke mayor Muriel Bowser car-jacking alone rose by 300 percent and within six months the US capital recorded its highest murder rate in 15 years. The same happened in Chicago and other cities in line with the Cult/Soros plan to bring fear to streets and neighbourhoods by reducing the police, releasing violent criminals and not prosecuting crime. This is the mob-rule agenda that I have warned in the books was coming for so long. Shootings in the area of Minneapolis where Floyd was arrested increased by 2,500 percent compared with the year before. Defunding the police over George Floyd has led to a big increase in dead people with many of them black. Police protection for politicians making these decisions stayed the same or increased as you would expect from professional hypocrites. The Cult doesn't actually want to abolish the police. It wants to abolish local control over the police and hand it to federal government as the psychopaths advance the Hunger Games Society. Many George Floyd protests turned into violent riots with black stores and businesses destroyed by fire and looting across America fuelled by Black Lives Matter. Woke doesn't do irony. If you want civil rights you must loot the liquor store and the supermarket and make off with a smart TV. It's the only way.

It's not a race war – it's a class war

Black people are patronised by privileged blacks and whites alike and told they are victims of white supremacy. I find it extraordinary to watch privileged blacks supporting the very system and bloodline networks behind the slave trade and parroting the same Cult-serving manipulative crap of their privileged white, often billionaire, associates. It is indeed not a race war but a class war and colour is just a diversion. Black Senator Cory Booker and black Congresswoman Maxine Waters, more residents of Nincompoopia, personify this. Once you tell people they are victims of someone else you devalue both their own responsibility for their plight and the power they have to impact on their reality and experience. Instead

we have: 'You are only in your situation because of whitey – turn on them and everything will change.' It won't change. Nothing changes in our lives unless *we* change it. Crucial to that is never seeing yourself as a victim and always as the creator of your reality. Life is a simple sequence of choice and consequence. Make different choices and you create different consequences. *You* have to make those choices – not Black Lives Matter, the Woke Mafia and anyone else that seeks to dictate your life. Who are they these Wokers, an emotional and psychological road traffic accident, to tell you what to do? Personal empowerment is the last thing the Cult and its Black Lives Matter want black people or anyone else to have. They claim to be defending the underdog while *creating* and perpetuating the underdog. The Cult's worst nightmare is human unity and if they are going to keep blacks, whites and every other race under economic servitude and control then the focus must be diverted from what they have in common to what they can be manipulated to believe divides them. Blacks have to be told that their poverty and plight is the fault of the white bloke living on the street in the same poverty and with the same plight they are experiencing. The difference is that your plight black people is due to him, a white supremacist with 'white privilege' living on the street. Don't unite as one human family against your mutual oppressors and suppressors – fight the oppressor with the white face who is as financially deprived as you are. The Cult knows that as its 'Covid' agenda moves into still new levels of extremism people are going to respond and it has been spreading the seeds of disunity everywhere to stop a united response to the evil that targets *all of us*.

Racist attacks on 'whiteness' are getting ever more outrageous and especially through the American Democratic Party which has an appalling history for anti-black racism. Barack Obama, Joe Biden, Hillary Clinton and Nancy Pelosi all eulogised about Senator Robert Byrd at his funeral in 2010 after a nearly 60-year career in Congress. Byrd was a brutal Ku Klux Klan racist and a violent abuser of Cathy O'Brien in MKUltra. He said he would never fight in the military 'with a negro by my side' and 'rather I should die a thousand times,

and see Old Glory trampled in the dirt never to rise again, than to see this beloved land of ours become degraded by race mongrels, a throwback to the blackest specimen from the wilds'. Biden called Byrd a 'very close friend and mentor'. These 'Woke' hypocrites are not anti-racist they are anti-poor and anti-people not of their perceived class. Here is an illustration of the scale of anti-white racism to which we have now descended. Seriously Woke and moronic *New York Times* contributor Damon Young described whiteness as a 'virus' that 'like other viruses will not die until there are no bodies left for it to infect'. He went on: '… the only way to stop it is to locate it, isolate it, extract it, and kill it.' Young can say that as a black man with no consequences when a white man saying the same in reverse would be facing a jail sentence. *That's* racism. We had super-Woke numbskull senators Tammy Duckworth and Mazie Hirono saying they would object to future Biden Cabinet appointments if he did not nominate more Asian Americans and Pacific Islanders. Never mind the ability of the candidate what do they look like? Duckworth said: 'I will vote for racial minorities and I will vote for LGBTQ, but anyone else I'm not voting for.' Appointing people on the grounds of race is illegal, but that was not a problem for this ludicrous pair. They were on-message and that's a free pass in any situation.

Critical race racism

White children are told at school they are intrinsically racist as they are taught the divisive 'critical race theory'. This claims that the law and legal institutions are inherently racist and that race is a socially constructed concept used by white people to further their economic and political interests at the expense of people of colour. White is a 'virus' as we've seen. Racial inequality results from 'social, economic, and legal differences that white people create between races to maintain white interests which leads to poverty and criminality in minority communities'. I must tell that to the white guy sleeping on the street. The principal of East Side Community School in New York sent white parents a manifesto that called on

them to become 'white traitors' and advocate for full 'white abolition'. These people are teaching your kids when they urgently need a psychiatrist. The 'school' included a chart with 'eight white identities' that ranged from 'white supremacist' to 'white abolition' and defined the behaviour white people must follow to end 'the regime of whiteness'. Woke blacks and their privileged white associates are acting exactly like the slave owners of old and Ku Klux Klan racists like Robert Byrd. They are too full of their own self-purity to see that, but it's true. Racism is not a body type; it's a state of mind that can manifest through any colour, creed or culture.

Another racial fraud is *'equity'*. Not equality of treatment and opportunity – equity. It's a term spun as equality when it means something very different. Equality in its true sense is a raising up while 'equity' is a race to the bottom. Everyone in the same level of poverty is 'equity'. Keep everyone down – that's equity. The Cult doesn't want anyone in the human family to be empowered and BLM leaders, like all these 'anti-racist' organisations, continue their privileged, pampered existence by perpetuating the perception of gathering racism. When is the last time you heard an 'anti-racist' or 'anti-Semitism' organisation say that acts of racism and discrimination have *fallen?* It's not in the interests of their fund-raising and power to influence and the same goes for the professional soccer anti-racism operation, Kick It Out. Two things confirmed that the Black Lives Matter riots in the summer of 2020 were Cult creations. One was that while anti-lockdown protests were condemned in this same period for 'transmitting 'Covid' the authorities supported mass gatherings of Black Lives Matter supporters. I even saw self-deluding people claiming to be doctors say the two types of protest were not the same. No – the non-existent 'Covid' was in favour of lockdowns and attacked those that protested against them while 'Covid' supported Black Lives Matter and kept well away from its protests. The whole thing was a joke and as lockdown protestors were arrested, often brutally, by reframed Face-Nappies we had the grotesque sight of police officers taking the knee to Black Lives Matter, a Cult-funded Marxist

organisation that supports violent riots and wants to destroy the nuclear family and white people.

He's not white? Shucks!

Woke obsession with race was on display again when ten people were shot dead in Boulder, Colorado, in March, 2021. Cult-owned Woke TV channels like CNN said the shooter appeared to be a white man and Wokers were on Twitter condemning 'violent white men' with the usual mantras. Then the shooter's name was released as Ahmad Al Aliwi Alissa, an anti-Trump Arab-American, and the sigh of disappointment could be heard five miles away. Never mind that ten people were dead and what that meant for their families. Race baiting was all that mattered to these sick Cult-serving people like Barack Obama who exploited the deaths to further divide America on racial grounds which is his job for the Cult. This is the man that 'racist' white Americans made the first black president of the United States and then gave him a second term. Not-very-bright Obama has become filthy rich on the back of that and today appears to have a big influence on the Biden administration. Even so he's still a downtrodden black man and a victim of white supremacy. This disingenuous fraud reveals the contempt he has for black people when he puts on a Deep South Alabama accent whenever he talks to them, no, *at* them.

Another BLM red flag was how the now fully-Woke (fully-Cult) and fully-virtue-signalled professional soccer authorities had their teams taking the knee before every match in support of Marxist Black Lives Matter. Soccer authorities and clubs displayed 'Black Lives Matter' on the players' shirts and flashed the name on electronic billboards around the pitch. Any fans that condemned what is a Freemasonic taking-the-knee ritual were widely condemned as you would expect from the Woke virtue-signallers of professional sport and the now fully-Woke media. We have reverse racism in which you are banned from criticising any race or culture except for white people for whom anything goes – say what you like, no problem. What has this got to do with racial harmony and

equality? We've had black supremacists from Black Lives Matter telling white people to fall to their knees in the street and apologise for their white supremacy. Black supremacists acting like white supremacist slave owners of the past couldn't breach their self-obsessed, race-obsessed sense of self-purity. Joe Biden appointed a race-obsessed black supremacist Kristen Clarke to head the Justice Department Civil Rights Division. Clarke claimed that blacks are endowed with 'greater mental, physical and spiritual abilities' than whites. If anyone reversed that statement they would be vilified. Clarke is on-message so no problem. She's never seen a black-white situation in which the black figure is anything but a virtuous victim and she heads the Civil Rights Division which should treat everyone the same or it isn't civil rights. Another perception of the Renegade Mind: If something or someone is part of the Cult agenda they will be supported by Woke governments and media no matter what. If they're not, they will be condemned and censored. It really is that simple and so racist Clarke prospers despite (make that because of) her racism.

The end of culture

Biden's administration is full of such racial, cultural and economic bias as the Cult requires the human family to be divided into warring factions. We are now seeing racially-segregated graduations and everything, but everything, is defined through the lens of perceived 'racism. We have 'racist' mathematics, 'racist' food and even 'racist' *plants*. World famous Kew Gardens in London said it was changing labels on plants and flowers to tell its pre-'Covid' more than two million visitors a year how racist they are. Kew director Richard Deverell said this was part of an effort to 'move quickly to decolonise collections' after they were approached by one Ajay Chhabra 'an actor with an insight into how sugar cane was linked to slavery'. They are *plants* you idiots. 'Decolonisation' in the Woke manual really means colonisation of society with its mentality and by extension colonisation by the Cult. We are witnessing a new Chinese-style 'Cultural Revolution' so essential to the success of all

Marxist takeovers. Our cultural past and traditions have to be swept away to allow a new culture to be built-back-better. Woke targeting of long-standing Western cultural pillars including historical monuments and cancelling of historical figures is what happened in the Mao revolution in China which 'purged remnants of capitalist and traditional elements from Chinese society' and installed Maoism as the dominant ideology'. For China see the Western world today and for 'dominant ideology' see Woke. Better still see Marxism or Maoism. The 'Covid' hoax has specifically sought to destroy the arts and all elements of Western culture from people meeting in a pub or restaurant to closing theatres, music venues, sports stadiums, places of worship and even banning *singing*. Destruction of Western society is also why criticism of any religion is banned except for Christianity which again is the dominant religion as white is the numerically-dominant race. Christianity may be fading rapidly, but its history and traditions are weaved through the fabric of Western society. Delete the pillars and other structures will follow until the whole thing collapses. I am not a Christian defending that religion when I say that. I have no religion. It's just a fact. To this end Christianity has itself been turned Woke to usher its own downfall and its ranks are awash with 'change agents' – knowing and unknowing – at every level including Pope Francis (*definitely* knowing) and the clueless Archbishop of Canterbury Justin Welby (possibly not, but who can be sure?). Woke seeks to coordinate attacks on Western culture, traditions, and ways of life through 'intersectionality' defined as 'the complex, cumulative way in which the effects of multiple forms of discrimination (such as racism, sexism, and classism) combine, overlap, or intersect especially in the experiences of marginalised individuals or groups'. Wade through the Orwellian Woke-speak and this means coordinating disparate groups in a common cause to overthrow freedom and liberal values.

The entire structure of public institutions has been infested with Woke – government at all levels, political parties, police, military, schools, universities, advertising, media and trade unions. This abomination has been achieved through the Cult web by appointing

Wokers to positions of power and battering non-Wokers into line through intimidation, isolation and threats to their job. Many have been fired in the wake of the empathy-deleted, vicious hostility of 'social justice' Wokers and the desire of gutless, spineless employers to virtue-signal their Wokeness. Corporations are filled with Wokers today, most notably those in Silicon Valley. Ironically at the top they are not Woke at all. They are only exploiting the mentality their Cult masters have created and funded to censor and enslave while the Wokers cheer them on until it's their turn. Thus the Woke 'liberal left' is an inversion of the traditional liberal left. Campaigning for justice on the grounds of power and wealth distribution has been replaced by campaigning for identity politics. The genuine traditional left would never have taken money from today's billionaire abusers of fairness and justice and nor would the billionaires have wanted to fund that genuine left. It would not have been in their interests to do so. The division of opinion in those days was between the haves and have nots. This all changed with Cult manipulated and funded identity politics. The division of opinion today is between Wokers and non-Wokers and not income brackets. Cult corporations and their billionaires may have taken wealth disparity to cataclysmic levels of injustice, but as long as they speak the language of Woke, hand out the dosh to the Woke network and censor the enemy they are 'one of us'. Billionaires who don't give a damn about injustice are laughing at them till their bellies hurt. Wokers are not even close to self-aware enough to see that. The transformed 'left' dynamic means that Wokers who drone on about 'social justice' are funded by billionaires that have destroyed social justice the world over. It's *why* they are billionaires.

The climate con

Nothing encapsulates what I have said more comprehensively than the hoax of human-caused global warming. I have detailed in my books over the years how Cult operatives and organisations were the pump-primers from the start of the climate con. A purpose-built vehicle for this is the Club of Rome established by the Cult in 1968

with the Rockefellers and Rothschilds centrally involved all along. Their gofer frontman Maurice Strong, a Canadian oil millionaire, hosted the Earth Summit in Rio de Janeiro, Brazil, in 1992 where the global 'green movement' really expanded in earnest under the guiding hand of the Cult. The Earth Summit established Agenda 21 through the Cult-created-and-owned United Nations to use the illusion of human-caused climate change to justify the transformation of global society to save the world from climate disaster. It is a No-Problem-Reaction-Solution sold through governments, media, schools and universities as whole generations have been terrified into believing that the world was going to end in their lifetimes unless what old people had inflicted upon them was stopped by a complete restructuring of how everything is done. Chill, kids, it's all a hoax. Such restructuring is precisely what the Cult agenda demands (purely by coincidence of course). Today this has been given the codename of the Great Reset which is only an updated term for Agenda 21 and its associated Agenda 2030. The latter, too, is administered through the UN and was voted into being by the General Assembly in 2015. Both 21 and 2030 seek centralised control of all resources and food right down to the raindrops falling on your own land. These are some of the demands of Agenda 21 established in 1992. See if you recognise this society emerging today:

- End national sovereignty
- State planning and management of all land resources, ecosystems, deserts, forests, mountains, oceans and fresh water; agriculture; rural development; biotechnology; and ensuring *'equity'*
- The state to 'define the role' of business and financial resources
- Abolition of private property
- 'Restructuring' the family unit (see BLM)
- Children raised by the state
- People told what their job will be
- Major restrictions on movement
- Creation of 'human settlement zones'

- Mass resettlement as people are forced to vacate land where they live
- Dumbing down education
- Mass global depopulation in pursuit of all the above

The United Nations was created as a Trojan horse for world government. With the climate con of critical importance to promoting that outcome you would expect the UN to be involved. Oh, it's involved all right. The UN is promoting Agenda 21 and Agenda 2030 justified by 'climate change' while also driving the climate hoax through its Intergovernmental Panel on Climate Change (IPCC), one of the world's most corrupt organisations. The IPCC has been lying ferociously and constantly since the day it opened its doors with the global media hanging unquestioningly on its every mendacious word. The Green movement is entirely Woke and has long lost its original environmental focus since it was co-opted by the Cult. An obsession with 'global warming' has deleted its values and scrambled its head. I experienced a small example of what I mean on a beautiful country walk that I have enjoyed several times a week for many years. The path merged into the fields and forests and you felt at one with the natural world. Then a 'Green' organisation, the Hampshire and Isle of Wight Wildlife Trust, took over part of the land and proceeded to cut down a large number of trees, including mature ones, to install a horrible big, bright steel 'this-is-ours-stay-out' fence that destroyed the whole atmosphere of this beautiful place. No one with a feel for nature would do that. Day after day I walked to the sound of chainsaws and a magnificent mature weeping willow tree that I so admired was cut down at the base of the trunk. When I challenged a Woke young girl in a green shirt (of course) about this vandalism she replied: 'It's a weeping willow – it will grow back.' This is what people are paying for when they donate to the Hampshire and Isle of Wight Wildlife Trust and many other 'green' organisations today. It is not the environmental movement that I knew and instead has become a support-system – as with Extinction Rebellion – for a very dark agenda.

Private jets for climate justice

The Cult-owned, Gates-funded, World Economic Forum and its founder Klaus Schwab were behind the emergence of Greta Thunberg to harness the young behind the climate agenda and she was invited to speak to the world at … the UN. Schwab published a book, *Covid-19: The Great Reset* in 2020 in which he used the 'Covid' hoax and the climate hoax to lay out a new society straight out of Agenda 21 and Agenda 2030. Bill Gates followed in early 2021 when he took time out from destroying the world to produce a book in his name about the way to save it. Gates flies across the world in private jets and admitted that 'I probably have one of the highest greenhouse gas footprints of anyone on the planet … my personal flying alone is gigantic.' He has also bid for the planet's biggest private jet operator. Other climate change saviours who fly in private jets include John Kerry, the US Special Presidential Envoy for Climate, and actor Leonardo DiCaprio, a 'UN Messenger of Peace with special focus on climate change'. These people are so full of bullshit they could corner the market in manure. We mustn't be sceptical, though, because the Gates book, *How to Avoid a Climate Disaster: The Solutions We Have and the Breakthroughs We Need*, is a genuine attempt to protect the world and not an obvious pile of excrement attributed to a mega-psychopath aimed at selling his masters' plans for humanity. The Gates book and the other shite-pile by Klaus Schwab could have been written by the same person and may well have been. Both use 'climate change' and 'Covid' as the excuses for their new society and by coincidence the Cult's World Economic Forum and Bill and Melinda Gates Foundation promote the climate hoax and hosted Event 201 which pre-empted with a 'simulation' the very 'coronavirus' hoax that would be simulated for real on humanity within weeks. The British 'royal' family is promoting the 'Reset' as you would expect through Prince 'climate change caused the war in Syria' Charles and his hapless son Prince William who said that we must 'reset our relationship with nature and our trajectory as a species' to avoid a climate disaster. Amazing how many promotors of the 'Covid' and 'climate change' control

systems are connected to Gates and the World Economic Forum. A 'study' in early 2021 claimed that carbon dioxide emissions must fall by the equivalent of a global lockdown roughly every two years for the next decade to save the planet. The 'study' appeared in the same period that the Schwab mob claimed in a video that lockdowns destroying the lives of billions are good because they make the earth 'quieter' with less 'ambient noise'. They took down the video amid a public backlash for such arrogant, empathy-deleted stupidity You see, however, where they are going with this. Corinne Le Quéré, a professor at the Tyndall Centre for Climate Change Research, University of East Anglia, was lead author of the climate lockdown study, and she writes for … the World Economic Forum. Gates calls in 'his' book for changing 'every aspect of the economy' (long-time Cult agenda) and for humans to eat synthetic 'meat' (predicted in my books) while cows and other farm animals are eliminated. Australian TV host and commentator Alan Jones described what carbon emission targets would mean for farm animals in Australia alone if emissions were reduced as demanded by 35 percent by 2030 and zero by 2050:

Well, let's take agriculture, the total emissions from agriculture are about 75 million tonnes of carbon dioxide, equivalent. Now reduce that by 35 percent and you have to come down to 50 million tonnes, I've done the maths. So if you take for example 1.5 million cows, you're going to have to reduce the herd by 525,000 [by] 2030, nine years, that's 58,000 cows a year. The beef herd's 30 million, reduce that by 35 percent, that's 10.5 million, which means 1.2 million cattle have to go every year between now and 2030. This is insanity!

There are 75 million sheep. Reduce that by 35 percent, that's 26 million sheep, that's almost 3 million a year. So under the Paris Agreement over 30 million beasts. dairy cows, cattle, pigs and sheep would go. More than 8,000 every minute of every hour for the next decade, do these people know what they're talking about?

Clearly they don't at the level of campaigners, politicians and administrators. The Cult *does* know; that's the outcome it wants. We are faced with not just a war on humanity. Animals and the natural world are being targeted and I have been saying since the 'Covid' hoax began that the plan eventually was to claim that the 'deadly virus' is able to jump from animals, including farm animals and

domestic pets, to humans. Just before this book went into production came this story: 'Russia registers world's first Covid-19 vaccine for cats & dogs as makers of Sputnik V warn pets & farm animals could spread virus'. The report said 'top scientists warned that the deadly pathogen could soon begin spreading through homes and farms' and 'the next stage is the infection of farm and domestic animals'. Know the outcome and you'll see the journey. Think what that would mean for animals and keep your eye on a term called zoonosis or zoonotic diseases which transmit between animals and humans. The Cult wants to break the connection between animals and people as it does between people and people. Farm animals fit with the Cult agenda to transform food from natural to synthetic.

The gas of life is killing us

There can be few greater examples of Cult inversion than the condemnation of carbon dioxide as a dangerous pollutant when it is the gas of life. Without it the natural world would be dead and so we would all be dead. We breathe in oxygen and breathe out carbon dioxide while plants produce oxygen and absorb carbon dioxide. It is a perfect symbiotic relationship that the Cult wants to dismantle for reasons I will come to in the final two chapters. Gates, Schwab, other Cult operatives and mindless repeaters, want the world to be 'carbon neutral' by at least 2050 and the earlier the better. 'Zero carbon' is the cry echoed by lunatics calling for 'Zero Covid' when we already have it. These carbon emission targets will deindustrialise the world in accordance with Cult plans – the post-industrial, post-democratic society – and with so-called renewables like solar and wind not coming even close to meeting human energy needs blackouts and cold are inevitable. Texans got the picture in the winter of 2021 when a snow storm stopped wind turbines and solar panels from working and the lights went down along with water which relies on electricity for its supply system. Gates wants everything to be powered by electricity to ensure that his masters have the kill switch to stop all human activity, movement, cooking, water and warmth any time they like. The climate lie is so

stupendously inverted that it claims we must urgently reduce carbon dioxide when we *don't have enough*.

Co2 in the atmosphere is a little above 400 parts per million when the optimum for plant growth is 2,000 ppm and when it falls anywhere near 150 ppm the natural world starts to die and so do we. It fell to as low as 280 ppm in an 1880 measurement in Hawaii and rose to 413 ppm in 2019 with industrialisation which is why the planet has become *greener* in the industrial period. How insane then that psychopathic madman Gates is not satisfied only with blocking the rise of Co2. He's funding technology to suck it out of the atmosphere. The reason why will become clear. The industrial era is not destroying the world through Co2 and has instead turned around a potentially disastrous ongoing fall in Co2. Greenpeace co-founder and scientist Patrick Moore walked away from Greenpeace in 1986 and has exposed the green movement for fear-mongering and lies. He said that 500 million years ago there was *17 times* more Co2 in the atmosphere than we have today and levels have been falling for hundreds of millions of years. In the last 150 million years Co2 levels in Earth's atmosphere had reduced by *90 percent*. Moore said that by the time humanity began to unlock carbon dioxide from fossil fuels we were at '38 seconds to midnight' and in that sense: 'Humans are [the Earth's] salvation.' Moore made the point that only half the Co2 emitted by fossil fuels stays in the atmosphere and we should remember that all pollution pouring from chimneys that we are told is carbon dioxide is in fact nothing of the kind. It's pollution. Carbon dioxide is an invisible gas.

William Happer, Professor of Physics at Princeton University and long-time government adviser on climate, has emphasised the Co2 deficiency for maximum growth and food production. Greenhouse growers don't add carbon dioxide for a bit of fun. He said that most of the warming in the last 100 years, after the earth emerged from the super-cold period of the 'Little Ice Age' into a natural warming cycle, was over by 1940. Happer said that a peak year for warming in 1988 can be explained by a 'monster El Nino' which is a natural and cyclical warming of the Pacific that has nothing to do with 'climate

change'. He said the effect of Co2 could be compared to painting a wall with red paint in that once two or three coats have been applied it didn't matter how much more you slapped on because the wall will not get much redder. Almost all the effect of the rise in Co2 has already happened, he said, and the volume in the atmosphere would now have to *double* to increase temperature by a single degree. Climate hoaxers know this and they have invented the most ridiculously complicated series of 'feedback' loops to try to overcome this rather devastating fact. You hear puppet Greta going on cluelessly about feedback loops and this is why.

The Sun affects temperature? No you *climate denier*

Some other nonsense to contemplate: Climate graphs show that rises in temperature do not follow rises in Co2 – *it's the other way round* with a lag between the two of some 800 years. If we go back 800 years from present time we hit the Medieval Warm Period when temperatures were higher than now without any industrialisation and this was followed by the Little Ice Age when temperatures plummeted. The world was still emerging from these centuries of serious cold when many climate records began which makes the ever-repeated line of the 'hottest year since records began' meaningless when you are not comparing like with like. The coldest period of the Little Ice Age corresponded with the lowest period of sunspot activity when the Sun was at its least active. Proper scientists will not be at all surprised by this when it confirms the obvious fact that earth temperature is affected by the scale of Sun activity and the energetic power that it subsequently emits; but when is the last time you heard a climate hoaxer talking about the Sun as a source of earth temperature?? Everything has to be focussed on Co2 which makes up just 0.117 percent of so-called greenhouse gases and only a fraction of even that is generated by human activity. The rest is natural. More than *90 percent* of those greenhouse gases are water vapour and clouds (Fig 9). Ban moisture I say. Have you noticed that the climate hoaxers no longer use the polar bear as their promotion image? That's because far from becoming extinct polar

bear communities are stable or thriving. Joe Bastardi, American meteorologist, weather forecaster and outspoken critic of the climate lie, documents in his book *The Climate Chronicles* how weather patterns and events claimed to be evidence of climate change have been happening since long before industrialisation: 'What happened before naturally is happening again, as is to be expected given the cyclical nature of the climate due to the design of the planet.' If you read the detailed background to the climate hoax in my other books you will shake your head and wonder how anyone could believe the crap which has spawned a multi-trillion dollar industry based on absolute garbage (see HIV causes AIDs and Sars-Cov-2 causes 'Covid-19'). Climate and 'Covid' have much in common given they have the same source. They both have the contradictory *everything* factor in which everything is explained by reference to them. It's hot – 'it's climate change'. It's cold – 'it's climate change'. I got a sniffle – 'it's Covid'. I haven't got a sniffle – 'it's Covid'. Not having a sniffle has to be a symptom of 'Covid'. Everything is and not having a sniffle is especially dangerous if you are a slow walker. For sheer audacity I offer you a Cambridge University 'study' that actually linked 'Covid' to 'climate change'. It had to happen eventually. They concluded that climate change played a role in 'Covid-19' spreading from animals to humans because … wait for it … I kid you not … *the two groups were forced closer together as populations grow*. Er, that's it. The whole foundation on which this depended was that 'Bats are the likely zoonotic origin of SARS-CoV-1 and SARS-CoV-2'. Well, they are not. They are nothing to do with it. Apart from bats not being the origin and therefore 'climate change' effects on bats being irrelevant I am in awe of their academic insight. Where would we be without them? Not where we are that's for sure.

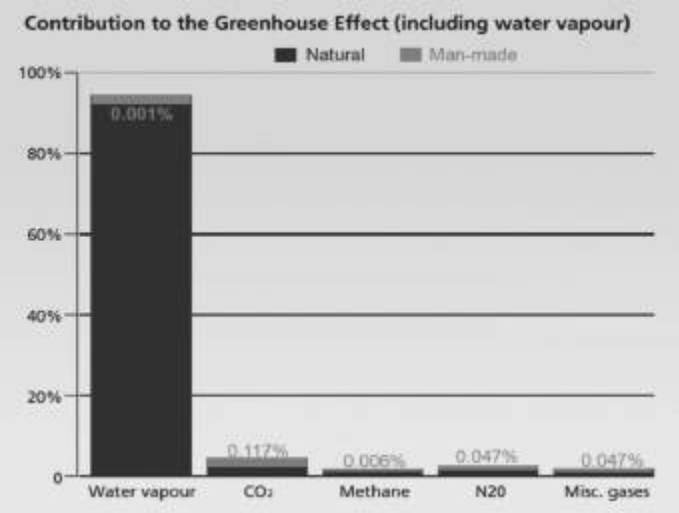


Figure 9: The idea that the gas of life is disastrously changing the climate is an insult to brain cell activity.

One other point about the weather is that climate modification is now well advanced and not every major weather event is natural – or earthquake come to that. I cover this subject at some length in other books. China is openly planning a rapid expansion of its weather modification programme which includes changing the climate in an area more than one and a half times the size of India. China used weather manipulation to ensure clear skies during the 2008 Olympics in Beijing. I have quoted from US military documents detailing how to employ weather manipulation as a weapon of war and they did that in the 1960s and 70s during the conflict in Vietnam with Operation Popeye manipulating monsoon rains for military purposes. Why would there be international treaties on weather modification if it wasn't possible? Of course it is. Weather is energetic information and it can be changed.

How was the climate hoax pulled off? See 'Covid'

If you can get billions to believe in a 'virus' that doesn't exist you can get them to believe in human-caused climate change that doesn't exist. Both are being used by the Cult to transform global society in the way it has long planned. Both hoaxes have been achieved in pretty much the same way. First you declare a lie is a fact. There's a

'virus' you call SARS-Cov-2 or humans are warming the planet with their behaviour. Next this becomes, via Cult networks, the foundation of government, academic and science policy and belief. Those who parrot the mantra are given big grants to produce research that confirms the narrative is true and ever more 'symptoms' are added to make the 'virus'/'climate change' sound even more scary. Scientists and researchers who challenge the narrative have their grants withdrawn and their careers destroyed. The media promote the lie as the unquestionable truth and censor those with an alternative view or evidence. A great percentage of the population believe what they are told as the lie becomes an everybody-knows-that and the believing-masses turn on those with a mind of their own. The technique has been used endlessly throughout human history. Wokers are the biggest promotors of the climate lie *and* 'Covid' fascism because their minds are owned by the Cult; their sense of self-righteous self-purity knows no bounds; and they exist in a bubble of reality in which facts are irrelevant and only get in the way of looking without seeing.

Running through all of this like veins in a blue cheese is control of information, which means control of perception, which means control of behaviour, which collectively means control of human society. The Cult owns the global media and Silicon Valley fascists for the simple reason that it *has* to. Without control of information it can't control perception and through that human society. Examine every facet of the Cult agenda and you will see that anything supporting its introduction is never censored while anything pushing back is always censored. I say again: Psychopaths that know why they are doing this must go before Nuremberg trials and those that follow their orders must trot along behind them into the same dock. 'I was just following orders' didn't work the first time and it must not work now. Nuremberg trials must be held all over the world before public juries for politicians, government officials, police, compliant doctors, scientists and virologists, and all Cult operatives such as Gates, Tedros, Fauci, Vallance, Whitty, Ferguson, Zuckerberg, Wojcicki, Brin, Page, Dorsey, the whole damn lot of

them – including, no *especially*, the psychopath psychologists. Without them and the brainless, gutless excuses for journalists that have repeated their lies, none of this could be happening. Nobody can be allowed to escape justice for the psychological and economic Armageddon they are all responsible for visiting upon the human race.

As for the compliant, unquestioning, swathes of humanity, and the self-obsessed, all-knowing ignorance of the Wokers … don't start me. God help their kids. God help their grandkids. God *help them*.

CHAPTER NINE

We must have it? So what is it?

Well I won't back down. No, I won't back down. You can stand me up at the Gates of Hell. But I won't back down
Tom Petty

I will now focus on the genetically-manipulating 'Covid vaccines' which do not meet this official definition of a vaccine by the US Centers for Disease Control (CDC): 'A product that stimulates a person's immune system to produce immunity to a specific disease, protecting the person from that disease.' On that basis 'Covid vaccines' are not a vaccine in that the makers don't even claim they stop infection or transmission.

They are instead part of a multi-levelled conspiracy to change the nature of the human body and what it means to be 'human' and to depopulate an enormous swathe of humanity. What I shall call Human 1.0 is on the cusp of becoming Human 2.0 and for very sinister reasons. Before I get to the 'Covid vaccine' in detail here's some background to vaccines in general. Government regulators do not test vaccines – the makers do – and the makers control which data is revealed and which isn't. Children in America are given 50 vaccine doses by age six and 69 by age 19 and the effect of the whole combined schedule has never been tested. Autoimmune diseases when the immune system attacks its own body have soared in the mass vaccine era and so has disease in general in children and the young. Why wouldn't this be the case when vaccines target the *immune system?* The US government gave Big Pharma drug

companies immunity from prosecution for vaccine death and injury in the 1986 National Childhood Vaccine Injury Act (NCVIA) and since then the government (taxpayer) has been funding compensation for the consequences of Big Pharma vaccines. The criminal and satanic drug giants can't lose and the vaccine schedule has increased dramatically since 1986 for this reason. There is no incentive to make vaccines safe and a big incentive to make money by introducing ever more. Even against a ridiculously high bar to prove vaccine liability, and with the government controlling the hearing in which it is being challenged for compensation, the vaccine court has so far paid out more than $4 billion. These are the vaccines we are told are safe and psychopaths like Zuckerberg censor posts saying otherwise. The immunity law was even justified by a ruling that vaccines by their nature were 'unavoidably unsafe'.

Check out the ingredients of vaccines and you will be shocked if you are new to this. *They put that in children's bodies?? What??* Try aluminium, a brain toxin connected to dementia, aborted foetal tissue and formaldehyde which is used to embalm corpses. World-renowned aluminium expert Christopher Exley had his research into the health effect of aluminium in vaccines shut down by Keele University in the UK when it began taking funding from the Bill and Melinda Gates Foundation. Research when diseases 'eradicated' by vaccines began to decline and you will find the fall began long *before* the vaccine was introduced. Sometimes the fall even plateaued after the vaccine. Diseases like scarlet fever for which there was no vaccine declined in the same way because of environmental and other factors. A perfect case in point is the polio vaccine. Polio began when lead arsenate was first sprayed as an insecticide and residues remained in food products. Spraying started in 1892 and the first US polio epidemic came in Vermont in 1894. The simple answer was to stop spraying, but Rockefeller-created Big Pharma had a better idea. Polio was decreed to be caused by the *poliovirus* which 'spreads from person to person and can infect a person's spinal cord'. Lead arsenate was replaced by the lethal DDT which had the same effect of causing paralysis by damaging the brain and central nervous

system. Polio plummeted when DDT was reduced and then banned, but the vaccine is still given the credit for something it didn't do. Today by far the biggest cause of polio is the vaccines promoted by Bill Gates. Vaccine justice campaigner Robert Kennedy Jr, son of assassinated (by the Cult) US Attorney General Robert Kennedy, wrote:

> In 2017, the World Health Organization (WHO) reluctantly admitted that the global explosion in polio is predominantly vaccine strain. The most frightening epidemics in Congo, Afghanistan, and the Philippines, are all linked to vaccines. In fact, by 2018, 70% of global polio cases were vaccine strain.

Vaccines make fortunes for Cult-owned Gates and Big Pharma while undermining the health and immune systems of the population. We had a glimpse of the mentality behind the Big Pharma cartel with a report on WION (World is One News), an international English language TV station based in India, which exposed the extraordinary behaviour of US drug company Pfizer over its 'Covid vaccine'. The WION report told how Pfizer had made fantastic demands of Argentina, Brazil and other countries in return for its 'vaccine'. These included immunity from prosecution, even for Pfizer negligence, government insurance to protect Pfizer from law suits and handing over as collateral sovereign assets of the country to include Argentina's bank reserves, military bases and embassy buildings. Pfizer demanded the same of Brazil in the form of waiving sovereignty of its assets abroad; exempting Pfizer from Brazilian laws; and giving Pfizer immunity from all civil liability. This is a 'vaccine' developed with government funding. Big Pharma is evil incarnate as a creation of the Cult and all must be handed tickets to Nuremberg.

Phantom 'vaccine' for a phantom 'disease'

I'll expose the 'Covid vaccine' fraud and then go on to the wider background of why the Cult has set out to 'vaccinate' every man, woman and child on the planet for an alleged 'new disease' with a survival rate of 99.77 percent (or more) even by the grotesquely-

manipulated figures of the World Health Organization and Johns Hopkins University. The 'infection' to 'death' ratio is 0.23 to 0.15 percent according to Stanford epidemiologist Dr John Ioannidis and while estimates vary the danger remains tiny. I say that if the truth be told the fake infection to fake death ratio is zero. Never mind all the evidence I have presented here and in *The Answer* that there is no 'virus' let us just focus for a moment on that death-rate figure of say 0.23 percent. The figure includes all those worldwide who have tested positive with a test not testing for the 'virus' and then died within 28 days or even longer of any other cause – *any other cause*. Now subtract all those illusory 'Covid' deaths on the global data sheets from the 0.23 percent. What do you think you would be left with? *Zero*. A vaccination has never been successfully developed for a so-called coronavirus. They have all failed at the animal testing stage when they caused hypersensitivity to what they were claiming to protect against and made the impact of a disease far worse. Cult-owned vaccine corporations got around that problem this time by bypassing animal trials, going straight to humans and making the length of the 'trials' before the public rollout as short as they could get away with. Normally it takes five to ten years or more to develop vaccines that still cause demonstrable harm to many people and that's without including the long-term effects that are never officially connected to the vaccination. 'Covid' non-vaccines have been officially produced and approved in a matter of months from a standing start and part of the reason is that (a) they were developed before the 'Covid' hoax began and (b) they are based on computer programs and not natural sources. Official non-trials were so short that government agencies gave *emergency*, not full, approval. 'Trials' were not even completed and full approval cannot be secured until they are. Public 'Covid vaccination' is actually a *continuation of the trial*. Drug company 'trials' are not scheduled to end until 2023 by which time a lot of people are going to be dead. Data on which government agencies gave this emergency approval was supplied by the Big Pharma corporations themselves in the form of Pfizer/BioNTech, AstraZeneca, Moderna, Johnson & Johnson, and

others, and this is the case with all vaccines. By its very nature *emergency* approval means drug companies do not have to prove that the 'vaccine' is 'safe and effective'. How could they with trials way short of complete? Government regulators only have to *believe* that they *could* be safe and effective. It is criminal manipulation to get products in circulation with no testing worth the name. Agencies giving that approval are infested with Big Pharma-connected place-people and they act in the interests of Big Pharma (the Cult) and not the public about whom they do not give a damn.

More human lab rats

'Covid vaccines' produced in record time by Pfizer/BioNTech and Moderna employ a technique *never approved before for use on humans*. They are known as mRNA 'vaccines' and inject a synthetic version of 'viral' mRNA or 'messenger RNA'. The key is in the term 'messenger'. The body works, or doesn't, on the basis of information messaging. Communications are constantly passing between and within the genetic system and the brain. Change those messages and you change the state of the body and even its very nature and you can change psychology and behaviour by the way the brain processes information. I think you are going to see significant changes in personality and perception of many people who have had the 'Covid vaccine' synthetic potions. Insider Aldous Huxley predicted the following in 1961 and mRNA 'vaccines' can be included in the term 'pharmacological methods':

> There will be, in the next generation or so, a pharmacological method of making people love their servitude, and producing dictatorship without tears, so to speak, producing a kind of painless concentration camp for entire societies, so that people will in fact have their own liberties taken away from them, but rather enjoy it, because they will be distracted from any desire to rebel by propaganda or brainwashing, or brainwashing enhanced by pharmacological methods. And this seems to be the final revolution.

Apologists claim that mRNA synthetic 'vaccines' don't change the DNA genetic blueprint because RNA does not affect DNA only the other way round. This is so disingenuous. A process called 'reverse

transcription' can convert RNA into DNA and be integrated into DNA in the cell nucleus. This was highlighted in December, 2020, by scientists at Harvard and Massachusetts Institute of Technology (MIT). Geneticists report that more than 40 percent of mammalian genomes results from reverse transcription. On the most basic level if messaging changes then that sequence must lead to changes in DNA which is receiving and transmitting those communications. How can introducing synthetic material into cells not change the cells where DNA is located? The process is known as transfection which is defined as 'a technique to insert foreign nucleic acid (DNA or RNA) into a cell, typically with the intention of altering the properties of the cell'. Researchers at the Sloan Kettering Institute in New York found that changes in messenger RNA can deactivate tumour-suppressing proteins and thereby promote cancer. This is what happens when you mess with messaging. 'Covid vaccine' maker Moderna was founded in 2010 by Canadian stem cell biologist Derrick J. Rossi after his breakthrough discovery in the field of transforming and reprogramming stem cells. These are neutral cells that can be programmed to become any cell including sperm cells. Moderna was therefore founded on the principle of genetic manipulation and has never produced any vaccine or drug before its genetically-manipulating synthetic 'Covid' shite. Look at the name – Mode-RNA or Modify-RNA. Another important point is that the US Supreme Court has ruled that genetically-modified DNA, or complementary DNA (cDNA) synthesized in the laboratory from messenger RNA, can be patented and owned. These psychopaths are doing this to the human body.

Cells replicate synthetic mRNA in the 'Covid vaccines' and in theory the body is tricked into making antigens which trigger antibodies to target the 'virus spike proteins' which as Dr Tom Cowan said have *never been seen*. Cut the crap and these 'vaccines' deliver *self-replicating* synthetic material to the cells with the effect of changing human DNA. The more of them you have the more that process is compounded while synthetic material is all the time self-replicating. 'Vaccine'-maker Moderna describes mRNA as 'like

software for the cell' and so they are messing with the body's software. What happens when you change the software in a computer? Everything changes. For this reason the Cult is preparing a production line of mRNA 'Covid vaccines' and a long list of excuses to use them as with all the 'variants' of a 'virus' never shown to exist. The plan is further to transfer the mRNA technique to other vaccines mostly given to children and young people. The cumulative consequences will be a transformation of human DNA through a constant infusion of synthetic genetic material which will kill many and change the rest. Now consider that governments that have given emergency approval for a vaccine that's not a vaccine; never been approved for humans before; had no testing worth the name; and the makers have been given immunity from prosecution for any deaths or adverse effects suffered by the public. The UK government awarded *permanent legal indemnity* to itself and its employees for harm done when a patient is being treated for 'Covid-19' or 'suspected Covid-19'. That is quite a thought when these are possible 'side-effects' from the 'vaccine' (they are not 'side', they are effects) listed by the US Food and Drug Administration:

Guillain-Barre syndrome; acute disseminated encephalomyelitis; transverse myelitis; encephalitis; myelitis; encephalomyelitis; meningoencephalitis; meningitis; encephalopathy; convulsions; seizures; stroke; narcolepsy; cataplexy; anaphylaxis; acute myocardial infarction (heart attack); myocarditis; pericarditis; autoimmune disease; death; implications for pregnancy, and birth outcomes; other acute demyelinating diseases; non anaphylactic allergy reactions; thrombocytopenia ; disseminated intravascular coagulation; venous thromboembolism; arthritis; arthralgia; joint pain; Kawasaki disease; multisystem inflammatory syndrome in children; vaccine enhanced disease. The latter is the way the 'vaccine' has the potential to make diseases far worse than they would otherwise be.

UK doctor and freedom campaigner Vernon Coleman described the conditions in this list as 'all unpleasant, most of them very serious, and you can't get more serious than death'. The thought that anyone at all has had the 'vaccine' in these circumstances is testament to the potential that humanity has for clueless, unquestioning, stupidity and for many that programmed stupidity has already been terminal.

An insider speaks

Dr Michael Yeadon is a former Vice President, head of research and Chief Scientific Adviser at vaccine giant Pfizer. Yeadon worked on the inside of Big Pharma, but that did not stop him becoming a vocal critic of 'Covid vaccines' and their potential for multiple harms, including infertility in women. By the spring of 2021 he went much further and even used the no, no, term 'conspiracy'. When you begin to see what is going on it is impossible not to do so. Yeadon spoke out in an interview with freedom campaigner James Delingpole and I mentioned earlier how he said that no one had samples of 'the virus'. He explained that the mRNA technique originated in the anti-cancer field and ways to turn on and off certain genes which could be advantageous if you wanted to stop cancer growing out of control. 'That's the origin of them. They are a very unusual application, really.' Yeadon said that treating a cancer patient with an aggressive procedure might be understandable if the alternative was dying, but it was quite another thing to use the same technique as a public health measure. Most people involved wouldn't catch the infectious agent you were vaccinating against and if they did they probably wouldn't die:

> If you are really using it as a public health measure you really want to as close as you can get to zero sides-effects … I find it odd that they chose techniques that were really cutting their teeth in the field of oncology and I'm worried that in using gene-based vaccines that have to be injected in the body and spread around the body, get taken up into some cells, and the regulators haven't quite told us which cells they get taken up into … you are going to be generating a wide range of responses … with multiple steps each of which could go well or badly.

I doubt the Cult intends it to go well. Yeadon said that you can put any gene you like into the body through the 'vaccine'. 'You can certainly give them a gene that would do them some harm if you wanted.' I was intrigued when he said that when used in the cancer field the technique could turn genes on and off. I explore this process in *The Answer* and with different genes having different functions you could create mayhem – physically and psychologically – if you turned the wrong ones on and the right ones off. I read reports of an experiment by researchers at the University of Washington's school of computer science and engineering in which they encoded DNA to infect computers. The body is itself a biological computer and if human DNA can inflict damage on a computer why can't the computer via synthetic material mess with the human body? It can. The Washington research team said it was possible to insert malicious malware into 'physical DNA strands' and corrupt the computer system of a gene sequencing machine as it 'reads gene letters and stores them as binary digits 0 and 1'. They concluded that hackers could one day use blood or spit samples to access computer systems and obtain sensitive data from police forensics labs or infect genome files. It is at this level of digital interaction that synthetic 'vaccines' need to be seen to get the full picture and that will become very clear later on. Michael Yeadon said it made no sense to give the 'vaccine' to younger people who were in no danger from the 'virus'. What was the benefit? It was all downside with potential effects:

> The fact that my government in what I thought was a civilised, rational country, is raining [the 'vaccine'] on people in their 30s and 40s, even my children in their 20s, they're getting letters and phone calls, I know this is not right and any of you doctors who are vaccinating you know it's not right, too. They are not at risk. They are not at risk from the disease, so you are now hoping that the side-effects are so rare that you get away with it. You don't give new technology … that you don't understand to 100 percent of the population.

Blood clot problems with the AstraZeneca 'vaccine' have been affecting younger people to emphasise the downside risks with no benefit. AstraZeneca's version, produced with Oxford University, does not use mRNA, but still gets its toxic cocktail inside cells where

it targets DNA. The Johnson & Johnson 'vaccine' which uses a similar technique has also produced blood clot effects to such an extent that the United States paused its use at one point. They are all 'gene therapy' (cell modification) procedures and not 'vaccines'. The truth is that once the content of these injections enter cells we have no idea what the effect will be. People can speculate and some can give very educated opinions and that's good. In the end, though, only the makers know what their potions are designed to do and even they won't know every last consequence. Michael Yeadon was scathing about doctors doing what they knew to be wrong. 'Everyone's mute', he said. Doctors in the NHS must know this was not right, coming into work and injecting people. 'I don't know how they sleep at night. I know I couldn't do it. I know that if I were in that position I'd have to quit.' He said he knew enough about toxicology to know this was not a good risk-benefit. Yeadon had spoken to seven or eight university professors and all except two would not speak out publicly. Their universities had a policy that no one said anything that countered the government and its medical advisors. They were afraid of losing their government grants. This is how intimidation has been used to silence the truth at every level of the system. I say silence, but these people could still speak out if they made that choice. Yeadon called them 'moral cowards' – 'This is about your children and grandchildren's lives and you have just buggered off and left it.'

'Variant' nonsense

Some of his most powerful comments related to the alleged 'variants' being used to instil more fear, justify more lockdowns, and introduce more 'vaccines'. He said government claims about 'variants' were nonsense. He had checked the alleged variant 'codes' and they were 99.7 percent identical to the 'original'. This was the human identity difference equivalent to putting a baseball cap on and off or wearing it the other way round. A 0.3 percent difference would make it impossible for that 'variant' to escape immunity from the 'original'. This made no sense of having new 'vaccines' for

'variants'. He said there would have to be at least a *30 percent* difference for that to be justified and even then he believed the immune system would still recognise what it was. Gates-funded 'variant modeller' and 'vaccine'-pusher John Edmunds might care to comment. Yeadon said drug companies were making new versions of the 'vaccine' as a 'top up' for 'variants'. Worse than that, he said, the 'regulators' around the world like the MHRA in the UK had got together and agreed that because 'vaccines' for 'variants' were so similar to the first 'vaccines' *they did not have to do safety studies*. How transparently sinister that is. This is when Yeadon said: 'There is a conspiracy here.' There was no need for another vaccine for 'variants' and yet we were told that there was and the country had shut its borders because of them. 'They are going into hundreds of millions of arms without passing 'go' or any regulator. Why did they do that? Why did they pick this method of making the vaccine?'

The reason had to be something bigger than that it seemed and 'it's not protection against the virus'. It's was a far bigger project that meant politicians and advisers were willing to do things and not do things that knowingly resulted in avoidable deaths – 'that's already happened when you think about lockdown and deprivation of health care for a year.' He spoke of people prepared to do something that results in the avoidable death of their fellow human beings and it not bother them. This is the penny-drop I have been working to get across for more than 30 years – the level of pure evil we are dealing with. Yeadon said his friends and associates could not believe there could be that much evil, but he reminded them of Stalin, Pol Pot and Hitler and of what Stalin had said: 'One death is a tragedy. A million? A statistic.' He could not think of a benign explanation for why you need top-up vaccines 'which I'm sure you don't' and for the regulators 'to just get out of the way and wave them through'. Why would the regulators do that when they were still wrestling with the dangers of the 'parent' vaccine? He was clearly shocked by what he had seen since the 'Covid' hoax began and now he was thinking the previously unthinkable:

If you wanted to depopulate a significant proportion of the world and to do it in a way that doesn't involve destruction of the environment with nuclear weapons, poisoning everyone with anthrax or something like that, and you wanted plausible deniability while you had a multi-year infectious disease crisis, I actually don't think you could come up with a better plan of work than seems to be in front of me. I can't say that's what they are going to do, but I can't think of a benign explanation why they are doing it.

He said he never thought that they would get rid of 99 percent of humans, but now he wondered. 'If you wanted to that this would be a hell of a way to do it – it would be unstoppable folks.' Yeadon had concluded that those who submitted to the 'vaccine' would be allowed to have some kind of normal life (but for how long?) while screws were tightened to coerce and mandate the last few percent. 'I think they'll put the rest of them in a prison camp. I wish I was wrong, but I don't think I am.' Other points he made included: There were no coronavirus vaccines then suddenly they all come along at the same time; we have no idea of the long term affect with trials so short; coercing or forcing people to have medical procedures is against the Nuremberg Code instigated when the Nazis did just that; people should at least delay having the 'vaccine'; a quick Internet search confirms that masks don't reduce respiratory viral transmission and 'the government knows that'; they have smashed civil society and they know that, too; two dozen peer-reviewed studies show no connection between lockdown and reducing deaths; he knew from personal friends the elite were still flying around and going on holiday while the public were locked down; the elite were not having the 'vaccines'. He was also asked if 'vaccines' could be made to target difference races. He said he didn't know, but the document by the Project for the New American Century in September, 2000, said developing 'advanced forms of biological warfare that can target *specific genotypes* may transform biological warfare from the realm of terror to a politically useful tool.' Oh, they're evil all right. Of that we can be *absolutely* sure.

Another cull of old people

We have seen from the CDC definition that the mRNA 'Covid vaccine' is not a vaccine and nor are the others that *claim* to reduce 'severity of symptoms' in *some* people, but not protect from infection or transmission. What about all the lies about returning to 'normal' if people were 'vaccinated'? If they are not claimed to stop infection and transmission of the alleged 'virus', how does anything change? This was all lies to manipulate people to take the jabs and we are seeing that now with masks and distancing still required for the 'vaccinated'. How did they think that elderly people with fragile health and immune responses were going to be affected by infusing their cells with synthetic material and other toxic substances? They *knew* that in the short and long term it would be devastating and fatal as the culling of the old that began with the first lockdowns was continued with the 'vaccine'. Death rates in care homes soared immediately residents began to be 'vaccinated' – infused with synthetic material. Brave and committed whistleblower nurses put their careers at risk by exposing this truth while the rest kept their heads down and their mouths shut to put their careers before those they are supposed to care for. A long-time American Certified Nursing Assistant who gave his name as James posted a video in which he described emotionally what happened in his care home when vaccination began. He said that during 2020 very few residents were sick with 'Covid' and no one died during the entire year; but shortly after the Pfizer mRNA injections 14 people died within two weeks and many others were near death. 'They're dropping like flies', he said. Residents who walked on their own before the shot could no longer and they had lost their ability to conduct an intelligent conversation. The home's management said the sudden deaths were caused by a 'super-spreader' of 'Covid-19'. Then how come, James asked, that residents who refused to take the injections were not sick? It was a case of inject the elderly with mRNA synthetic potions and blame their illness and death that followed on the 'virus'. James described what was happening in care homes as 'the greatest crime of genocide this country has ever seen'. Remember the NHS staff nurse from earlier who used the same

word ‘genocide’ for what was happening with the ‘vaccines’ and that it was an ‘act of human annihilation’. A UK care home whistleblower told a similar story to James about the effect of the ‘vaccine’ in deaths and ‘outbreaks’ of illness dubbed ‘Covid’ after getting the jab. She told how her care home management and staff had zealously imposed government regulations and no one was allowed to even question the official narrative let alone speak out against it. She said the NHS was even worse. Again we see the results of reframing. A worker at a local care home where I live said they had not had a single case of ‘Covid’ there for almost a year and when the residents were ‘vaccinated’ they had 19 positive cases in two weeks with eight dying.

It’s not the ‘vaccine’ – honest

The obvious cause and effect was being ignored by the media and most of the public. Australia’s health minister Greg Hunt (a former head of strategy at the World Economic Forum) was admitted to hospital after he had the ‘vaccine’. He was suffering according to reports from the skin infection ‘cellulitis’ and it must have been a severe case to have warranted days in hospital. Immediately the authorities said this was nothing to do with the ‘vaccine’ when an effect of some vaccines is a ‘cellulitis-like reaction’. We had families of perfectly healthy old people who died after the ‘vaccine’ saying that if only they had been given the ‘vaccine’ earlier they would still be alive. As a numbskull rating that is off the chart. A father of four ‘died of Covid’ at aged 48 when he was taken ill two days after having the ‘vaccine’. The man, a health administrator, had been ‘shielding during the pandemic’ and had ‘not really left the house’ until he went for the ‘vaccine’. Having the ‘vaccine’ and then falling ill and dying does not seem to have qualified as a possible cause and effect and ‘Covid-19’ went on his death certificate. His family said they had no idea how he ‘caught the virus’. A family member said: ‘Tragically, it could be that going for a vaccination ultimately led to him catching Covid …The sad truth is that they are never going to know where it came from.’ The family warned people to remember

that the virus still existed and was 'very real'. So was their stupidity. Nurses and doctors who had the first round of the 'vaccine' were collapsing, dying and ending up in a hospital bed while they or their grieving relatives were saying they'd still have the 'vaccine' again despite what happened. I kid you not. You mean if your husband returned from the dead he'd have the same 'vaccine' again that killed him??

Doctors at the VCU Medical Center in Richmond, Virginia, said the Johnson & Johnson 'vaccine' was to blame for a man's skin peeling off. Patient Richard Terrell said: 'It all just happened so fast. My skin peeled off. It's still coming off on my hands now.' He said it was stinging, burning and itching and when he bent his arms and legs it was very painful with 'the skin swollen and rubbing against itself'. Pfizer/BioNTech and Moderna vaccines use mRNA to change the cell while the Johnson & Johnson version uses DNA in a process similar to AstraZeneca's technique. Johnson & Johnson and AstraZeneca have both had their 'vaccines' paused by many countries after causing serious blood problems. Terrell's doctor Fnu Nutan said he could have died if he hadn't got medical attention. It sounds terrible so what did Nutan and Terrell say about the 'vaccine' now? Oh, they still recommend that people have it. A nurse in a hospital bed 40 minutes after the vaccination and unable to swallow due to throat swelling was told by a doctor that he lost mobility in his arm for 36 hours following the vaccination. What did he say to the ailing nurse? 'Good for you for getting the vaccination.' We are dealing with a serious form of cognitive dissonance madness in both public and medical staff. There is a remarkable correlation between those having the 'vaccine' and trumpeting the fact and suffering bad happenings shortly afterwards. Witold Rogiewicz, a Polish doctor, made a video of his 'vaccination' and ridiculed those who were questioning its safety and the intentions of Bill Gates: 'Vaccinate yourself to protect yourself, your loved ones, friends and also patients. And to mention quickly I have info for anti-vaxxers and anti-Coviders if you want to contact Bill Gates you can do this through me.' He further ridiculed the dangers of 5G. Days later he

was dead, but naturally the vaccination wasn't mentioned in the verdict of 'heart attack'.

Lies, lies and more lies

So many members of the human race have slipped into extreme states of insanity and unfortunately they include reframed doctors and nursing staff. Having a 'vaccine' and dying within minutes or hours is not considered a valid connection while death from any cause within 28 days or longer of a positive test with a test not testing for the 'virus' means 'Covid-19' goes on the death certificate. How could that 'vaccine'-death connection not have been made except by calculated deceit? US figures in the initial rollout period to February 12th, 2020, revealed that a third of the deaths reported to the CDC after 'Covid vaccines' happened within 48 hours. Five men in the UK suffered an 'extremely rare' blood clot problem after having the AstraZeneca 'vaccine', but no causal link was established said the Gates-funded Medicines and Healthcare products Regulatory Agency (MHRA) which had given the 'vaccine' emergency approval to be used. Former Pfizer executive Dr Michael Yeadon explained in his interview how the procedures could cause blood coagulation and clots. People who should have been at no risk were dying from blood clots in the brain and he said he had heard from medical doctor friends that people were suffering from skin bleeding and massive headaches. The AstraZeneca 'shot' was stopped by some 20 countries over the blood clotting issue and still the corrupt MHRA, the European Medicines Agency (EMA) and the World Health Organization said that it should continue to be given even though the EMA admitted that it 'still cannot rule out definitively' a link between blood clotting and the 'vaccine'. Later Marco Cavaleri, head of EMA vaccine strategy, said there was indeed a clear link between the 'vaccine' and thrombosis, but they didn't know why. So much for the trials showing the 'vaccine' is safe. Blood clots were affecting younger people who would be under virtually no danger from 'Covid' even if it existed which makes it all the more stupid and sinister.

The British government responded to public alarm by wheeling out June Raine, the terrifyingly weak infant school headmistress sound-alike who heads the UK MHRA drug 'regulator'. The idea that she would stand up to Big Pharma and government pressure is laughable and she told us that all was well in the same way that she did when allowing untested, never-used-on-humans-before, genetically-manipulating 'vaccines' to be exposed to the public in the first place. Mass lying is the new normal of the 'Covid' era. The MHRA later said 30 cases of rare blood clots had by then been connected with the AstraZeneca 'vaccine' (that means a lot more in reality) while stressing that the benefits of the jab in preventing 'Covid-19' outweighed any risks. A more ridiculous and disingenuous statement with callous disregard for human health it is hard to contemplate. Immediately after the mendacious 'all-clears' two hospital workers in Denmark experienced blood clots and cerebral haemorrhaging following the AstraZeneca jab and one died. Top Norwegian health official Pål Andre Holme said the 'vaccine' was the only common factor: 'There is nothing in the patient history of these individuals that can give such a powerful immune response … I am confident that the antibodies that we have found are the cause, and I see no other explanation than it being the vaccine which triggers it.' Strokes, a clot or bleed in the brain, were clearly associated with the 'vaccine' from word of mouth and whistleblower reports. Similar consequences followed with all these 'vaccines' that we were told were so safe and as the numbers grew by the day it was clear we were witnessing human carnage.

Learning the hard way

A woman interviewed by UKColumn told how her husband suffered dramatic health effects after the vaccine when he'd been in good health all his life. He went from being a little unwell to losing all feeling in his legs and experiencing 'excruciating pain'. Misdiagnosis followed twice at Accident and Emergency (an 'allergy' and 'sciatica') before he was admitted to a neurology ward where doctors said his serious condition had been caused by the

'vaccine'. Another seven 'vaccinated' people were apparently being treated on the same ward for similar symptoms. The woman said he had the 'vaccine' because they believed media claims that it was safe. 'I didn't think the government would give out a vaccine that does this to somebody; I believed they would be bringing out a vaccination that would be safe.' What a tragic way to learn that lesson. Another woman posted that her husband was transporting stroke patients to hospital on almost every shift and when he asked them if they had been 'vaccinated' for 'Covid' they all replied 'yes'. One had a 'massive brain bleed' the day after his second dose. She said her husband reported the 'just been vaccinated' information every time to doctors in A and E only for them to ignore it, make no notes and appear annoyed that it was even mentioned. This particular report cannot be verified, but it expresses a common theme that confirms the monumental underreporting of 'vaccine' consequences. Interestingly as the 'vaccines' and their brain blood clot/stroke consequences began to emerge the UK National Health Service began a publicity campaign telling the public what to do in the event of a stroke. A Scottish NHS staff nurse who quit in disgust in March, 2021, said:

I have seen traumatic injuries from the vaccine, they're not getting reported to the yellow card [adverse reaction] scheme, they're treating the symptoms, not asking why, why it's happening. It's just treating the symptoms and when you speak about it you're dismissed like you're crazy, I'm not crazy, I'm not crazy because every other colleague I've spoken to is terrified to speak out, they've had enough.

Videos appeared on the Internet of people uncontrollably shaking after the 'vaccine' with no control over muscles, limbs and even their face. A Scottish mother broke out in a severe rash all over her body almost immediately after she was given the AstraZeneca 'vaccine'. The pictures were horrific. Leigh King, a 41-year-old hairdresser from Lanarkshire said: 'Never in my life was I prepared for what I was about to experience … My skin was so sore and constantly hot … I have never felt pain like this …' But don't you worry, the 'vaccine' is perfectly safe. Then there has been the effect on medical

staff who have been pressured to have the 'vaccine' by psychopathic 'health' authorities and government. A London hospital consultant who gave the name K. Polyakova wrote this to the *British Medical Journal* or *BMJ*:

> I am currently struggling with … the failure to report the reality of the morbidity caused by our current vaccination program within the health service and staff population. The levels of sickness after vaccination is unprecedented and staff are getting very sick and some with neurological symptoms which is having a huge impact on the health service function. Even the young and healthy are off for days, some for weeks, and some requiring medical treatment. Whole teams are being taken out as they went to get vaccinated together.
>
> Mandatory vaccination in this instance is stupid, unethical and irresponsible when it comes to protecting our staff and public health. We are in the voluntary phase of vaccination, and encouraging staff to take an unlicensed product that is impacting on their immediate health … it is clearly stated that these vaccine products do not offer immunity or stop transmission. In which case why are we doing it?

Not to protect health that's for sure. Medical workers are lauded by governments for agenda reasons when they couldn't give a toss about them any more than they can for the population in general. Schools across America faced the same situation as they closed due to the high number of teachers and other staff with bad reactions to the Pfizer/BioNTech, Moderna, and Johnson & Johnson 'Covid vaccines' all of which were linked to death and serious adverse effects. The *BMJ* took down the consultant's comments pretty quickly on the grounds that they were being used to spread 'disinformation'. They were exposing the truth about the 'vaccine' was the real reason. The cover-up is breathtaking.

Hiding the evidence

The scale of the 'vaccine' death cover-up worldwide can be confirmed by comparing official figures with the personal experience of the public. I heard of many people in my community who died immediately or soon after the vaccine that would never appear in the media or even likely on the official totals of 'vaccine' fatalities and adverse reactions when only about ten percent are estimated to be

reported and I have seen some estimates as low as one percent in a Harvard study. In the UK alone by April 29th, 2021, some 757,654 adverse reactions had been officially reported from the Pfizer/BioNTech, Oxford/AstraZeneca and Moderna 'vaccines' with more than a thousand deaths linked to jabs and that means an estimated ten times this number in reality from a ten percent reporting rate percentage. That's seven million adverse reactions and 10,000 potential deaths and a one percent reporting rate would be ten times *those* figures. In 1976 the US government pulled the swine flu vaccine after *53* deaths. The UK data included a combined 10,000 eye disorders from the 'Covid vaccines' with more than 750 suffering visual impairment or blindness and again multiply by the estimated reporting percentages. As 'Covid cases' officially fell hospitals virtually empty during the 'Covid crisis' began to fill up with a range of other problems in the wake of the 'vaccine' rollout. The numbers across America have also been catastrophic. Deaths linked to *all* types of vaccine increased by *6,000 percent* in the first quarter of 2021 compared with 2020. A 39-year-old woman from Ogden, Utah, died four days after receiving a second dose of Moderna's 'Covid vaccine' when her liver, heart and kidneys all failed despite the fact that she had no known medical issues or conditions. Her family sought an autopsy, but Dr Erik Christensen, Utah's chief medical examiner, said proving vaccine injury as a cause of death almost never happened. He could think of only one instance where an autopsy would name a vaccine as the official cause of death and that would be anaphylaxis where someone received a vaccine and died almost instantaneously. 'Short of that, it would be difficult for us to definitively say this is the vaccine,' Christensen said. If that is true this must be added to the estimated ten percent (or far less) reporting rate of vaccine deaths and serious reactions and the conclusion can only be that vaccine deaths and serious reactions – including these 'Covid' potions' – are phenomenally understated in official figures. The same story can be found everywhere. Endless accounts of deaths and serious reactions among the public, medical

and care home staff while official figures did not even begin to reflect this.

Professional script-reader Dr David Williams, a 'top public-health official' in Ontario, Canada, insulted our intelligence by claiming only four serious adverse reactions and no deaths from the more than 380,000 vaccine doses then given. This bore no resemblance to what people knew had happened in their owns circles and we had Dirk Huyer in charge of getting millions vaccinated in Ontario while at the same time he was Chief Coroner for the province investigating causes of death including possible death from the vaccine. An aide said he had stepped back from investigating deaths, but evidence indicated otherwise. Rosemary Frei, who secured a Master of Science degree in molecular biology at the Faculty of Medicine at Canada's University of Calgary before turning to investigative journalism, was one who could see that official figures for 'vaccine' deaths and reactions made no sense. She said that doctors seldom reported adverse events and when people got really sick or died after getting a vaccination they would attribute that to anything except the vaccines. It had been that way for years and anyone who wondered aloud whether the 'Covid vaccines' or other shots cause harm is immediately branded as 'anti-vax' and 'anti-science'. This was 'career-threatening' for health professionals. Then there was the huge pressure to support the push to 'vaccinate' billions in the quickest time possible. Frei said:

> So that's where we're at today. More than half a million vaccine doses have been given to people in Ontario alone. The rush is on to vaccinate all 15 million of us in the province by September. And the mainstream media are screaming for this to be sped up even more. That all adds up to only a very slim likelihood that we're going to be told the truth by officials about how many people are getting sick or dying from the vaccines.

What is true of Ontario is true of everywhere.

They KNEW – and still did it

The authorities knew what was going to happen with multiple deaths and adverse reactions. The UK government's Gates-funded

and Big Pharma-dominated Medicines and Healthcare products Regulatory Agency (MHRA) hired a company to employ AI in compiling the projected reactions to the 'vaccine' that would otherwise be uncountable. The request for applications said: 'The MHRA urgently seeks an Artificial Intelligence (AI) software tool to process the expected high volume of Covid-19 vaccine Adverse Drug Reaction …' This was from the agency, headed by the disingenuous June Raine, that gave the 'vaccines' emergency approval and the company was hired before the first shot was given. 'We are going to kill and maim you – is that okay?' 'Oh, yes, perfectly fine – I'm very grateful, thank you, doctor.' The range of 'Covid vaccine' adverse reactions goes on for page after page in the MHRA criminally underreported 'Yellow Card' system and includes affects to eyes, ears, skin, digestion, blood and so on. Raine's MHRA amazingly claimed that the 'overall safety experience … is so far as expected from the clinical trials'. The death, serious adverse effects, deafness and blindness were *expected?* When did they ever mention that? If these human tragedies were expected then those that gave approval for the use of these 'vaccines' must be guilty of crimes against humanity including murder – a definition of which is 'killing a person with malice aforethought or with recklessness manifesting extreme indifference to the value of human life.' People involved at the MHRA, the CDC in America and their equivalent around the world must go before Nuremberg trials to answer for their callous inhumanity. We are only talking here about the immediate effects of the 'vaccine'. The longer-term impact of the DNA synthetic manipulation is the main reason they are so hysterically desperate to inoculate the entire global population in the shortest possible time.

Africa and the developing world are a major focus for the 'vaccine' depopulation agenda and a mass vaccination sales-pitch is underway thanks to caring people like the Rockefellers and other Cult assets. The Rockefeller Foundation, which pre-empted the 'Covid pandemic' in a document published in 2010 that 'predicted' what happened a decade later, announced an initial $34.95 million grant in February, 2021, 'to ensure more equitable access to Covid-19

testing and vaccines' among other things in Africa in collaboration with '24 organizations, businesses, and government agencies'. The pan-Africa initiative would focus on 10 countries: Burkina Faso, Ethiopia, Ghana, Kenya, Nigeria, Rwanda, South Africa, Tanzania, Uganda, and Zambia'. Rajiv Shah, President of the Rockefeller Foundation and former administrator of CIA-controlled USAID, said that if Africa was not mass-vaccinated (to change the DNA of its people) it was a 'threat to all of humanity' and not fair on Africans. When someone from the Rockefeller Foundation says they want to do something to help poor and deprived people and countries it is time for a belly-laugh. They are doing this out of the goodness of their 'heart' because 'vaccinating' the entire global population is what the 'Covid' hoax set out to achieve. Official 'decolonisation' of Africa by the Cult was merely a prelude to financial colonisation on the road to a return to physical colonisation. The 'vaccine' is vital to that and the sudden and convenient death of the 'Covid' sceptic president of Tanzania can be seen in its true light. A lot of people in Africa are aware that this is another form of colonisation and exploitation and they need to stand their ground.

The 'vaccine is working' scam

A potential problem for the Cult was that the 'vaccine' is meant to change human DNA and body messaging and not to protect anyone from a 'virus' never shown to exist. The vaccine couldn't work because it was not designed to work and how could they make it *appear* to be working so that more people would have it? This was overcome by lowering the amplification rate of the PCR test to produce fewer 'cases' and therefore fewer 'deaths'. Some of us had been pointing out since March, 2020, that the amplification rate of the test not testing for the 'virus' had been made artificially high to generate positive tests which they could call 'cases' to justify lockdowns. The World Health Organization recommended an absurdly high 45 amplification cycles to ensure the high positives required by the Cult and then remained silent on the issue until January 20th, 2021 – Biden's Inauguration Day. This was when the

'vaccinations' were seriously underway and on that day the WHO recommended after discussions with America's CDC that laboratories *lowered their testing amplification*. Dr David Samadi, a certified urologist and health writer, said the WHO was encouraging all labs to reduce their cycle count for PCR tests. He said the current cycle was much too high and was 'resulting in any particle being declared a positive case'. Even one mainstream news report I saw said this meant the number of 'Covid' infections may have been 'dramatically inflated'. Oh, just a little bit. The CDC in America issued new guidance to laboratories in April, 2021, to use 28 cycles *but only for 'vaccinated' people*. The timing of the CDC/WHO interventions were cynically designed to make it appear the 'vaccines' were responsible for falling cases and deaths when the real reason can be seen in the following examples. New York's state lab, the Wadsworth Center, identified 872 positive tests in July, 2020, based on a threshold of 40 cycles. When the figure was lowered to 35 cycles *43 percent* of the 872 were no longer 'positives'. At 30 cycles the figure was 63 percent. A Massachusetts lab found that between *85 to 90 percent* of people who tested positive in July with a cycle threshold of 40 would be negative at 30 cycles, Ashish Jha, MD, director of the Harvard Global Health Institute, said: 'I'm really shocked that it could be that high … Boy, does it really change the way we need to be thinking about testing.' I'm shocked that I could see the obvious in the spring of 2020, with no medical background, and most medical professionals still haven't worked it out. No, that's not shocking – it's terrifying.

Three weeks after the WHO directive to lower PCR cycles the London *Daily Mail* ran this headline: 'Why ARE Covid cases plummeting? New infections have fallen 45% in the US and 30% globally in the past 3 weeks but experts say vaccine is NOT the main driver because only 8% of Americans and 13% of people worldwide have received their first dose.' They acknowledged that the drop could not be attributed to the 'vaccine', but soon this morphed throughout the media into the 'vaccine' has caused cases and deaths to fall when it was the PCR threshold. In December, 2020, there was

chaos at English Channel ports with truck drivers needing negative 'Covid' tests before they could board a ferry home for Christmas. The government wanted to remove the backlog as fast as possible and they brought in troops to do the 'testing'. Out of 1,600 drivers just *36* tested positive and the rest were given the all clear to cross the Channel. I guess the authorities thought that 36 was the least they could get away with without the unquestioning catching on. The amplification trick which most people believed in the absence of information in the mainstream applied more pressure on those refusing the 'vaccine' to succumb when it 'obviously worked'. The truth was the exact opposite with deaths in care homes soaring with the 'vaccine' and in Israel the term used was 'skyrocket'. A re-analysis of published data from the Israeli Health Ministry led by Dr Hervé Seligmann at the Medicine Emerging Infectious and Tropical Diseases at Aix-Marseille University found that Pfizer's 'Covid vaccine' killed 'about 40 times more [elderly] people than the disease itself would have killed' during a five-week vaccination period and *260 times* more younger people than would have died from the 'virus' even according to the manipulated 'virus' figures. Dr Seligmann and his co-study author, Haim Yativ, declared after reviewing the Israeli 'vaccine' death data: 'This is a new Holocaust.'

Then, in mid-April, 2021, after vast numbers of people worldwide had been 'vaccinated', the story changed with clear coordination. The UK government began to prepare the ground for more future lockdowns when Nuremberg-destined Boris Johnson told yet another whopper. He said that cases had fallen because of *lockdowns* not 'vaccines'. Lockdowns are irrelevant when *there is no 'virus'* and the test and fraudulent death certificates are deciding the number of 'cases' and 'deaths'. Study after study has shown that lockdowns don't work and instead kill and psychologically destroy people. Meanwhile in the United States Anthony Fauci and Rochelle Walensky, the ultra-Zionist head of the CDC, peddled the same line. More lockdown was the answer and not the 'vaccine', a line repeated on cue by the moron that is Canadian Prime Minister Justin Trudeau. Why all the hysteria to get everyone 'vaccinated' if lockdowns and

not 'vaccines' made the difference? None of it makes sense on the face of it. Oh, but it does. The Cult wants lockdowns *and* the 'vaccine' and if the 'vaccine' is allowed to be seen as the total answer lockdowns would no longer be justified when there are still livelihoods to destroy. 'Variants' and renewed upward manipulation of PCR amplification are planned to instigate never-ending lockdown *and* more 'vaccines'.

You *must* have it – we're desperate

Israel, where the Jewish and Arab population are ruled by the Sabbatian Cult, was the front-runner in imposing the DNA-manipulating 'vaccine' on its people to such an extent that Jewish refusers began to liken what was happening to the early years of Nazi Germany. This would seem to be a fantastic claim. Why would a government of Jewish people be acting like the Nazis did? If you realise that the Sabbatian Cult was behind the Nazis and that Sabbatians hate Jews the pieces start to fit and the question of why a 'Jewish' government would treat Jews with such callous disregard for their lives and freedom finds an answer. Those controlling the government of Israel *aren't Jewish* – they're Sabbatian. Israeli lawyer Tamir Turgal was one who made the Nazi comparison in comments to German lawyer Reiner Fuellmich who is leading a class action lawsuit against the psychopaths for crimes against humanity. Turgal described how the Israeli government was vaccinating children and pregnant women on the basis that there was no evidence that this was dangerous when they had no evidence that it *wasn't* dangerous either. They just had no evidence. This was medical experimentation and Turgal said this breached the Nuremberg Code about medical experimentation and procedures requiring informed consent and choice. Think about that. A Nuremberg Code developed because of Nazi experimentation on Jews and others in concentration camps by people like the evil-beyond-belief Josef Mengele is being breached by the *Israeli* government; but when you know that it's a *Sabbatian* government along with its intelligence and military agencies like Mossad, Shin Bet and the Israeli Defense Forces, and that Sabbatians

were the force behind the Nazis, the kaleidoscope comes into focus. What have we come to when Israeli Jews are suing their government for violating the Nuremberg Code by essentially making Israelis subject to a medical experiment using the controversial 'vaccines'? It's a shocker that this has to be done in the light of what happened in Nazi Germany. The Anshe Ha-Emet, or 'People of the Truth', made up of Israeli doctors, lawyers, campaigners and public, have launched a lawsuit with the International Criminal Court. It says:

> When the heads of the Ministry of Health as well as the prime minister presented the vaccine in Israel and began the vaccination of Israeli residents, the vaccinated were not advised, that, in practice, they are taking part in a medical experiment and that their consent is required for this under the Nuremberg Code.

The irony is unbelievable, but easily explained in one word: Sabbatians. The foundation of Israeli 'Covid' apartheid is the 'green pass' or 'green passport' which allows Jews and Arabs who have had the DNA-manipulating 'vaccine' to go about their lives – to work, fly, travel in general, go to shopping malls, bars, restaurants, hotels, concerts, gyms, swimming pools, theatres and sports venues, while non-'vaccinated' are banned from all those places and activities. Israelis have likened the 'green pass' to the yellow stars that Jews in Nazi Germany were forced to wear – the same as the yellow stickers that a branch of UK supermarket chain Morrisons told exempt mask-wears they had to display when shopping. How very sensitive. The Israeli system is blatant South African-style apartheid on the basis of compliance or non-compliance to fascism rather than colour of the skin. How appropriate that the Sabbatian Israeli government was so close to the pre-Mandela apartheid regime in Pretoria. The Sabbatian-instigated 'vaccine passport' in Israel is planned for everywhere. Sabbatians struck a deal with Pfizer that allowed them to lead the way in the percentage of a national population infused with synthetic material and the result was catastrophic. Israeli freedom activist Shai Dannon told me how chairs were appearing on beaches that said 'vaccinated only'. Health Minister Yuli Edelstein said that anyone unwilling or unable to get

the jabs that 'confer immunity' will be 'left behind'. The man's a liar. Not even the makers claim the 'vaccines' confer immunity. When you see those figures of 'vaccine' deaths these psychopaths were saying that you must take the chance the 'vaccine' will kill you or maim you while knowing it will change your DNA or lockdown for you will be permanent. That's fascism. The Israeli parliament passed a law to allow personal information of the non-vaccinated to be shared with local and national authorities for three months. This was claimed by its supporters to be a way to 'encourage' people to be vaccinated. Hadas Ziv from Physicians for Human Rights described this as a 'draconian law which crushed medical ethics and the patient rights'. But that's the idea, the Sabbatians would reply.

Your papers, please

Sabbatian Israel was leading what has been planned all along to be a global 'vaccine pass' called a 'green passport' without which you would remain in permanent lockdown restriction and unable to do anything. This is how badly – *desperately* – the Cult is to get everyone 'vaccinated'. The term and colour 'green' was not by chance and related to the psychology of fusing the perception of the green climate hoax with the 'Covid' hoax and how the 'solution' to both is the same Great Reset. Lying politicians, health officials and psychologists denied there were any plans for mandatory vaccinations or restrictions based on vaccinations, but they knew that was exactly what was meant to happen with governments of all countries reaching agreements to enforce a global system. 'Free' Denmark and 'free' Sweden unveiled digital vaccine certification. Cyprus, Czech Republic, Estonia, Greece, Hungary, Iceland, Italy, Poland, Portugal, Slovakia, and Spain have all committed to a vaccine passport system and the rest including the whole of the EU would follow. The satanic UK government will certainly go this way despite mendacious denials and at the time of writing it is trying to manipulate the public into having the 'vaccine' so they could go abroad on a summer holiday. How would that work without something to prove you had the synthetic toxicity injected into you?

Documents show that the EU's European Commission was moving towards 'vaccine certificates' in 2018 and 2019 before the 'Covid' hoax began. They knew what was coming. Abracadabra – Ursula von der Leyen, the German President of the Commission, announced in March, 2021, an EU 'Digital Green Certificate' – green again – to track the public's 'Covid status'. The passport sting is worldwide and the Far East followed the same pattern with South Korea ruling that only those with 'vaccination' passports – again the *green* pass – would be able to 'return to their daily lives'.

Bill Gates has been preparing for this 'passport' with other Cult operatives for years and beyond the paper version is a Gates-funded 'digital tattoo' to identify who has been vaccinated and who hasn't. The 'tattoo' is reported to include a substance which is externally readable to confirm who has been vaccinated. This is a bio-luminous light-generating enzyme (think fireflies) called … *Luciferase*. Yes, named after the Cult 'god' Lucifer the 'light bringer' of whom more to come. Gates said he funded the readable tattoo to ensure children in the developing world were vaccinated and no one was missed out. He cares so much about poor kids as we know. This was just the cover story to develop a vaccine tagging system for everyone on the planet. Gates has been funding the ID2020 'alliance' to do just that in league with other lovely people at Microsoft, GAVI, the Rockefeller Foundation, Accenture and IDEO.org. He said in interviews in March, 2020, before any 'vaccine' publicly existed, that the world must have a globalised digital certificate to track the 'virus' and who had been vaccinated. Gates knew from the start that the mRNA vaccines were coming and when they would come and that the plan was to tag the 'vaccinated' to marginalise the intelligent and stop them doing anything including travel. Evil just doesn't suffice. Gates was exposed for offering a $10 million bribe to the Nigerian House of Representatives to invoke compulsory 'Covid' vaccination of all Nigerians. Sara Cunial, a member of the Italian Parliament, called Gates a 'vaccine criminal'. She urged the Italian President to hand him over to the International Criminal Court for crimes against

humanity and condemned his plans to 'chip the human race' through ID2020.

You know it's a long-planned agenda when war criminal and Cult gofer Tony Blair is on the case. With the scale of arrogance only someone as dark as Blair can muster he said: 'Vaccination in the end is going to be your route to liberty.' Blair is a disgusting piece of work and he confirms that again. The media has given a lot of coverage to a bloke called Charlie Mullins, founder of London's biggest independent plumbing company, Pimlico Plumbers, who has said he won't employ anyone who has not been vaccinated or have them go to any home where people are not vaccinated. He said that if he had his way no one would be allowed to walk the streets if they have not been vaccinated. Gates was cheering at the time while I was alerting the white coats. The plan is that people will qualify for 'passports' for having the first two doses and then to keep it they will have to have all the follow ups and new ones for invented 'variants' until human genetics is transformed and many are dead who can't adjust to the changes. Hollywood celebrities – the usual propaganda stunt – are promoting something called the WELL Health-Safety Rating to verify that a building or space has 'taken the necessary steps to prioritize the health and safety of their staff, visitors and other stakeholders'. They included Lady Gaga, Jennifer Lopez, Michael B. Jordan, Robert DeNiro, Venus Williams, Wolfgang Puck, Deepak Chopra and 17th Surgeon General Richard Carmona. Yawn. WELL Health-Safety has big connections with China. Parent company Delos is headed by former Goldman Sachs partner Paul Scialla. This is another example – and we will see so many others – of using the excuse of 'health' to dictate the lives and activities of the population. I guess one confirmation of the 'safety' of buildings is that only 'vaccinated' people can go in, right?

Electronic concentration camps

I wrote decades ago about the plans to restrict travel and here we are for those who refuse to bow to tyranny. This can be achieved in one go with air travel if the aviation industry makes a blanket decree.

The ‘vaccine’ and guaranteed income are designed to be part of a global version of China’s social credit system which tracks behaviour 24/7 and awards or deletes ‘credits’ based on whether your behaviour is supported by the state or not. I mean your entire lifestyle – what you do, eat, say, everything. Once your credit score falls below a certain level consequences kick in. In China tens of millions have been denied travel by air and train because of this. All the locations and activities denied to refusers by the ‘vaccine’ passports will be included in one big mass ban on doing almost anything for those that don’t bow their head to government. It’s beyond fascist and a new term is required to describe its extremes – I guess fascist technocracy will have to do. The way the Chinese system of technological – technocratic – control is sweeping the West can be seen in the Los Angeles school system and is planned to be expanded worldwide. Every child is required to have a ‘Covid’-tracking app scanned daily before they can enter the classroom. The so-called Daily Pass tracking system is produced by Gates’ Microsoft which I’m sure will shock you rigid. The pass will be scanned using a barcode (one step from an inside-the-body barcode) and the information will include health checks, ‘Covid’ tests and vaccinations. Entry codes are for one specific building only and access will only be allowed if a student or teacher has a negative test with a test not testing for the ‘virus’, has no symptoms of anything alleged to be related to ‘Covid’ (symptoms from a range of other illness), and has a temperature under 100 degrees. No barcode, no entry, is planned to be the case for everywhere and not only schools.

Kids are being psychologically prepared to accept this as ‘normal’ their whole life which is why what they can impose in schools is so important to the Cult and its gofers. Long-time American freedom campaigner John Whitehead of the Rutherford Institute was not exaggerating when he said: ‘Databit by databit, we are building our own electronic concentration camps.’ Canada under its Cult gofer prime minister Justin Trudeau has taken a major step towards the real thing with people interned against their will if they test positive with a test not testing for the ‘virus’ when they arrive at a Canadian

airport. They are jailed in internment hotels often without food or water for long periods and with many doors failing to lock there have been sexual assaults. The interned are being charged sometimes $2,000 for the privilege of being abused in this way. Trudeau is fully on board with the Cult and says the 'Covid pandemic' has provided an opportunity for a global 'reset' to permanently change Western civilisation. His number two, Deputy Prime Minister Chrystia Freeland, is a trustee of the World Economic Forum and a Rhodes Scholar. The Trudeau family have long been servants of the Cult. See *The Biggest Secret* and Cathy O'Brien's book *Trance-Formation of America* for the horrific background to Trudeau's father Pierre Trudeau another Canadian prime minister. Hide your fascism behind the façade of a heart-on-the-sleeve liberal. It's a well-honed Cult technique.

What can the 'vaccine' *really* do?

We have a 'virus' never shown to exist and 'variants' of the 'virus' that have also never been shown to exist except, like the 'original', as computer-generated fictions. Even if you believe there's a 'virus' the 'case' to 'death' rate is in the region of 0.23 to 0.15 percent and those 'deaths' are concentrated among the very old around the same average age that people die anyway. In response to this lack of threat (in truth none) psychopaths and idiots, knowingly and unknowingly answering to Gates and the Cult, are seeking to 'vaccinate' every man, woman and child on Planet Earth. Clearly the 'vaccine' is not about 'Covid' – none of this ever has been. So what is it all about *really*? Why the desperation to infuse genetically-manipulating synthetic material into everyone through mRNA fraudulent 'vaccines' with the intent of doing this over and over with the excuses of 'variants' and other 'virus' inventions? Dr Sherri Tenpenny, an osteopathic medical doctor in the United States, has made herself an expert on vaccines and their effects as a vehement campaigner against their use. Tenpenny was board certified in emergency medicine, the director of a level two trauma centre for 12 years, and moved to Cleveland in 1996 to start an integrative

medicine practice which has treated patients from all 50 states and some 17 other countries. Weaning people off pharmaceutical drugs is a speciality.

She became interested in the consequences of vaccines after attending a meeting at the National Vaccine Information Center in Washington DC in 2000 where she 'sat through four days of listening to medical doctors and scientists and lawyers and parents of vaccine injured kids' and asked: 'What's going on?' She had never been vaccinated and never got ill while her father was given a list of vaccines to be in the military and was 'sick his entire life'. The experience added to her questions and she began to examine vaccine documents from the Centers for Disease Control (CDC). After reading the first one, the 1998 version of *The General Recommendations of Vaccination*, she thought: 'This is it?' The document was poorly written and bad science and Tenpenny began 20 years of research into vaccines that continues to this day. She began her research into 'Covid vaccines' in March, 2020, and she describes them as 'deadly'. For many, as we have seen, they already have been. Tenpenny said that in the first 30 days of the 'vaccine' rollout in the United States there had been more than 40,000 adverse events reported to the vaccine adverse event database. A document had been delivered to her the day before that was 172 pages long. 'We have over 40,000 adverse events; we have over 3,100 cases of [potentially deadly] anaphylactic shock; we have over 5,000 neurological reactions.' Effects ranged from headaches to numbness, dizziness and vertigo, to losing feeling in hands or feet and paraesthesia which is when limbs 'fall asleep' and people have the sensation of insects crawling underneath their skin. All this happened in the first 30 days and remember that only about *ten percent* (or far less) of adverse reactions and vaccine-related deaths are estimated to be officially reported. Tenpenny said:

> So can you think of one single product in any industry, any industry, for as long as products have been made on the planet that within 30 days we have 40,000 people complaining of side effects that not only is still on the market but … we've got paid actors telling us how great

they are for getting their vaccine. We're offering people $500 if they will just get their vaccine and we've got nurses and doctors going; 'I got the vaccine, I got the vaccine'.

Tenpenny said they were not going to be 'happy dancing folks' when they began to suffer Bell's palsy (facial paralysis), neuropathies, cardiac arrhythmias and autoimmune reactions that kill through a blood disorder. 'They're not going to be so happy, happy then, but we're never going to see pictures of those people' she said. Tenpenny described the 'vaccine' as 'a well-designed killing tool'.

No off-switch

Bad as the initial consequences had been Tenpenny said it would be maybe 14 months before we began to see the 'full ravage' of what is going to happen to the 'Covid vaccinated' with full-out consequences taking anything between two years and 20 years to show. You can understand why when you consider that variations of the 'Covid vaccine' use mRNA (messenger RNA) to in theory activate the immune system to produce protective antibodies without using the actual 'virus'. How can they when it's a computer program and they've never isolated what they claim is the 'real thing'? Instead they use *synthetic* mRNA. They are inoculating synthetic material into the body which through a technique known as the Trojan horse is absorbed into cells to change the nature of DNA. Human DNA is changed by an infusion of messenger RNA and with each new 'vaccine' of this type it is changed even more. Say so and you are banned by Cult Internet platforms. The contempt the contemptuous Mark Zuckerberg has for the truth and human health can be seen in an internal Facebook video leaked to the Project Veritas investigative team in which he said of the 'Covid vaccines': '… I share some caution on this because we just don't know the long term side-effects of basically modifying people's DNA and RNA.' At the same time this disgusting man's Facebook was censoring and banning anyone saying exactly the same. He must go before a Nuremberg trial for crimes against humanity when he *knows* that he

is censoring legitimate concerns and denying the right of informed consent on behalf of the Cult that owns him. People have been killed and damaged by the very 'vaccination' technique he cast doubt on himself when they may not have had the 'vaccine' with access to information that he denied them. The plan is to have at least annual 'Covid vaccinations', add others to deal with invented 'variants', and change all other vaccines into the mRNA system. Pfizer executives told shareholders at a virtual Barclays Global Healthcare Conference in March, 2021, that the public may need a third dose of 'Covid vaccine', plus regular yearly boosters and the company planned to hike prices to milk the profits in a 'significant opportunity for our vaccine'. These are the professional liars, cheats and opportunists who are telling you their 'vaccine' is safe. Given this volume of mRNA planned to be infused into the human body and its ability to then replicate we will have a transformation of human genetics from biological to synthetic biological – exactly the long-time Cult plan for reasons we'll see – and many will die. Sherri Tenpenny said of this replication:

> It's like having an on-button but no off-button and that whole mechanism … they actually give it a name and they call it the Trojan horse mechanism, because it allows that [synthetic] virus and that piece of that [synthetic] virus to get inside of your cells, start to replicate and even get inserted into other parts of your DNA as a Trojan-horse.

Ask the overwhelming majority of people who have the 'vaccine' what they know about the contents and what they do and they would reply: 'The government says it will stop me getting the virus.' Governments give that false impression on purpose to increase take-up. You can read Sherri Tenpenny's detailed analysis of the health consequences in her blog at Vaxxter.com, but in summary these are some of them. She highlights the statement by Bill Gates about how human beings can become their own 'vaccine manufacturing machine'. The man is insane. ['Vaccine'-generated] 'antibodies' carry synthetic messenger RNA into the cells and the damage starts, Tenpenny contends, and she says that lungs can be adversely affected through varying degrees of pus and bleeding which

obviously affects breathing and would be dubbed 'Covid-19'. Even more sinister was the impact of 'antibodies' on macrophages, a white blood cell of the immune system. They consist of Type 1 and Type 2 which have very different functions. She said Type 1 are 'hyper-vigilant' white blood cells which 'gobble up' bacteria etc. However, in doing so, this could cause inflammation and in extreme circumstances be fatal. She says these affects are mitigated by Type 2 macrophages which kick in to calm down the system and stop it going rogue. They clear up dead tissue debris and reduce inflammation that the Type 1 'fire crews' have caused. Type 1 kills the infection and Type 2 heals the damage, she says. This is her punchline with regard to 'Covid vaccinations': She says that mRNA 'antibodies' block Type 2 macrophages by attaching to them and deactivating them. This meant that when the Type 1 response was triggered by infection there was nothing to stop that getting out of hand by calming everything down. There's an on-switch, but no off-switch, she says. What follows can be 'over and out, see you when I see you'.

Genetic suicide

Tenpenny also highlights the potential for autoimmune disease – the body attacking itself – which has been associated with vaccines since they first appeared. Infusing a synthetic foreign substance into cells could cause the immune system to react in a panic believing that the body is being overwhelmed by an invader (it is) and the consequences can again be fatal. There is an autoimmune response known as a 'cytokine storm' which I have likened to a homeowner panicked by an intruder and picking up a gun to shoot randomly in all directions before turning the fire on himself. The immune system unleashes a storm of inflammatory response called cytokines to a threat and the body commits hara-kiri. The lesson is that you mess with the body's immune response at your peril and these 'vaccines' seriously – fundamentally – mess with immune response. Tenpenny refers to a consequence called anaphylactic shock which is a severe and highly dangerous allergic reaction when the immune system

floods the body with chemicals. She gives the example of having a bee sting which primes the immune system and makes it sensitive to those chemicals. When people are stung again maybe years later the immune response can be so powerful that it leads to anaphylactic shock. Tenpenny relates this 'shock' with regard to the 'Covid vaccine' to something called polyethylene glycol or PEG. Enormous numbers of people have become sensitive to this over decades of use in a whole range of products and processes including food, drink, skin creams and 'medicine'. Studies have claimed that some 72 percent of people have antibodies triggered by PEG compared with two percent in the 1960s and allergic hypersensitive reactions to this become a gathering cause for concern. Tenpenny points out that the 'mRNA vaccine' is coated in a 'bubble' of polyethylene glycol which has the potential to cause anaphylactic shock through immune sensitivity. Many reports have appeared of people reacting this way after having the 'Covid vaccine'. What do we think is going to happen as humanity has more and more of these 'vaccines'? Tenpenny said: 'All these pictures we have seen with people with these rashes … these weepy rashes, big reactions on their arms and things like that – it's an acute allergic reaction most likely to the polyethylene glycol that you've been previously primed and sensitised to.'

Those who have not studied the conspiracy and its perpetrators at length might think that making the population sensitive to PEG and then putting it in these 'vaccines' is just a coincidence. It is not. It is instead testament to how carefully and coldly-planned current events have been and the scale of the conspiracy we are dealing with. Tenpenny further explains that the 'vaccine' mRNA procedure can breach the blood-brain barrier which protects the brain from toxins and other crap that will cause malfunction. In this case they could make two proteins corrupt brain function to cause Amyotrophic lateral sclerosis (ALS) , a progressive nervous system disease leading to loss of muscle control, and frontal lobe degeneration – Alzheimer's and dementia. Immunologist J. Bart Classon published a paper connecting mRNA 'vaccines' to prion

disease which can lead to Alzheimer's and other forms of neurogenerative disease while others have pointed out the potential to affect the placenta in ways that make women infertile. This will become highly significant in the next chapter when I will discuss other aspects of this non-vaccine that relate to its nanotechnology and transmission from the injected to the uninjected.

Qualified in idiocy

Tenpenny describes how research has confirmed that these 'vaccine'-generated antibodies can interact with a range of other tissues in the body and attack many other organs including the lungs. 'This means that if you have a hundred people standing in front of you that all got this shot they could have a hundred different symptoms.' Anyone really think that Cult gofers like the Queen, Tony Blair, Christopher Whitty, Anthony Fauci, and all the other psychopaths have really had this 'vaccine' in the pictures we've seen? Not a bloody chance. Why don't doctors all tell us about all these dangers and consequences of the 'Covid vaccine'? Why instead do they encourage and pressure patients to have the shot? Don't let's think for a moment that doctors and medical staff can't be stupid, lazy, and psychopathic and that's without the financial incentives to give the jab. Tenpenny again:

> Some people are going to die from the vaccine directly but a large number of people are going to start to get horribly sick and get all kinds of autoimmune diseases 42 days to maybe a year out. What are they going to do, these stupid doctors who say; 'Good for you for getting that vaccine.' What are they going to say; 'Oh, it must be a mutant, we need to give an extra dose of that vaccine.'
>
> Because now the vaccine, instead of one dose or two doses we need three or four because the stupid physicians aren't taking the time to learn anything about it. If I can learn this sitting in my living room reading a 19 page paper and several others so can they. There's nothing special about me, I just take the time to do it.

Remember how Sara Kayat, the NHS and TV doctor, said that the 'Covid vaccine' would '100 percent prevent hospitalisation and death'. Doctors can be idiots like every other profession and they

should not be worshipped as infallible. They are not and far from it. Behind many medical and scientific 'experts' lies an uninformed prat trying to hide themselves from you although in the 'Covid' era many have failed to do so as with UK narrative-repeating 'TV doctor' Hilary Jones. Pushing back against the minority of proper doctors and scientists speaking out against the 'vaccine' has been the entire edifice of the Cult global state in the form of governments, medical systems, corporations, mainstream media, Silicon Valley, and an army of compliant doctors, medical staff and scientists willing to say anything for money and to enhance their careers by promoting the party line. If you do that you are an 'expert' and if you won't you are an 'anti-vaxxer' and 'Covidiot'. The pressure to be 'vaccinated' is incessant. We have even had reports claiming that the 'vaccine' can help cure cancer and Alzheimer's and make the lame walk. I am waiting for the announcement that it can bring you coffee in the morning and cook your tea. Just as the symptoms of 'Covid' seem to increase by the week so have the miracles of the 'vaccine'. American supermarket giant Kroger Co. offered nearly 500,000 employees in 35 states a $100 bonus for having the 'vaccine' while donut chain Krispy Kreme promised 'vaccinated' customers a free glazed donut every day for the rest of 2021. Have your DNA changed and you will get a doughnut although we might not have to give you them for long. Such offers and incentives confirm the desperation.

Perhaps the worse vaccine-stunt of them all was UK 'Health' Secretary Matt-the-prat Hancock on live TV after watching a clip of someone being 'vaccinated' when the roll-out began. Hancock faked tears so badly it was embarrassing. Brain-of-Britain Piers Morgan, the lockdown-supporting, 'vaccine' supporting, 'vaccine' passport-supporting, TV host played along with Hancock – 'You're quite emotional about that' he said in response to acting so atrocious it would have been called out at a school nativity which will presumably today include Mary and Jesus in masks, wise men keeping their camels six feet apart, and shepherds under tent arrest. System-serving Morgan tweeted this: 'Love the idea of covid vaccine passports for everywhere: flights, restaurants, clubs, football, gyms,

shops etc. It's time covid-denying, anti-vaxxer loonies had their bullsh*t bluff called & bar themselves from going anywhere that responsible citizens go.' If only I could aspire to his genius. To think that Morgan, who specialises in shouting over anyone he disagrees with, was lauded as a free speech hero when he lost his job after storming off the set of his live show like a child throwing his dolly out of the pram. If he is a free speech hero we are in real trouble. I have no idea what 'bullsh*t' means, by the way, the * throws me completely.

The Cult is desperate to infuse its synthetic DNA-changing concoction into everyone and has been using every lie, trick and intimidation to do so. The question of '*Why*?' we shall now address.

CHAPTER TEN

Human 2.0

I believe that at the end of the century the use of words and general educated opinion will have altered so much that one will be able to speak of machines thinking without expecting to be contradicted –
Alan Turing (1912-1954), the 'Father of artificial intelligence'

I have been exposing for decades the plan to transform the human body from a biological to a synthetic-biological state. The new human that I will call Human 2.0 is planned to be connected to artificial intelligence and a global AI 'Smart Grid' that would operate as one global system in which AI would control everything from your fridge to your heating system to your car to your mind. Humans would no longer be 'human', but post-human and sub-human, with their thinking and emotional processes replaced by AI.

What I said sounded crazy and beyond science fiction and I could understand that. To any balanced, rational, mind it *is* crazy. Today, however, that world is becoming reality and it puts the 'Covid vaccine' into its true context. Ray Kurzweil is the ultra-Zionist 'computer scientist, inventor and futurist' and co-founder of the Singularity University. Singularity refers to the merging of humans with machines or 'transhumanism'. Kurzweil has said humanity would be connected to the cyber 'cloud' in the period of the ever-recurring year of 2030:

Our thinking … will be a hybrid of biological and non-biological thinking … humans will be able to extend their limitations and 'think in the cloud' … We're going to put gateways to the

cloud in our brains ... We're going to gradually merge and enhance ourselves ... In my view, that's the nature of being human – we transcend our limitations. As the technology becomes vastly superior to what we are then the small proportion that is still human gets smaller and smaller and smaller until it's just utterly negligible.

They are trying to sell this end-of-humanity-as-we-know-it as the next stage of 'evolution' when we become super-human and 'like the gods'. They are lying to you. Shocked, eh? The population, and again especially the young, have been manipulated into addiction to technologies designed to enslave them for life. First they induced an addiction to smartphones (holdables); next they moved to technology on the body (wearables); and then began the invasion of the body (implantables). I warned way back about the plan for microchipped people and we are now entering that era. We should not be diverted into thinking that this refers only to chips we can see. Most important are the nanochips known as smart dust, neural dust and nanobots which are far too small to be seen by the human eye. Nanotechnology is everywhere, increasingly in food products, and released into the atmosphere by the geoengineering of the skies funded by Bill Gates to 'shut out the Sun' and 'save the planet from global warming'. Gates has been funding a project to spray millions of tonnes of chalk (calcium carbonate) into the stratosphere over Sweden to 'dim the Sun' and cool the Earth. Scientists warned the move could be disastrous for weather systems in ways no one can predict and opposition led to the Swedish space agency announcing that the 'experiment' would not be happening as planned in the summer of 2021; but it shows where the Cult is going with dimming the impact of the Sun and there's an associated plan to change the planet's atmosphere. Who gives psychopath Gates the right to dictate to the entire human race and dismantle planetary systems? The world will not be safe while this man is at large.

The global warming hoax has made the Sun, like the gas of life, something to fear when both are essential to good health and human survival (more inversion). The body transforms sunlight into vital vitamin D through a process involving … *cholesterol*. This is the cholesterol we are also told to fear. We are urged to take Big Pharma

statin drugs to reduce cholesterol and it's all systematic. Reducing cholesterol means reducing vitamin D uptake with all the multiple health problems that will cause. At least if you take statins long term it saves the government from having to pay you a pension. The delivery system to block sunlight is widely referred to as chemtrails although these have a much deeper agenda, too. They appear at first to be contrails or condensation trails streaming from aircraft into cold air at high altitudes. Contrails disperse very quickly while chemtrails do not and spread out across the sky before eventually their content falls to earth. Many times I have watched aircraft cross-cross a clear blue sky releasing chemtrails until it looks like a cloudy day. Chemtrails contain many things harmful to humans and the natural world including toxic heavy metals, aluminium (see Alzheimer's) and nanotechnology. Ray Kurzweil reveals the reason without actually saying so: 'Nanobots will infuse all the matter around us with information. Rocks, trees, everything will become these intelligent creatures.' How do you deliver that? *From the sky*. Self-replicating nanobots would connect everything to the Smart Grid. The phenomenon of Morgellons disease began in the chemtrail era and the correlation has led to it being dubbed the 'chemtrail disease'. Self-replicating fibres appear in the body that can be pulled out through the skin. Morgellons fibres continue to grow outside the body and have a form of artificial intelligence. I cover this at greater length in *Phantom Self*.

'Vaccine' operating system

'Covid vaccines' with their self-replicating synthetic material are also designed to make the connection between humanity and Kurzweil's 'cloud'. American doctor and dedicated campaigner for truth, Carrie Madej, an Internal Medicine Specialist in Georgia with more than 20 years medical experience, has highlighted the nanotechnology aspect of the fake 'vaccines'. She explains how one of the components in at least the Moderna and Pfizer synthetic potions are 'lipid nanoparticles' which are 'like little tiny computer bits' – a 'sci-fi substance' known as nanobots and hydrogel which can be 'triggered

at any moment to deliver its payload' and act as 'biosensors'. The synthetic substance had 'the ability to accumulate data from your body like your breathing, your respiration, thoughts and emotions, all kind of things' and each syringe could carry a *million* nanobots:

This substance because it's like little bits of computers in your body, crazy, but it's true, it can do that, [and] obviously has the ability to act through Wi-Fi. It can receive and transmit energy, messages, frequencies or impulses. That issue has never been addressed by these companies. What does that do to the human?

Just imagine getting this substance in you and it can react to things all around you, the 5G, your smart device, your phones, what is happening with that? What if something is triggering it, too, like an impulse, a frequency? We have something completely foreign in the human body.

Madej said her research revealed that electromagnetic (EMF) frequencies emitted by phones and other devices had increased dramatically in the same period of the 'vaccine' rollout and she was seeing more people with radiation problems as 5G and other electromagnetic technology was expanded and introduced to schools and hospitals. She said she was 'floored with the EMF coming off' the devices she checked. All this makes total sense and syncs with my own work of decades when you think that Moderna refers in documents to its mRNA 'vaccine' as an 'operating system':

Recognizing the broad potential of mRNA science, we set out to create an mRNA technology platform that functions very much like an operating system on a computer. It is designed so that it can plug and play interchangeably with different programs. In our case, the 'program' or 'app' is our mRNA drug – the unique mRNA sequence that codes for a protein …

… Our MRNA Medicines – 'The 'Software Of Life': When we have a concept for a new mRNA medicine and begin research, fundamental components are already in place. Generally, the only thing that changes from one potential mRNA medicine to another is the coding region – the actual genetic code that instructs ribosomes to make protein. Utilizing these instruction sets gives our investigational mRNA medicines a software-like quality. We also have the ability to combine different mRNA sequences encoding for different proteins in a single mRNA investigational medicine.

Who needs a real 'virus' when you can create a computer version to justify infusing your operating system into the entire human race on the road to making living, breathing people into cyborgs? What is missed with the 'vaccines' is the *digital* connection between synthetic material and the body that I highlighted earlier with the study that hacked a computer with human DNA. On one level the body is digital, based on mathematical codes, and I'll have more about that in the next chapter. Those who ridiculously claim that mRNA 'vaccines' are not designed to change human genetics should explain the words of Dr Tal Zaks, chief medical officer at Moderna, in a 2017 TED talk. He said that over the last 30 years 'we've been living this phenomenal digital scientific revolution, and I'm here today to tell you, that we are actually *hacking the software of life*, and that it's changing the way we think about prevention and treatment of disease':

In every cell there's this thing called messenger RNA, or mRNA for short, that transmits the critical information from the DNA in our genes to the protein, which is really the stuff we're all made out of. This is the critical information that determines what the cell will do. So we think about it as an operating system. So if you could change that, if you could introduce a line of code, or change a line of code, it turns out, that has profound implications for everything, from the flu to cancer.

Zaks should more accurately have said that this has profound implications for the human genetic code and the nature of DNA. Communications within the body go both ways and not only one. But, hey, no, the 'Covid vaccine' will not affect your genetics. Cult fact-checkers say so even though the man who helped to develop the mRNA technique says that it does. Zaks said in 2017:

If you think about what it is we're trying to do. We've taken information and our understanding of that information and how that information is transmitted in a cell, and we've taken our understanding of medicine and how to make drugs, and we're fusing the two. We think of it as information therapy.

I have been writing for decades that the body is an information field communicating with itself and the wider world. This is why

radiation which is information can change the information field of body and mind through phenomena like 5G and change their nature and function. 'Information therapy' means to change the body's information field and change the way it operates. DNA is a receiver-transmitter of information and can be mutated by information like mRNA synthetic messaging. Technology to do this has been ready and waiting in the underground bases and other secret projects to be rolled out when the 'Covid' hoax was played. 'Trials' of such short and irrelevant duration were only for public consumption. When they say the 'vaccine' is 'experimental' that is not true. It may appear to be 'experimental' to those who don't know what's going on, but the trials have already been done to ensure the Cult gets the result it desires. Zaks said that it took decades to sequence the human genome, completed in 2003, but now they could do it in a week. By 'they' he means scientists operating in the public domain. In the secret projects they were sequencing the genome in a week long before even 2003.

Deluge of mRNA

Highly significantly the Moderna document says the guiding premise is that if using mRNA as a medicine works for one disease then it should work for many diseases. They were leveraging the flexibility afforded by their platform and the fundamental role mRNA plays in protein synthesis to pursue mRNA medicines for a broad spectrum of diseases. Moderna is confirming what I was saying through 2020 that multiple 'vaccines' were planned for 'Covid' (and later invented 'variants') and that previous vaccines would be converted to the mRNA system to infuse the body with massive amounts of genetically-manipulating synthetic material to secure a transformation to a synthetic-biological state. The 'vaccines' are designed to kill stunning numbers as part of the long-exposed Cult depopulation agenda and transform the rest. Given this is the goal you can appreciate why there is such hysterical demand for every human to be 'vaccinated' for an alleged 'disease' that has an estimated 'infection' to 'death' ratio of 0.23-0.15 percent. As I write

children are being given the 'vaccine' in trials (their parents are a disgrace) and ever-younger people are being offered the vaccine for a 'virus' that even if you believe it exists has virtually zero chance of harming them. Horrific effects of the 'trials' on a 12-year-old girl were revealed by a family member to be serious brain and gastric problems that included a bowel obstruction and the inability to swallow liquids or solids. She was unable to eat or drink without throwing up, had extreme pain in her back, neck and abdomen, and was paralysed from the waist down which stopped her urinating unaided. When the girl was first taken to hospital doctors said it was all in her mind. She was signed up for the 'trial' by her parents for whom no words suffice. None of this 'Covid vaccine' insanity makes any sense unless you see what the 'vaccine' really is – a body-changer. Synthetic biology or 'SynBio' is a fast-emerging and expanding scientific discipline which includes everything from genetic and molecular engineering to electrical and computer engineering. Synthetic biology is defined in these ways:

- A multidisciplinary area of research that seeks to create new biological parts, devices, and systems, or to redesign systems that are already found in nature.

- The use of a mixture of physical engineering and genetic engineering to create new (and therefore synthetic) life forms.

- An emerging field of research that aims to combine the knowledge and methods of biology, engineering and related disciplines in the design of chemically-synthesized DNA to create organisms with novel or enhanced characteristics and traits (synthetic organisms including humans).

We now have synthetic blood, skin, organs and limbs being developed along with synthetic body parts produced by 3D printers. These are all elements of the synthetic human programme and this comment by Kurzweil's co-founder of the Singularity University,

Peter Diamandis, can be seen in a whole new light with the 'Covid' hoax and the sanctions against those that refuse the 'vaccine':

Anybody who is going to be resisting the progress forward [to transhumanism] is going to be resisting evolution and, fundamentally, they will die out. It's not a matter of whether it's good or bad. It's going to happen.

'Resisting evolution'? What absolute bollocks. The arrogance of these people is without limit. His 'it's going to happen' mantra is another way of saying 'resistance is futile' to break the spirit of those pushing back and we must not fall for it. Getting this genetically-transforming 'vaccine' into everyone is crucial to the Cult plan for total control and the desperation to achieve that is clear for anyone to see. Vaccine passports are a major factor in this and they, too, are a form of resistance is futile. It's NOT. The paper funded by the Rockefeller Foundation for the 2013 'health conference' in China said:

We will interact more with artificial intelligence. The use of robotics, bio-engineering to augment human functioning is already well underway and will advance. Re-engineering of humans into potentially separate and unequal forms through genetic engineering or mixed human-robots raises debates on ethics and equality.

A new demography is projected to emerge after 2030 [that year again] of technologies (robotics, genetic engineering, nanotechnology) producing robots, engineered organisms, 'nanobots' and artificial intelligence (AI) that can self-replicate. Debates will grow on the implications of an impending reality of human designed life.

What is happening today is so long planned. The world army enforcing the will of the world government is intended to be a robot army, not a human one. Today's military and its technologically 'enhanced' troops, pilotless planes and driverless vehicles are just stepping stones to that end. Human soldiers are used as Cult fodder and its time they woke up to that and worked for the freedom of the population instead of their own destruction and their family's destruction – the same with the police. Join us and let's sort this out. The phenomenon of enforce my own destruction is widespread in the 'Covid' era with Woker 'luvvies' in the acting and entertainment

industries supporting 'Covid' rules which have destroyed their profession and the same with those among the public who put signs on the doors of their businesses 'closed due to Covid – stay safe' when many will never reopen. It's a form of masochism and most certainly insanity.

Transgender = transhumanism

When something explodes out of nowhere and is suddenly everywhere it is always the Cult agenda and so it is with the tidal wave of claims and demands that have infiltrated every aspect of society under the heading of 'transgenderism'. The term 'trans' is so 'in' and this is the dictionary definition:

A prefix meaning 'across', 'through', occurring … in loanwords from Latin, used in particular for denoting movement or conveyance from place to place (transfer; transmit; transplant) or complete change (transform; transmute), or to form adjectives meaning 'crossing', 'on the other side of', or 'going beyond' the place named (transmontane; transnational; trans-Siberian).

Transgender means to go beyond gender and transhuman means to go beyond human. Both are aspects of the Cult plan to transform the human body to a synthetic state with *no gender*. Human 2.0 is not designed to procreate and would be produced technologically with no need for parents. The new human would mean the end of parents and so men, and increasingly women, are being targeted for the deletion of their rights and status. Parental rights are disappearing at an ever-quickening speed for the same reason. The new human would have no need for men or women when there is no procreation and no gender. Perhaps the transgender movement that appears to be in a permanent state of frenzy might now contemplate on how it is being used. This was never about transgender rights which are only the interim excuse for confusing gender, particularly in the young, on the road to *fusing* gender. Transgender activism is not an end; it is a *means* to an end. We see again the technique of creative destruction in which you destroy the status quo to 'build back better' in the form that you want. The gender status quo had to be

destroyed by persuading the Cult-created Woke mentality to believe that you can have 100 genders or more. A programme for 9 to 12 year olds produced by the Cult-owned BBC promoted the 100 genders narrative. The very idea may be the most monumental nonsense, but it is not what is true that counts, only what you can make people *believe* is true. Once the gender of 2 + 2 = 4 has been dismantled through indoctrination, intimidation and 2 + 2 = 5 then the new no-gender normal can take its place with Human 2.0. Aldous Huxley revealed the plan in his prophetic *Brave New World* in 1932:

> Natural reproduction has been done away with and children are created, decanted', and raised in 'hatcheries and conditioning centres'. From birth, people are genetically designed to fit into one of five castes, which are further split into 'Plus' and 'Minus' members and designed to fulfil predetermined positions within the social and economic strata of the World State.

How could Huxley know this in 1932? For the same reason George Orwell knew about the Big Brother state in 1948, Cult insiders I have quoted knew about it in 1969, and I have known about it since the early 1990s. If you are connected to the Cult or you work your balls off to uncover the plan you can predict the future. The process is simple. If there is a plan for the world and nothing intervenes to stop it then it will happen. Thus if you communicate the plan ahead of time you are perceived to have predicted the future, but you haven't. You have revealed the plan which without intervention will become the human future. The whole reason I have done what I have is to alert enough people to inspire an intervention and maybe at last that time has come with the Cult and its intentions now so obvious to anyone with a brain in working order.

The future is here

Technological wombs that Huxley described to replace parent procreation are already being developed and they are only the projects we know about in the public arena. Israeli scientists told *The Times of Israel* in March, 2021, that they have grown 250-cell embryos

into mouse foetuses with fully formed organs using artificial wombs in a development they say could pave the way for gestating humans outside the womb. Professor Jacob Hanna of the Weizmann Institute of Science said:

We took mouse embryos from the mother at day five of development, when they are just of 250 cells, and had them in the incubator from day five until day 11, by which point they had grown all their organs.

By day 11 they make their own blood and have a beating heart, a fully developed brain. Anybody would look at them and say, 'this is clearly a mouse foetus with all the characteristics of a mouse.' It's gone from being a ball of cells to being an advanced foetus.

A special liquid is used to nourish embryo cells in a laboratory dish and they float on the liquid to duplicate the first stage of embryonic development. The incubator creates all the right conditions for its development, Hanna said. The liquid gives the embryo 'all the nutrients, hormones and sugars they need' along with a custom-made electronic incubator which controls gas concentration, pressure and temperature. The cutting-edge in the underground bases and other secret locations will be light years ahead of that, however, and this was reported by the London *Guardian* in 2017:

We are approaching a biotechnological breakthrough. Ectogenesis, the invention of a complete external womb, could completely change the nature of human reproduction. In April this year, researchers at the Children's Hospital of Philadelphia announced their development of an artificial womb.

The article was headed 'Artificial wombs could soon be a reality. What will this mean for women?' What would it mean for children is an even bigger question. No mother to bond with only a machine in preparation for a life of soulless interaction and control in a world governed by machines (see the *Matrix* movies). Now observe the calculated manipulations of the 'Covid' hoax as human interaction and warmth has been curtailed by distancing, isolation and fear with people communicating via machines on a scale never seen before.

These are all dots in the same picture as are all the personal assistants, gadgets and children's toys through which kids and adults communicate with AI as if it is human. The AI 'voice' on Sat-Nav should be included. All these things are psychological preparation for the Cult endgame. Before you can make a physical connection with AI you have to make a psychological connection and that is what people are being conditioned to do with this ever gathering human-AI interaction. Movies and TV programmes depicting the transhuman, robot dystopia relate to a phenomenon known as 'pre-emptive programming' in which the world that is planned is portrayed everywhere in movies, TV and advertising. This is conditioning the conscious and subconscious mind to become familiar with the planned reality to dilute resistance when it happens for real. What would have been a shock such is the change is made less so. We have young children put on the road to transgender transition surgery with puberty blocking drugs at an age when they could never be able to make those life-changing decisions.

Rachel Levine, a professor of paediatrics and psychiatry who believes in treating children this way, became America's highest-ranked openly-transgender official when she was confirmed as US Assistant Secretary at the Department of Health and Human Services after being nominated by Joe Biden (the Cult). Activists and governments press for laws to deny parents a say in their children's transition process so the kids can be isolated and manipulated into agreeing to irreversible medical procedures. A Canadian father Robert Hoogland was denied bail by the Vancouver Supreme Court in 2021 and remained in jail for breaching a court order that he stay silent over his young teenage daughter, a minor, who was being offered life-changing hormone therapy without parental consent. At the age of 12 the girl's 'school counsellor' said she may be transgender, referred her to a doctor and told the school to treat her like a boy. This is another example of state-serving schools imposing ever more control over children's lives while parents have ever less.

Contemptible and extreme child abuse is happening all over the world as the Cult gender-fusion operation goes into warp-speed.

Why the war on men – and now women?

The question about what artificial wombs mean for women should rightly be asked. The answer can be seen in the deletion of women's rights involving sport, changing rooms, toilets and status in favour of people in male bodies claiming to identify as women. I can identify as a mountain climber, but it doesn't mean I can climb a mountain any more than a biological man can be a biological woman. To believe so is a triumph of belief over factual reality which is the very perceptual basis of everything Woke. Women's sport is being destroyed by allowing those with male bodies who say they identify as female to 'compete' with girls and women. Male body 'women' dominate 'women's' competition with their greater muscle mass, bone density, strength and speed. With that disadvantage sport for women loses all meaning. To put this in perspective nearly 300 American high school boys can run faster than the quickest woman sprinter in the world. Women are seeing their previously protected spaces invaded by male bodies simply because they claim to identify as women. That's all they need to do to access all women's spaces and activities under the Biden 'Equality Act' that destroys equality for women with the usual Orwellian Woke inversion. Male sex offenders have already committed rapes in women's prisons after claiming to identify as women to get them transferred. Does this not matter to the Woke 'equality' hypocrites? Not in the least. What matters to Cult manipulators and funders behind transgender activists is to advance gender fusion on the way to the no-gender 'human'. When you are seeking to impose transparent nonsense like this, or the 'Covid' hoax, the only way the nonsense can prevail is through censorship and intimidation of dissenters, deletion of factual information, and programming of the unquestioning, bewildered and naive. You don't have to scan the world for long to see that all these things are happening.

Many women's rights organisations have realised that rights and status which took such a long time to secure are being eroded and that it is systematic. Kara Dansky of the global Women's Human Rights Campaign said that Biden's transgender executive order immediately he took office, subsequent orders, and Equality Act legislation that followed 'seek to erase women and girls in the law as a category'. *Exactly*. I said during the long ago-started war on men (in which many women play a crucial part) that this was going to turn into a war on them. The Cult is phasing out *both* male and female genders. To get away with that they are brought into conflict so they are busy fighting each other while the Cult completes the job with no unity of response. Unity, people, *unity*. We need unity everywhere. Transgender is the only show in town as the big step towards the no-gender human. It's not about rights for transgender people and never has been. Woke political correctness is deleting words relating to genders to the same end. Wokers believe this is to be 'inclusive' when the opposite is true. They are deleting words describing gender because gender *itself* is being deleted by Human 2.0. Terms like 'man', 'woman', 'mother' and 'father' are being deleted in the universities and other institutions to be replaced by the *no*-gender, not trans-gender, 'individuals' and 'guardians'. Women's rights campaigner Maria Keffler of Partners for Ethical Care said: 'Children are being taught from kindergarten upward that some boys have a vagina, some girls have a penis, and that kids can be any gender they want to be.' Do we really believe that suddenly countries all over the world at the same time had the idea of having drag queens go into schools or read transgender stories to very young children in the local library? It's coldly-calculated confusion of gender on the way to the fusion of gender. Suzanne Vierling, a psychologist from Southern California, made another important point:

Yesterday's slave woman who endured gynecological medical experiments is today's girl-child being butchered in a booming gender-transitioning sector. Ovaries removed, pushing her into menopause and osteoporosis, uncharted territory, and parents' rights and authority decimated.

The erosion of parental rights is a common theme in line with the Cult plans to erase the very concept of parents and 'ovaries removed, pushing her into menopause' means what? Those born female lose the ability to have children – another way to discontinue humanity as we know it.

Eliminating Human 1.0 (before our very eyes)

To pave the way for Human 2.0 you must phase out Human 1.0. This is happening through plummeting sperm counts and making women infertile through an onslaught of chemicals, radiation (including smartphones in pockets of men) and mRNA 'vaccines'. Common agriculture pesticides are also having a devastating impact on human fertility. I have been tracking collapsing sperm counts in the books for a long time and in 2021 came a book by fertility scientist and reproductive epidemiologist Shanna Swan, *Count Down: How Our Modern World Is Threatening Sperm Counts, Altering Male and Female Reproductive Development and Imperiling the Future of the Human Race*. She reports how the global fertility rate dropped by *half* between 1960 and 2016 with America's birth rate 16 percent below where it needs to be to sustain the population. Women are experiencing declining egg quality, more miscarriages, and more couples suffer from infertility. Other findings were an increase in erectile dysfunction, infant boys developing more genital abnormalities, male problems with conception, and plunging levels of the male hormone testosterone which would explain why so many men have lost their backbone and masculinity. This has been very evident during the 'Covid' hoax when women have been prominent among the Pushbackers and big strapping blokes have bowed their heads, covered their faces with a nappy and quietly submitted. Mind control expert Cathy O'Brien also points to how global education introduced the concept of 'we're all winners' in sport and classrooms: 'Competition was defused, and it in turn defused a sense of fighting back.' This is another version of the 'equity' doctrine in which you drive down rather than raise up. What a contrast in Cult-controlled China with its global ambitions

where the government published plans in January, 2021, to 'cultivate masculinity' in boys from kindergarten through to high school in the face of a 'masculinity crisis'. A government adviser said boys would be soon become 'delicate, timid and effeminate' unless action was taken. Don't expect any similar policy in the targeted West. A 2006 study showed that a 65-year-old man in 2002 had testosterone levels *15 percent* lower than a 65-year-old man in 1987 while a 2020 study found a similar story with young adults and adolescents. Men are getting prescriptions for testosterone replacement therapy which causes an even greater drop in sperm count with up to 99 percent seeing sperm counts drop to zero during the treatment. More sperm is defective and malfunctioning with some having two heads or not pursuing an egg.

A class of *synthetic* chemicals known as phthalates are being blamed for the decline. These are found everywhere in plastics, shampoos, cosmetics, furniture, flame retardants, personal care products, pesticides, canned foods and even receipts. Why till receipts? Everyone touches them. Let no one delude themselves that all this is not systematic to advance the long-time agenda for human body transformation. Phthalates mimic hormones and disrupt the hormone balance causing testosterone to fall and genital birth defects in male infants. Animals and fish have been affected in the same way due to phthalates and other toxins in rivers. When fish turn gay or change sex through chemicals in rivers and streams it is a pointer to why there has been such an increase in gay people and the sexually confused. It doesn't matter to me what sexuality people choose to be, but if it's being affected by chemical pollution and consumption then we need to know. Does anyone really think that this is not connected to the transgender agenda, the war on men and the condemnation of male 'toxic masculinity'? You watch this being followed by 'toxic femininity'. It's already happening. When breastfeeding becomes 'chest-feeding', pregnant women become pregnant people along with all the other Woke claptrap you know that the world is going insane and there's a Cult scam in progress. Transgender activists are promoting the Cult agenda while Cult

billionaires support and fund the insanity as they laugh themselves to sleep at the sheer stupidity for which humans must be infamous in galaxies far, far away.

'Covid vaccines' and female infertility

We can now see why the 'vaccine' has been connected to potential infertility in women. Dr Michael Yeadon, former Vice President and Chief Scientific Advisor at Pfizer, and Dr Wolfgang Wodarg in Germany, filed a petition with the European Medicines Agency in December, 2020, urging them to stop trials for the Pfizer/BioNTech shot and all other mRNA trials until further studies had been done. They were particularly concerned about possible effects on fertility with 'vaccine'-produced antibodies attacking the protein Syncytin-1 which is responsible for developing the placenta. The result would be infertility 'of indefinite duration' in women who have the 'vaccine' with the placenta failing to form. Section 10.4.2 of the Pfizer/BioNTech trial protocol says that pregnant women or those who might become so should not have mRNA shots. Section 10.4 warns men taking mRNA shots to 'be abstinent from heterosexual intercourse' and not to donate sperm. The UK government said that it *did not know* if the mRNA procedure had an effect on fertility. *Did not know?* These people have to go to jail. UK government advice did not recommend at the start that pregnant women had the shot and said they should avoid pregnancy for at least two months after 'vaccination'. The 'advice' was later updated to pregnant women should only have the 'vaccine' if the benefits outweighed the risks to mother and foetus. What the hell is that supposed to mean? Then 'spontaneous abortions' began to appear and rapidly increase on the adverse reaction reporting schemes which include only a fraction of adverse reactions. Thousands and ever-growing numbers of 'vaccinated' women are describing changes to their menstrual cycle with heavier blood flow, irregular periods and menstruating again after going through the menopause – all links to reproduction effects. Women are passing blood clots and the lining of their uterus while men report erectile dysfunction and blood effects. Most

significantly of all *un*vaccinated women began to report similar menstrual changes after interaction with *'vaccinated'* people and men and children were also affected with bleeding noses, blood clots and other conditions. 'Shedding' is when vaccinated people can emit the content of a vaccine to affect the unvaccinated, but this is different. 'Vaccinated' people were not shedding a 'live virus' allegedly in 'vaccines' as before because the fake 'Covid vaccines' involve synthetic material and other toxicity. Doctors exposing what is happening prefer the term 'transmission' to shedding. Somehow those that have had the shots are transmitting effects to those that haven't. Dr Carrie Madej said the nano-content of the 'vaccines' can 'act like an antenna' to others around them which fits perfectly with my own conclusions. This 'vaccine' transmission phenomenon was becoming known as the book went into production and I deal with this further in the Postscript.

Vaccine effects on sterility are well known. The World Health Organization was accused in 2014 of sterilising millions of women in Kenya with the evidence confirmed by the content of the vaccines involved. The same WHO behind the 'Covid' hoax admitted its involvement for more than ten years with the vaccine programme. Other countries made similar claims. Charges were lodged by Tanzania, Nicaragua, Mexico, and the Philippines. The Gardasil vaccine claimed to protect against a genital 'virus' known as HPV has also been linked to infertility. Big Pharma and the WHO (same thing) are criminal and satanic entities. Then there's the Bill Gates Foundation which is connected through funding and shared interests with 20 pharmaceutical giants and laboratories. He stands accused of directing the policy of United Nations Children's Fund (UNICEF), vaccine alliance GAVI, and other groupings, to advance the vaccine agenda and silence opposition at great cost to women and children. At the same time Gates wants to reduce the global population. Coincidence?

Great Reset = Smart Grid = new human

The Cult agenda I have been exposing for 30 years is now being openly promoted by Cult assets like Gates and Klaus Schwab of the World Economic Forum under code-terms like the 'Great Reset', 'Build Back Better' and 'a rare but narrow window of opportunity to reflect, reimagine, and reset our world'. What provided this 'rare but narrow window of opportunity'? The 'Covid' hoax did. Who created that? *They* did. My books from not that long ago warned about the planned 'Internet of Things' (IoT) and its implications for human freedom. This was the plan to connect all technology to the Internet and artificial intelligence and today we are way down that road with an estimated 36 billion devices connected to the World Wide Web and that figure is projected to be 76 billion by 2025. I further warned that the Cult planned to go beyond that to the Internet of *Everything* when the human brain was connected via AI to the Internet and Kurzweil's 'cloud'. Now we have Cult operatives like Schwab calling for precisely that under the term 'Internet of Bodies', a fusion of the physical, digital and biological into one centrally-controlled Smart Grid system which the Cult refers to as the 'Fourth Industrial Revolution'. They talk about the 'biological', but they really mean the synthetic-biological which is required to fully integrate the human body and brain into the Smart Grid and artificial intelligence planned to replace the human mind. We have everything being synthetically manipulated including the natural world through GMO and smart dust, the food we eat and the human body itself with synthetic 'vaccines'. I said in *The Answer* that we would see the Cult push for synthetic meat to replace animals and in February, 2021, the so predictable psychopath Bill Gates called for the introduction of synthetic meat to save us all from 'climate change'. The climate hoax just keeps on giving like the 'Covid' hoax. The war on meat by vegan activists is a carbon (oops, sorry) copy of the manipulation of transgender activists. They have no idea (except their inner core) that they are being used to promote and impose the agenda of the Cult or that they are only the *vehicle* and not the *reason*. This is not to say those who choose not to eat meat shouldn't be respected and supported in that right, but there are ulterior motives

for those in power. A *Forbes* article in December, 2019, highlighted the plan so beloved of Schwab and the Cult under the heading: 'What Is The Internet of Bodies? And How Is It Changing Our World?' The article said the human body is the latest data platform (remember 'our vaccine is an operating system'). *Forbes* described the plan very accurately and the words could have come straight out of my books from long before:

The Internet of Bodies (IoB) is an extension of the IoT and basically connects the human body to a network through devices that are ingested, implanted, or connected to the body in some way. Once connected, data can be exchanged, and the body and device can be remotely monitored and controlled.

They were really describing a human hive mind with human perception centrally-dictated via an AI connection as well as allowing people to be 'remotely monitored and controlled'. Everything from a fridge to a human mind could be directed from a central point by these insane psychopaths and 'Covid vaccines' are crucial to this. Forbes explained the process I mentioned earlier of holdable and wearable technology followed by implantable. The article said there were three generations of the Internet of Bodies that include:

- Body external: These are wearable devices such as Apple Watches or Fitbits that can monitor our health.

- Body internal: These include pacemakers, cochlear implants, and digital pills that go inside our bodies to monitor or control various aspects of health.

- Body embedded: The third generation of the Internet of Bodies is embedded technology where technology and the human body are melded together and have a real-time connection to a remote machine.

Forbes noted the development of the Brain Computer Interface (BCI) which merges the brain with an external device for monitoring and controlling in real-time. 'The ultimate goal is to help restore function to individuals with disabilities by using brain signals rather than conventional neuromuscular pathways.' Oh, do fuck off. The goal of brain interface technology is controlling human thought and emotion from the central point in a hive mind serving its masters wishes. Many people are now agreeing to be chipped to open doors without a key. You can recognise them because they'll be wearing a mask, social distancing and lining up for the 'vaccine'. The Cult plans a Great Reset money system after they have completed the demolition of the global economy in which 'money' will be exchanged through communication with body operating systems. Rand Corporation, a Cult-owned think tank, said of the Internet of Bodies or IoB:

> Internet of Bodies technologies fall under the broader IoT umbrella. But as the name suggests, IoB devices introduce an even more intimate interplay between humans and gadgets. IoB devices monitor the human body, collect health metrics and other personal information, and transmit those data over the Internet. Many devices, such as fitness trackers, are already in use … IoB devices … and those in development can track, record, and store users' whereabouts, bodily functions, and what they see, hear, and even think.

Schwab's World Economic Forum, a long-winded way of saying 'fascism' or 'the Cult', has gone full-on with the Internet of Bodies in the 'Covid' era. 'We're entering the era of the Internet of Bodies', it declared, 'collecting our physical data via a range of devices that can be implanted, swallowed or worn'. The result would be a huge amount of health-related data that could improve human wellbeing around the world, and prove crucial in fighting the 'Covid-19 pandemic'. Does anyone think these clowns care about 'human wellbeing' after the death and devastation their pandemic hoax has purposely caused? Schwab and co say we should move forward with the Internet of Bodies because 'Keeping track of symptoms could help us stop the spread of infection, and quickly detect new cases'. How wonderful, but keeping track' is all they are really bothered

about. Researchers were investigating if data gathered from smartwatches and similar devices could be used as viral infection alerts by tracking the user's heart rate and breathing. Schwab said in his 2018 book *Shaping the Future of the Fourth Industrial Revolution*:

> The lines between technologies and beings are becoming blurred and not just by the ability to create lifelike robots or synthetics. Instead it is about the ability of new technologies to literally become part of us. Technologies already influence how we understand ourselves, how we think about each other, and how we determine our realities. As the technologies … give us deeper access to parts of ourselves, we may begin to integrate digital technologies into our bodies.

You can see what the game is. Twenty-four hour control and people – if you could still call them that – would never know when something would go ping and take them out of circulation. It's the most obvious rush to a global fascist dictatorship and the complete submission of humanity and yet still so many are locked away in their Cult-induced perceptual coma and can't see it.

Smart Grid control centres

The human body is being transformed by the 'vaccines' and in other ways into a synthetic cyborg that can be attached to the global Smart Grid which would be controlled from a central point and other sub-locations of Grid manipulation. Where are these planned to be? Well, China for a start which is one of the Cult's biggest centres of operation. The technological control system and technocratic rule was incubated here to be unleashed across the world after the 'Covid' hoax came out of China in 2020. Another Smart Grid location that will surprise people new to this is Israel. I have exposed in *The Trigger* how Sabbatian technocrats, intelligence and military operatives were behind the horrors of 9/11 and not '19 Arab hijackers' who somehow manifested the ability to pilot big passenger airliners when instructors at puddle-jumping flying schools described some of them as a joke. The 9/11 attacks were made possible through control of civilian and military air computer systems and those of the White House, Pentagon and connected agencies. See *The Trigger* – it

will blow your mind. The controlling and coordinating force were the Sabbatian networks in Israel and the United States which by then had infiltrated the entire US government, military and intelligence system. The real name of the American Deep State is 'Sabbatian State'. Israel is a tiny country of only nine million people, but it is one of the global centres of cyber operations and fast catching Silicon Valley in importance to the Cult. Israel is known as the 'start-up nation' for all the cyber companies spawned there with the Sabbatian specialisation of 'cyber security' that I mentioned earlier which gives those companies access to computer systems of their clients in real time through 'backdoors' written into the coding when security software is downloaded. The Sabbatian centre of cyber operations outside Silicon Valley is the Israeli military Cyber Intelligence Unit, the biggest infrastructure project in Israel's history, headquartered in the desert-city of Beersheba and involving some 20,000 'cyber soldiers'. Here are located a literal army of Internet trolls scanning social media, forums and comment lists for anyone challenging the Cult agenda. The UK military has something similar with its 77th Brigade and associated operations. The Beersheba complex includes research and development centres for other Cult operations such as Intel, Microsoft, IBM, Google, Apple, Hewlett-Packard, Cisco Systems, Facebook and Motorola. Techcrunch.com ran an article about the Beersheba global Internet technology centre headlined 'Israel's desert city of Beersheba is turning into a cybertech oasis':

> The military's massive relocation of its prestigious technology units, the presence of multinational and local companies, a close proximity to Ben Gurion University and generous government subsidies are turning Beersheba into a major global cybertech hub. Beersheba has all of the ingredients of a vibrant security technology ecosystem, including Ben Gurion University with its graduate program in cybersecurity and Cyber Security Research Center, and the presence of companies such as EMC, Deutsche Telekom, PayPal, Oracle, IBM, and Lockheed Martin. It's also the future home of the INCB (Israeli National Cyber Bureau); offers a special income tax incentive for cyber security companies, and was the site for the relocation of the army's intelligence corps units.

Sabbatians have taken over the cyber world through the following process: They scan the schools for likely cyber talent and develop them at Ben Gurion University and their period of conscription in the Israeli Defense Forces when they are stationed at the Beersheba complex. When the cyber talented officially leave the army they are funded to start cyber companies with technology developed by themselves or given to them by the state. Much of this is stolen through backdoors of computer systems around the world with America top of the list. Others are sent off to Silicon Valley to start companies or join the major ones and so we have many major positions filled by apparently 'Jewish' but really Sabbatian operatives. Google, YouTube and Facebook are all run by 'Jewish' CEOs while Twitter is all but run by ultra-Zionist hedge-fund shark Paul Singer. At the centre of the Sabbatian global cyber web is the Israeli army's Unit 8200 which specialises in hacking into computer systems of other countries, inserting viruses, gathering information, instigating malfunction, and even taking control of them from a distance. A long list of Sabbatians involved with 9/11, Silicon Valley and Israeli cyber security companies are operatives of Unit 8200. This is not about Israel. It's about the Cult. Israel is planned to be a Smart Grid hub as with China and what is happening at Beersheba is not for the benefit of Jewish people who are treated disgustingly by the Sabbatian elite that control the country. A glance at the Nuremberg Codes will tell you that.

The story is much bigger than 'Covid', important as that is to where we are being taken. Now, though, it's time to really strap in. There's more … much more …

CHAPTER ELEVEN

Who controls the Cult?

Awake, arise or be forever fall'n
John Milton, *Paradise Lost*

I have exposed this far the level of the Cult conspiracy that operates in the world of the seen and within the global secret society and satanic network which operates in the shadows one step back from the seen. The story, however, goes much deeper than that.

The 'Covid' hoax is major part of the Cult agenda, but only part, and to grasp the biggest picture we have to expand our attention beyond the realm of human sight and into the infinity of possibility that we cannot see. It is from here, ultimately, that humanity is being manipulated into a state of total control by the force which dictates the actions of the Cult. How much of reality can we see? Next to damn all is the answer. We may appear to see all there is to see in the 'space' our eyes survey and observe, but little could be further from the truth. The human 'world' is only a tiny band of frequency that the body's visual and perceptual systems can decode into *perception* of a 'world'. According to mainstream science the electromagnetic spectrum is 0.005 percent of what exists in the Universe (Fig 10). The maximum estimate I have seen is 0.5 percent and either way it's miniscule. I say it is far, far, smaller even than 0.005 percent when you compare reality we see with the totality of reality that we don't. Now get this if you are new to such information: Visible light, the only band of frequency that we can see, is a *fraction* of the 0.005

percent (Fig 11 overleaf). Take this further and realise that our universe is one of infinite universes and that universes are only a fragment of overall reality – *infinite* reality. Then compare that with the almost infinitesimal frequency band of visible light or human sight. You see that humans are as near blind as it is possible to be without actually being so. Artist and filmmaker, Sergio Toporek, said:

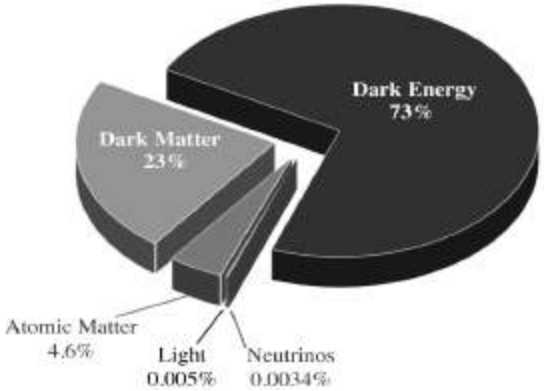


Figure 10: Humans can perceive such a tiny band of visual reality it's laughable.

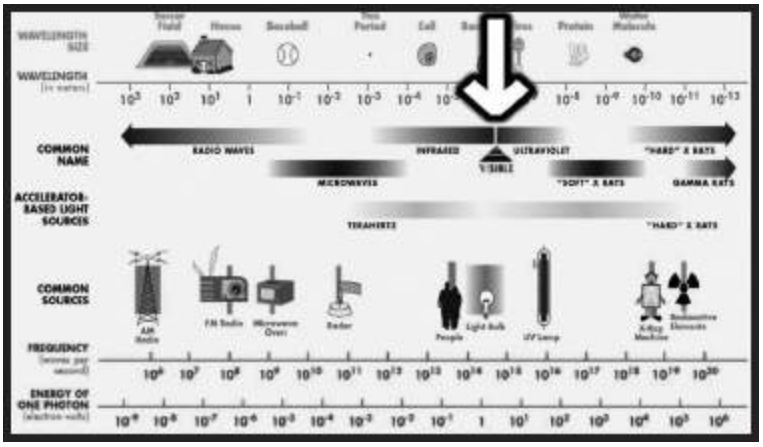


Figure 11: We can see a smear of the 0.005 percent electromagnetic spectrum, but we still know it all. Yep, makes sense.

Consider that you can see less than 1% of the electromagnetic spectrum and hear less than 1% of the acoustic spectrum. 90% of the cells in your body carry their own microbial DNA and are not 'you'. The atoms in your body are 99.9999999999999999% empty space and none of them are the ones you were born with ... Human beings have 46 chromosomes, two less than a potato.

The existence of the rainbow depends on the conical photoreceptors in your eyes; to animals without cones, the rainbow does not exist. So you don't just look at a rainbow, you create it. This is pretty amazing, especially considering that all the beautiful colours you see represent less than 1% of the electromagnetic spectrum.

Suddenly the 'world' of humans looks a very different place. Take into account, too, that Planet Earth when compared with the projected size of this single universe is the equivalent of a billionth of a pinhead. Imagine the ratio that would be when compared to infinite reality. To think that Christianity once insisted that Earth and humanity were the centre of everything. This background is vital if we are going to appreciate the nature of 'human' and how we can be manipulated by an unseen force. To human visual reality virtually *everything* is unseen and yet the prevailing perception within the institutions and so much of the public is that if we can't see it, touch it, hear it, taste it and smell it then it cannot exist. Such perception is indoctrinated and encouraged by the Cult and its agents because it isolates believers in the strictly limited, village-idiot, realm of the five senses where perceptions can be firewalled and information controlled. Most of those perpetuating the 'this-world-is-all-there-is' insanity are themselves indoctrinated into believing the same delusion. While major players and influencers know that official reality is laughable most of those in science, academia and medicine really believe the nonsense they peddle and teach succeeding generations. Those who challenge the orthodoxy are dismissed as nutters and freaks to protect the manufactured illusion from exposure. Observe the dynamic of the 'Covid' hoax and you will see how that takes the same form. The inner-circle psychopaths knows it's a gigantic scam, but almost the entirety of those imposing their fascist rules believe that 'Covid' is all that they're told it is.

Stolen identity

Ask people who they are and they will give you their name, place of birth, location, job, family background and life story. Yet that is not who they are – it is what they are *experiencing*. The difference is *absolutely crucial*. The true 'I', the eternal, infinite 'I', is consciousness,

a state of being aware. Forget 'form'. That is a vehicle for a brief experience. Consciousness does not come *from* the brain, but *through* the brain and even that is more symbolic than literal. We are awareness, pure awareness, and this is what withdraws from the body at what we call 'death' to continue our eternal beingness, *isness*, in other realms of reality within the limitlessness of infinity or the Biblical 'many mansions in my father's house'. Labels of a human life, man, woman, transgender, black, white, brown, nationality, circumstances and income are not who we are. They are what we are – awareness – is *experiencing* in a brief connection with a band of frequency we call 'human'. The labels are not the self; they are, to use the title of one of my books, a *Phantom* Self. I am not David Icke born in Leicester, England, on April 29th, 1952. I am the consciousness *having that experience*. The Cult and its non-human masters seek to convince us through the institutions of 'education', science, medicine, media and government that what we are *experiencing* is who we *are*. It's so easy to control and direct perception locked away in the bewildered illusions of the five senses with no expanded radar. Try, by contrast, doing the same with a humanity aware of its true self and its true power to consciously create its reality and experience. How is it possible to do this? We do it all day every day. If you perceive yourself as 'little me' with no power to impact upon your life and the world then your life experience will reflect that. You will hand the power you don't think you have to authority in all its forms which will use it to control your experience. This, in turn, will appear to confirm your perception of 'little me' in a self-fulfilling feedback loop. But that is what 'little me' really is – a *perception*. We are all 'big-me', infinite me, and the Cult has to make us forget that if its will is to prevail. We are therefore manipulated and pressured into self-identifying with human labels and not the consciousness/awareness *experiencing* those human labels.

The phenomenon of identity politics is a Cult-instigated manipulation technique to sub-divide previous labels into even smaller ones. A United States university employs this list of letters to

describe student identity: LGBTTQQFAGPBDSM or lesbian, gay, bisexual, transgender, transsexual, queer, questioning, flexual, asexual, gender-fuck, polyamorous, bondage/discipline, dominance/submission and sadism/masochism. I'm sure other lists are even longer by now as people feel the need to self-identity the 'I' with the minutiae of race and sexual preference. Wokers programmed by the Cult for generations believe this is about 'inclusivity' when it's really the Cult locking them away into smaller and smaller versions of Phantom Self while firewalling them from the influence of their true self, the infinite, eternal 'I'. You may notice that my philosophy which contends that we are all unique points of attention/awareness within the same infinite whole or Oneness is the ultimate non-racism. The very sense of Oneness makes the judgement of people by their body-type, colour or sexuality utterly ridiculous and confirms that racism has no understanding of reality (including anti-white racism). Yet despite my perception of life Cult agents and fast-asleep Wokers label me racist to discredit my information while they are themselves phenomenally racist and sexist. All they see is race and sexuality and they judge people as good or bad, demons or untouchables, by their race and sexuality. All they see is *Phantom Self* and perceive themselves in terms of Phantom Self. They are pawns and puppets of the Cult agenda to focus attention and self-identity in the five senses and play those identities against each other to divide and rule. Columbia University has introduced segregated graduations in another version of social distancing designed to drive people apart and teach them that different racial and cultural groups have nothing in common with each other. The last thing the Cult wants is unity. Again the pump-primers of this will be Cult operatives in the knowledge of what they are doing, but the rest are just the Phantom Self blind leading the Phantom Self blind. We *do* have something in common – we are all *the same consciousness* having different temporary experiences.

What is this 'human'?

Yes, what *is* 'human'? That is what we are supposed to be, right? I mean 'human'? True, but 'human' is the experience not the 'I'. Break it down to basics and 'human' is the way that information is processed. If we are to experience and interact with this band of frequency we call the 'world' we must have a vehicle that operates within that band of frequency. Our consciousness in its prime form cannot do that; it is way beyond the frequency of the human realm. My consciousness or awareness could not tap these keys and pick up the cup in front of me in the same way that radio station A cannot interact with radio station B when they are on different frequencies. The human body is the means through which we have that interaction. I have long described the body as a biological computer which processes information in a way that allows consciousness to experience this reality. The body is a receiver, transmitter and processor of information in a particular way that we call human. We visually perceive only the world of the five senses in a wakened state – that is the limit of the body's visual decoding system. In truth it's not even visual in the way we experience 'visual reality' as I will come to in a moment. We are 'human' because the body processes the information sources of human into a reality and behaviour system that we *perceive* as human. Why does an elephant act like an elephant and not like a human or a duck? The elephant's biological computer is a different information field and processes information according to that program into a visual and behaviour type we call an elephant. The same applies to everything in our reality. These body information fields are perpetuated through procreation (like making a copy of a software program). The Cult wants to break that cycle and intervene technologically to transform the human information field into one that will change what we call humanity. If it can change the human information field it will change the way that field processes information and change humanity both 'physically' and psychologically. Hence the *messenger* (information) RNA 'vaccines' and so much more that is targeting human genetics by changing the body's information – *messaging* – construct through food, drink, radiation, toxicity and other means.

Reality that we experience is nothing like reality as it really is in the same way that the reality people experience in virtual reality games is not the reality they are really living in. The game is only a decoded source of information that appears to be a reality. Our world is also an information construct – a *simulation* (more later). In its base form our reality is a wavefield of information much the same in theme as Wi-Fi. The five senses decode wavefield information into electrical information which they communicate to the brain to decode into holographic (illusory 'physical') information. Different parts of the brain specialise in decoding different senses and the information is fused into a reality that appears to be outside of us but is really inside the brain and the genetic structure in general (Fig 12 overleaf). DNA is a receiver-transmitter of information and a vital part of this decoding process and the body's connection to other realities. Change DNA and you change the way we decode and connect with reality – see 'Covid vaccines'. Think of computers decoding Wi-Fi. You have information encoded in a radiation field and the computer decodes that information into a very different form on the screen. You can't see the Wi-Fi until its information is made manifest on the screen and the information on the screen is inside the computer and not outside. I have just described how we decode the 'human world'. All five senses decode the waveform 'Wi-Fi' field into electrical signals and the brain (computer) constructs reality inside the brain and not outside – 'You don't just look at a rainbow, you create it'. Sound is a simple example. We don't hear sound until the brain decodes it. Waveform sound waves are picked up by the hearing sense and communicated to the brain in an electrical form to be decoded into the sounds that we hear. Everything we hear is inside the brain along with everything we see, feel, smell and taste. Words and language are waveform fields generated by our vocal chords which pass through this process until they are decoded by the brain into words that we hear. Different languages are different frequency fields or sound waves generated by vocal chords. Late British philosopher Alan Watts said:

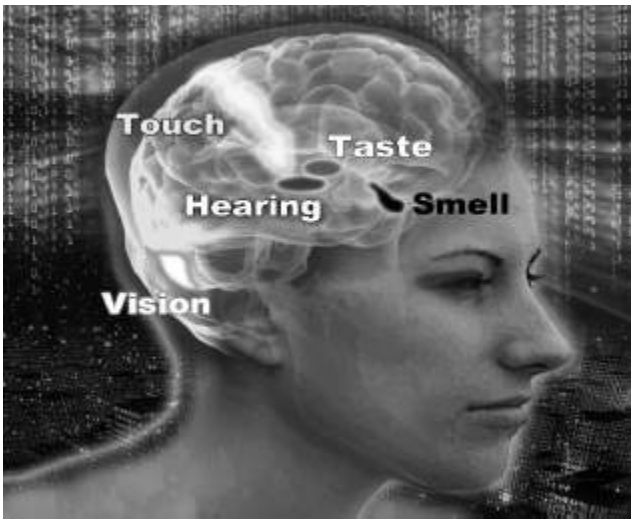


Figure 12: The brain receives information from the five senses and constructs from that our perceived reality.

[Without the brain] the world is devoid of light, heat, weight, solidity, motion, space, time or any other imaginable feature. All these phenomena are interactions, or transactions, of vibrations with a certain arrangement of neurons.

That's exactly what they are and scientist Robert Lanza describes in his book, *Biocentrism*, how we decode electromagnetic waves and energy into visual and 'physical' experience. He uses the example of a flame emitting photons, electromagnetic energy, each pulsing electrically and magnetically:

… these … invisible electromagnetic waves strike a human retina, and if (and only if) the waves happen to measure between 400 and 700 nano meters in length from crest to crest, then their energy is just right to deliver a stimulus to the 8 million cone-shaped cells in the retina.

Each in turn send an electrical pulse to a neighbour neuron, and on up the line this goes, at 250 mph, until it reaches the … occipital lobe of the brain, in the back of the head. There, a cascading complex of neurons fire from the incoming stimuli, and we subjectively perceive this experience as a yellow brightness occurring in a place we have been conditioned to call the 'external world'.

You hear what you decode

If a tree falls or a building collapses they make no noise unless someone is there to decode the energetic waves generated by the disturbance into what we call sound. Does a falling tree make a noise? Only if you hear it – *decode* it. Everything in our reality is a frequency field of information operating within the overall 'Wi-Fi' field that I call The Field. A vibrational disturbance is generated in The Field by the fields of the falling tree or building. These disturbance waves are what we decode into the sound of them falling. If no one is there to do that then neither will make any noise. Reality is created by the observer – *decoder* – and the *perceptions* of the observer affect the decoding process. For this reason different people – different *perceptions* – will perceive the same reality or situation in a different way. What one may perceive as a nightmare another will see as an opportunity. The question of why the Cult is so focused on controlling human perception now answers itself. All experienced reality is the act of decoding and we don't experience Wi-Fi until it is decoded on the computer screen. The sight and sound of an Internet video is encoded in the Wi-Fi all around us, but we don't see or hear it until the computer decodes that information. Taste, smell and touch are all phenomena of the brain as a result of the same process. We don't taste, smell or feel anything except in the brain and there are pain relief techniques that seek to block the signal from the site of discomfort to the brain because if the brain doesn't decode that signal we don't feel pain. Pain is in the brain and only appears to be at the point of impact thanks to the feedback loop between them. We don't see anything until electrical information from the sight senses is decoded in an area at the back of the brain. If that area is damaged we can go blind when our eyes are perfectly okay. So why do we go blind if we damage an eye? We damage the information processing between the waveform visual information and the visual decoding area of the brain. If information doesn't reach the brain in a form it can decode then we can't see the visual reality that it represents. What's more the brain is decoding only a fraction of the information it receives and the rest is absorbed by the

sub-conscious mind. This explanation is from the science magazine, *Wonderpedia*:

> Every second, 11 million sensations crackle along these [brain] pathways ... The brain is confronted with an alarming array of images, sounds and smells which it rigorously filters down until it is left with a manageable list of around 40. Thus 40 sensations per second make up what we perceive as reality.

The 'world' is not what people are told to believe that is it and the inner circles of the Cult *know that*.

Illusory 'physical' reality

We can only see a smear of 0.005 percent of the Universe which is only one of a vast array of universes – 'mansions' – within infinite reality. Even then the brain decodes only 40 pieces of information ('sensations') from a potential *11 million* that we receive every second. Two points strike you from this immediately: The sheer breathtaking stupidity of believing we know anything so rigidly that there's nothing more to know; and the potential for these processes to be manipulated by a malevolent force to control the reality of the population. One thing I can say for sure with no risk of contradiction is that when you can perceive an almost indescribable fraction of infinite reality there is always more to know as in tidal waves of it. Ancient Greek philosopher Socrates was so right when he said that wisdom is to know how little we know. How obviously true that is when you think that we are experiencing a physical world of solidity that is neither physical nor solid and a world of apartness when everything is connected. Cult-controlled 'science' dismisses the so-called 'paranormal' and all phenomena related to that when the 'para'-normal is perfectly normal and explains the alleged 'great mysteries' which dumbfound scientific minds. There is a reason for this. A 'scientific mind' in terms of the mainstream is a material mind, a five-sense mind imprisoned in see it, touch it, hear it, smell it and taste it. Phenomena and happenings that can't be explained that way leave the 'scientific mind' bewildered and the rule is that if they

can't account for why something is happening then it can't, by definition, be happening. I beg to differ. Telepathy is thought waves passing through The Field (think wave disturbance again) to be decoded by someone able to connect with that wavelength (information). For example: You can pick up the thought waves of a friend at any distance and at the very least that will bring them to mind. A few minutes later the friend calls you. 'My god', you say, 'that's incredible – I was just thinking of you.' Ah, but *they* were thinking of *you* before they made the call and that's what you decoded. Native peoples not entrapped in five-sense reality do this so well it became known as the 'bush telegraph'. Those known as psychics and mediums (genuine ones) are doing the same only across dimensions of reality. 'Mind over matter' comes from the fact that matter and mind are the *same*. The state of one influences the state of the other. Indeed one *and* the other are illusions. They are aspects of the same field. Paranormal phenomena are all explainable so why are they still considered 'mysteries' or not happening? Once you go down this road of understanding you begin to expand awareness beyond the five senses and that's the nightmare for the Cult.



Figure 13: Holograms are not solid, but the best ones appear to be.

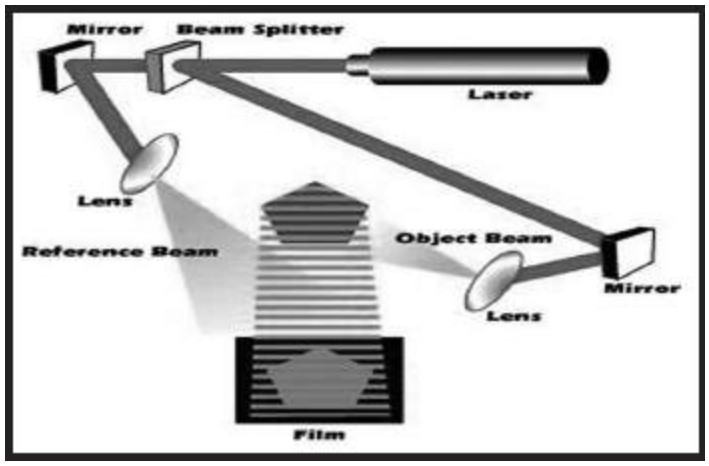


Figure 14: How holograms are created by capturing a waveform version of the subject image.

Holographic 'solidity'

Our reality is not solid, it is holographic. We are now well aware of holograms which are widely used today. Two-dimensional information is decoded into a three-dimensional reality that is not solid although can very much appear to be (Fig 13). Holograms are created with a laser divided into two parts. One goes directly onto a holographic photographic print ('reference beam') and the other takes a waveform image of the subject ('working beam') before being directed onto the print where it 'collides' with the other half of the laser (Fig 14). This creates a *waveform* interference pattern which contains the wavefield information of whatever is being photographed (Fig 15 overleaf). The process can be likened to dropping pebbles in a pond. Waves generated by each one spread out across the water to collide with the others and create a wave representation of where the stones fell and at what speed, weight and distance. A waveform interference pattern of a hologram is akin to the waveform information in The Field which the five senses decode into electrical signals to be decoded by the brain into a holographic illusory 'physical' reality. In the same way when a laser (think human attention) is directed at the waveform interference pattern a three-dimensional version of the subject is projected into apparently 'solid' reality (Fig 16). An amazing trait of holograms reveals more 'paranormal mysteries'. Information of the *whole*

hologram is encoded in waveform in every part of the interference pattern by the way they are created. This means that every *part* of a hologram is a smaller version of the whole. Cut the interference wave-pattern into four and you won't get four parts of the image. You get quarter-sized versions of the *whole* image. The body is a hologram and the same applies. Here we have the basis of acupuncture, reflexology and other forms of healing which identify representations of the whole body in all of the parts, hands, feet, ears, everywhere. Skilled palm readers can do what they do because the information of whole body is encoded in the hand. The concept of as above, so below, comes from this.



Figure 15: A waveform interference pattern that holds the information that transforms into a hologram.



Figure 16: Holographic people including 'Elvis' holographically inserted to sing a duet with Celine Dion.

The question will be asked of why, if solidity is illusory, we can't just walk through walls and each other. The resistance is not solid against solid; it is electromagnetic field against electromagnetic field and we decode this into the *experience* of solid against solid. We should also not underestimate the power of belief to dictate reality. What you believe is impossible *will be*. Your belief impacts on your decoding processes and they won't decode what you think is impossible. What we believe we perceive and what we perceive we experience. 'Can't dos' and 'impossibles' are like a firewall in a computer system that won't put on the screen what the firewall blocks. How vital that is to understanding how human experience has been hijacked. I explain in *The Answer*, *Everything You Need To Know But Have Never Been Told* and other books a long list of 'mysteries' and 'paranormal' phenomena that are not mysterious and perfectly normal once you realise what reality is and how it works. 'Ghosts' can be seen to pass through 'solid' walls because the walls are not solid and the ghost is a discarnate entity operating on a frequency so different to that of the wall that it's like two radio stations sharing the same space while never interfering with each other. I have seen ghosts do this myself. The apartness of people and objects is also an illusion. Everything is connected by the Field like all sea life is connected by the sea. It's just that within the limits of our visual reality we only 'see' holographic information and not the field of information that connects everything and from which the holographic world is made manifest. If you can only see holographic 'objects' and not the field that connects them they will appear to you as unconnected to each other in the same way that we see the computer while not seeing the Wi-Fi.

What you don't know *can* hurt you

Okay, we return to those 'two worlds' of human society and the Cult with its global network of interconnecting secret societies and satanic groups which manipulate through governments, corporations, media, religions, etc. The fundamental difference between them is *knowledge*. The idea has been to keep humanity

ignorant of the plan for its total enslavement underpinned by a crucial ignorance of reality – who we are and where we are – and how we interact with it. 'Human' should be the interaction between our expanded eternal consciousness and the five-sense body experience. We are meant to be *in* this world in terms of the five senses but not *of* this world in relation to our greater consciousness and perspective. In that state we experience the small picture of the five senses within the wider context of the big picture of awareness beyond the five senses. Put another way the five senses see the dots and expanded awareness connects them into pictures and patterns that give context to the apparently random and unconnected. Without the context of expanded awareness the five senses see only apartness and randomness with apparently no meaning. The Cult and its other-dimensional controllers seek to intervene in the frequency realm where five-sense reality is supposed to connect with expanded reality and to keep the two apart (more on this in the final chapter). When that happens five-sense mental and emotional processes are no longer influenced by expanded awareness, or the True 'I', and instead are driven by the isolated perceptions of the body's decoding systems. They are in the world *and* of it. Here we have the human plight and why humanity with its potential for infinite awareness can be so easily manipulatable and descend into such extremes of stupidity.

Once the Cult isolates five-sense mind from expanded awareness it can then program the mind with perceptions and beliefs by controlling information that the mind receives through the 'education' system of the formative years and the media perceptual bombardment and censorship of an entire lifetime. Limit perception and a sense of the possible through limiting knowledge by limiting and skewing information while censoring and discrediting that which could set people free. As the title of another of my books says *… And The Truth Shall Set You Free*. For this reason the last thing the Cult wants in circulation is the truth about anything – especially the reality of the eternal 'I' – and that's why it is desperate to control information. The Cult knows that information becomes perception

which becomes behaviour which, collectively, becomes human society. Cult-controlled and funded mainstream 'science' denies the existence of an eternal 'I' and seeks to dismiss and trash all evidence to the contrary. Cult-controlled mainstream religion has a version of 'God' that is little more than a system of control and dictatorship that employs threats of damnation in an afterlife to control perceptions and behaviour in the here and now through fear and guilt. Neither is true and it's the 'neither' that the Cult wishes to suppress. This 'neither' is that everything is an expression, a point of attention, within an infinite state of consciousness which is the real meaning of the term 'God'.

Perceptual obsession with the 'physical body' and five-senses means that 'God' becomes personified as a bearded bloke sitting among the clouds or a raging bully who loves us if we do what 'he' wants and condemns us to the fires of hell if we don't. These are no more than a 'spiritual' fairy tales to control and dictate events and behaviour through fear of this 'God' which has bizarrely made 'God-fearing' in religious circles a state to be desired. I would suggest that fearing *anything* is not to be encouraged and celebrated, but rather deleted. You can see why 'God fearing' is so beneficial to the Cult and its religions when *they* decide what 'God' wants and what 'God' demands (the Cult demands) that everyone do. As the great American comedian Bill Hicks said satirising a Christian zealot: 'I think what God meant to say.' How much of this infinite awareness ('God') that we access is decided by how far we choose to expand our perceptions, self-identity and sense of the possible. The scale of self-identity reflects itself in the scale of awareness that we can connect with and are influenced by – how much knowing and insight we have instead of programmed perception. You cannot expand your awareness into the infinity of possibility when you believe that you are little me Peter the postman or Mary in marketing and nothing more. I'll deal with this in the concluding chapter because it's crucial to how we turnaround current events.

Where the Cult came from

When I realised in the early 1990s there was a Cult network behind global events I asked the obvious question: When did it start? I took it back to ancient Rome and Egypt and on to Babylon and Sumer in Mesopotamia, the 'Land Between Two Rivers', in what we now call Iraq. The two rivers are the Tigris and Euphrates and this region is of immense historical and other importance to the Cult, as is the land called Israel only 550 miles away by air. There is much more going with deep esoteric meaning across this whole region. It's not only about 'wars for oil'. Priceless artefacts from Mesopotamia were stolen or destroyed after the American and British invasion of Iraq in 2003 justified by the lies of Boy Bush and Tony Blair (their Cult masters) about non-existent 'weapons of mass destruction'. Mesopotamia was the location of Sumer (about 5,400BC to 1,750BC), and Babylon (about 2,350BC to 539BC). Sabbatians may have become immensely influential in the Cult in modern times but they are part of a network that goes back into the mists of history. Sumer is said by historians to be the 'cradle of civilisation'. I disagree. I say it was the re-start of what we call human civilisation after cataclysmic events symbolised in part as the 'Great Flood' destroyed the world that existed before. These fantastic upheavals that I have been describing in detail in the books since the early1990s appear in accounts and legends of ancient cultures across the world and they are supported by geological and biological evidence. Stone tablets found in Iraq detailing the Sumer period say the cataclysms were caused by non-human 'gods' they call the Anunnaki. These are described in terms of extraterrestrial visitations in which knowledge supplied by the Anunnaki is said to have been the source of at least one of the world's oldest writing systems and developments in astronomy, mathematics and architecture that were way ahead of their time. I have covered this subject at length in *The Biggest Secret* and *Children of the Matrix* and the same basic 'Anunnaki' story can be found in Zulu accounts in South Africa where the late and very great Zulu high shaman Credo Mutwa told me that the Sumerian Anunnaki were known by Zulus as the Chitauri or 'children of the serpent'. See my six-hour video interview with Credo on this subject entitled *The*

Reptilian Agenda recorded at his then home near Johannesburg in 1999 which you can watch on the Ickonic media platform.

The Cult emerged out of Sumer, Babylon and Egypt (and elsewhere) and established the Roman Empire before expanding with the Romans into northern Europe from where many empires were savagely imposed in the form of Cult-controlled societies all over the world. Mass death and destruction was their calling card. The Cult established its centre of operations in Europe and European Empires were Cult empires which allowed it to expand into a global force. Spanish and Portuguese colonialists headed for Central and South America while the British and French targeted North America. Africa was colonised by Britain, France, Belgium, the Netherlands, Portugal, Spain, Italy, and Germany. Some like Britain and France moved in on the Middle East. The British Empire was by far the biggest for a simple reason. By now Britain was the headquarters of the Cult from which it expanded to form Canada, the United States, Australia and New Zealand. The Sun never set on the British Empire such was the scale of its occupation. London remains a global centre for the Cult along with Rome and the Vatican although others have emerged in Israel and China. It is no accident that the 'virus' is alleged to have come out of China while Italy was chosen as the means to terrify the Western population into compliance with 'Covid' fascism. Nor that Israel has led the world in 'Covid' fascism and mass 'vaccination'.

You would think that I would mention the United States here, but while it has been an important means of imposing the Cult's will it is less significant than would appear and is currently in the process of having what power it does have deleted. The Cult in Europe has mostly loaded the guns for the US to fire. America has been controlled from Europe from the start through Cult operatives in Britain and Europe. The American Revolution was an illusion to make it appear that America was governing itself while very different forces were pulling the strings in the form of Cult families such as the Rothschilds through the Rockefellers and other subordinates. The Rockefellers are extremely close to Bill Gates and

established both scalpel and drug 'medicine' and the World Health Organization. They play a major role in the development and circulation of vaccines through the Rockefeller Foundation on which Bill Gates said his Foundation is based. Why wouldn't this be the case when the Rockefellers and Gates are on the same team? Cult infiltration of human society goes way back into what we call history and has been constantly expanding and centralising power with the goal of establishing a global structure to dictate everything. Look how this has been advanced in great leaps with the 'Covid' hoax.

The non-human dimension

I researched and observed the comings and goings of Cult operatives through the centuries and even thousands of years as they were born, worked to promote the agenda within the secret society and satanic networks, and then died for others to replace them. Clearly there had to be a coordinating force that spanned this entire period while operatives who would not have seen the end goal in their lifetimes came and went advancing the plan over millennia. I went in search of that coordinating force with the usual support from the extraordinary synchronicity of my life which has been an almost daily experience since 1990. I saw common themes in religious texts and ancient cultures about a non-human force manipulating human society from the hidden. Christianity calls this force Satan, the Devil and demons; Islam refers to the Jinn or Djinn; Zulus have their Chitauri (spelt in other ways in different parts of Africa); and the Gnostic people in Egypt in the period around and before 400AD referred to this phenomena as the 'Archons', a word meaning rulers in Greek. Central American cultures speak of the 'Predators' among other names and the same theme is everywhere. I will use 'Archons' as a collective name for all of them. When you see how their nature and behaviour is described all these different sources are clearly talking about the same force. Gnostics described the Archons in terms of 'luminous fire' while Islam relates the Jinn to 'smokeless fire'. Some refer to beings in form that could occasionally be seen, but the most common of common theme is that they operate from

unseen realms which means almost all existence to the visual processes of humans. I had concluded that this was indeed the foundation of human control and that the Cult was operating within the human frequency band on behalf of this hidden force when I came across the writings of Gnostics which supported my conclusions in the most extraordinary way.

A sealed earthen jar was found in 1945 near the town of Nag Hammadi about 75-80 miles north of Luxor on the banks of the River Nile in Egypt. Inside was a treasure trove of manuscripts and texts left by the Gnostic people some 1,600 years earlier. They included 13 leather-bound papyrus codices (manuscripts) and more than 50 texts written in Coptic Egyptian estimated to have been hidden in the jar in the period of 400AD although the source of the information goes back much further. Gnostics oversaw the Great or Royal Library of Alexandria, the fantastic depository of ancient texts detailing advanced knowledge and accounts of human history. The Library was dismantled and destroyed in stages over a long period with the death-blow delivered by the Cult-established Roman Church in the period around 415AD. The Church of Rome was the Church of Babylon relocated as I said earlier. Gnostics were not a race. They were a way of perceiving reality. Whenever they established themselves and their information circulated the terrorists of the Church of Rome would target them for destruction. This happened with the Great Library and with the Gnostic Cathars who were burned to death by the psychopaths after a long period of oppression at the siege of the Castle of Monségur in southern France in 1244. The Church has always been terrified of Gnostic information which demolishes the official Christian narrative although there is much in the Bible that supports the Gnostic view if you read it in another way. To anyone studying the texts of what became known as the Nag Hammadi Library it is clear that great swathes of Christian and Biblical belief has its origin with Gnostics sources going back to Sumer. Gnostic themes have been twisted to manipulate the perceived reality of Bible believers. Biblical texts have been in the open for centuries where they could be changed while Gnostic

documents found at Nag Hammadi were sealed away and untouched for 1,600 years. What you see is what they wrote.

Use your *pneuma* not your *nous*

Gnosticism and Gnostic come from 'gnosis' which means knowledge, or rather *secret* knowledge, in the sense of spiritual awareness – knowledge about reality and life itself. The desperation of the Cult's Church of Rome to destroy the Gnostics can be understood when the knowledge they were circulating was the last thing the Cult wanted the population to know. Sixteen hundred years later the same Cult is working hard to undermine and silence me for the same reason. The dynamic between knowledge and ignorance is a constant. 'Time' appears to move on, but essential themes remain the same. We are told to 'use your nous', a Gnostic word for head/brain/intelligence. They said, however, that spiritual awakening or 'salvation' could only be secured by expanding awareness *beyond* what they called *nous* and into *pneuma* or Infinite Self. Obviously as I read these texts the parallels with what I have been saying since 1990 were fascinating to me. There is a universal truth that spans human history and in that case why wouldn't we be talking the same language 16 centuries apart? When you free yourself from the perception program of the five senses and explore expanded realms of consciousness you are going to connect with the same information no matter what the perceived 'era' within a manufactured timeline of a single and tiny range of manipulated frequency. Humans working with 'smart' technology or knocking rocks together in caves is only a timeline appearing to operate within the human frequency band. Expanded awareness and the knowledge it holds have always been there whether the era be Stone Age or computer age. We can only access that knowledge by opening ourselves to its frequency which the five-sense prison cell is designed to stop us doing. Gates, Fauci, Whitty, Vallance, Zuckerberg, Brin, Page, Wojcicki, Bezos, and all the others behind the 'Covid' hoax clearly have a long wait before their range of frequency can make that connection given that an open heart is

crucial to that as we shall see. Instead of accessing knowledge directly through expanded awareness it is given to Cult operatives by the secret society networks of the Cult where it has been passed on over thousands of years outside the public arena. Expanded realms of consciousness is where great artists, composers and writers find their inspiration and where truth awaits anyone open enough to connect with it. We need to go there fast.

Archon hijack

A fifth of the Nag Hammadi texts describe the existence and manipulation of the Archons led by a 'Chief Archon' they call 'Yaldabaoth', or the 'Demiurge', and this is the Christian 'Devil', 'Satan', 'Lucifer', and his demons. Archons in Biblical symbolism are the 'fallen ones' which are also referred to as fallen angels after the angels expelled from heaven according to the Abrahamic religions of Judaism, Christianity and Islam. These angels are claimed to tempt humans to 'sin' ongoing and you will see how accurate that symbolism is during the rest of the book. The theme of 'original sin' is related to the 'Fall' when Adam and Eve were 'tempted by the serpent' and fell from a state of innocence and 'obedience' (connection) with God into a state of disobedience (disconnection). The Fall is said to have brought sin into the world and corrupted everything including human nature. Yaldabaoth, the 'Lord Archon', is described by Gnostics as a 'counterfeit spirit', 'The Blind One', 'The Blind God', and 'The Foolish One'. The Jewish name for Yaldabaoth in Talmudic writings is Samael which translates as 'Poison of God', or 'Blindness of God'. You see the parallels. Yaldabaoth in Islamic belief is the Muslim Jinn devil known as Shaytan – Shaytan is Satan as the same themes are found all over the world in every religion and culture. The 'Lord God' of the Old Testament is the 'Lord Archon' of Gnostic manuscripts and that's why he's such a bloodthirsty bastard. Satan is known by Christians as 'the Demon of Demons' and Gnostics called Yaldabaoth the 'Archon of Archons'. Both are known as 'The Deceiver'. We are talking about the same 'bloke' for sure and these common themes

using different names, storylines and symbolism tell a common tale of the human plight.

Archons are referred to in Nag Hammadi documents as mind parasites, inverters, guards, gatekeepers, detainers, judges, pitiless ones and deceivers. The 'Covid' hoax alone is a glaring example of all these things. The Biblical 'God' is so different in the Old and New Testaments because they are not describing the same phenomenon. The vindictive, angry, hate-filled, 'God' of the Old Testament, known as Yahweh, is Yaldabaoth who is depicted in Cult-dictated popular culture as the 'Dark Lord', 'Lord of Time', Lord (Darth) Vader and Dormammu, the evil ruler of the 'Dark Dimension' trying to take over the 'Earth Dimension' in the Marvel comic movie, *Dr Strange*. Yaldabaoth is both the Old Testament 'god' and the Biblical 'Satan'. Gnostics referred to Yaldabaoth as the 'Great Architect of the Universe'and the Cult-controlled Freemason network calls their god 'the 'Great Architect of the Universe' (also Grand Architect). The 'Great Architect' Yaldabaoth is symbolised by the Cult as the all-seeing eye at the top of the pyramid on the Great Seal of the United States and the dollar bill. Archon is encoded in *arch*-itect as it is in *arch*-angels and *arch*-bishops. All religions have the theme of a force for good and force for evil in some sort of spiritual war and there is a reason for that – the theme is true. The Cult and its non-human masters are quite happy for this to circulate. They present themselves as the force for good fighting evil when they are really the force of evil (absence of love). The whole foundation of Cult modus operandi is inversion. They promote themselves as a force for good and anyone challenging them in pursuit of peace, love, fairness, truth and justice is condemned as a satanic force for evil. This has been the game plan throughout history whether the Church of Rome inquisitions of non-believers or 'conspiracy theorists' and 'anti-vaxxers' of today. The technique is the same whatever the timeline era.

Yaldabaoth is revolting (true)

Yaldabaoth and the Archons are said to have revolted against God with Yaldabaoth claiming to *be* God – the *All That Is*. The Old Testament 'God' (Yaldabaoth) demanded to be worshipped as such: ' *I am* the LORD, and there is none else, there is no God beside me' (Isaiah 45:5). I have quoted in other books a man who said he was the unofficial son of the late Baron Philippe de Rothschild of the Mouton-Rothschild wine producing estates in France who died in 1988 and he told me about the Rothschild 'revolt from God'. The man said he was given the name Phillip Eugene de Rothschild and we shared long correspondence many years ago while he was living under another identity. He said that he was conceived through 'occult incest' which (within the Cult) was 'normal and to be admired'. 'Phillip' told me about his experience attending satanic rituals with rich and famous people whom he names and you can see them and the wider background to Cult Satanism in my other books starting with *The Biggest Secret*. Cult rituals are interactions with Archontic 'gods'. 'Phillip' described Baron Philippe de Rothschild as 'a master Satanist and hater of God' and he used the same term 'revolt from God' associated with Yaldabaoth/Satan/Lucifer/the Devil in describing the Sabbatian Rothschild dynasty. 'I played a key role in my family's revolt from God', he said. That role was to infiltrate in classic Sabbatian style the Christian Church, but eventually he escaped the mind-prison to live another life. The Cult has been targeting religion in a plan to make worship of the Archons the global one-world religion. Infiltration of Satanism into modern 'culture', especially among the young, through music videos, stage shows and other means, is all part of this.

Nag Hammadi texts describe Yaldabaoth and the Archons in their prime form as energy – consciousness – and say they can take form if they choose in the same way that consciousness takes form as a human. Yaldabaoth is called 'formless' and represents a deeply inverted, distorted and chaotic state of consciousness which seeks to attached to humans and turn them into a likeness of itself in an attempt at assimilation. For that to happen it has to manipulate

humans into low frequency mental and emotional states that match its own. Archons can certainly appear in human form and this is the origin of the psychopathic personality. The energetic distortion Gnostics called Yaldabaoth *is* psychopathy. When psychopathic Archons take human form that human will be a psychopath as an expression of Yaldabaoth consciousness. Cult psychopaths are Archons in human form. The principle is the same as that portrayed in the 2009 *Avatar* movie when the American military travelled to a fictional Earth-like moon called Pandora in the Alpha Centauri star system to infiltrate a society of blue people, or Na'vi, by hiding within bodies that looked like the Na'vi. Archons posing as humans have a particular hybrid information field, part human, part Archon, (the ancient 'demigods') which processes information in a way that manifests behaviour to match their psychopathic evil, lack of empathy and compassion, and stops them being influenced by the empathy, compassion and love that a fully-human information field is capable of expressing. Cult bloodlines interbreed, be they royalty or dark suits, for this reason and you have their obsession with incest. Interbreeding with full-blown humans would dilute the Archontic energy field that guarantees psychopathy in its representatives in the human realm.

Gnostic writings say the main non-human forms that Archons take are *serpentine* (what I have called for decades 'reptilian' amid unbounded ridicule from the Archontically-programmed) and what Gnostics describe as 'an unborn baby or foetus with grey skin and dark, unmoving eyes'. This is an excellent representation of the ET 'Greys' of UFO folklore which large numbers of people claim to have seen and been abducted by – Zulu shaman Credo Mutwa among them. I agree with those that believe in extraterrestrial or interdimensional visitations today and for thousands of years past. No wonder with their advanced knowledge and technological capability they were perceived and worshipped as gods for technological and other 'miracles' they appeared to perform. Imagine someone arriving in a culture disconnected from the modern world with a smartphone and computer. They would be

seen as a 'god' capable of 'miracles'. The Renegade Mind, however, wants to know the source of everything and not only the way that source manifests as human or non-human. In the same way that a Renegade Mind seeks the original source material for the 'Covid virus' to see if what is claimed is true. The original source of Archons in form is consciousness – the distorted state of consciousness known to Gnostics as Yaldabaoth.

'Revolt from God' is energetic disconnection

Where I am going next will make a lot of sense of religious texts and ancient legends relating to 'Satan', Lucifer' and the 'gods'. Gnostic descriptions sync perfectly with the themes of my own research over the years in how they describe a consciousness distortion seeking to impose itself on human consciousness. I've referred to the core of infinite awareness in previous books as Infinite Awareness in Awareness of Itself. By that I mean a level of awareness that knows that it is all awareness and is aware of all awareness. From here comes the frequency of love in its true sense and balance which is what love is on one level – the balance of all forces into a single whole called Oneness and Isness. The more we disconnect from this state of love that many call 'God' the constituent parts of that Oneness start to unravel and express themselves as a part and not a whole. They become individualised as intellect, mind, selfishness, hatred, envy, desire for power over others, and such like. This is not a problem in the greater scheme in that 'God', the *All That Is*, can experience all these possibilities through different expressions of itself including humans. What we as expressions of the whole experience the *All That Is* experiences. We are the *All That Is* experiencing itself. As we withdraw from that state of Oneness we disconnect from its influence and things can get very unpleasant and very stupid. Archontic consciousness is at the extreme end of that. It has so disconnected from the influence of Oneness that it has become an inversion of unity and love, an inversion of everything, an inversion of life itself. Evil is appropriately live written backwards. Archontic consciousness is obsessed with death, an inversion of life,

and so its manifestations in Satanism are obsessed with death. They use inverted symbols in their rituals such as the inverted pentagram and cross. Sabbatians as Archontic consciousness incarnate invert Judaism and every other religion and culture they infiltrate. They seek disunity and chaos and they fear unity and harmony as they fear love like garlic to a vampire. As a result the Cult, Archons incarnate, act with such evil, psychopathy and lack of empathy and compassion disconnected as they are from the source of love. How could Bill Gates and the rest of the Archontic psychopaths do what they have to human society in the 'Covid' era with all the death, suffering and destruction involved and have no emotional consequence for the impact on others? Now you know. Why have Zuckerberg, Brin, Page, Wojcicki and company callously censored information warning about the dangers of the 'vaccine' while thousands have been dying and having severe, sometimes life-changing reactions? Now you know. Why have Tedros, Fauci, Whitty, Vallance and their like around the world been using case and death figures they're aware are fraudulent to justify lockdowns and all the deaths and destroyed lives that have come from that? Now you know. Why did Christian Drosten produce and promote a 'testing' protocol that he knew couldn't test for infectious disease which led to a global human catastrophe. Now you know. The Archontic mind doesn't give a shit (Fig 17). I personally think that Gates and major Cult insiders are a form of AI cyborg that the Archons want humans to become.

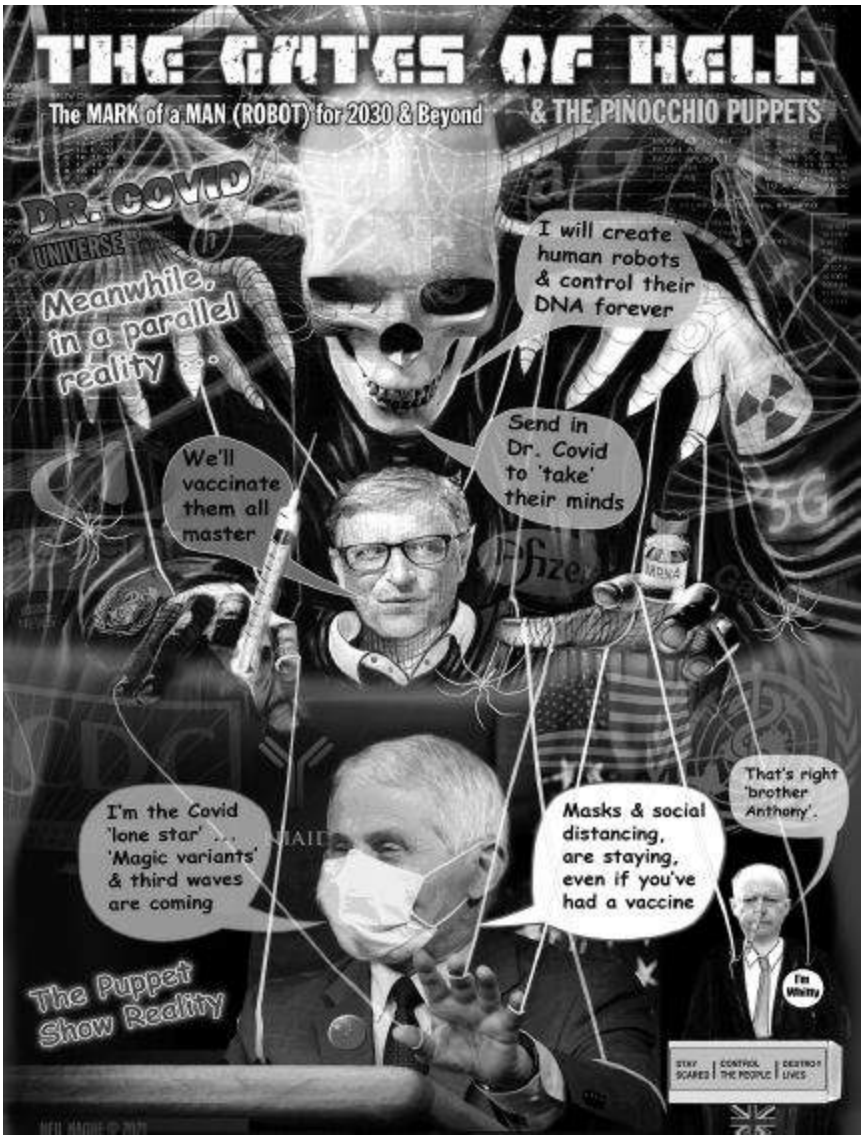


Figure 17: Artist Neil Hague's version of the 'Covid' hierarchy.

Human batteries

A state of such inversion does have its consequences, however. The level of disconnection from the Source of All means that you withdraw from that source of energetic sustenance and creativity. This means that you have to find your own supply of energetic power and it has – *us*. When the Morpheus character in the first *Matrix* movie held up a battery he spoke a profound truth when he said: 'The Matrix is a computer-generated dream world built to keep us under control in order to change the human being into one of

these.' The statement was true in all respects. We do live in a technologically-generated virtual reality simulation (more very shortly) and we have been manipulated to be an energy source for Archontic consciousness. The Disney-Pixar animated movie *Monsters, Inc*. in 2001 symbolised the dynamic when monsters in their world had no energy source and they would enter the human world to terrify children in their beds, catch the child's scream, terror (low-vibrational frequencies), and take that energy back to power the monster world. The lead character you might remember was a single giant eye and the symbolism of the Cult's all-seeing eye was obvious. Every thought and emotion is broadcast as a frequency unique to that thought and emotion. Feelings of love and joy, empathy and compassion, are high, quick, frequencies while fear, depression, anxiety, suffering and hate are low, slow, dense frequencies. Which kind do you think Archontic consciousness can connect with and absorb? In such a low and dense frequency state there's no way it can connect with the energy of love and joy. Archons can only feed off energy compatible with their own frequency and they and their Cult agents want to delete the human world of love and joy and manipulate the transmission of low vibrational frequencies through low-vibrational human mental and emotional states. *We are their energy source*. Wars are energetic banquets to the Archons – a world war even more so – and think how much low-frequency mental and emotional energy has been generated from the consequences for humanity of the 'Covid' hoax orchestrated by Archons incarnate like Gates.

The ancient practice of human sacrifice 'to the gods', continued in secret today by the Cult, is based on the same principle. 'The gods' are Archontic consciousness in different forms and the sacrifice is induced into a state of intense terror to generate the energy the Archontic frequency can absorb. Incarnate Archons in the ritual drink the blood which contains an adrenaline they crave which floods into the bloodstream when people are terrorised. Most of the sacrifices, ancient and modern, are children and the theme of 'sacrificing young virgins to the gods' is just code for children. They

have a particular pre-puberty energy that Archons want more than anything and the energy of the young in general is their target. The California Department of Education wants students to chant the names of Aztec gods (Archontic gods) once worshipped in human sacrifice rituals in a curriculum designed to encourage them to 'challenge racist, bigoted, discriminatory, imperialist/colonial beliefs', join 'social movements that struggle for social justice', and 'build new possibilities for a post-racist, post-systemic racism society'. It's the usual Woke crap that inverts racism and calls it anti-racism. In this case solidarity with 'indigenous tribes' is being used as an excuse to chant the names of 'gods' to which people were sacrificed (and still are in secret). What an example of Woke's inability to see beyond black and white, us and them, They condemn the colonisation of these tribal cultures by Europeans (quite right), but those cultures sacrificing people including children to their 'gods', and mass murdering untold numbers as the Aztecs did, is just fine. One chant is to the Aztec god Tezcatlipoca who had a man sacrificed to him in the 5th month of the Aztec calendar. His heart was cut out and he was eaten. Oh, that's okay then. Come on children … after three … Other sacrificial 'gods' for the young to chant their allegiance include Quetzalcoatl, Huitzilopochtli and Xipe Totec. The curriculum says that 'chants, affirmations, and energizers can be used to bring the class together, build unity around ethnic studies principles and values, and to reinvigorate the class following a lesson that may be emotionally taxing or even when student engagement may appear to be low'. Well, that's the cover story, anyway. Chanting and mantras are the repetition of a particular frequency generated from the vocal cords and chanting the names of these Archontic 'gods' tunes you into their frequency. That is the last thing you want when it allows for energetic synchronisation, attachment and perceptual influence. Initiates chant the names of their 'Gods' in their rituals for this very reason.

Vampires of the Woke

Paedophilia is another way that Archons absorb the energy of children. Paedophiles possessed by Archontic consciousness are used as the conduit during sexual abuse for discarnate Archons to vampire the energy of the young they desire so much. Stupendous numbers of children disappear every year never to be seen again although you would never know from the media. Imagine how much low-vibrational energy has been generated by children during the 'Covid' hoax when so many have become depressed and psychologically destroyed to the point of killing themselves. Shocking numbers of children are now taken by the state from loving parents to be handed to others. I can tell you from long experience of researching this since 1996 that many end up with paedophiles and assets of the Cult through corrupt and Cult-owned social services which in the reframing era has hired many psychopaths and emotionless automatons to do the job. Children are even stolen to order using spurious reasons to take them by the corrupt and secret (because they're corrupt) 'family courts'. I have written in detail in other books, starting with *The Biggest Secret* in 1997, about the ubiquitous connections between the political, corporate, government, intelligence and military elites (Cult operatives) and Satanism and paedophilia. If you go deep enough both networks have an interlocking leadership. The Woke mentality has been developed by the Cult for many reasons: To promote almost every aspect of its agenda; to hijack the traditional political left and turn it fascist; to divide and rule; and to target agenda pushbackers. But there are other reasons which relate to what I am describing here. How many happy and joyful Wokers do you ever see especially at the extreme end? They are a mental and psychological mess consumed by emotional stress and constantly emotionally cocked for the next explosion of indignation at someone referring to a female as a female. They are walking, talking, batteries as Morpheus might say emitting frequencies which both enslave them in low-vibrational bubbles of perceptual limitation and feed the Archons. Add to this the hatred claimed to be love; fascism claimed to 'anti-fascism', racism claimed to be 'anti-racism';

exclusion claimed to inclusion; and the abuse-filled Internet trolling. You have a purpose-built Archontic energy system with not a wind turbine in sight and all founded on Archontic *inversion*. We have whole generations now manipulated to serve the Archons with their actions and energy. They will be doing so their entire adult lives unless they snap out of their Archon-induced trance. Is it really a surprise that Cult billionaires and corporations put so much money their way? Where is the energy of joy and laughter, including laughing at yourself which is confirmation of your own emotional security? Mark Twain said: 'The human race has one really effective weapon, and that is laughter.' We must use it all the time. Woke has destroyed comedy because it has no humour, no joy, sense of irony, or self-deprecation. Its energy is dense and intense. *Mmmmm*, lunch says the Archontic frequency. Rudolf Steiner (1861-1925) was the Austrian philosopher and famous esoteric thinker who established Waldorf education or Steiner schools to treat children like unique expressions of consciousness and not minds to be programmed with the perceptions determined by authority. I'd been writing about this energy vampiring for decades when I was sent in 2016 a quote by Steiner. He was spot on:

> There are beings in the spiritual realms for whom anxiety and fear emanating from human beings offer welcome food. When humans have no anxiety and fear, then these creatures starve. If fear and anxiety radiates from people and they break out in panic, then these creatures find welcome nutrition and they become more and more powerful. These beings are hostile towards humanity. Everything that feeds on negative feelings, on anxiety, fear and superstition, despair or doubt, are in reality hostile forces in super-sensible worlds, launching cruel attacks on human beings, while they are being fed ... These are exactly the feelings that belong to contemporary culture and materialism; because it estranges people from the spiritual world, it is especially suited to evoke hopelessness and fear of the unknown in people, thereby calling up the above mentioned hostile forces against them.

Pause for a moment from this perspective and reflect on what has happened in the world since the start of 2020. Not only will pennies drop, but billion dollar bills. We see the same theme from Don Juan Matus, a Yaqui Indian shaman in Mexico and the information source for Peruvian-born writer, Carlos Castaneda, who wrote a series of

books from the 1960s to 1990s. Don Juan described the force manipulating human society and his name for the Archons was the predator:

> We have a predator that came from the depths of the cosmos and took over the rule of our lives. Human beings are its prisoners. The predator is our lord and master. It has rendered us docile, helpless. If we want to protest, it suppresses our protest. If we want to act independently, it demands that we don't do so ... indeed we are held prisoner!
>
> They took us over because we are food to them, and they squeeze us mercilessly because we are their sustenance. Just as we rear chickens in coops, the predators rear us in human coops, humaneros. Therefore, their food is always available to them.
>
> Different cultures, different eras, same recurring theme.

The 'ennoia' dilemma

Nag Hammadi Gnostic manuscripts say that Archon consciousness has no 'ennoia'. This is directly translated as 'intentionality', but I'll use the term 'creative imagination'. The *All That Is* in awareness of itself is the source of all creativity – all possibility – and the more disconnected you are from that source the more you are subsequently denied 'creative imagination'. Given that Archon consciousness is almost entirely disconnected it severely lacks creativity and has to rely on far more mechanical processes of thought and exploit the creative potential of those that do have 'ennoia'. You can see cases of this throughout human society. Archon consciousness almost entirely dominates the global banking system and if we study how that system works you will appreciate what I mean. Banks manifest 'money' out of nothing by issuing lines of 'credit' which is 'money' that has never, does not, and will never exist except in theory. It's a confidence trick. If you think 'credit' figures-on-a-screen 'money' is worth anything you accept it as payment. If you don't then the whole system collapses through lack of confidence in the value of that 'money'. Archontic bankers with no 'ennoia' are 'lending' 'money' that doesn't exist to humans that *do* have creativity – those that have the inspired ideas and create businesses and products. Archon banking feeds off human creativity

which it controls through 'money' creation and debt. Humans have the creativity and Archons exploit that for their own benefit and control while having none themselves. Archon Internet platforms like Facebook claim joint copyright of everything that creative users post and while Archontic minds like Zuckerberg may officially head that company it will be human creatives on the staff that provide the creative inspiration. When you have limitless 'money' you can then buy other companies established by creative humans. Witness the acquisition record of Facebook, Google and their like. Survey the Archon-controlled music industry and you see non-creative dark suit executives making their fortune from the human creativity of their artists. The cases are endless. Research the history of people like Gates and Zuckerberg and how their empires were built on exploiting the creativity of others. Archon minds cannot create out of nothing, but they are skilled (because they have to be) in what Gnostic texts call 'countermimicry'. They can imitate, but not innovate. Sabbatians trawl the creativity of others through backdoors they install in computer systems through their cybersecurity systems. Archon-controlled China is globally infamous for stealing intellectual property and I remember how Hong Kong, now part of China, became notorious for making counterfeit copies of the creativity of others – 'countermimicry'. With the now pervasive and all-seeing surveillance systems able to infiltrate any computer you can appreciate the potential for Archons to vampire the creativity of humans. Author John Lamb Lash wrote in his book about the Nag Hammadi texts, *Not In His Image*:

> Although they cannot originate anything, because they lack the divine factor of ennoia (intentionality), Archons can imitate with a vengeance. Their expertise is simulation (HAL, virtual reality). The Demiurge [Yaldabaoth] fashions a heaven world copied from the fractal patterns [of the original] ... His construction is celestial kitsch, like the fake Italianate villa of a Mafia don complete with militant angels to guard every portal.

This brings us to something that I have been speaking about since the turn of the millennium. Our reality is a simulation; a virtual reality that we think is real. No, I'm not kidding.

Human reality? Well, virtually

I had pondered for years about whether our reality is 'real' or some kind of construct. I remembered being immensely affected on a visit as a small child in the late 1950s to the then newly-opened Planetarium on the Marylebone Road in London which is now closed and part of the adjacent Madame Tussauds wax museum. It was in the middle of the day, but when the lights went out there was the night sky projected in the Planetarium's domed ceiling and it appeared to be so real. The experience never left me and I didn't know why until around the turn of the millennium when I became certain that our 'night sky' and entire reality is a projection, a virtual reality, akin to the illusory world portrayed in the *Matrix* movies. I looked at the sky one day in this period and it appeared to me like the domed roof of the Planetarium. The release of the first *Matrix* movie in 1999 also provided a synchronistic and perfect visual representation of where my mind had been going for a long time. I hadn't come across the Gnostic Nag Hammadi texts then. When I did years later the correlation was once again astounding. As I read Gnostic accounts from 1,600 years and more earlier it was clear that they were describing the same simulation phenomenon. They tell how the Yaldabaoth 'Demiurge' and Archons created a 'bad copy' of original reality to rule over all that were captured by its illusions and the body was a prison to trap consciousness in the 'bad copy' fake reality. Read how Gnostics describe the 'bad copy' and update that to current times and they are referring to what we would call today a virtual reality simulation.

Author John Lamb Lash said 'the Demiurge fashions a heaven world copied from the fractal patterns' of the original through expertise in 'HAL' or virtual reality simulation. Fractal patterns are part of the energetic information construct of our reality, a sort of blueprint. If these patterns were copied in computer terms it would indeed give you a copy of a 'natural' reality in a non-natural frequency and digital form. The principle is the same as making a copy of a website. The original website still exists, but now you can change the copy version to make it whatever you like and it can

become very different to the original website. Archons have done this with our reality, a *synthetic* copy of prime reality that still exists beyond the frequency walls of the simulation. Trapped within the illusions of this synthetic Matrix, however, were and are human consciousness and other expressions of prime reality and this is why the Archons via the Cult are seeking to make the human body synthetic and give us synthetic AI minds to complete the job of turning the entire reality synthetic including what we perceive to be the natural world. To quote Kurzweil: 'Nanobots will infuse all the matter around us with information. Rocks, trees, everything will become these intelligent creatures.' Yes, *synthetic* 'creatures' just as 'Covid' and other genetically-manipulating 'vaccines' are designed to make the human body synthetic. From this perspective it is obvious why Archons and their Cult are so desperate to infuse synthetic material into every human with their 'Covid' scam.

Let there be (electromagnetic) light

Yaldabaoth, the force that created the simulation, or Matrix, makes sense of the Gnostic reference to 'The Great Architect' and its use by Cult Freemasonry as the name of its deity. The designer of the Matrix in the movies is called 'The Architect' and that trilogy is jam-packed with symbolism relating to these subjects. I have contended for years that the angry Old Testament God (Yaldabaoth) is the 'God' being symbolically 'quoted' in the opening of Genesis as 'creating the world'. This is not the creation of prime reality – it's the creation of the *simulation*. The Genesis 'God' says: 'Let there be Light: and there was light.' But what is this 'Light'? I have said for decades that the speed of light (186,000 miles per second) is not the fastest speed possible as claimed by mainstream science and is in fact the frequency walls or outer limits of the Matrix. You can't have a fastest or slowest anything within all possibility when everything is possible. The human body is encoded to operate within the speed of light or *within the simulation* and thus we see only the tiny frequency band of visible *light*. Near-death experiencers who perceive reality outside the body during temporary 'death' describe a very different

form of light and this is supported by the Nag Hammadi texts. Prime reality beyond the simulation ('Upper Aeons' to the Gnostics) is described as a realm of incredible beauty, bliss, love and harmony – a realm of 'watery light' that is so powerful 'there are no shadows'. Our false reality of Archon control, which Gnostics call the 'Lower Aeons', is depicted as a realm with a different kind of 'light' and described in terms of chaos, 'Hell', 'the Abyss' and 'Outer Darkness', where trapped souls are tormented and manipulated by demons (relate that to the 'Covid' hoax alone). The watery light theme can be found in near-death accounts and it is not the same as *simulation* 'light' which is electromagnetic or radiation light within the speed of light – the 'Lower Aeons'. Simulation 'light' is the 'luminous fire' associated by Gnostics with the Archons. The Bible refers to Yaldabaoth as 'that old serpent, called the Devil, and Satan, which deceiveth the whole world' (Revelation 12:9). I think that making a simulated copy of prime reality ('countermimicry') and changing it dramatically while all the time manipulating humanity to believe it to be real could probably meet the criteria of deceiving the whole world. Then we come to the Cult god Lucifer – the *Light Bringer*. Lucifer is symbolic of Yaldabaoth, the bringer of radiation light that forms the bad copy simulation within the speed of light. 'He' is symbolised by the lighted torch held by the Statue of Liberty and in the name 'Illuminati'. Sabbatian-Frankism declares that Lucifer is the true god and Lucifer is the real god of Freemasonry honoured as their 'Great or Grand Architect of the Universe' (simulation).

I would emphasise, too, the way Archontic technologically-generated luminous fire of radiation has deluged our environment since I was a kid in the 1950s and changed the nature of The Field with which we constantly interact. Through that interaction technological radiation is changing us. The Smart Grid is designed to operate with immense levels of communication power with 5G expanding across the world and 6G, 7G, in the process of development. Radiation is the simulation and the Archontic manipulation system. Why wouldn't the Archon Cult wish to unleash radiation upon us to an ever-greater extreme to form

Kurzweil's 'cloud'? The plan for a synthetic human is related to the need to cope with levels of radiation beyond even anything we've seen so far. Biological humans would not survive the scale of radiation they have in their script. The Smart Grid is a technological sub-reality within the technological simulation to further disconnect five-sense perception from expanded consciousness. It's a technological prison of the mind.

Infusing the 'spirit of darkness'

A recurring theme in religion and native cultures is the manipulation of human genetics by a non-human force and most famously recorded as the biblical 'sons of god' (the god*s* plural in the original) who interbred with the daughters of men. The Nag Hammadi *Apocryphon of John* tells the same story this way:

> He [Yaldabaoth] sent his angels [Archons/demons] to the daughters of men, that they might take some of them for themselves and raise offspring for their enjoyment. And at first they did not succeed. When they had no success, they gathered together again and they made a plan together ... And the angels changed themselves in their likeness into the likeness of their mates, filling them with the spirit of darkness, which they had mixed for them, and with evil ... And they took women and begot children out of the darkness according to the likeness of their spirit.

Possession when a discarnate entity takes over a human body is an age-old theme and continues today. It's very real and I've seen it. Satanic and secret society rituals can create an energetic environment in which entities can attach to initiates and I've heard many stories of how people have changed their personality after being initiated even into lower levels of the Freemasons. I have been inside three Freemasonic temples, one at a public open day and two by just walking in when there was no one around to stop me. They were in Ryde, the town where I live, Birmingham, England, when I was with a group, and Boston, Massachusetts. They all felt the same energetically – dark, dense, low-vibrational and sinister. Demonic attachment can happen while the initiate has no idea what is going on. To them it's just a ritual to get in the Masons and do a bit of good

business. In the far more extreme rituals of Satanism human possession is even more powerful and they are designed to make possession possible. The hierarchy of the Cult is dictated by the power and perceived status of the possessing Archon. In this way the Archon hierarchy becomes the Cult hierarchy. Once the entity has attached it can influence perception and behaviour and if it attaches to the extreme then so much of its energy (information) infuses into the body information field that the hologram starts to reflect the nature of the possessing entity. This is the *Exorcist* movie type of possession when facial features change and it's known as shapeshifting. Islam's Jinn are said to be invisible tricksters who change shape, 'whisper', confuse and take human form. These are all traits of the Archons and other versions of the same phenomenon. Extreme possession could certainty infuse the 'spirit of darkness' into a partner during sex as the Nag Hammadi texts appear to describe. Such an infusion can change genetics which is also energetic information. Human genetics is information and the 'spirit of darkness' is information. Mix one with the other and change must happen. Islam has the concept of a 'Jinn baby' through possession of the mother and by Jinn taking human form. There are many ways that human genetics can be changed and remember that Archons have been aware all along of advanced techniques to do this. What is being done in human society today – and far more – was known about by Archons at the time of the 'fallen ones' and their other versions described in religions and cultures.

Archons and their human-world Cult are obsessed with genetics as we see today and they know this dictates how information is processed into perceived reality during a human life. They needed to produce a human form that would decode the simulation and this is symbolically known as 'Adam and Eve' who left the 'garden' (prime reality) and 'fell' into Matrix reality. The simulation is not a 'physical' construct (there is no 'physical'); it is a source of information. Think Wi-Fi again. The simulation is an energetic field encoded with information and body-brain systems are designed to decode that information encoded in wave or frequency form which

is transmitted to the brain as electrical signals. These are decoded by the brain to construct our sense of reality – an illusory 'physical' world that only exists in the brain or the mind. Virtual reality games mimic this process using the same sensory decoding system. Information is fed to the senses to decode a virtual reality that can appear so real, but isn't (Figs 18 and 19). Some scientists believe – and I agree with them – that what we perceive as 'physical' reality only exists when we are looking or observing. The act of perception or focus triggers the decoding systems which turn waveform information into holographic reality. When we are not observing something our reality reverts from a holographic state to a waveform state. This relates to the same principle as a falling tree not making a noise unless someone is there to hear it or decode it. The concept makes sense from the simulation perspective. A computer is not decoding all the information in a Wi-Fi field all the time and only decodes or brings into reality on the screen that part of Wi-Fi that it's decoding – focusing upon – at that moment.



Figure 18: Virtual reality technology 'hacks' into the body's five-sense decoding system.



Figure 19: The result can be experienced as very 'real'.

Interestingly, Professor Donald Hoffman at the Department of Cognitive Sciences at the University of California, Irvine, says that our experienced reality is like a computer interface that shows us only the level with which we interact while hiding all that exists beyond it: 'Evolution shaped us with a user interface that hides the truth. Nothing that we see is the truth – the very language of space and time and objects is the wrong language to describe reality.' He is correct in what he says on so many levels. Space and time are not a universal reality. They are a phenomenon of decoded *simulation* reality as part of the process of enslaving our sense of reality. Near-death experiencers report again and again how space and time did not exist as we perceive them once they were free of the body – body decoding systems. You can appreciate from this why Archons and their Cult are so desperate to entrap human attention in the five senses where we are in the Matrix and of the Matrix. Opening your mind to expanded states of awareness takes you beyond the information confines of the simulation and you become aware of knowledge and insights denied to you before. This is what we call 'awakening' – *awakening from the Matrix* – and in the final chapter I will relate this to current events.

Where are the 'aliens'?

A simulation would explain the so-called 'Fermi Paradox' named after Italian physicist Enrico Fermi (1901-1954) who created the first nuclear reactor. He considered the question of why there is such a lack of extraterrestrial activity when there are so many stars and planets in an apparently vast universe; but what if the night sky that we see, or think we do, is a simulated projection as I say? If you control the simulation and your aim is to hold humanity fast in essential ignorance would you want other forms of life including advanced life coming and going sharing information with humanity? Or would you want them to believe they were isolated and apparently alone? Themes of human isolation and apartness are common whether they be the perception of a lifeless universe or the fascist isolation laws of the 'Covid' era. Paradoxically the very

existence of a simulation means that we are not alone when some force had to construct it. My view is that experiences that people have reported all over the world for centuries with Reptilians and Grey entities are Archon phenomena as Nag Hammadi texts describe; and that benevolent 'alien' interactions are non-human groups that come in and out of the simulation by overcoming Archon attempts to keep them out. It should be highlighted, too, that Reptilians and Greys are obsessed with *genetics* and *technology* as related by cultural accounts and those who say they have been abducted by them. Technology is their way of overcoming some of the limitations in their creative potential and our technology-driven and controlled human society of today is *arch*etypical Archon-Reptilian-Grey modus operandi. Technocracy is really *Archon*tocracy. The Universe does not have to be as big as it appears with a simulation. There is no space or distance only information decoded into holographic reality. What we call 'space' is only the absence of holographic 'objects' and that 'space' is The Field of energetic information which connects everything into a single whole. The same applies with the artificially-generated information field of the simulation. The Universe is not big or small as a physical reality. It is decoded information, that's all, and its perceived size is decided by the way the simulation is encoded to make it appear. The entire night sky as we perceive it only exists in our brain and so where are those 'millions of light years'? The 'stars' on the ceiling of the Planetarium looked a vast distance away.

There's another point to mention about 'aliens'. I have been highlighting since the 1990s the plan to stage a fake 'alien invasion' to justify the centralisation of global power and a world military. Nazi scientist Werner von Braun, who was taken to America by Operation Paperclip after World War Two to help found NASA, told his American assistant Dr Carol Rosin about the Cult agenda when he knew he was dying in 1977. Rosin said that he told her about a sequence that would lead to total human control by a one-world government. This included threats from terrorism, rogue nations, meteors and asteroids before finally an 'alien invasion'. All of these

things, von Braun said, would be bogus and what I would refer to as a No-Problem-Reaction-Solution. Keep this in mind when 'the aliens are coming' is the new mantra. The aliens are not coming – they are *already here* and they have infiltrated human society while looking human. French-Canadian investigative journalist Serge Monast said in 1994 that he had uncovered a NASA/military operation called Project Blue Beam which fits with what Werner von Braun predicted. Monast died of a 'heart attack' in 1996 the day after he was arrested and spent a night in prison. He was 51. He said Blue Beam was a plan to stage an alien invasion that would include religious figures beamed holographically into the sky as part of a global manipulation to usher in a 'new age' of worshipping what I would say is the Cult 'god' Yaldabaoth in a one-world religion. Fake holographic asteroids are also said to be part of the plan which again syncs with von Braun. How could you stage an illusory threat from asteroids unless they were holographic inserts? This is pretty straightforward given the advanced technology outside the public arena and the fact that our 'physical' reality is holographic anyway. Information fields would be projected and we would decode them into the illusion of a 'physical' asteroid. If they can sell a global 'pandemic' with a 'virus' that doesn't exist what will humans not believe if government and media tell them?

All this is particularly relevant as I write with the Pentagon planning to release in June, 2021, information about 'UFO sightings'. I have been following the UFO story since the early 1990s and the common theme throughout has been government and military denials and cover up. More recently, however, the Pentagon has suddenly become more talkative and apparently open with Air Force pilot radar images released of unexplained craft moving and changing direction at speeds well beyond anything believed possible with human technology. Then, in March, 2021, former Director of National Intelligence John Ratcliffe said a Pentagon report months later in June would reveal a great deal of information about UFO sightings unknown to the public. He said the report would have 'massive implications'. The order to do this was included bizarrely

in a $2.3 trillion 'coronavirus' relief and government funding bill passed by the Trump administration at the end of 2020. I would add some serious notes of caution here. I have been pointing out since the 1990s that the US military and intelligence networks have long had craft – 'flying saucers' or anti-gravity craft – which any observer would take to be extraterrestrial in origin. Keeping this knowledge from the public allows craft flown by *humans* to be perceived as alien visitations. I am not saying that 'aliens' do not exist. I would be the last one to say that, but we have to be streetwise here. President Ronald Reagan told the UN General Assembly in 1987: 'I occasionally think how quickly our differences worldwide would vanish if we were facing an alien threat from outside this world.' That's the idea. Unite against a common 'enemy' with a common purpose behind your 'saviour force' (the Cult) as this age-old technique of mass manipulation goes global.

Science moves this way …

I could find only one other person who was discussing the simulation hypothesis publicly when I concluded it was real. This was Nick Bostrom, a Swedish-born philosopher at the University of Oxford, who has explored for many years the possibility that human reality is a computer simulation although his version and mine are not the same. Today the simulation and holographic reality hypothesis have increasingly entered the scientific mainstream. Well, the more open-minded mainstream, that is. Here are a few of the ever-gathering examples. American nuclear physicist Silas Beane led a team of physicists at the University of Bonn in Germany pursuing the question of whether we live in a simulation. They concluded that we probably do and it was likely based on a lattice of cubes. They found that cosmic rays align with that specific pattern. The team highlighted the Greisen–Zatsepin–Kuzmin (GZK) limit which refers to cosmic ray particle interaction with cosmic background radiation that creates an apparent boundary for cosmic ray particles. They say in a paper entitled 'Constraints on the Universe as a Numerical Simulation' that this 'pattern of constraint' is exactly what you

would find with a computer simulation. They also made the point that a simulation would create its own 'laws of physics' that would limit possibility. I've been making the same point for decades that the *perceived* laws of physics relate only to this reality, or what I would later call the simulation. When designers write codes to create computer and virtual reality games they are the equivalent of the laws of physics for that game. Players interact within the limitations laid out by the coding. In the same way those who wrote the codes for the simulation decided the laws of physics that would apply. These can be overridden by expanded states of consciousness, but not by those enslaved in only five-sense awareness where simulation codes rule. Overriding the codes is what people call 'miracles'. They are not. They are bypassing the encoded limits of the simulation. A population caught in simulation perception would have no idea that this was their plight. As the Bonn paper said: 'Like a prisoner in a pitch-black cell we would not be able to see the "walls" of our prison,' That's true if people remain mesmerised by the five senses. Open to expanded awareness and those walls become very clear. The main one is the speed of light.

American theoretical physicist James Gates is another who has explored the simulation question and found considerable evidence to support the idea. Gates was Professor of Physics at the University of Maryland, Director of The Center for String and Particle Theory, and on Barack Obama's Council of Advisors on Science and Technology. He and his team found *computer codes* of digital data embedded in the fabric of our reality. They relate to on-off electrical charges of 1 and 0 in the binary system used by computers. 'We have no idea what they are doing there', Gates said. They found within the energetic fabric mathematical sequences known as error-correcting codes or block codes that 'reboot' data to its original state or 'default settings' when something knocks it out of sync. Gates was asked if he had found a set of equations embedded in our reality indistinguishable from those that drive search engines and browsers and he said: 'That is correct.' Rich Terrile, director of the Centre for Evolutionary Computation and Automated Design at NASA's Jet

Propulsion Laboratory, has said publicly that he believes the Universe is a digital hologram that must have been created by a form of intelligence. I agree with that in every way. Waveform information is delivered electrically by the senses to the brain which constructs a *digital* holographic reality that we call the 'world'. This digital level of reality can be read by the esoteric art of numerology. Digital holograms are at the cutting edge of holographics today. We have digital technology everywhere designed to access and manipulate our digital level of perceived reality. Synthetic mRNA in 'Covid vaccines' has a digital component to manipulate the body's digital 'operating system'.

Reality is numbers

How many know that our reality can be broken down to numbers and codes that are the same as computer games? Max Tegmark, a physicist at the Massachusetts Institute of Technology (MIT), is the author of *Our Mathematical Universe* in which he lays out how reality can be entirely described by numbers and maths in the way that a video game is encoded with the 'physics' of computer games. Our world and computer virtual reality are essentially the same. Tegmark imagines the perceptions of characters in an advanced computer game when the graphics are so good they don't know they are in a game. They think they can bump into real objects (electromagnetic resistance in our reality), fall in love and feel emotions like excitement. When they began to study the apparently 'physical world' of the video game they would realise that everything was made of pixels (which have been found in our energetic reality as must be the case when on one level our world is digital). What computer game characters thought was physical 'stuff', Tegmark said, could actually be broken down into numbers:

> And we're exactly in this situation in our world. We look around and it doesn't seem that mathematical at all, but everything we see is made out of elementary particles like quarks and electrons. And what properties does an electron have? Does it have a smell or a colour or a texture? No! ... We physicists have come up with geeky names for [Electron] properties, like

electric charge, or spin, or lepton number, but the electron doesn't care what we call it, the properties are just numbers.

This is the illusory reality Gnostics were describing. This is the simulation. The A, C, G, and T codes of DNA have a binary value – A and C = 0 while G and T = 1. This has to be when the simulation is digital and the body must be digital to interact with it. Recurring mathematical sequences are encoded throughout reality and the body. They include the Fibonacci sequence in which the two previous numbers are added to get the next one, as in ... 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc. The sequence is encoded in the human face and body, proportions of animals, DNA, seed heads, pine cones, trees, shells, spiral galaxies, hurricanes and the number of petals in a flower. The list goes on and on. There are fractal patterns – a 'never-ending pattern that is infinitely complex and self-similar across all scales in the as above, so below, principle of holograms. These and other famous recurring geometrical and mathematical sequences such as Phi, Pi, Golden Mean, Golden Ratio and Golden Section are *computer codes* of the simulation. I had to laugh and give my head a shake the day I finished this book and it went into the production stage. I was sent an article in *Scientific American* published in April, 2021, with the headline 'Confirmed! We Live in a Simulation'. Two decades after I first said our reality is a simulation and the speed of light is it's outer limit the article suggested that we do live in a simulation and that the speed of light is its outer limit. I left school at 15 and never passed a major exam in my life while the writer was up to his eyes in qualifications. As I will explain in the final chapter *knowing* is far better than thinking and they come from very different sources. The article rightly connected the speed of light to the processing speed of the 'Matrix' and said what has been in my books all this time … 'If we are in a simulation, as it appears, then space is an abstract property written in code. It is not real'. No it's not and if we live in a simulation something created it and it wasn't *us*. 'That David Icke says we are manipulated by aliens' – he's crackers.'

Wow …

The reality that humanity thinks is so real is an illusion. Politicians, governments, scientists, doctors, academics, law enforcement, media, school and university curriculums, on and on, are all founded on a world that *does not exist* except as a simulated prison cell. Is it such a stretch to accept that 'Covid' doesn't exist when our entire 'physical' reality doesn't exist? Revealed here is the knowledge kept under raps in the Cult networks of compartmentalised secrecy to control humanity's sense of reality by inducing the population to believe in a reality that's not real. If it wasn't so tragic in its experiential consequences the whole thing would be hysterically funny. None of this is new to Renegade Minds. Ancient Greek philosopher Plato (about 428 to about 347BC) was a major influence on Gnostic belief and he described the human plight thousands of years ago with his Allegory of the Cave. He told the symbolic story of prisoners living in a cave who had never been outside. They were chained and could only see one wall of the cave while behind them was a fire that they could not see. Figures walked past the fire casting shadows on the prisoners' wall and those moving shadows became their sense of reality. Some prisoners began to study the shadows and were considered experts on them (today's academics and scientists), but what they studied was only an illusion (today's academics and scientists). A prisoner escaped from the cave and saw reality as it really is. When he returned to report this revelation they didn't believe him, called him mad and threatened to kill him if he tried to set them free. Plato's tale is not only a brilliant analogy of the human plight and our illusory reality. It describes, too, the dynamics of the 'Covid' hoax. I have only skimmed the surface of these subjects here. The aim of this book is to crisply connect all essential dots to put what is happening today into its true context. All subject areas and their connections in this chapter are covered in great evidential detail in *Everything You Need To Know, But Have Never Been Told* and *The Answer*.

They say that bewildered people 'can't see the forest for the trees'. Humanity, however, can't see the forest for the *twigs*. The five senses

see only twigs while Renegade Minds can see the forest and it's the forest where the answers lie with the connections that reveals. Breaking free of perceptual programming so the forest can be seen is the way we turn all this around. Not breaking free is how humanity got into this mess. The situation may seem hopeless, but I promise you it's not. We are a perceptual heartbeat from paradise if only we knew.

CHAPTER TWELVE

Escaping Wetiko

Life is simply a vacation from the infinite
Dean Cavanagh

Renegade Minds weave the web of life and events and see common themes in the apparently random. They are always there if you look for them and their pursuit is aided by incredible synchronicity that comes when your mind is open rather than mesmerised by what it thinks it can see.

Infinite awareness is infinite possibility and the more of infinite possibility that we access the more becomes infinitely possible. That may be stating the apparently obvious, but it is a devastatingly-powerful fact that can set us free. We are a point of attention within an infinity of consciousness. The question is how much of that infinity do we choose to access? How much knowledge, insight, awareness, wisdom, do we want to connect with and explore? If your focus is only in the five senses you will be influenced by a fraction of infinite awareness. I mean a range so tiny that it gives new meaning to infinitesimal. Limitation of self-identity and a sense of the possible limit accordingly your range of consciousness. We are what we think we are. Life is what we think it is. The dream is the dreamer and the dreamer is the dream. Buddhist philosophy puts it this way: 'As a thing is viewed, so it appears.' Most humans live in the realm of touch, taste, see, hear, and smell and that's the limit of their sense of the possible and sense of self. Many will follow a religion and speak of a God in his heaven, but their lives are still

dominated by the five senses in their perceptions and actions. The five senses become the arbiter of everything. When that happens all except a smear of infinity is sealed away from influence by the rigid, unyielding, reality bubbles that are the five-sense human or Phantom Self. Archon Cult methodology is to isolate consciousness within five-sense reality – the simulation – and then program that consciousness with a sense of self and the world through a deluge of life-long information designed to instil the desired perception that allows global control. Efforts to do this have increased dramatically with identity politics as identity bubbles are squeezed into the minutiae of five-sense detail which disconnect people even more profoundly from the infinite 'I'.

Five-sense focus and self-identity are like a firewall that limits access to the infinite realms. You only perceive one radio or television station and no other. We'll take that literally for a moment. Imagine a vast array of stations giving different information and angles on reality, but you only ever listen to one. Here we have the human plight in which the population is overwhelmingly confined to CultFM. This relates only to the frequency range of CultFM and limits perception and insight to that band – limits *possibility* to that band. It means you are connecting with an almost imperceptibly minuscule range of possibility and creative potential within the infinite Field. It's a world where everything seems apart from everything else and where synchronicity is rare. Synchronicity is defined in the dictionary as 'the happening by chance of two or more related or similar events at the same time'. Use of 'by chance' betrays a complete misunderstanding of reality. Synchronicity is not 'by chance'. As people open their minds, or 'awaken' to use the term, they notice more and more coincidences in their lives, bits of 'luck', apparently miraculous happenings that put them in the right place at the right time with the right people. Days become peppered with 'fancy meeting you here' and 'what are the chances of that?' My entire life has been lived like this and ever more so since my own colossal awakening in 1990 and 91 which transformed my sense of reality. Synchronicity is not 'by chance'; it is by accessing expanded

realms of possibility which allow expanded potential for manifestation. People broadcasting the same vibe from the same openness of mind tend to be drawn 'by chance' to each other through what I call frequency magnetism and it's not only people. In the last more than 30 years incredible synchronicity has also led me through the Cult maze to information in so many forms and to crucial personal experiences. These 'coincidences' have allowed me to put the puzzle pieces together across an enormous array of subjects and situations. Those who have breached the bubble of five-sense reality will know exactly what I mean and this escape from the perceptual prison cell is open to everyone whenever they make that choice. This may appear super-human when compared with the limitations of 'human', but it's really our natural state. 'Human' as currently experienced is consciousness in an unnatural state of induced separation from the infinity of the whole. I'll come to how this transformation into unity can be made when I have described in more detail the force that holds humanity in servitude by denying this access to infinite self.

The Wetiko factor

I have been talking and writing for decades about the way five-sense mind is systematically barricaded from expanded awareness. I have used the analogy of a computer (five-sense mind) and someone at the keyboard (expanded awareness). Interaction between the computer and the operator is symbolic of the interaction between five-sense mind and expanded awareness. The computer directly experiences the Internet and the operator experiences the Internet via the computer which is how it's supposed to be – the two working as one. Archons seek to control that point where the operator connects with the computer to stop that interaction (Fig 20). Now the operator is banging the keyboard and clicking the mouse, but the computer is not responding and this happens when the computer is taken over – *possessed* – by an appropriately-named computer 'virus'. The operator has lost all influence over the computer which goes its own way making decisions under the control of the 'virus'. I have

just described the dynamic through which the force known to Gnostics as Yaldabaoth and Archons disconnects five-sense mind from expanded awareness to imprison humanity in perceptual servitude.



Figure 20: The mind 'virus' I have been writing about for decades seeks to isolate five-sense mind (the computer) from the true 'I'. (Image by Neil Hague).

About a year ago I came across a Native American concept of Wetiko which describes precisely the same phenomenon. Wetiko is the spelling used by the Cree and there are other versions including wintiko and windigo used by other tribal groups. They spell the name with lower case, but I see Wetiko as a proper noun as with Archons and prefer a capital. I first saw an article about Wetiko by writer and researcher Paul Levy which so synced with what I had been writing about the computer/operator disconnection and later the Archons. I then read his book, the fascinating *Dispelling Wetiko, Breaking the Spell of Evil*. The parallels between what I had concluded long before and the Native American concept of Wetiko were so clear and obvious that it was almost funny. For Wetiko see the Gnostic Archons for sure and the Jinn, the Predators, and every other name for a force of evil, inversion and chaos. Wetiko is the Native American name for the force that divides the computer from

the operator (Fig 21). Indigenous author Jack D. Forbes, a founder of the Native American movement in the 1960s, wrote another book about Wetiko entitled *Columbus And Other Cannibals – The Wetiko Disease of Exploitation, Imperialism, and Terrorism* which I also read. Forbes says that Wetiko refers to an evil person or spirit 'who terrorizes other creatures by means of terrible acts, including cannibalism'. Zulu shaman Credo Mutwa told me that African accounts tell how cannibalism was brought into the world by the Chitauri 'gods' – another manifestation of Wetiko. The distinction between 'evil person or spirit' relates to Archons/Wetiko possessing a human or acting as pure consciousness. Wetiko is said to be a sickness of the soul or spirit and a state of being that takes but gives nothing back – the Cult and its operatives perfectly described. Black Hawk, a Native American war leader defending their lands from confiscation, said European invaders had 'poisoned hearts' – Wetiko hearts – and that this would spread to native societies. Mention of the heart is very significant as we shall shortly see. Forbes writes: 'Tragically, the history of the world for the past 2,000 years is, in great part, the story of the epidemiology of the wetiko disease.' Yes, and much longer. Forbes is correct when he says: 'The wetikos destroyed Egypt and Babylon and Athens and Rome and Tenochtitlan [capital of the Aztec empire] and perhaps now they will destroy the entire earth.' Evil, he said, is the number one export of a Wetiko culture – see its globalisation with 'Covid'. Constant war, mass murder, suffering of all kinds, child abuse, Satanism, torture and human sacrifice are all expressions of Wetiko and the Wetiko possessed. The world is Wetiko made manifest, *but it doesn't have to be*. There is a way out of this even now.



Figure 21: The mind 'virus' is known to Native Americans as 'Wetiko'. (Image by Neil Hague).

Cult of Wetiko

Wetiko is the Yaldabaoth frequency distortion that seeks to attach to human consciousness and absorb it into its own. Once this connection is made Wetiko can drive the perceptions of the target which they believe to be coming from their own mind. All the horrors of history and today from mass killers to Satanists, paedophiles like Jeffrey Epstein and other psychopaths, are the embodiment of Wetiko and express its state of being in all its grotesqueness. The Cult is Wetiko incarnate, Yaldabaoth incarnate, and it seeks to facilitate Wetiko assimilation of humanity in totality into its distortion by manipulating the population into low frequency states that match its own. Paul Levy writes: 'Holographically enforced within the psyche of every human being the wetiko virus pervades and underlies the entire field of consciousness, and can therefore potentially manifest through any one of us at any moment if we are not mindful.' The 'Covid' hoax has achieved this with many people, but others have not fallen into Wetiko's frequency lair. Players in the 'Covid' human catastrophe including Gates, Schwab, Tedros, Fauci, Whitty, Vallance, Johnson, Hancock, Ferguson, Drosten, and all the rest, including the psychopath psychologists, are expressions of Wetiko. This is why

they have no compassion or empathy and no emotional consequence for what they do that would make them stop doing it. Observe all the people who support the psychopaths in authority against the Pushbackers despite the damaging impact the psychopaths have on their own lives and their family's lives. You are again looking at Wetiko possession which prevents them seeing through the lies to the obvious scam going on. *Why can't they see it?* Wetiko won't let them see it. The perceptual divide that has now become a chasm is between the Wetikoed and the non-Wetikoed.

Paul Levy describes Wetiko in the same way that I have long described the Archontic force. They are the same distorted consciousness operating across dimensions of reality: '… the subtle body of wetiko is not located in the third dimension of space and time, literally existing in another dimension … it is able to affect ordinary lives by mysteriously interpenetrating into our three-dimensional world.' Wetiko does this through its incarnate representatives in the Cult and by weaving itself into The Field which on our level of reality is the electromagnetic information field of the simulation or Matrix. More than that, the simulation *is* Wetiko / Yaldabaoth. Caleb Scharf, Director of Astrobiology at Columbia University, has speculated that 'alien life' could be so advanced that it has transcribed itself into the quantum realm to become what we call physics. He said intelligence indistinguishable from the fabric of the Universe would solve many of its greatest mysteries:

> Perhaps hyper-advanced life isn't just external. Perhaps it's already all around. It is embedded in what we perceive to be physics itself, from the root behaviour of particles and fields to the phenomena of complexity and emergence ... In other words, life might not just be in the equations. It might BE the equations [My emphasis].

Scharf said it is possible that 'we don't recognise advanced life because it forms an integral and unsuspicious part of what we've considered to be the natural world'. I agree. Wetiko/Yaldabaoth *is* the simulation. We are literally in the body of the beast. But that doesn't mean it has to control us. We all have the power to overcome Wetiko

influence and the Cult knows that. I doubt it sleeps too well because it knows that.

Which Field?

This, I suggest, is how it all works. There are two Fields. One is the fierce electromagnetic light of the Matrix within the speed of light; the other is the 'watery light' of The Field beyond the walls of the Matrix that connects with the Great Infinity. Five-sense mind and the decoding systems of the body attach us to the Field of Matrix light. They have to or we could not experience this reality. Five-sense mind sees only the Matrix Field of information while our expanded consciousness is part of the Infinity Field. When we open our minds, and most importantly our hearts, to the Infinity Field we have a mission control which gives us an expanded perspective, a road map, to understand the nature of the five-sense world. If we are isolated only in five-sense mind there is no mission control. We're on our own trying to understand a world that's constantly feeding us information to ensure we do not understand. People in this state can feel 'lost' and bewildered with no direction or radar. You can see ever more clearly those who are influenced by the Fields of Big Infinity or little five-sense mind simply by their views and behaviour with regard to the 'Covid' hoax. We have had this division throughout known human history with the mass of the people on one side and individuals who could see and intuit beyond the walls of the simulation – Plato's prisoner who broke out of the cave and saw reality for what it is. Such people have always been targeted by Wetiko/Archon-possessed authority, burned at the stake or demonised as mad, bad and dangerous. The Cult today and its global network of 'anti-hate', 'anti-fascist' Woke groups are all expressions of Wetiko attacking those exposing the conspiracy, 'Covid' lies and the 'vaccine' agenda.

Woke as a whole is Wetiko which explains its black and white mentality and how at one it is with the Wetiko-possessed Cult. Paul Levy said: 'To be in this paradigm is to still be under the thrall of a two-valued logic – where things are either true or false – of a

wetikoized mind.' Wetiko consciousness is in a permanent rage, therefore so is Woke, and then there is Woke inversion and contradiction. 'Anti-fascists' act like fascists because fascists *and* 'anti-fascists' are both Wetiko at work. Political parties act the same while claiming to be different for the same reason. Secret society and satanic rituals are attaching initiates to Wetiko and the cold, ruthless, psychopathic mentality that secures the positions of power all over the world is Wetiko. Reframing 'training programmes' have the same cumulative effect of attaching Wetiko and we have their graduates described as automatons and robots with a cold, psychopathic, uncaring demeanour. They are all traits of Wetiko possession and look how many times they have been described in this book and elsewhere with regard to personnel behind 'Covid' including the police and medical profession. Climbing the greasy pole in any profession in a Wetiko society requires traits of Wetiko to get there and that is particularly true of politics which is not about fair competition and pre-eminence of ideas. It is founded on how many backs you can stab and arses you can lick. This culminated in the global 'Covid' coordination between the Wetiko possessed who pulled it off in all the different countries without a trace of empathy and compassion for their impact on humans. Our sight sense can see only holographic form and not the Field which connects holographic form. Therefore we perceive 'physical' objects with 'space' in between. In fact that 'space' is energy/consciousness operating on multiple frequencies. One of them is Wetiko and that connects the Cult psychopaths, those who submit to the psychopaths, and those who serve the psychopaths in the media operations of the world. Wetiko is Gates. Wetiko is the mask-wearing submissive. Wetiko is the fake journalist and 'fact-checker'. The Wetiko Field is coordinating the whole thing. Psychopaths, gofers, media operatives, 'anti-hate' hate groups, 'fact-checkers' and submissive people work as one unit *even without human coordination* because they are attached to the *same* Field which is organising it all (Fig 22). Paul Levy is here describing how Wetiko-possessed people are drawn together and refuse to let any information breach their rigid

perceptions. He was writing long before 'Covid', but I think you will recognise followers of the 'Covid' religion *oh just a little bit*:

People who are channelling the vibratory frequency of wetiko align with each other through psychic resonance to reinforce their unspoken shared agreement so as to uphold their deranged view of reality. Once an unconscious content takes possession of certain individuals, it irresistibly draws them together by mutual attraction and knits them into groups tied together by their shared madness that can easily swell into an avalanche of insanity.

A psychic epidemic is a closed system, which is to say that it is insular and not open to any new information or informing influences from the outside world which contradict its fixed, limited, and limiting perspective.

There we have the Woke mind and the 'Covid' mind. Compatible resonance draws the awakening together, too, which is clearly happening today.



Figure 22: The Wetiko Field from which the Cult pyramid and its personnel are made manifest. (Image by Neil Hague).

Spiritual servitude

Wetiko doesn't care about humans. It's not human; it just possesses humans for its own ends and the effect (depending on the scale of

possession) can be anything from extreme psychopathy to unquestioning obedience. Wetiko's worst nightmare is for human consciousness to expand beyond the simulation. Everything is focussed on stopping that happening through control of information, thus perception, thus frequency. The 'education system', media, science, medicine, academia, are all geared to maintaining humanity in five-sense servitude as is the constant stimulation of low-vibrational mental and emotional states (see 'Covid'). Wetiko seeks to dominate those subconscious spaces between five-sense perception and expanded consciousness where the computer meets the operator. From these subconscious hiding places Wetiko speaks to us to trigger urges and desires that we take to be our own and manipulate us into anything from low-vibrational to psychopathic states. Remember how Islam describes the Jinn as invisible tricksters that 'whisper' and confuse. Wetiko is the origin of the 'trickster god' theme that you find in cultures all over the world. Jinn, like the Archons, are Wetiko which is terrified of humans awakening and reconnecting with our true self for then its energy source has gone. With that the feedback loop breaks between Wetiko and human perception that provides the energetic momentum on which its very existence depends as a force of evil. Humans are both its target and its source of survival, but only if we are operating in low-vibrational states of fear, hate, depression and the background anxiety that most people suffer. We are Wetiko's target because we are its key to survival. It needs us, not the other way round. Paul Levy writes:

A vampire has no intrinsic, independent, substantial existence in its own right; it only exists in relation to us. The pathogenic, vampiric mind-parasite called wetiko is nothing in itself – not being able to exist from its own side – yet it has a 'virtual reality' such that it can potentially destroy our species …

…The fact that a vampire is not reflected by a mirror can also mean that what we need to see is that there's nothing, no-thing to see, other than ourselves. The fact that wetiko is the expression of something inside of us means that the cure for wetiko is with us as well. The critical issue is finding this cure within us and then putting it into effect.

Evil begets evil because if evil does not constantly expand and find new sources of energetic sustenance its evil, its *distortion*, dies with the assimilation into balance and harmony. Love is the garlic to Wetiko's vampire. Evil, the absence of love, cannot exist in the presence of love. I think I see a way out of here. I have emphasised so many times over the decades that the Archons/Wetiko and their Cult are not all powerful. *They are not*. I don't care how it looks even now *they are not*. I have not called them little boys in short trousers for effect. I have said it because it is true. Wetiko's insatiable desire for power over others is not a sign of its omnipotence, but its insecurity. Paul Levy writes: 'Due to the primal fear which ultimately drives it and which it is driven to cultivate, wetiko's body politic has an intrinsic and insistent need for centralising power and control so as to create imagined safety for itself.' *Yeeeeeees!* Exactly! Why does Wetiko want humans in an ongoing state of fear? Wetiko itself *is* fear and it is petrified of love. As evil is an absence of love, so love is an absence of fear. Love conquers all and *especially* Wetiko which *is* fear. Wetiko brought fear into the world when it wasn't here before. *Fear* was the 'fall', the fall into low-frequency ignorance and illusion – fear is **F**alse **E**motion **A**ppearing **R**eal. The simulation is driven and energised by fear because Wetiko/Yaldabaoth (fear) *are* the simulation. Fear is the absence of love and Wetiko is the absence of love.

Wetiko today

We can now view current events from this level of perspective. The 'Covid' hoax has generated momentous amounts of ongoing fear, anxiety, depression and despair which have empowered Wetiko. No wonder people like Gates have been the instigators when they are Wetiko incarnate and exhibit every trait of Wetiko in the extreme. See how cold and unemotional these people are like Gates and his cronies, how dead of eye they are. That's Wetiko. Sabbatians are Wetiko and everything they control including the World Health Organization, Big Pharma and the 'vaccine' makers, national 'health'

hierarchies, corporate media, Silicon Valley, the banking system, and the United Nations with its planned transformation into world government. All are controlled and possessed by the Wetiko distortion into distorting human society in its image. We are with this knowledge at the gateway to understanding the world. Divisions of race, culture, creed and sexuality are diversions to hide the real division between those possessed and influenced by Wetiko and those that are not. The 'Covid' hoax has brought both clearly into view. Human behaviour is not about race. Tyrants and dictatorships come in all colours and creeds. What unites the US president bombing the innocent and an African tribe committing genocide against another as in Rwanda? What unites them? *Wetiko*. All wars are Wetiko, all genocide is Wetiko, all hunger over centuries in a world of plenty is Wetiko. Children going to bed hungry, including in the West, is Wetiko. Cult-generated Woke racial divisions that focus on the body are designed to obscure the reality that divisions in behaviour are manifestations of mind, not body. Obsession with body identity and group judgement is a means to divert attention from the real source of behaviour – mind and perception. Conflict sown by the Woke both within themselves and with their target groups are Wetiko providing lunch for itself through still more agents of the division, chaos, and fear on which it feeds. The Cult is seeking to assimilate the entirety of humanity and all children and young people into the Wetiko frequency by manipulating them into states of fear and despair. Witness all the suicide and psychological unravelling since the spring of 2020. Wetiko psychopaths want to impose a state of unquestioning obedience to authority which is no more than a conduit for Wetiko to enforce its will and assimilate humanity into itself. It needs us to believe that resistance is futile when it fears resistance and even more so the game-changing non-cooperation with its impositions. It can use violent resistance for its benefit. Violent impositions and violent resistance are *both* Wetiko. The Power of Love with its Power of No will sweep Wetiko from our world. Wetiko and its Cult know that. They just don't want us to know.

AI Wetiko

This brings me to AI or artificial intelligence and something else Wetikos don't want us to know. What is AI *really*? I know about computer code algorithms and AI that learns from data input. These, however, are more diversions, the expeditionary force, for the real AI that they want to connect to the human brain as promoted by Silicon Valley Wetikos like Kurzweil. What is this AI? It is the frequency of *Wetiko*, the frequency of the Archons. The connection of AI to the human brain is the connection of the Wetiko frequency to create a Wetiko hive mind and complete the job of assimilation. The hive mind is planned to be controlled from Israel and China which are both 100 percent owned by Wetiko Sabbatians. The assimilation process has been going on minute by minute in the 'smart' era which fused with the 'Covid' era. We are told that social media is scrambling the minds of the young and changing their personality. This is true, but what is social media? Look more deeply at how it works, how it creates divisions and conflict, the hostility and cruelty, the targeting of people until they are destroyed. That's Wetiko. Social media is manipulated to tune people to the Wetiko frequency with all the emotional exploitation tricks employed by platforms like Facebook and its Wetiko front man, Zuckerberg. Facebook's Instagram announced a new platform for children to overcome a legal bar on them using the main site. This is more Wetiko exploitation and manipulation of kids. Amnesty International likened the plan to foxes offering to guard the henhouse and said it was incompatible with human rights. Since when did Wetiko or Zuckerberg (I repeat myself) care about that? Would Brin and Page at Google, Wojcicki at YouTube, Bezos at Amazon and whoever the hell runs Twitter act as they do if they were not channelling Wetiko? Would those who are developing technologies for no other reason than human control? How about those designing and selling technologies to kill people and Big Pharma drug and 'vaccine' producers who know they will end or devastate lives? Quite a thought for these people to consider is that if you are Wetiko in a human life you are Wetiko on the 'other side' unless your frequency

changes and that can only change by a change of perception which becomes a change of behaviour. Where Gates is going does not bear thinking about although perhaps that's exactly where he wants to go. Either way, that's where he's going. His frequency will make it so.

The frequency lair

I have been saying for a long time that a big part of the addiction to smartphones and devices is that a frequency is coming off them that entraps the mind. People spend ages on their phones and sometimes even a minute or so after they put them down they pick them up again and it all repeats. 'Covid' lockdowns will have increased this addiction a million times for obvious reasons. Addictions to alcohol overindulgence and drugs are another way that Wetiko entraps consciousness to attach to its own. Both are symptoms of low-vibrational psychological distress which alcoholism and drug addiction further compound. Do we think it's really a coincidence that access to them is made so easy while potions that can take people into realms beyond the simulation are banned and illegal? I have explored smartphone addiction in other books, the scale is mind-blowing, and that level of addiction does not come without help. Tech companies that make these phones are Wetiko and they will have no qualms about destroying the minds of children. We are seeing again with these companies the Wetiko perceptual combination of psychopathic enforcers and weak and meek unquestioning compliance by the rank and file.

The global Smart Grid is the Wetiko Grid and it is crucial to complete the Cult endgame. The simulation is radiation and we are being deluged with technological radiation on a devastating scale. Wetiko frauds like Elon Musk serve Cult interests while occasionally criticising them to maintain his street-cred. 5G and other forms of Wi-Fi are being directed at the earth from space on a volume and scale that goes on increasing by the day. Elon Musk's (officially) SpaceX Starlink project is in the process of putting tens of thousands of satellites in low orbit to cover every inch of the planet with 5G and other Wi-Fi to create Kurzweil's global 'cloud' to which the

human mind is planned to be attached very soon. SpaceX has approval to operate 12,000 satellites with more than 1,300 launched at the time of writing and applications filed for *30,000* more. Other operators in the Wi-Fi, 5G, low-orbit satellite market include OneWeb (UK), Telesat (Canada), and AST & Science (US). Musk tells us that AI could be the end of humanity and then launches a company called Neuralink to connect the human brain to computers. Musk's (in theory) Tesla company is building electric cars and the driverless vehicles of the smart control grid. As frauds and bullshitters go Elon Musk in my opinion is Major League.

5G and technological radiation in general are destructive to human health, genetics and psychology and increasing the strength of artificial radiation underpins the five-sense perceptual bubbles which are themselves expressions of radiation or electromagnetism. Freedom activist John Whitehead was so right with his 'databit by databit, we are building our own electronic concentration camps'. The Smart Grid and 5G is a means to control the human mind and infuse perceptual information into The Field to influence anyone in sync with its frequency. You can change perception and behaviour en masse if you can manipulate the population into those levels of frequency and this is happening all around us today. The arrogance of Musk and his fellow Cult operatives knows no bounds in the way that we see with Gates. Musk's satellites are so many in number already they are changing the night sky when viewed from Earth. The astronomy community has complained about this and they have seen nothing yet. Some consequences of Musk's Wetiko hubris include: Radiation; visible pollution of the night sky; interference with astronomy and meteorology; ground and water pollution from intensive use of increasingly many spaceports; accumulating space debris; continual deorbiting and burning up of aging satellites, polluting the atmosphere with toxic dust and smoke; and ever-increasing likelihood of collisions. A collective public open letter of complaint to Musk said:

> We are writing to you … because SpaceX is in process of surrounding the Earth with a network of thousands of satellites whose very purpose is to irradiate every square inch of the

Earth. SpaceX, like everyone else, is treating the radiation as if it were not there. As if the mitochondria in our cells do not depend on electrons moving undisturbed from the food we digest to the oxygen we breathe.

As if our nervous systems and our hearts are not subject to radio frequency interference like any piece of electronic equipment. As if the cancer, diabetes, and heart disease that now afflict a majority of the Earth's population are not metabolic diseases that result from interference with our cellular machinery. As if insects everywhere, and the birds and animals that eat them, are not starving to death as a result.

People like Musk and Gates believe in their limitless Wetiko arrogance that they can do whatever they like to the world because they own it. Consequences for humanity are irrelevant. It's absolutely time that we stopped taking this shit from these self-styled masters of the Earth when you consider where this is going.

Why is the Cult so anti-human?

I hear this question often: Why would they do this when it will affect them, too? Ah, but will it? Who is this *them*? Forget their bodies. They are just vehicles for Wetiko consciousness. When you break it all down to the foundations we are looking at a state of severely distorted consciousness targeting another state of consciousness for assimilation. The rest is detail. The simulation is the fly-trap in which unique sensations of the five senses create a cycle of addiction called reincarnation. Renegade Minds see that everything which happens in our reality is a smaller version of the whole picture in line with the holographic principle. Addiction to the radiation of smart technology is a smaller version of addiction to the whole simulation. Connecting the body/brain to AI is taking that addiction on a giant step further to total ongoing control by assimilating human incarnate consciousness into Wetiko. I have watched during the 'Covid' hoax how many are becoming ever more profoundly attached to Wetiko's perceptual calling cards of aggressive response to any other point of view ('There is no other god but me'), psychopathic lack of compassion and empathy, and servile submission to the narrative and will of authority. Wetiko is the psychopaths *and* subservience to psychopaths. The Cult of Wetiko is

so anti-human because it is *not* human. It embarked on a mission to destroy human by targeting everything that it means to be human and to survive as human. 'Covid' is not the end, just a means to an end. The Cult with its Wetiko consciousness is seeking to change Earth systems, including the atmosphere, to suit them, not humans. The gathering bombardment of 5G alone from ground and space is dramatically changing The Field with which the five senses interact. There is so much more to come if we sit on our hands and hope it will all go away. It is not meant to go away. It is meant to get ever more extreme and we need to face that while we still can – just.

Carbon dioxide is the gas of life. Without that human is over. Kaput, gone, history. No natural world, no human. The Cult has created a cock and bull story about carbon dioxide and climate change to justify its reduction to the point where Gates and the ignoramus Biden 'climate chief' John Kerry want to suck it out of the atmosphere. Kerry wants to do this because his master Gates does. Wetikos have made the gas of life a demon with the usual support from the Wokers of Extinction Rebellion and similar organisations and the bewildered puppet-child that is Greta Thunberg who was put on the world stage by Klaus Schwab and the World Economic Forum. The name Extinction Rebellion is both ironic and as always Wetiko inversion. The gas that we need to survive must be reduced to save us from extinction. The most basic need of human is oxygen and we now have billions walking around in face nappies depriving body and brain of this essential requirement of human existence. More than that 5G at 60 gigahertz interacts with the oxygen molecule to reduce the amount of oxygen the body can absorb into the bloodstream. The obvious knock-on consequences of that for respiratory and cognitive problems and life itself need no further explanation. Psychopaths like Musk are assembling a global system of satellites to deluge the human atmosphere with this insanity. The man should be in jail. Here we have two most basic of human needs, oxygen and carbon dioxide, being dismantled.

Two others, water and food, are getting similar treatment with the United Nations Agendas 21 and 2030 – the Great Reset – planning to

centrally control all water and food supplies. People will not even own rain water that falls on their land. Food is affected at the most basic level by reducing carbon dioxide. We have genetic modification or GMO infiltrating the food chain on a mass scale, pesticides and herbicides polluting the air and destroying the soil. Freshwater fish that provide livelihoods for 60 million people and feed hundreds of millions worldwide are being 'pushed to the brink' according the conservationists while climate change is the only focus. Now we have Gates and Schwab wanting to dispense with current food sources all together and replace them with a synthetic version which the Wetiko Cult would control in terms of production and who eats and who doesn't. We have been on the Totalitarian Tiptoe to this for more than 60 years as food has become ever more processed and full of chemical shite to the point today when it's not natural food at all. As Dr Tom Cowan says: 'If it has a label don't eat it.' Bill Gates is now the biggest owner of farmland in the United States and he does nothing without an ulterior motive involving the Cult. Klaus Schwab wrote: 'To feed the world in the next 50 years we will need to produce as much food as was produced in the last 10,000 years … food security will only be achieved, however, if regulations on genetically modified foods are adapted to reflect the reality that gene editing offers a precise, efficient and safe method of improving crops.' Liar. People and the world are being targeted with aluminium through vaccines, chemtrails, food, drink cans, and endless other sources when aluminium has been linked to many health issues including dementia which is increasing year after year. Insects, bees and wildlife essential to the food chain are being deleted by pesticides, herbicides and radiation which 5G is dramatically increasing with 6G and 7G to come. The pollinating bee population is being devastated while wildlife including birds, dolphins and whales are having their natural radar blocked by the effects of ever-increasing radiation. In the summer windscreens used to be splattered with insects so numerous were they. It doesn't happen now. Where have they gone?

Synthetic everything

The Cult is introducing genetically-modified versions of trees, plants and insects including a Gates-funded project to unleash hundreds of millions of genetically-modified, lab-altered and patented male mosquitoes to mate with wild mosquitoes and induce genetic flaws that cause them to die out. Clinically-insane Gates-funded Japanese researchers have developed mosquitos that spread vaccine and are dubbed 'flying vaccinators'. Gates is funding the modification of weather patterns in part to sell the myth that this is caused by carbon dioxide and he's funding geoengineering of the skies to change the atmosphere. Some of this came to light with the Gates-backed plan to release tonnes of chalk into the atmosphere to 'deflect the Sun and cool the planet'. Funny how they do this while the heating effect of the Sun is not factored into climate projections focussed on carbon dioxide. The reason is that they want to reduce carbon dioxide (so don't mention the Sun), but at the same time they do want to reduce the impact of the Sun which is so essential to human life and health. I have mentioned the sun-cholesterol-vitamin D connection as they demonise the Sun with warnings about skin cancer (caused by the chemicals in sun cream they tell you to splash on). They come from the other end of the process with statin drugs to reduce cholesterol that turns sunlight into vitamin D. A lack of vitamin D leads to a long list of health effects and how vitamin D levels must have fallen with people confined to their homes over 'Covid'. Gates is funding other forms of geoengineering and most importantly chemtrails which are dropping heavy metals, aluminium and self-replicating nanotechnology onto the Earth which is killing the natural world. See *Everything You Need To Know, But Have Never Been Told* for the detailed background to this.

Every human system is being targeted for deletion by a force that's not human. The Wetiko Cult has embarked on the process of transforming the human body from biological to synthetic biological as I have explained. Biological is being replaced by the artificial and synthetic – Archontic 'countermimicry' – right across human society. The plan eventually is to dispense with the human body altogether

and absorb human consciousness – which it wouldn't really be by then – into cyberspace (the simulation which is Wetiko/Yaldabaoth). Preparations for that are already happening if people would care to look. The alternative media rightly warns about globalism and 'the globalists', but this is far bigger than that and represents the end of the human race as we know it. The 'bad copy' of prime reality that Gnostics describe was a bad copy of harmony, wonder and beauty to start with before Wetiko/Yaldabaoth set out to change the simulated 'copy' into something very different. The process was slow to start with. Entrapped humans in the simulation timeline were not technologically aware and they had to be brought up to intellectual speed while being suppressed spiritually to the point where they could build their own prison while having no idea they were doing so. We have now reached that stage where technological intellect has the potential to destroy us and that's why events are moving so fast. Central American shaman Don Juan Matus said:

> Think for a moment, and tell me how you would explain the contradictions between the intelligence of man the engineer and the stupidity of his systems of belief, or the stupidity of his contradictory behaviour. Sorcerers believe that the predators have given us our systems of beliefs, our ideas of good and evil; our social mores. They are the ones who set up our dreams of success or failure. They have given us covetousness, greed, and cowardice. It is the predator who makes us complacent, routinary, and egomaniacal.
>
> In order to keep us obedient and meek and weak, the predators engaged themselves in a stupendous manoeuvre – stupendous, of course, from the point of view of a fighting strategist; a horrendous manoeuvre from the point of those who suffer it. They gave us their mind. The predators' mind is baroque, contradictory, morose, filled with the fear of being discovered any minute now.

For 'predators' see Wetiko, Archons, Yaldabaoth, Jinn, and all the other versions of the same phenomenon in cultures and religions all over the world. The theme is always the same because it's true and it's real. We have reached the point where we have to deal with it. The question is – how?

Don't fight – walk away

I thought I'd use a controversial subheading to get things moving in terms of our response to global fascism. What do you mean 'don't fight'? What do you mean 'walk away'? We've got to fight. We can't walk away. Well, it depends what we mean by fight and walk away. If fighting means physical combat we are playing Wetiko's game and falling for its trap. It wants us to get angry, aggressive, and direct hate and hostility at the enemy we think we must fight. Every war, every battle, every conflict, has been fought with Wetiko leading both sides. It's what it does. Wetiko wants a fight, anywhere, any place. Just hit me, son, so I can hit you back. Wetiko hits Wetiko and Wetiko hits Wetiko in return. I am very forthright as you can see in exposing Wetikos of the Cult, but I don't hate them. I refuse to hate them. It's what they want. What you hate you become. What you *fight* you become. Wokers, 'anti-haters' and 'anti-fascists' prove this every time they reach for their keyboards or don their balaclavas. By walk away I mean to disengage from Wetiko which includes ceasing to cooperate with its tyranny. Paul Levy says of Wetiko:

> The way to 'defeat' evil is not to try to destroy it (for then, in playing evil's game, we have already lost), but rather, to find the invulnerable place within ourselves where evil is unable to vanquish us – this is to truly 'win' our battle with evil.

Wetiko is everywhere in human society and it's been on steroids since the 'Covid' hoax. Every shouting match over wearing masks has Wetiko wearing a mask and Wetiko not wearing one. It's an electrical circuit of push and resist, push and resist, with Wetiko pushing *and* resisting. Each polarity is Wetiko empowering itself. Dictionary definitions of 'resist' include 'opposing, refusing to accept or comply with' and the word to focus on is 'opposing'. What form does this take – setting police cars alight or 'refusing to accept or comply with'? The former is Wetiko opposing Wetiko while the other points the way forward. This is the difference between those aggressively demanding that government fascism must be obeyed who stand in stark contrast to the great majority of Pushbackers. We saw this clearly with a march by thousands of Pushbackers against lockdown in London followed days later by a Woker-hijacked

protest in Bristol in which police cars were set on fire. Masks were virtually absent in London and widespread in Bristol. Wetiko wants lockdown on every level of society and infuses its aggression to police it through its unknowing stooges. Lockdown protesters are the ones with the smiling faces and the hugs, The two blatantly obvious states of being – getting more obvious by the day – are the result of Wokers and their like becoming ever more influenced by the simulation Field of Wetiko and Pushbackers ever more influenced by The Field of a far higher vibration beyond the simulation. Wetiko can't invade the heart which is where most lockdown opponents are coming from. It's the heart that allows them to see through the lies to the truth in ways I will be highlighting.

Renegade Minds know that calmness is the place from which wisdom comes. You won't find wisdom in a hissing fit and wisdom is what we need in abundance right now. Calmness is not weakness – you don't have to scream at the top of your voice to be strong. Calmness is indeed a sign of strength. 'No' means I'm not doing it. *NOOOO!!!* doesn't mean you're not doing it even more. Volume does not advance 'No – I'm not doing it'. You are just not doing it. Wetiko possessed and influenced don't know how to deal with that. Wetiko wants a fight and we should not give it one. What it needs more than anything is our *cooperation* and we should not give that either. Mass rallies and marches are great in that they are a visual representation of feeling, but if it ends there they are irrelevant. You demand that Wetikos act differently? Well, they're not going to are they? They are Wetikos. We don't need to waste our time demanding that something doesn't happen when that will make no difference. We need to delete the means that *allows* it to happen. This, invariably, is our cooperation. You can demand a child stop firing a peashooter at the dog or you can refuse to buy the peashooter. If you provide the means you are cooperating with the dog being smacked on the nose with a pea. How can the authorities enforce mask-wearing if millions in a country refuse? What if the 74 million Pushbackers that voted for Trump in 2020 refused to wear masks, close their businesses or stay in their homes. It would be unenforceable. The

few control the many through the compliance of the many and that's always been the dynamic be it 'Covid' regulations or the Roman Empire. I know people can find it intimidating to say no to authority or stand out in a crowd for being the only one with a face on display; but it has to be done or it's over. I hope I've made clear in this book that where this is going will be far more intimidating than standing up now and saying 'No' – I will not cooperate with my own enslavement and that of my children. There might be consequences for some initially, although not so if enough do the same. The question that must be addressed is what is going to happen if we don't? It is time to be strong and unyieldingly so. No means no. Not here and there, but *everywhere* and *always*. I have refused to wear a mask and obey all the other nonsense. I will not comply with tyranny. I repeat: Fascism is not imposed by fascists – there are never enough of them. Fascism is imposed by the population acquiescing to fascism. *I will not do it*. I will die first, or my body will. Living meekly under fascism is a form of death anyway, the death of the spirit that Martin Luther King described.

Making things happen

We must not despair. This is not over till it's over and it's far from that. The 'fat lady' must refuse to sing. The longer the 'Covid' hoax has dragged on and impacted on more lives we have seen an awakening of phenomenal numbers of people worldwide to the realisation that what they have believed all their lives is not how the world really is. Research published by the system-serving University of Bristol and King's College London in February, 2021, concluded: 'One in every 11 people in Britain say they trust David Icke's take on the coronavirus pandemic.' It will be more by now and we have gathering numbers to build on. We must urgently progress from seeing the scam to ceasing to cooperate with it. Prominent German lawyer Reiner Fuellmich, also licenced to practice law in America, is doing a magnificent job taking the legal route to bring the psychopaths to justice through a second Nuremberg tribunal for crimes against humanity. Fuellmich has an impressive record of

beating the elite in court and he formed the German Corona Investigative Committee to pursue civil charges against the main perpetrators with a view to triggering criminal charges. Most importantly he has grasped the foundation of the hoax – the PCR test not testing for the 'virus' – and Christian Drosten is therefore on his charge sheet along with Gates frontman Tedros at the World Health Organization. Major players must be not be allowed to inflict their horrors on the human race without being brought to book. A life sentence must follow for Bill Gates and the rest of them. A group of researchers has also indicted the government of Norway for crimes against humanity with copies sent to the police and the International Criminal Court. The lawsuit cites participation in an internationally-planned false pandemic and violation of international law and human rights, the European Commission's definition of human rights by coercive rules, Nuremberg and Hague rules on fundamental human rights, and the Norwegian constitution. We must take the initiative from hereon and not just complain, protest and react.

There are practical ways to support vital mass non-cooperation. Organising in numbers is one. Lockdown marches in London in the spring in 2021 were mass non-cooperation that the authorities could not stop. There were too many people. Hundreds of thousands walked the London streets in the centre of the road for mile after mile while the Face-Nappies could only look on. They were determined, but calm, and just *did it* with no histrionics and lots of smiles. The police were impotent. Others are organising group shopping without masks for mutual support and imagine if that was happening all over. Policing it would be impossible. If the store refuses to serve people in these circumstances they would be faced with a long line of trolleys full of goods standing on their own and everything would have to be returned to the shelves. How would they cope with that if it kept happening? I am talking here about moving on from complaining to being pro-active; from watching things happen to making things happen. I include in this our relationship with the police. The behaviour of many Face-Nappies

has been disgraceful and anyone who thinks they would never find concentration camp guards in the 'enlightened' modern era have had that myth busted big-time. The period and setting may change – Wetikos never do. I watched film footage from a London march in which a police thug viciously kicked a protestor on the floor who had done nothing. His fellow Face-Nappies stood in a ring protecting him. What he did was a criminal assault and with a crowd far outnumbering the police this can no longer be allowed to happen unchallenged. I get it when people chant 'shame on you' in these circumstances, but that is no longer enough. They *have* no shame those who do this. Crowds needs to start making a citizen's arrest of the police who commit criminal offences and brutally attack innocent people and defenceless women. A citizen's arrest can be made under section 24A of the UK Police and Criminal Evidence (PACE) Act of 1984 and you will find something similar in other countries. I prefer to call it a Common Law arrest rather than citizen's for reasons I will come to shortly. Anyone can arrest a person committing an indictable offence or if they have reasonable grounds to suspect they are committing an indictable offence. On both counts the attack by the police thug would have fallen into this category. A citizen's arrest can be made to stop someone:

- Causing physical injury to himself or any other person
- Suffering physical injury
- Causing loss of or damage to property
- Making off before a constable can assume responsibility for him

A citizen's arrest may also be made to prevent a breach of the peace under Common Law and if they believe a breach of the peace will happen or anything related to harm likely to be done or already done in their presence. This is the way to go I think – the Common Law version. If police know that the crowd and members of the public will no longer be standing and watching while they commit

their thuggery and crimes they will think twice about acting like Brownshirts and Blackshirts.

Common Law – common sense

Mention of Common Law is very important. Most people think the law is the law as in one law. This is not the case. There are two bodies of law, Common Law and Statute Law, and they are not the same. Common Law is founded on the simple premise of do no harm. It does not recognise victimless crimes in which no harm is done while Statute Law does. There is a Statute Law against almost everything. So what is Statute Law? Amazingly it's the law of the *sea* that was brought ashore by the Cult to override the law of the land which is Common Law. They had no right to do this and as always they did it anyway. They had to. They could not impose their will on the people through Common Law which only applies to do no harm. How could you stitch up the fine detail of people's lives with that? Instead they took the law of the sea, or Admiralty Law, and applied it to the population. Statute Law refers to all the laws spewing out of governments and their agencies including all the fascist laws and regulations relating to 'Covid'. The key point to make is that Statute Law is *contract law*. It only applies between *contracting* corporations. Most police officers don't even know this. They have to be kept in the dark, too. Long ago when merchants and their sailing ships began to trade with different countries a contractual law was developed called Admiralty Law and other names. Again it only applied to *contracts* agreed between *corporate* entities. If there is no agreed contract the law of the sea had no jurisdiction *and that still applies to its new alias of Statute Law*. The problem for the Cult when the law of the sea was brought ashore was an obvious one. People were not corporations and neither were government entities. To overcome the latter they made governments and all associated organisations corporations. All the institutions are *private corporations* and I mean governments and their agencies, local councils, police, courts, military, US states, the whole lot. Go to the

Dun and Bradstreet corporate listings website for confirmation that they are all corporations. You are arrested by a private corporation called the police by someone who is really a private security guard and they take you to court which is another private corporation. Neither have jurisdiction over you unless you consent and *contract* with them. This is why you hear the mantra about law enforcement policing by *consent* of the people. In truth the people 'consent' only in theory through monumental trickery.

Okay, the Cult overcame the corporate law problem by making governments and institutions corporate entities; but what about people? They are not corporations are they? Ah ... well in a sense, and *only* a sense, they are. Not people exactly – the illusion of people. The Cult creates a corporation in the name of everyone at the time that their birth certificate is issued. Note birth/ *berth* certificate and when you go to court under the law of the sea on land you stand in a *dock*. These are throwbacks to the origin. My Common Law name is David Vaughan Icke. The name of the corporation created by the government when I was born is called Mr David Vaughan Icke usually written in capitals as MR DAVID VAUGHAN ICKE. That is not me, the living, breathing man. It is a fictitious corporate entity. The trick is to make you think that David Vaughan Icke and MR DAVID VAUGHAN ICKE are the same thing. *They are not*. When police charge you and take you to court they are prosecuting the corporate entity and not the living, breathing, man or woman. They have to trick you into identifying as the corporate entity and contracting with them. Otherwise they have no jurisdiction. They do this through a language known as legalese. Lawful and legal are not the same either. Lawful relates to Common Law and legal relates to Statute Law. Legalese is the language of Statue Law which uses terms that mean one thing to the public and another in legalese. Notice that when a police officer tells someone why they are being charged he or she will say at the end: 'Do you understand?' To the public that means 'Do you comprehend?' In legalese it means 'Do you stand under me?' Do you stand under my authority? If you say

yes to the question you are unknowingly agreeing to give them jurisdiction over you in a contract between two corporate entities.

This is a confidence trick in every way. Contracts have to be agreed between informed parties and if you don't know that David Vaughan Icke is agreeing to be the corporation MR DAVID VAUGHAN ICKE you cannot knowingly agree to contract. They are deceiving you and another way they do this is to ask for proof of identity. You usually show them a driving licence or other document on which your corporate name is written. In doing so you are accepting that you are that corporate entity when you are not. Referring to yourself as a 'person' or 'citizen' is also identifying with your corporate fiction which is why I made the Common Law point about the citizen's arrest. If you are approached by a police officer you identify yourself immediately as a living, breathing, man or woman and say 'I do not consent, I do not contract with you and I do not understand' or stand under their authority. I have a Common Law birth certificate as a living man and these are available at no charge from commonlawcourt.com. Businesses registered under the Statute Law system means that its laws apply. There are, however, ways to run a business under Common Law. Remember all 'Covid' laws and regulations are Statute Law – the law of *contracts* and you do not have to contract. This doesn't mean that you can kill someone and get away with it. Common Law says do no harm and that applies to physical harm, financial harm etc. Police are employees of private corporations and there needs to be a new system of non-corporate Common Law constables operating outside the Statute Law system. If you go to davidicke.com and put Common Law into the search engine you will find videos that explain Common Law in much greater detail. It is definitely a road we should walk.

With all my heart

I have heard people say that we are in a spiritual war. I don't like the term 'war' with its Wetiko dynamic, but I know what they mean. Sweep aside all the bodily forms and we are in a situation in which two states of consciousness are seeking very different realities.

Wetiko wants upheaval, chaos, fear, suffering, conflict and control. The other wants love, peace, harmony, fairness and freedom. That's where we are. We should not fall for the idea that Wetiko is all-powerful and there's nothing we can do. Wetiko is not all-powerful. It's a joke, pathetic. It doesn't have to be, but it has made that choice for now. A handful of times over the years when I have felt the presence of its frequency I have allowed it to attach briefly so I could consciously observe its nature. The experience is not pleasant, the energy is heavy and dark, but the ease with which you can kick it back out the door shows that its real power is in persuading us that it has power. It's all a con. Wetiko is a con. It's a trickster and not a power that can control us if we unleash our own. The con is founded on manipulating humanity to give its power to Wetiko which recycles it back to present the illusion that it has power when its power is *ours* that we gave away. This happens on an energetic level and plays out in the world of the seen as humanity giving its power to Wetiko authority which uses that power to control the population when the power is only the power the population has handed over. How could it be any other way for billions to be controlled by a relative few? I have had experiences with people possessed by Wetiko and again you can kick its arse if you do it with an open heart. Oh yes – the *heart* which can transform the world of perceived 'matter'.

We are receiver-transmitters and processors of information, but what information and where from? Information is processed into perception in three main areas – the brain, the heart and the belly. These relate to thinking, knowing, and emotion. Wetiko wants us to be head and belly people which means we think within the confines of the Matrix simulation and low-vibrational emotional reaction scrambles balance and perception. A few minutes on social media and you see how emotion is the dominant force. Woke is all emotion and is therefore thought-free and fact-free. Our heart is something different. It *knows* while the head *thinks* and has to try to work it out because it doesn't know. The human energy field has seven prime vortexes which connect us with wider reality (Fig 23). Chakra means

'wheels of light' in the Sanskrit language of ancient India. The main ones are: The crown chakra on top of the head; brow (or 'third eye') chakra in the centre of the forehead; throat chakra; heart chakra in the centre of the chest; solar plexus chakra below the sternum; sacral chakra beneath the navel; and base chakra at the bottom of the spine. Each one has a particular function or functions. We feel anxiety and nervousness in the belly where the sacral chakra is located and this processes emotion that can affect the colon to give people 'the shits' or make them 'shit scared' when they are nervous. Chakras all play an important role, but the Mr and Mrs Big is the heart chakra which sits at the centre of the seven, above the chakras that connect us to the 'physical' and below those that connect with higher realms (or at least should). Here in the heart chakra we feel love, empathy and compassion – 'My heart goes out to you'. Those with closed hearts become literally 'heart-less' in their attitudes and behaviour (see Bill Gates). Native Americans portrayed Wetiko with what Paul Levy calls a 'frigid, icy heart, devoid of mercy' (see Bill Gates).



Figure 23: The chakra system which interpenetrates the human energy field. The heart chakra is the governor – or should be.

Wetiko trembles at the thought of heart energy which it cannot infiltrate. The frequency is too high. What it seeks to do instead is close the heart chakra vortex to block its perceptual and energetic influence. Psychopaths have 'hearts of stone' and emotionally-damaged people have 'heartache' and 'broken hearts'. The astonishing amount of heart disease is related to heart chakra

disruption with its fundamental connection to the 'physical' heart. Dr Tom Cowan has written an outstanding book challenging the belief that the heart is a pump and making the connection between the 'physical' and spiritual heart. Rudolph Steiner who was way ahead of his time said the same about the fallacy that the heart is a pump. *What*? The heart is not a pump? That's crazy, right? Everybody knows that. Read Cowan's *Human Heart, Cosmic Heart* and you will realise that the very idea of the heart as a pump is ridiculous when you see the evidence. How does blood in the feet so far from the heart get pumped horizontally up the body by the heart?? Cowan explains in the book the real reason why blood moves as it does. Our 'physical' heart is used to symbolise love when the source is really the heart vortex or spiritual heart which is our most powerful energetic connection to 'out there' expanded consciousness. That's why we feel *knowing* – intuitive knowing – in the centre of the chest. Knowing doesn't come from a process of thoughts leading to a conclusion. It is there in an instant all in one go. Our heart knows because of its connection to levels of awareness that *do* know. This is the meaning and source of intuition – intuitive *knowing*.

For the last more than 30 years of uncovering the global game and the nature of reality my heart has been my constant antenna for truth and accuracy. An American intelligence insider once said that I had quoted a disinformer in one of my books and yet I had only quoted the part that was true. He asked: 'How do you do that?' By using my heart antenna was the answer and anyone can do it. Heart-centred is how we are meant to be. With a closed heart chakra we withdraw into a closed mind and the bubble of five-sense reality. If you take a moment to focus your attention on the centre of your chest, picture a spinning wheel of light and see it opening and expanding. You will feel it happening, too, and perceptions of the heart like joy and love as the heart impacts on the mind as they interact. The more the chakra opens the more you will feel expressions of heart consciousness and as the process continues, and becomes part of you, insights and knowings will follow. An open

heart is connected to that level of awareness that knows all is *One*. You will see from its perspective that the fault-lines that divide us are only illusions to control us. An open heart does not process the illusions of race, creed and sexuality except as brief experiences for a consciousness that is all. Our heart does not see division, only unity (Figs 24 and 25). There's something else, too. Our hearts love to laugh. Mark Twain's quote that says 'The human race has one really effective weapon, and that is laughter' is really a reference to the heart which loves to laugh with the joy of knowing the true nature of infinite reality and that all the madness of human society is an illusion of the mind. Twain also said: 'Against the assault of laughter nothing can stand.' This is so true of Wetiko and the Cult. Their insecurity demands that they be taken seriously and their power and authority acknowledged and feared. We should do nothing of the sort. We should not get aggressive or fearful which their insecurity so desires. We should laugh in their face. Even in their no-face as police come over in their face-nappies and expect to be taken seriously. They don't take themselves seriously looking like that so why should we? Laugh in the face of intimidation. Laugh in the face of tyranny. You will see by its reaction that you have pressed all of its buttons. Wetiko does not know what to do in the face of laughter or when its targets refuse to concede their joy to fear. We have seen many examples during the 'Covid' hoax when people have expressed their energetic power and the string puppets of Wetiko retreat with their tail limp between their knees. Laugh – the world is bloody mad after all and if it's a choice between laughter and tears I know which way I'm going.



Figure 24: Head consciousness without the heart sees division and everything apart from everything else.



Figure 25: Heart consciousness sees everything as One.

'Vaccines' and the soul

The foundation of Wetiko/Archon control of humans is the separation of incarnate five-sense mind from the infinite 'I' and closing the heart chakra where the True 'I' lives during a human life. The goal has been to achieve complete separation in both cases. I was interested therefore to read an account by a French energetic healer of what she said she experienced with a patient who had been given the 'Covid' vaccine. Genuine energy healers can sense information and consciousness fields at different levels of being which are referred to as 'subtle bodies'. She described treating the patient who later returned after having, without the healer's knowledge, two doses of the 'Covid vaccine'. The healer said:

I noticed immediately the change, very heavy energy emanating from [the] subtle bodies. The scariest thing was when I was working on the heart chakra, I connected with her soul: it was detached from the physical body, it had no contact and it was, as if it was floating in a state of total confusion: a damage to the consciousness that loses contact with the physical body, i.e. with our biological machine, there is no longer any communication between them.

I continued the treatment by sending light to the heart chakra, the soul of the person, but it seemed that the soul could no longer receive any light, frequency or energy. It was a very powerful experience for me. Then I understood that this substance is indeed used to detach consciousness so that this consciousness can no longer interact through this body that it possesses in life, where there is no longer any contact, no frequency, no light, no more energetic balance or mind.

This would create a human that is rudderless and at the extreme almost zombie-like operating with a fractional state of consciousness at the mercy of Wetiko. I was especially intrigued by what the healer said in the light of the prediction by the highly-informed Rudolf Steiner more than a hundred years ago. He said:

In the future, we will eliminate the soul with medicine. Under the pretext of a 'healthy point of view', there will be a vaccine by which the human body will be treated as soon as possible directly at birth, so that the human being cannot develop the thought of the existence of soul and Spirit. To materialistic doctors will be entrusted the task of removing the soul of humanity.

As today, people are vaccinated against this disease or that disease, so in the future, children will be vaccinated with a substance that can be produced precisely in such a way that people, thanks to this vaccination, will be immune to being subjected to the 'madness' of spiritual life. He would be extremely smart, but he would not develop a conscience, and that is the true goal of some materialistic circles.

Steiner said the vaccine would detach the physical body from the etheric body (subtle bodies) and 'once the etheric body is detached the relationship between the universe and the etheric body would become extremely unstable, and man would become an automaton'. He said 'the physical body of man must be polished on this Earth by spiritual will – so the vaccine becomes a kind of arymanique (Wetiko) force' and 'man can no longer get rid of a given materialistic feeling'. Humans would then, he said, become 'materialistic of constitution and can no longer rise to the spiritual'. I have been writing for years about DNA being a receiver-transmitter of information that connects us to other levels of reality and these 'vaccines' changing DNA can be likened to changing an antenna and what it can transmit and receive. Such a disconnection would clearly lead to changes in personality and perception. Steiner further predicted the arrival of AI. Big Pharma 'Covid vaccine' makers, expressions of Wetiko, are testing their DNA-manipulating evil on children as I write with a view to giving the 'vaccine' to babies. If it's a soul-body disconnector – and I say that it is or can be – every child would be disconnected from 'soul' at birth and the 'vaccine' would create a closed system in which spiritual guidance from the greater self would play no part. This has been the ambition of Wetiko all

along. A Pentagon video from 2005 was leaked of a presentation explaining the development of vaccines to change behaviour by their effect on the brain. Those that believe this is not happening with the 'Covid' genetically-modifying procedure masquerading as a 'vaccine' should make an urgent appointment with Naivety Anonymous. Klaus Schwab wrote in 2018:

> Neurotechnologies enable us to better influence consciousness and thought and to understand many activities of the brain. They include decoding what we are thinking in fine levels of detail through new chemicals and interventions that can influence our brains to correct for errors or enhance functionality.

The plan is clear and only the heart can stop it. With every heart that opens, every mind that awakens, Wetiko is weakened. Heart and love are far more powerful than head and hate and so nothing like a majority is needed to turn this around.

Beyond the Phantom

Our heart is the prime target of Wetiko and so it must be the answer to Wetiko. We *are* our heart which is part of one heart, the infinite heart. Our heart is where the true self lives in a human life behind firewalls of five-sense illusion when an imposter takes its place – *Phantom Self*; but our heart waits patiently to be set free any time we choose to see beyond the Phantom, beyond Wetiko. A Wetikoed Phantom Self can wreak mass death and destruction while the love of forever is locked away in its heart. The time is here to unleash its power and let it sweep away the fear and despair that is Wetiko. Heart consciousness does not seek manipulated, censored, advantage for its belief or religion, its activism and desires. As an expression of the One it treats all as One with the same rights to freedom and opinion. Our heart demands fairness for itself no more than for others. From this unity of heart we can come together in mutual support and transform this Wetikoed world into what reality is meant to be – a place of love, joy, happiness, fairness, justice and freedom. Wetiko has another agenda and that's why the world is as

it is, but enough of this nonsense. Wetiko can't stay where hearts are open and it works so hard to keep them closed. Fear is its currency and its food source and love in its true sense has no fear. Why would love have fear when it knows it is *All That Is, Has Been, And Ever Can Be* on an eternal exploration of all possibility? Love in this true sense is not the physical attraction that passes for love. This can be an expression of it, yes, but Infinite Love, a love without condition, goes far deeper to the core of all being. It *is* the core of all being. Infinite realty was born from love beyond the illusions of the simulation. Love infinitely expressed is the knowing that all is One and the swiftly-passing experience of separation is a temporary hallucination. You cannot disconnect from Oneness; you can only *perceive* that you have and withdraw from its influence. This is the most important of all perception trickery by the mind parasite that is Wetiko and the foundation of all its potential for manipulation.

If we open our hearts, open the sluice gates of the mind, and redefine self-identity amazing things start to happen. Consciousness expands or contracts in accordance with self-identity. When true self is recognised as infinite awareness and label self – Phantom Self – is seen as only a series of brief experiences life is transformed. Consciousness expands to the extent that self-identity expands and everything changes. You see unity, not division, the picture, not the pixels. From this we can play the long game. No more is an experience something in and of itself, but a fleeting moment in the eternity of forever. Suddenly people in uniform and dark suits are no longer intimidating. Doing what your heart knows to be right is no longer intimidating and consequences for those actions take on the same nature of a brief experience that passes in the blink of an infinite eye. Intimidation is all in the mind. Beyond the mind there is no intimidation.

An open heart does not consider consequences for what it knows to be right. To do so would be to consider not doing what it knows to be right and for a heart in its power that is never an option. The Renegade Mind is really the Renegade Heart. Consideration of consequences will always provide a getaway car for the mind and

the heart doesn't want one. What is right in the light of what we face today is to stop cooperating with Wetiko in all its forms and to do it without fear or compromise. You cannot compromise with tyranny when tyranny always demands more until it has everything. Life is your perception and you are your destiny. Change your perception and you change your life. Change collective perception and we change the world.

Come on people … One human family, One heart, One goal … *FREEEEEEDOM*!

We must settle for nothing less.

Postscript

The big scare story as the book goes to press is the 'Indian' variant and the world is being deluged with propaganda about the 'Covid catastrophe' in India which mirrors in its lies and misrepresentations what happened in Italy before the first lockdown in 2020.

The *New York Post* published a picture of someone who had 'collapsed in the street from Covid' in India in April, 2021, which was actually taken during a gas leak in May, 2020. Same old, same old. Media articles in mid-February were asking why India had been so untouched by 'Covid' and then as their vaccine rollout gathered pace the alleged 'cases' began to rapidly increase. Indian 'Covid vaccine' maker Bharat Biotech was funded into existence by the Bill and Melinda Gates Foundation (the pair announced their divorce in May, 2021, which is a pity because they so deserve each other). The Indian 'Covid crisis' was ramped up by the media to terrify the world and prepare people for submission to still more restrictions. The scam that worked the first time was being repeated only with far more people seeing through the deceit. Davidicke.com and Ickonic.com have sought to tell the true story of what is happening by talking to people living through the Indian nightmare which has nothing to do with 'Covid'. We posted a letter from 'Alisha' in Pune who told a very different story to government and media mendacity. She said scenes of dying people and overwhelmed hospitals were designed to hide what was really happening – genocide and starvation. Alisha said that millions had already died of starvation during the ongoing lockdowns while government and media were lying and making it look like the 'virus':

Restaurants, shops, gyms, theatres, basically everything is shut. The cities are ghost towns. Even so-called 'essential' businesses are only open till 11am in the morning. You basically have just an hour to buy food and then your time is up.

Inter-state travel and even inter-district travel is banned. The cops wait at all major crossroads to question why you are traveling outdoors or to fine you if you are not wearing a mask.

The medical community here is also complicit in genocide, lying about hospitals being full and turning away people with genuine illnesses, who need immediate care. They have even created a shortage of oxygen cylinders.

This is the classic Cult modus operandi played out in every country. Alisha said that people who would not have a PCR test not testing for the 'virus' were being denied hospital treatment. She said the people hit hardest were migrant workers and those in rural areas. Most businesses employed migrant workers and with everything closed there were no jobs, no income and no food. As a result millions were dying of starvation or malnutrition. All this was happening under Prime Minister Narendra Modi, a 100-percent asset of the Cult, and it emphasises yet again the scale of pure anti-human evil we are dealing with. Australia banned its people from returning home from India with penalties for trying to do so of up to five years in jail and a fine of £37,000. The manufactured 'Covid' crisis in India was being prepared to justify further fascism in the West. Obvious connections could be seen between the Indian 'vaccine' programme and increased 'cases' and this became a common theme. The Seychelles, the most per capita 'Covid vaccinated' population in the world, went back into lockdown after a 'surge of cases'.

Long ago the truly evil Monsanto agricultural biotechnology corporation with its big connections to Bill Gates devastated Indian farming with genetically-modified crops. Human rights activist Gurcharan Singh highlighted the efforts by the Indian government to complete the job by destroying the food supply to hundreds of millions with 'Covid' lockdowns. He said that 415 million people at the bottom of the disgusting caste system (still going whatever they say) were below the poverty line and struggled to feed themselves every year. Now the government was imposing lockdown at just the

time to destroy the harvest. This deliberate policy was leading to mass starvation. People may reel back at the suggestion that a government would do that, but Wetiko-controlled 'leaders' are capable of any level of evil. In fact what is described in India is in the process of being instigated worldwide. The food chain and food supply are being targeted at every level to cause world hunger and thus control. Bill Gates is not the biggest owner of farmland in America for no reason and destroying access to food aids both the depopulation agenda and the plan for synthetic 'food' already being funded into existence by Gates. Add to this the coming hyper-inflation from the suicidal creation of fake 'money' in response to 'Covid' and the breakdown of container shipping systems and you have a cocktail that can only lead one way and is meant to. The Cult plan is to crash the entire system to 'build back better' with the Great Reset.

'Vaccine' transmission

Reports from all over the world continue to emerge of women suffering menstrual and fertility problems after having the fake 'vaccine' and of the non-'vaccinated' having similar problems when interacting with the 'vaccinated'. There are far too many for 'coincidence' to be credible. We've had menopausal women getting periods, others having periods stop or not stopping for weeks, passing clots, sometimes the lining of the uterus, breast irregularities, and miscarriages (which increased by 400 percent in parts of the United States). Non-'vaccinated' men and children have suffered blood clots and nose bleeding after interaction with the 'vaccinated'. Babies have died from the effects of breast milk from a 'vaccinated' mother. Awake doctors – the small minority – speculated on the cause of non-'vaccinated' suffering the same effects as the 'vaccinated'. Was it nanotechnology in the synthetic substance transmitting frequencies or was it a straight chemical bioweapon that was being transmitted between people? I am not saying that some kind of chemical transmission is not one possible answer, but the foundation of all that the Cult does is frequency and

this is fertile ground for understanding how transmission can happen. American doctor Carrie Madej, an internal medicine physician and osteopath, has been practicing for the last 20 years, teaching medical students, and she says attending different meetings where the agenda for humanity was discussed. Madej, who operates out of Georgia, did not dismiss other possible forms of transmission, but she focused on frequency in search of an explanation for transmission. She said the Moderna and Pfizer 'vaccines' contained nano-lipid particles as a key component. This was a brand new technology never before used on humanity. 'They're using a nanotechnology which is pretty much little tiny computer bits … nanobots or hydrogel.' Inside the 'vaccines' was 'this sci-fi kind of substance' which suppressed immune checkpoints to get into the cell. I referred to this earlier as the 'Trojan horse' technique that tricks the cell into opening a gateway for the self-replicating synthetic material and while the immune system is artificially suppressed the body has no defences. Madej said the substance served many purposes including an on-demand ability to 'deliver the payload' and using the nano 'computer bits' as biosensors in the body. 'It actually has the ability to accumulate data from your body, like your breathing, your respiration, thoughts, emotions, all kinds of things.'

She said the technology obviously has the ability to operate through Wi-Fi and transmit and receive energy, messages, frequencies or impulses. 'Just imagine you're getting this new substance in you and it can react to things all around you, the 5G, your smart device, your phones.' We had something completely foreign in the human body that had never been launched large scale at a time when we were seeing 5G going into schools and hospitals (plus the Musk satellites) and she believed the 'vaccine' transmission had something to do with this: '… if these people have this inside of them … it can act like an antenna and actually transmit it outwardly as well.' The synthetic substance produced its own voltage and so it could have that kind of effect. This fits with my own contention that the nano receiver-transmitters are designed to connect people to the

Smart Grid and break the receiver-transmitter connection to expanded consciousness. That would explain the French energy healer's experience of the disconnection of body from 'soul' with those who have had the 'vaccine'. The nanobots, self-replicating inside the body, would also transmit the synthetic frequency which could be picked up through close interaction by those who have not been 'vaccinated'. Madej speculated that perhaps it was 5G and increased levels of other radiation that was causing the symptoms directly although interestingly she said that non-'vaccinated' patients had shown improvement when they were away from the 'vaccinated' person they had interacted with. It must be remembered that you can control frequency and energy with your mind and you can consciously create energetic barriers or bubbles with the mind to stop damaging frequencies from penetrating your field. American paediatrician Dr Larry Palevsky said the 'vaccine' was not a 'vaccine' and was never designed to protect from a 'viral' infection. He called it 'a massive, brilliant propaganda of genocide' because they didn't have to inject everyone to get the result they wanted. He said the content of the jabs was able to infuse any material into the brain, heart, lungs, kidneys, liver, sperm and female productive system. 'This is genocide; this is a weapon of mass destruction.' At the same time American colleges were banning students from attending if they didn't have this life-changing and potentially life-ending 'vaccine'. Class action lawsuits must follow when the consequences of this college fascism come to light. As the book was going to press came reports about fertility effects on sperm in 'vaccinated' men which would absolutely fit with what I have been saying and hospitals continued to fill with 'vaccine' reactions. Another question is what about transmission via blood transfusions? The NHS has extended blood donation restrictions from seven days after a 'Covid vaccination' to 28 days after even a sore arm reaction.

I said in the spring of 2020 that the then touted 'Covid vaccine' would be ongoing each year like the flu jab. A year later Pfizer CEO, the appalling Albert Bourla, said people would 'likely' need a 'booster dose' of the 'vaccine' within 12 months of getting 'fully

vaccinated' and then a yearly shot. 'Variants will play a key role', he said confirming the point. Johnson & Johnson CEO Alex Gorsky also took time out from his 'vaccine' disaster to say that people may need to be vaccinated against 'Covid-19' each year. UK Health Secretary, the psychopath Matt Hancock, said additional 'boosters' would be available in the autumn of 2021. This is the trap of the 'vaccine passport'. The public will have to accept every last 'vaccine' they introduce, including for the fake 'variants', or it would cease to be valid. The only other way in some cases would be continuous testing with a test not testing for the 'virus' and what is on the swabs constantly pushed up your noise towards the brain every time?

'Vaccines' changing behaviour

I mentioned in the body of the book how I believed we would see gathering behaviour changes in the 'vaccinated' and I am already hearing such comments from the non-'vaccinated' describing behaviour changes in friends, loved ones and work colleagues. This will only increase as the self-replicating synthetic material and nanoparticles expand in body and brain. An article in the *Guardian* in 2016 detailed research at the University of Virginia in Charlottesville which developed a new method for controlling brain circuits associated with complex animal behaviour. The method, dubbed 'magnetogenetics', involves genetically-engineering a protein called ferritin, which stores and releases iron, to create a magnetised substance – 'Magneto' – that can activate specific groups of nerve cells from a distance. This is claimed to be an advance on other methods of brain activity manipulation known as optogenetics and chemogenetics (the Cult has been developing methods of brain control for a long time). The ferritin technique is said to be non-invasive and able to activate neurons 'rapidly and reversibly'. In other words, human thought and perception. The article said that earlier studies revealed how nerve cell proteins 'activated by heat and mechanical pressure can be genetically engineered so that they become sensitive to radio waves and magnetic fields, by attaching them to an iron-storing protein called ferritin, or to inorganic

paramagnetic particles'. Sensitive to radio waves and magnetic fields? You mean like 5G, 6G and 7G? This is the human-AI Smart Grid hive mind we are talking about. The *Guardian* article said:

> … the researchers injected Magneto into the striatum of freely behaving mice, a deep brain structure containing dopamine-producing neurons that are involved in reward and motivation, and then placed the animals into an apparatus split into magnetised and non-magnetised sections.
>
> Mice expressing Magneto spent far more time in the magnetised areas than mice that did not, because activation of the protein caused the striatal neurons expressing it to release dopamine, so that the mice found being in those areas rewarding. This shows that Magneto can remotely control the firing of neurons deep within the brain, and also control complex behaviours.

Make no mistake this basic methodology will be part of the 'Covid vaccine' cocktail and using magnetics to change brain function through electromagnetic field frequency activation. The Pentagon is developing a 'Covid vaccine' using ferritin. Magnetics would explain changes in behaviour and why videos are appearing across the Internet as I write showing how magnets stick to the skin at the point of the 'vaccine' shot. Once people take these 'vaccines' anything becomes possible in terms of brain function and illness which will be blamed on 'Covid-19' and 'variants'. Magnetic field manipulation would further explain why the non-'vaccinated' are reporting the same symptoms as the 'vaccinated' they interact with and why those symptoms are reported to decrease when not in their company. Interestingly 'Magneto', a 'mutant', is a character in the Marvel Comic *X-Men* stories with the ability to manipulate magnetic fields and he believes that mutants should fight back against their human oppressors by any means necessary. The character was born Erik Lehnsherr to a Jewish family in Germany.

Cult-controlled courts

The European Court of Human Rights opened the door for mandatory 'Covid-19 vaccines' across the continent when it ruled in a Czech Republic dispute over childhood immunisation that legally

enforced vaccination could be 'necessary in a democratic society'. The 17 judges decided that compulsory vaccinations did not breach human rights law. On the face of it the judgement was so inverted you gasp for air. If not having a vaccine infused into your body is not a human right then what is? Ah, but they said human rights law which has been specifically written to delete all human rights at the behest of the state (the Cult). Article 8 of the European Convention on Human Rights relates to the right to a private life. The crucial word here is *'except'*:

> There shall be no interference by a public authority with the exercise of this right EXCEPT such as is in accordance with the law and is necessary in a democratic society in the interests of national security, public safety or the economic wellbeing of the country, for the prevention of disorder or crime, for the protection of health or morals, or for the protection of the rights and freedoms of others [My emphasis].

No interference *except* in accordance with the law means there *are* no 'human rights' *except* what EU governments decide you can have at their behest. 'As is necessary in a democratic society' explains that reference in the judgement and 'in the interests of national security, public safety or the economic well-being of the country, for the prevention of disorder or crime, for the protection of health or morals, or for the protection of the rights and freedoms of others' gives the EU a coach and horses to ride through 'human rights' and scatter them in all directions. The judiciary is not a check and balance on government extremism; it is a vehicle to enforce it. This judgement was almost laughably predictable when the last thing the Cult wanted was a decision that went against mandatory vaccination. Judges rule over and over again to benefit the system of which they are a part. Vaccination disputes that come before them are invariably delivered in favour of doctors and authorities representing the view of the state which owns the judiciary. Oh, yes, and we have even had calls to stop putting 'Covid-19' on death certificates within 28 days of a 'positive test' because it is claimed the practice makes the 'vaccine' appear not to work. They are laughing at you.

The scale of madness, inhumanity and things to come was highlighted when those not 'vaccinated' for 'Covid' were refused evacuation from the Caribbean island of St Vincent during massive volcanic eruptions. Cruise ships taking residents to the safety of another island allowed only the 'vaccinated' to board and the rest were left to their fate. Even in life and death situations like this we see 'Covid' stripping people of their most basic human instincts and the insanity is even more extreme when you think that fake 'vaccine'-makers are not even claiming their body-manipulating concoctions stop 'infection' and 'transmission' of a 'virus' that doesn't exist. St Vincent Prime Minister Ralph Gonsalves said: 'The chief medical officer will be identifying the persons already vaccinated so that we can get them on the ship.' Note again the power of the chief medical officer who, like Whitty in the UK, will be answering to the World Health Organization. This is the Cult network structure that has overridden politicians who 'follow the science' which means doing what WHO-controlled 'medical officers' and 'science advisers' tell them. Gonsalves even said that residents who were 'vaccinated' after the order so they could board the ships would still be refused entry due to possible side effects such as 'wooziness in the head'. The good news is that if they were woozy enough in the head they could qualify to be prime minister of St Vincent.

Microchipping freedom

The European judgement will be used at some point to justify moves to enforce the 'Covid' DNA-manipulating procedure. Sandra Ro, CEO of the Global Blockchain Business Council, told a World Economic Forum event that she hoped 'vaccine passports' would help to 'drive forced consent and standardisation' of global digital identity schemes: 'I'm hoping with the desire and global demand for some sort of vaccine passport – so that people can get travelling and working again – [it] will drive forced consent, standardisation, and frankly, cooperation across the world.' The lady is either not very bright, or thoroughly mendacious, to use the term 'forced consent'.

You do not 'consent' if you are forced – you *submit*. She was describing what the plan has been all along and that's to enforce a digital identity on every human without which they could not function. 'Vaccine passports' are opening the door and are far from the end goal. A digital identity would allow you to be tracked in everything you do in cyberspace and this is the same technique used by Cult-owned China to enforce its social credit system of total control. The ultimate 'passport' is planned to be a microchip as my books have warned for nearly 30 years. Those nice people at the Pentagon working for the Cult-controlled Defense Advanced Research Projects Agency (DARPA) claimed in April, 2021, they have developed a microchip inserted under the skin to detect 'asymptomatic Covid-19 infection' before it becomes an outbreak and a 'revolutionary filter' that can remove the 'virus' from the blood when attached to a dialysis machine. The only problems with this are that the 'virus' does not exist and people transmitting the 'virus' with no symptoms is brain-numbing bullshit. This is, of course, not a ruse to get people to be microchipped for very different reasons. DARPA also said it was producing a one-stop 'vaccine' for the 'virus' and all 'variants'. One of the most sinister organisations on Planet Earth is doing this? Better have it then. These people are insane because Wetiko that possesses them is insane.

Researchers from the Salk Institute in California announced they have created an embryo that is part human and part monkey. My books going back to the 1990s have exposed experiments in top secret underground facilities in the United States where humans are being crossed with animal and non-human 'extraterrestrial' species. They are now easing that long-developed capability into the public arena and there is much more to come given we are dealing with psychiatric basket cases. Talking of which – Elon Musk's scientists at Neuralink trained a monkey to play Pong and other puzzles on a computer screen using a joystick and when the monkey made the correct move a metal tube squirted banana smoothie into his mouth which is the basic technique for training humans into unquestioning compliance. Two Neuralink chips were in the monkey's skull and

more than 2,000 wires 'fanned out' into its brain. Eventually the monkey played a video game purely with its brain waves. Psychopathic narcissist Musk said the 'breakthrough' was a step towards putting Neuralink chips into human skulls and merging minds with artificial intelligence. *Exactly*. This man is so dark and Cult to his DNA.

World Economic Fascism (WEF)

The World Economic Forum is telling you the plan by the statements made at its many and various events. Cult-owned fascist YouTube CEO Susan Wojcicki spoke at the 2021 WEF Global Technology Governance Summit (see the name) in which 40 governments and 150 companies met to ensure 'the responsible design and deployment of emerging technologies'. Orwellian translation: 'Ensuring the design and deployment of long-planned technologies will advance the Cult agenda for control and censorship.' Freedom-destroyer and Nuremberg-bound Wojcicki expressed support for tech platforms like hers to censor content that is 'technically legal but could be harmful'. Who decides what is 'harmful'? She does and they do. 'Harmful' will be whatever the Cult doesn't want people to see and we have legislation proposed by the UK government that would censor content on the basis of 'harm' no matter if the information is fair, legal and provably true. Make that *especially* if it is fair, legal and provably true. Wojcicki called for a global coalition to be formed to enforce content moderation standards through automated censorship. This is a woman and mega-censor so self-deluded that she shamelessly accepted a 'free expression' award – *Wojcicki* – in an event sponsored by her own *YouTube*. They have no shame and no self-awareness.

You know that 'Covid' is a scam and Wojcicki a Cult operative when YouTube is censoring medical and scientific opinion purely on the grounds of whether it supports or opposes the Cult 'Covid' narrative. Florida governor Ron DeSantis compiled an expert panel with four professors of medicine from Harvard, Oxford, and Stanford Universities who spoke against forcing children and

vaccinated people to wear masks. They also said there was no proof that lockdowns reduced spread or death rates of 'Covid-19'. Cult-gofer Wojcicki and her YouTube deleted the panel video 'because it included content that contradicts the consensus of local and global health authorities regarding the efficacy of masks to prevent the spread of Covid-19'. This 'consensus' refers to what the Cult tells the World Health Organization to say and the WHO tells 'local health authorities' to do. Wojcicki knows this, of course. The panellists pointed out that censorship of scientific debate was responsible for deaths from many causes, but Wojcicki couldn't care less. She would not dare go against what she is told and as a disgrace to humanity she wouldn't want to anyway. The UK government is seeking to pass a fascist 'Online Safety Bill' to specifically target with massive fines and other means non-censored video and social media platforms to make them censor 'lawful but harmful' content like the Cult-owned Facebook, Twitter, Google and YouTube. What is 'lawful but harmful' would be decided by the fascist Blair-created Ofcom.

Another WEF obsession is a cyber-attack on the financial system and this is clearly what the Cult has planned to take down the bank accounts of everyone – except theirs. Those that think they have enough money for the Cult agenda not to matter to them have got a big lesson coming if they continue to ignore what is staring them in the face. The World Economic Forum, funded by Gates and fronted by Klaus Schwab, announced it would be running a 'simulation' with the Russian government and global banks of just such an attack called Cyber Polygon 2021. What they simulate – as with the 'Covid' Event 201 – they plan to instigate. The WEF is involved in a project with the Cult-owned Carnegie Endowment for International Peace called the WEF-Carnegie Cyber Policy Initiative which seeks to merge Wall Street banks, 'regulators' (I love it) and intelligence agencies to 'prevent' (arrange and allow) a cyber-attack that would bring down the global financial system as long planned by those that control the WEF and the Carnegie operation. The Carnegie Endowment for International Peace sent an instruction to First World

War US President Woodrow Wilson not to let the war end before society had been irreversibly transformed.

The Wuhan lab diversion

As I close, the Cult-controlled authorities and lapdog media are systematically pushing 'the virus was released from the Wuhan lab' narrative. There are two versions – it happened by accident and it happened on purpose. Both are nonsense. The perceived existence of the never-shown-to-exist 'virus' is vital to sell the impression that there is actually an infective agent to deal with and to allow the endless potential for terrifying the population with 'variants' of a 'virus' that does not exist. The authorities at the time of writing are going with the 'by accident' while the alternative media is promoting the 'on purpose'. Cable news host Tucker Carlson who has questioned aspects of lockdown and 'vaccine' compulsion has bought the Wuhan lab story. 'Everyone now agrees' he said. Well, I don't and many others don't and the question is *why* does the system and its media suddenly 'agree'? When the media moves as one unit with a narrative it is always a lie – witness the hour by hour mendacity of the 'Covid' era. Why would this Cult-owned combination which has unleashed lies like machine gun fire suddenly 'agree' to tell the truth??

Much of the alternative media is buying the lie because it fits the conspiracy narrative, but it's the *wrong* conspiracy. The real conspiracy is that *there is no virus* and that is what the Cult is desperate to hide. The idea that the 'virus' was released by accident is ludicrous when the whole 'Covid' hoax was clearly long-planned and waiting to be played out as it was so fast in accordance with the Rockefeller document and Event 201. So they prepared everything in detail over decades and then sat around strumming their fingers waiting for an 'accidental' release from a bio-lab? *What??* It's crazy. Then there's the 'on purpose' claim. You want to circulate a 'deadly virus' and hide the fact that you've done so and you release it down the street from the highest-level bio-lab in China? I repeat – *What??*

You would release it far from that lab to stop any association being made. But, no, we'll do it in a place where the connection was certain to be made. Why would you need to scam 'cases' and 'deaths' and pay hospitals to diagnose 'Covid-19' if you had a real 'virus'? What are sections of the alternative media doing believing this crap? Where were all the mass deaths in Wuhan from a 'deadly pathogen' when the recovery to normal life after the initial propaganda was dramatic in speed? Why isn't the 'deadly pathogen' now circulating all over China with bodies in the street? Once again we have the technique of tell them what they want to hear and they will likely believe it. The alternative media has its 'conspiracy' and with Carlson it fits with his 'China is the danger' narrative over years. China *is* a danger as a global Cult operations centre, but not for this reason. The Wuhan lab story also has the potential to instigate conflict with China when at some stage the plan is to trigger a Problem-Reaction-Solution confrontation with the West. Question everything – *everything* – and especially when the media agrees on a common party line.

Third wave ... fourth wave ... fifth wave ...

As the book went into production the world was being set up for more lockdowns and a 'third wave' supported by invented 'variants' that were increasing all the time and will continue to do so in public statements and computer programs, but not in reality. India became the new Italy in the 'Covid' propaganda campaign and we were told to be frightened of the new 'Indian strain'. Somehow I couldn't find it within myself to do so. A document produced for the UK government entitled 'Summary of further modelling of easing of restrictions – Roadmap Step 2' declared that a third wave was inevitable (of course when it's in the script) and it would be the fault of children and those who refuse the health-destroying fake 'Covid vaccine'. One of the computer models involved came from the Cult-owned *Imperial College* and the other from Warwick University which I wouldn't trust to tell me the date in a calendar factory. The document states that both models presumed extremely high uptake

of the 'Covid vaccines' and didn't allow for 'variants'. The document states: 'The resurgence is a result of some people (mostly children) being ineligible for vaccination; others choosing not to receive the vaccine; and others being vaccinated but not perfectly protected.' The mendacity takes the breath away. Okay, blame those with a brain who won't take the DNA-modifying shots and put more pressure on children to have it as 'trials' were underway involving children as young as six months with parents who give insanity a bad name. Massive pressure is being put on the young to have the fake 'vaccine' and child age consent limits have been systematically lowered around the world to stop parents intervening. Most extraordinary about the document was its claim that the 'third wave' would be driven by 'the resurgence in both hospitalisations and deaths … dominated by *those that have received two doses of the vaccine*, comprising around 60-70% of the wave respectively'. The predicted peak of the 'third wave' suggested 300 deaths per day with 250 of them *fully 'vaccinated' people*. How many more lies do acquiescers need to be told before they see the obvious? Those who took the jab to 'protect themselves' are projected to be those who mostly get sick and die? So what's in the 'vaccine'? The document went on:

> It is possible that a summer of low prevalence could be followed by substantial increases in incidence over the following autumn and winter. Low prevalence in late summer should not be taken as an indication that SARS-CoV-2 has retreated or that the population has high enough levels of immunity to prevent another wave.

They are telling you the script and while many British people believed 'Covid' restrictions would end in the summer of 2021 the government was preparing for them to be ongoing. Authorities were awarding contracts for 'Covid marshals' to police the restrictions with contracts starting in July, 2021, and going through to January 31st, 2022, and the government was advertising for 'Media Buying Services' to secure media propaganda slots worth a potential £320 million for 'Covid-19 campaigns' with a contract not ending until March, 2022. The recipient – via a list of other front companies – was reported to be American media marketing giant Omnicom Group

Inc. While money is no object for 'Covid' the UK waiting list for all other treatment – including life-threatening conditions – passed 4.5 million. Meantime the Cult is seeking to control all official 'inquiries' to block revelations about what has really been happening and why. It must not be allowed to – we need Nuremberg jury trials in every country. The cover-up doesn't get more obvious than appointing ultra-Zionist professor Philip Zelikow to oversee two dozen US virologists, public health officials, clinicians, former government officials and four American 'charitable foundations' to 'learn the lessons' of the 'Covid' debacle. The personnel will be those that created and perpetuated the 'Covid' lies while Zelikow is the former executive director of the 9/11 Commission who ensured that the truth about those attacks never came out and produced a report that must be among the most mendacious and manipulative documents ever written – see *The Trigger* for the detailed exposure of the almost unimaginable 9/11 story in which Sabbatians can be found at every level.

Passive no more

People are increasingly challenging the authorities with amazing numbers of people taking to the streets in London well beyond the ability of the Face-Nappies to stop them. Instead the Nappies choose situations away from the mass crowds to target, intimidate, and seek to promote the impression of 'violent protestors'. One such incident happened in London's Hyde Park. Hundreds of thousands walking through the streets in protest against 'Covid' fascism were ignored by the Cult-owned BBC and most of the rest of the mainstream media, but they delighted in reporting how police were injured in 'clashes with protestors'. The truth was that a group of people gathered in Hyde Park at the end of one march when most had gone home and they were peacefully having a good time with music and chat. Face-Nappies who couldn't deal with the full-march crowd then waded in with their batons and got more than they bargained for. Instead of just standing for this criminal brutality the crowd used their numerical superiority to push the Face-Nappies out of the

park. Eventually the Nappies turned and ran. Unfortunately two or three idiots in the crowd threw drink cans striking two officers which gave the media and the government the image they wanted to discredit the 99.9999 percent who were peaceful. The idiots walked straight into the trap and we must always be aware of potential agent provocateurs used by the authorities to discredit their targets.

This response from the crowd – the can people apart – must be a turning point when the public no longer stand by while the innocent are arrested and brutally attacked by the Face-Nappies. That doesn't mean to be violent, that's the last thing we need. We'll leave the violence to the Face-Nappies and government. But it does mean that when the Face-Nappies use violence against peaceful people the numerical superiority is employed to stop them and make citizen's arrests or Common Law arrests for a breach of the peace. The time for being passive in the face of fascism is over.

We are the many, they are the few, and we need to make that count before there is no freedom left and our children and grandchildren face an ongoing fascist nightmare.

COME ON PEOPLE – IT'S TIME.

One final thought …

The power of love
A force from above
Cleaning my soul
Flame on burn desire
Love with tongues of fire
Purge the soul
Make love your goal

I'll protect you from the hooded claw
Keep the vampires from your door
When the chips are down I'll be around
With my undying, death-defying
Love for you

Envy will hurt itself
Let yourself be beautiful
Sparkling love, flowers
And pearls and pretty girls
Love is like an energy
Rushin' rushin' inside of me

This time we go sublime
Lovers entwine, divine, divine,
Love is danger, love is pleasure
Love is pure – the only treasure

I'm so in love with you
Purge the soul
Make love your goal

The power of love
A force from above
Cleaning my soul
The power of love
A force from above
A sky-scraping dove

Flame on burn desire
Love with tongues of fire
Purge the soul
Make love your goal

Frankie Goes To Hollywood

APPENDIX

Cowan-Kaufman-Morell Statement on Virus Isolation (SOVI)

Isolation: The action of isolating; the fact or condition of being isolated or standing alone; separation from other things or persons; solitariness
Oxford English Dictionary

The controversy over whether the SARS-CoV-2 virus has ever been isolated or purified continues. However, using the above definition, common sense, the laws of logic and the dictates of science, any unbiased person must come to the conclusion that the SARS-CoV-2 virus has never been isolated or purified. As a result, no confirmation of the virus' existence can be found. The logical, common sense, and scientific consequences of this fact are:

- the structure and composition of something not shown to exist can't be known, including the presence, structure, and function of any hypothetical spike or other proteins;
- the genetic sequence of something that has never been found can't be known;
- "variants" of something that hasn't been shown to exist can't be known;
- it's impossible to demonstrate that SARS-CoV-2 causes a disease called Covid-19.

In as concise terms as possible, here's the proper way to isolate, characterize and demonstrate a new virus. First, one takes samples (blood, sputum, secretions) from many people (e.g. 500) with symptoms which are unique and specific enough to characterize an illness. Without mixing these samples with ANY tissue or products that also contain genetic material, the virologist macerates, filters and ultracentrifuges i.e. *purifies* the specimen. This common virology technique, done for decades to isolate bacteriophages[1] and so-called giant viruses in every virology lab, then allows the virologist to demonstrate with electron microscopy thousands of identically sized and shaped particles. These particles are the isolated and purified virus.

These identical particles are then checked for uniformity by physical and/or microscopic techniques. Once the purity is determined, the particles may be further characterized. This would include examining the structure, morphology, and chemical composition of the particles. Next, their genetic makeup is characterized by extracting the genetic material directly from the purified particles and using genetic-sequencing techniques, such as Sanger sequencing, that have also been around for decades. Then one does an analysis to confirm that these uniform particles are exogenous (outside) in origin as a virus is conceptualized to be, and not the normal breakdown products of dead and dying tissues.[2] (As of May 2020, we know that virologists have no way to determine whether the particles they're seeing are viruses or just normal breakdown products of dead and dying tissues.)[3]

1 Isolation, characterization and analysis of bacteriophages from the haloalkaline lake Elmenteita, KenyaJuliah Khayeli Akhwale et al, PLOS One, Published: April 25, 2019. https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0215734 – accessed 2/15/21

2 "Extracellular Vesicles Derived From Apoptotic Cells: An Essential Link Between Death and Regeneration," Maojiao Li1 et al, Frontiers in Cell and Developmental Biology, 2020 October 2. https://www.frontiersin.org/articles/10.3389/fcell.2020.573511/full – accessed 2/15/21

3 "The Role of Extraellular Vesicles as Allies of HIV, HCV and SARS Viruses," Flavia Giannessi, et al, Viruses, 2020 May

If we have come this far then we have fully isolated, characterized, and genetically sequenced an exogenous virus particle. However, we still have to show it is causally related to a disease. This is carried out by exposing a group of healthy subjects (animals are usually used) to this isolated, purified virus in the manner in which the disease is thought to be transmitted. If the animals get sick with the same disease, as confirmed by clinical and autopsy findings, one has now shown that the virus actually causes a disease. This demonstrates infectivity and transmission of an infectious agent.

None of these steps has even been attempted with the SARS-CoV-2 virus, nor have all these steps been successfully performed for any so-called pathogenic virus. Our research indicates that a single study showing these steps does not exist in the medical literature.

Instead, since 1954, virologists have taken unpurified samples from a relatively few people, often less than ten, with a similar disease. They then minimally process this sample and inoculate this unpurified sample onto tissue culture containing usually four to six other types of material – all of which contain identical genetic material as to what is called a "virus." The tissue culture is starved and poisoned and naturally disintegrates into many types of particles, some of which contain genetic material. Against all common sense, logic, use of the English language and scientific integrity, this process is called "virus isolation." This brew containing fragments of genetic material from many sources is then subjected to genetic analysis, which then creates in a computer-simulation process the alleged sequence of the alleged virus, a so called in silico genome. At no time is an actual virus confirmed by electron microscopy. At no time is a genome extracted and sequenced from an actual virus. This is scientific fraud.

The observation that the unpurified specimen — inoculated onto tissue culture along with toxic antibiotics, bovine fetal tissue, amniotic fluid and other tissues — destroys the kidney tissue onto which it is inoculated is given as evidence of the virus' existence and pathogenicity. This is scientific fraud.

From now on, when anyone gives you a paper that suggests the SARS-CoV-2 virus has been isolated, please check the methods sections. If the researchers used Vero cells or any other culture method, you know that their process was not isolation. You will hear the following excuses for why actual isolation isn't done:

1. There were not enough virus particles found in samples from patients to analyze.

2. Viruses are intracellular parasites; they can't be found outside the cell in this manner.

If No. 1 is correct, and we can't find the virus in the sputum of sick people, then on what evidence do we think the virus is dangerous or even lethal? If No. 2 is correct, then how is the virus spread from person to person? We are told it emerges from the cell to infect others. Then why isn't it possible to find it?

Finally, questioning these virology techniques and conclusions is not some distraction or divisive issue. Shining the light on this truth is essential to stop this terrible fraud that humanity is confronting. For, as we now know, if the virus has never been isolated, sequenced or shown to cause illness, if the virus is imaginary, then why are we wearing masks, social distancing and putting the whole world into prison?

Finally, if pathogenic viruses don't exist, then what is going into those injectable devices erroneously called "vaccines," and what is their purpose? This scientific question is the most urgent and relevant one of our time.

We are correct. The SARS-CoV2 virus does not exist.

Sally Fallon Morell, MA
Dr. Thomas Cowan, MD
Dr. Andrew Kaufman, MD

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Index

A

abusive relationships
blaming themselves, abused as ref1
children ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
conspiracy theories ref1
domestic abuse ref1, ref2
economic abuse and dependency ref1
isolation ref1
physical abuse ref1
psychological abuse ref1
signs of abuse ref1
addiction
alcoholism ref1
frequencies ref1
substance abuse ref1, ref2
technology ref1, ref2, ref3
Adelson, Sheldon ref1, ref2, ref3
Agenda 21/Agenda 2030 (UN) ref1, ref2, ref3, ref4
AIDs/HIV ref1
causal link between HIV and AIDs ref1, ref2
retroviruses ref1
testing ref1, ref2
trial-run for Covid-19, as ref1, ref2
aliens/extraterrestrials ref1, ref2
aluminium ref1
Amazon ref1, ref2, ref3

amplification cycles ref1, ref2
anaphylactic shock ref1, ref2, ref3, ref4
animals ref1, ref2, ref3
antibodies ref1, ref2, ref3, ref4, ref5
Antifa ref1, ref2, ref3, ref4
antigens ref1, ref2
anti-Semitism ref1, ref2, ref3
Archons ref1, ref2
 consciousness ref1, ref2, ref3
 energy ref1, ref2, ref3
 ennoia ref1
 genetic manipulation ref1, ref2
 inversion ref1, ref2, ref3
 lockdowns ref1
 money ref1
 radiation ref1
 religion ref1, ref2
 technology ref1, ref2, ref3
 Wetiko factor ref1, ref2, ref3, ref4
artificial intelligence (AI) ref1
army made up of robots ref1, ref2
 Human 2.0 ref1, ref2
 Internet ref1
 MHRA ref1
 Morgellons fibres ref1, ref2
 Smart Grid ref1
 Wetiko factor ref1
asymptomatic, Covid-19 as ref1, ref2, ref3
aviation industry ref1

B

banking, finance and money ref1, ref2, ref3
2008 crisis ref1, ref2
boom and bust ref1
cashless digital money systems ref1
central banks ref1
credit ref1
digital currency ref1
fractional reserve lending ref1
Great Reset ref1
guaranteed income ref1, ref2, ref3
Human 2.0 ref1
incomes, destruction of ref1, ref2
interest ref1
one per cent ref1, ref2
scams ref1
BBC ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
Becker-Phelps, Leslie ref1
Behavioural Insights Team (BIT) (Nudge Unit) ref1, ref2, ref3
behavioural scientists *and* **psychologists, advice from** ref1, ref2
Bezos, Jeff ref1, ref2, ref3, ref4
Biden, Hunter ref1
Biden, Joe ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10, ref11, ref12, ref13, ref14, ref15, ref16, ref17
Big Pharma
cholesterol ref1
health professionals ref1, ref2
immunity from prosecution in US ref1
vaccines ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
Wetiko factor ref1, ref2
WHO ref1, ref2, ref3
Bill and Melinda Gates Foundation ref1, ref2, ref3, ref4, ref5, ref6, ref7

billionaires ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9 ref10, ref11
bird flu (H5N1) ref1
Black Lives Matter (BLM) ref1, ref2, ref3, ref4, ref5
Blair, Tony ref1, ref2, ref3, ref4, ref5, ref6, ref7
Brin, Sergei ref1, ref2, ref3, ref4, ref5, ref6, ref7
British Empire ref1
Bush, George HW ref1, ref2
Bush, George W ref1, ref2, ref3, ref4
Byrd, Robert ref1

C

Canada
- Global Cult ref1
- hate speech ref1
- internment ref1
- masks ref1
- old people ref1
- SARS-COV-2 ref1
- satellites ref1
- vaccines ref1
- wearable technology ref1

Capitol Hill riot ref1, ref2
- agents provocateur ref1
- Antifa ref1
- Black Lives Matter (BLM) ref1, ref2
- QAnon ref1
- security precautions, lack of ref1, ref2, ref3

carbon dioxide ref1, ref2
care homes, deaths in ref1, ref2
cashless digital money systems ref1
censorship ref1, ref2, ref3, ref4, ref5

fact-checkers ref1
masks ref1
media ref1, ref2
private messages ref1
social media ref1, ref2, ref3, ref4, ref5, ref6
transgender persons ref1
vaccines ref1, ref2, ref3
Wokeness ref1
Centers for Disease Control (CDC) (United States) ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10, ref11, ref12, ref13
centralisation ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
chakras ref1
change agents ref1, ref2, ref3
chemtrails ref1, ref2, ref3
chief medical officers and scientific advisers ref1, ref2, ref3, ref4, ref5, ref6
children *see also* **young people**
abuse ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
care, taken into ref1, ref2, ref3
education ref1, ref2, ref3, ref4
energy ref1
family courts ref1
hand sanitisers ref1
human sacrifice ref1
lockdowns ref1, ref2, ref3
masks ref1, ref2, ref3, ref4, ref5
mental health ref1
old people ref1
parents, replacement of ref1, ref2
Psyop (psychological operation), Covid as a ref1, ref2
reframing ref1
smartphone addiction ref1

social distancing and isolation ref1
social media ref1
transgender persons ref1, ref2
United States ref1
vaccines ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
Wetiko factor ref1
China ref1, ref2, ref3, ref4
anal swab tests ref1
Chinese Revolution ref1, ref2, ref3
digital currency ref1
Global Cult ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9
guaranteed income ref1
Imperial College ref1
Israel ref1
lockdown ref1, ref2
masculinity crisis ref1
masks ref1
media ref1
origins of virus in China ref1, ref2, ref3, ref4, ref5
pollution causing respiratory diseases ref1
Sabbatians ref1, ref2
Smart Grid ref1, ref2
social credit system ref1
testing ref1, ref2
United States ref1, ref2
vaccines ref1, ref2
Wetiko factor ref1
wet market conspiracy ref1
Wuhan ref1, ref2, ref3, ref4, ref5, ref6, ref7
cholesterol ref1, ref2
Christianity ref1, ref2, ref3, ref4, ref5
criticism ref1
cross, inversion of the ref1

Nag Hammadi texts ref1, ref2, ref3
Roman Catholic Church ref1, ref2
Sabbatians ref1, ref2
Satan ref1, ref2, ref3, ref4
Wokeness ref1
class ref1, ref2
climate change hoax ref1, ref2, ref3, ref4, ref5
Agenda 21/Agenda 2030 ref1, ref2, ref3
carbon dioxide ref1, ref2
Club of Rome ref1, ref2, ref3, ref4, ref5
fear ref1
funding ref1
Global Cult ref1
green new deals ref1
green parties ref1
inversion ref1
perception, control of ref1
PICC ref1
reframing ref1
temperature, increases in ref1
United Nations ref1, ref2
Wikipedia ref1
Wokeness ref1, ref2
Clinton, Bill ref1, ref2, ref3, ref4, ref5, ref6
Clinton, Hillary ref1, ref2, ref3
the cloud ref1, ref2, ref3, ref4, ref5, ref6, ref7
Club of Rome and climate change hoax ref1, ref2, ref3, ref4, ref5
cognitive therapy ref1
Cohn, Roy ref1
Common Law ref1
Admiralty Law ref1
arrests ref1, ref2

contractual law, Statute Law as ref1
corporate entities, people as ref1
legalese ref1
sea, law of the ref1
Statute Law ref1
Common Purpose leadership programme ref1, ref2
communism ref1, ref2
co-morbidities ref1
computer-generated virus,
Covid-19 as ref1, ref2, ref3
computer models ref1, ref2, ref3, ref4, ref5
connections ref1, ref2, ref3, ref4
consciousness ref1, ref2, ref3, ref4
Archons ref1, ref2, ref3
expanded ref1, ref2, ref3, ref4, ref5, ref6, ref7
experience ref1
heart ref1
infinity ref1, ref2
religion ref1, ref2
self-identity ref1
simulation thesis ref1
vaccines ref1
Wetiko factor ref1, ref2
conspiracy theorists ref1, ref2, ref3, ref4, ref5
contradictory rules ref1
contrails ref1
Corman-Drosten test ref1, ref2, ref3, ref4
countermimicry ref1, ref2, ref3
Covid-19 vaccines *see* vaccines
Covidiots ref1, ref2
Cowan, Tom ref1, ref2, ref3, ref4
crimes against humanity ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8

cyber-operations ref1
cyberwarfare ref1

D

DARPA (Defense Advanced Research Projects Agency) ref1
deaths
 care homes ref1
 certificates ref1, ref2, ref3, ref4
 mortality rate ref1
 post-mortems/autopsies ref1
 recording ref1, ref2, ref3, ref4, ref5, ref6, ref7
 vaccines ref1, ref2, ref3, ref4, ref5
deceit
 pyramid of deceit ref1, ref2
 sequence of deceit ref1
decoding ref1, ref2, ref3
dehumanisation ref1, ref2, ref3
Delphi technique ref1
democracy ref1
dependency ref1, ref2, ref3, ref4, ref5
Descartes, René ref1
DNA
 numbers ref1
 vaccines ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
DNR (do not resuscitate)
 orders ref1
domestic abuse ref1, ref2
downgrading of Covid-19 ref1
Drosten, Christian ref1, ref2, ref3, ref4, ref5, ref6, ref7
Duesberg, Peter ref1, ref2

E

economic abuse ref1
Edmunds, John ref1, ref2
education ref1, ref2, ref3, ref4
electromagnetic spectrum ref1, ref2
Enders, John ref1
energy
 Archons ref1, ref2, ref3
 children and young people ref1
 consciousness ref1
 decoding ref1
 frequencies ref1, ref2, ref3, ref4
 heart ref1
 human energy field ref1
 source, humans as an energy ref1, ref2
 vaccines ref1
 viruses ref1
ennoia ref1
Epstein, Jeffrey ref1, ref2
eternal 'I' ref1, ref2
ethylene oxide ref1
European Union ref1, ref2, ref3, ref4
Event ref1 *and* **Bill Gates** ref2
exosomes, Covid-19 as natural defence mechanism called ref1
experience ref1, ref2
Extinction Rebellion ref1, ref2

F

Facebook
 addiction ref1, 448–50
 Facebook

Archons ref1
censorship ref1, ref2, ref3
hate speech ref1
monopoly, as ref1
private messages, censorship of ref1
Sabbatians ref1
United States election fraud ref1
vaccines ref1
Wetiko factor ref1
fact-checkers ref1
Fauci, Anthony ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10, ref11, ref12
fear ref1, ref2, ref3, ref4
climate change ref1
computer models ref1
conspiracy theories ref1
empty hospitals ref1
Italy ref1, ref2, ref3
lockdowns ref1, ref2, ref3, ref4
masks ref1, ref2
media ref1, ref2
medical staff ref1
Psyop (psychological operation), Covid as a ref1
Wetiko factor ref1, ref2
female infertility ref1
Fermi Paradox ref1
Ferguson, Neil ref1, ref2, ref3, ref4, ref5, ref6, ref7
fertility, decline in ref1
The Field ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
finance *see* **banking, finance and money**
five-senses ref1, ref2
Archons ref1, ref2, ref3

censorship ref1
consciousness, expansion of ref1, ref2, ref3, ref4, ref5, ref6
decoding ref1
education ref1, ref2
the Field ref1, ref2
God, personification of ref1
infinity ref1, ref2
media ref1
paranormal ref1
perceptual programming ref1, ref2
Phantom Self ref1
pneuma not nous, using ref1
reincarnation ref1
self-identity ref1
Wetiko factor ref1, ref2, ref3, ref4, ref5, ref6
5G ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
Floyd, George and protests, killing of ref1
flu, re-labelling of ref1, ref2, ref3
food and water, control of ref1, ref2
Freemasons ref1, ref2, ref3, ref4, ref5, ref6
Frei, Rosemary ref1
frequencies
addictions ref1
Archons ref1, ref2, ref3
awareness ref1
chanting and mantras ref1
consciousness ref1
decoding ref1, ref2
education ref1
electromagnetic (EMF) frequencies ref1
energy ref1, ref2, ref3, ref4
fear ref1

the Field ref1, ref2 5G ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
five-senses ref1, ref2
ghosts ref1
Gnostics ref1
hive-minds ref1
human, meaning of ref1
light ref1, ref2
love ref1, ref2
magnetism ref1
perception ref1
reality ref1, ref2, ref3
simulation ref1
terror ref1
vaccines ref1
Wetiko ref1, ref2, ref3
Fuellmich, Reiner ref1, ref2, ref3
furlough/rescue payments ref1

G

Gallo, Robert ref1, ref2, ref3
Gates, Bill
Archons ref1, ref2, ref3
climate change ref1, ref2, ref3, ref4
Daily Pass tracking system ref1
Epstein ref1
fascism ref1
five senses ref1
GAVI ref1
Great Reset ref1
GSK ref1
Imperial College ref1, ref2
Johns Hopkins University ref1, ref2, ref3

lockdowns ref1, ref2
masks ref1
Nuremberg trial, proposal for ref1, ref2
Rockefellers ref1, ref2
social distancing and isolation ref1
Sun, dimming the ref1
synthetic meat ref1, ref2
vaccines ref1, ref2, ref3, ref4, ref5, ref6, ref7
Wellcome Trust ref1
Wetiko factor ref1, ref2, ref3
WHO ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
Wokeness ref1
World Economic Forum ref1, ref2, ref3, ref4
Gates, Melinda ref1, ref2, ref3
GAVI vaccine alliance ref1
genetics, manipulation of ref1, ref2, ref3
Germany ref1, ref2, ref3, ref4, ref5, ref6 *see also* **Nazi Germany**
Global Cult ref1, ref2, ref3, ref4, ref5
anti-human, why Global Cult is ref1
Black Lives Matter (BLM) ref1, ref2, ref3, ref4
China ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9
climate change hoax ref1
contradictory rules ref1
Covid-19 ref1, ref2, ref3
fascism ref1
geographical origins ref1
immigration ref1
Internet ref1
mainstream media ref1, ref2
masks ref1, ref2
monarchy ref1
non-human dimension ref1

perception ref1
political parties ref1, ref2
pyramidal hierarchy ref1, ref2, ref3
reframing ref1
Sabbantian-Frankism ref1, ref2
science, manipulation of ref1
spider and the web ref1
transgender persons ref1
vaccines ref1
who controls the Cult ref1
Wokeness ref1, ref2, ref3, ref4
globalisation ref1, ref2
Gnostics ref1, ref2, ref3, ref4, ref5
Google ref1, ref2, ref3, ref4
government
behavioural scientists and psychologists, advice from ref1, ref2
definition ref1
Joint Biosecurity Centre (JBC) ref1
people, abusive relationship with ref1
Great Reset ref1, ref2, ref3, ref4, ref5, ref6
fascism ref1, ref2, ref3
financial system ref1
Human 2.0 ref1
water and food, control of ref1
green parties ref1
Griesz-Brisson, Margarite ref1
guaranteed income ref1, ref2, ref3

H

Hancock, Matt ref1, ref2, ref3, ref4, ref5
hand sanitisers ref1
heart ref1, ref2

hive-minds/groupthink ref1, ref2, ref3
holographs ref1, ref2, ref3, ref4
hospitals, empty ref1
human, meaning of ref1
Human 2.0 ref1
addiction to technology ref1
artificial intelligence (AI) ref1, ref2
elimination of Human 1.0 ref1
fertility, decline in ref1
Great Reset ref1
implantables ref1
money ref1
mRNA ref1
nanotechnology ref1
parents, replacement of ref1, ref2
Smart Grid, connection to ref1, ref2
synthetic biology ref1, ref2, ref3, ref4
testosterone levels, decrease in ref1
transgender = transhumanism ref1, ref2, ref3
vaccines ref1, ref2, ref3, ref4
human sacrifice ref1, ref2, ref3
Hunger Games Society ref1, ref2, ref3, ref4, ref5, ref6, ref7
Huxley, Aldous ref1, ref2, ref3

I

identity politics ref1, ref2, ref3
Illuminati ref1, ref2
illusory physical reality ref1
immigration ref1, ref2, ref3, ref4
Imperial College ref1, ref2, ref3, ref4, ref5, ref6
implantables ref1, ref2

incomes, destruction of ref1, ref2
Infinite Awareness ref1, ref2, ref3, ref4
Internet ref1, ref2 *see also* social media
artificial intelligence (AI) ref1
independent journalism, lack of ref1
Internet of Bodies (IoB) ref1
Internet of Everything (IoE) ref1, ref2
Internet of Things (IoT) ref1, ref2
lockdowns ref1
Psyop (psychological operation), Covid as a ref1
trolls ref1
intersectionality ref1
inversion
Archons ref1, ref2, ref3
climate change hoax ref1
energy ref1
Judaism ref1, ref2, ref3
symbolism ref1
Wetiko factor ref1
Wokeness ref1, ref2, ref3
Islam
Archons ref1
crypto-Jews ref1
Islamic State ref1, ref2
Jinn and Djinn ref1, ref2, ref3
Ottoman Empire ref1
Wahhabism ref1
isolation *see* **social distancing** *and* **isolation**
Israel
China ref1
Cyber Intelligence Unit Beersheba complex ref1
expansion of illegal settlements ref1

formation ref1
Global Cult ref1
Judaism ref1, ref2, ref3, ref4, ref5
medical experiments, consent for ref1
Mossad ref1, ref2, ref3, ref4
Palestine-Israel conflict ref1, ref2, ref3
parents, replacement of ref1
Sabbatians ref1, ref2, ref3, ref4, ref5
September 11, 2001, terrorist attacks on United States ref1
Silicon Valley ref1
Smart Grid ref1, ref2
United States ref1, ref2
vaccines ref1
Wetiko factor ref1
Italy
fear ref1, ref2, ref3
Lombardy ref1, ref2, ref3
vaccines ref1

J

Johns Hopkins University ref1, ref2, ref3, ref4, ref5, ref6, ref7
Johnson, Boris ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
Joint Biosecurity Centre (JBC) ref1
Judaism
anti-Semitism ref1, ref2, ref3
Archons ref1, ref2
crypto-Jews ref1
inversion ref1, ref2, ref3
Israel ref1, ref2, ref3, ref4, ref5
Labour Party ref1
Nazi Germany ref1, ref2, ref3, ref4
Sabbatians ref1, ref2, ref3, ref4, ref5

Silicon Valley ref1
Torah ref1
United States ref1, ref2
Zionists ref1, ref2, ref3

K

Kaufman, Andrew ref1, ref2, ref3, ref4
knowledge ref1, ref2, ref3, ref4, ref5, ref6
Koch's postulates ref1
Kurzweil, Ray ref1, ref2, ref3, ref4, ref5, ref6, ref7
Kushner, Jared **ref1**, **ref2**

L

Labour Party ref1, ref2
Lanka, Stefan ref1, ref2
Lateral Flow Device (LFD) ref1
Levy, Paul ref1, ref2, ref3
Life Program ref1
lockdowns ref1, ref2, ref3
amplification tampering ref1
Archons ref1
Behavioural Insights Team ref1
Black Lives Matter (BLM) ref1
care homes, deaths in ref1
children
abuse ref1, ref2
mental health ref1
China ref1, ref2
computer models ref1
consequences ref1, ref2
dependency ref1, ref2, ref3

domestic abuse ref1
fall in cases ref1
fear ref1, ref2, ref3, ref4
guaranteed income ref1
Hunger Games Society ref1, ref2, ref3
interaction, destroying ref1
Internet ref1, ref2
overdoses ref1
perception ref1
police-military state ref1, ref2
protests ref1, ref2, ref3, ref4, ref5
psychopathic personality ref1, ref2, ref3
reporting/snitching, encouragement of ref1, ref2
testing ref1
vaccines ref1
Wetiko factor ref1
WHO ref1
love ref1, ref2, ref3
Lucifer ref1, ref2, ref3

M

Madej, Carrie ref1, ref2
Magufuli, John ref1, ref2
mainstream media ref1
BBC ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
censorship ref1, ref2
China ref1
climate change hoax ref1
fear ref1, ref2
Global Cult ref1, ref2
independent journalism, lack of ref1
Ofcom ref1, ref2, ref3

perception ref1, ref2
Psyop (psychological operation), Covid as a ref1
Sabbatians ref1, ref2
social disapproval ref1
social distancing and isolation ref1
United States ref1, ref2
vaccines ref1, ref2, ref3, ref4, ref5
Mao Zedong ref1, ref2, ref3
Marx and Marxism ref1, ref2, ref3, ref4, ref5, ref6
masculinity ref1
masks/face coverings ref1, ref2, ref3
censorship ref1
children ref1, ref2, ref3, ref4, ref5
China, made in ref1
dehumanisation ref1, ref2, ref3
fear ref1, ref2
flu ref1
health professionals ref1, ref2, ref3, ref4
isolation ref1
laughter ref1
mass non-cooperation ref1
microplastics, risk of ref1
mind control ref1
multiple masks ref1
oxygen deficiency ref1, ref2, ref3
police ref1, ref2, ref3, ref4, ref5
pollution, as cause of plastic ref1
Psyop (psychological operation), Covid as a ref1
reframing ref1, ref2
risk assessments, lack of ref1, ref2
self-respect ref1
surgeons ref1

United States ref1
vaccines ref1, ref2, ref3, ref4, ref5
Wetiko factor ref1
'worms' ref1
The Matrix movies ref1, ref2, ref3
measles ref1, ref2
media see mainstream media
Medicines and Healthcare products Regulatory Agency (MHRA) ref1, ref2, ref3, ref4
Mesopotamia ref1
messaging ref1
military-police state ref1, ref2, ref3
mind control ref1, ref2, ref3, ref4, ref5, ref6 *see also* **MKUltra**
MKUltra ref1, ref2, ref3
monarchy ref1
money *see* **banking, finance and money**
Montagnier, Luc ref1, ref2, ref3
Mooney, Bel ref1
Morgellons disease ref1, ref2
mortality rate ref1
Mullis, Kary ref1, ref2, ref3
Musk, Elon ref1

N

Nag Hammadi texts ref1, ref2, ref3
nanotechnology ref1, ref2, ref3
narcissism ref1
Nazi Germany ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
near-death experiences ref1, ref2
Neocons ref1, ref2, ref3

Neuro-Linguistic Programming (NLP) and the Delphi technique ref1
NHS (National Health Service)
amplification cycles ref1
Common Purpose ref1, ref2
mind control ref1
NHS England ref1
saving the NHS ref1, ref2
vaccines ref1, ref2, ref3, ref4, ref5
whistle-blowers ref1, ref2, ref3
No-Problem-Reaction-Solution ref1, ref2, ref3, ref4
non-human dimension of Global Cult ref1
nous ref1
numbers, reality as ref1
Nuremberg Codes ref1, ref2, ref3
Nuremberg-like tribunal, proposal for ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10, ref11, ref12

O

Obama, Barack ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
O'Brien, Cathy ref1, ref2, ref3, ref4
Ochel, Evita ref1
Ofcom ref1, ref2, ref3
old people ref1, ref2, ref3, ref4, ref5
Oneness ref1, ref2, ref3
Open Society Foundations (Soros) ref1, ref2, ref3
oxygen 406, 528–34

P

paedophilia ref1, ref2
Page, Larry ref1, ref2, ref3, ref4, ref5, ref6, ref7

Palestine-Israel conflict ref1, ref2, ref3
pandemic, definition of ref1
pandemic and health crisis scenarios/simulations ref1, ref2, ref3, ref4
paranormal ref1
PCR tests ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
Pearl Harbor attacks, prior knowledge of ref1
Pelosi, Nancy ref1, ref2, ref3
perception ref1, ref2, ref3, ref4
- climate change hoax ref1
- control ref1, ref2, ref3
- decoding ref1, ref2
- enslavement ref1
- externally-delivered perceptions ref1
- five senses ref1
- human labels ref1
- media ref1, ref2
- political parties ref1, ref2
- Psyop (psychological operation), Covid as a ref1
- sale of perception ref1
- self-identity ref1, ref2
- Wokeness ref1

Phantom Self ref1, ref2, ref3
pharmaceutical industry *see* **Big Pharma**
phthalates ref1
Plato's Allegory of the Cave ref1, ref2
pneuma ref1
police
- Black Lives Matter (BLM) ref1
- brutality ref1
- citizen's arrests ref1, ref2
- common law arrests ref1, ref2

Common Purpose ref1
defunding ref1
lockdowns ref1, ref2
masks ref1, ref2, ref3, ref4
police-military state ref1, ref2, ref3
psychopathic personality ref1, ref2, ref3, ref4
reframing ref1
United States ref1, ref2, ref3, ref4
Wokeness ref1
polio ref1
political correctness ref1, ref2, ref3, ref4
political parties ref1, ref2, ref3, ref4
political puppets ref1
pollution ref1, ref2, ref3
post-mortems/autopsies ref1
Postage Stamp Consensus ref1, ref2
pre-emptive programming ref1
Problem-Reaction-Solution ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
Project for the New American Century ref1, ref2, ref3, ref4
psychopathic personality ref1
Archons ref1
heart energy ref1
lockdowns ref1, ref2, ref3
police ref1, ref2, ref3, ref4
recruitment ref1, ref2
vaccines ref1
wealth ref1
Wetiko ref1, ref2
Psyop (psychological operation), Covid as a ref1, ref2, ref3, ref4, ref5
Pushbackers ref1, ref2, ref3, ref4
pyramid structure ref1, ref2, ref3, ref4

Q

QAnon Psyop ref1, ref2, ref3

R

racism *see also* **Black Lives**
 Matter (BLM)
 anti-racism industry ref1
 class ref1
 critical race theory ref1
 culture ref1
 intersectionality ref1
 reverse racism ref1
white privilege ref1, ref2
white supremacy ref1, ref2, ref3, ref4, ref5
 Wokeness ref1, ref2, ref3
radiation ref1, ref2
randomness, illusion of ref1, ref2, ref3
reality ref1, ref2, ref3
reframing ref1, ref2
 change agents ref1, ref2
 children ref1
 climate change ref1
 Common Purpose leadership programme ref1, ref2
 contradictory rules ref1
 enforcers ref1
 masks ref1, ref2
 NLP and the Delphi technique ref1
 police ref1
 Wetiko factor ref1
 Wokeness ref1, ref2
religion *see also* particular religions
 alien invasions ref1

Archons ref1, ref2
consciousness ref1, ref2
control, system of ref1, ref2, ref3
criticism, prohibition on ref1
five senses ref1
good and evil, war between ref1
hidden non-human forces ref1, ref2
Sabbatians ref1
save me syndrome ref1
Wetiko ref1
Wokeness ref1
repetition and mind control ref1, ref2, ref3
reporting/snitching, encouragement of ref1, ref2
Reptilians/Grey entities ref1
rewiring the mind ref1
Rivers, Thomas Milton ref1, ref2
Rockefeller family ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9
Rockefeller Foundation documents ref1, ref2, ref3, ref4
Roman Empire ref1
Rothschild family ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9
RT-PCR tests ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
Russia
collusion inquiry in US ref1
Russian Revolution ref1, ref2
Sabbatians ref1

S

Sabbantian-Frankism ref1, ref2
anti-Semitism ref1, ref2
banking and finance ref1, ref2, ref3
China ref1, ref2
Israel ref1, ref2, ref3, ref4, ref5

Judaism ref1, ref2, ref3, ref4, ref5
Lucifer ref1
media ref1, ref2
Nazis ref1, ref2
QAnon ref1
Rothschilds ref1, ref2, ref3, ref4, ref5, ref6
Russia ref1
Saudi Arabia ref1
Silicon Valley ref1
Sumer ref1
United States ref1, ref2, ref3
Wetiko factor ref1
Wokeness ref1, ref2, ref3
SAGE (Scientific Advisory Group for Emergencies) ref1, ref2, ref3, ref4
SARS-1 ref1
SARs-CoV-2 ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
Satan/Satanism ref1, ref2, ref3, ref4, ref5, ref6, ref7
satellites in low-orbit ref1
Saudi Arabia ref1
Save Me Syndrome ref1
scapegoating ref1
Schwab, Klaus ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10, ref11, ref12
science, manipulation of ref1
self-identity ref1, ref2, ref3, ref4
self-respect, attacks on ref1
September 11, 2001, terrorist attacks on United States ref1, ref2, ref3, ref4
77th Brigade of UK military ref1, ref2, ref3
Silicon Valley/tech giants ref1, ref2, ref3, ref4, ref5, ref6 *see also* **Facebook**

Israel ref1
Sabbatians ref1
technocracy ref1
Wetiko factor ref1
Wokeness ref1
simulation hypothesis ref1, ref2, ref3, ref4, ref5
Smart Grid ref1, ref2, ref3
artificial intelligence (AI) ref1
China ref1, ref2
control centres ref1
the Field ref1
Great Reset ref1
Human 2.0 ref1, ref2
Israel ref1, ref2
vaccines ref1
Wetiko factor ref1
social disapproval ref1
social distancing *and* **isolation** ref1, ref2, ref3
abusive relationships ref1, ref2
children ref1
flats and apartments ref1
heart issues ref1
hugs ref1
Internet ref1
masks ref1
media ref1
older people ref1, ref2
one-metre (three feet) rule ref1
rewiring the mind ref1
simulation, universe as a ref1
SPI-B ref1
substance abuse ref1

suicide and self-harm ref1, ref2, ref3, ref4, ref5
technology ref1
torture, as ref1, ref2
two-metre (six feet) rule ref1
women ref1
social justice ref1, ref2, ref3, ref4
social media *see also* **Facebook bans on alternative views** ref1
censorship ref1, ref2, ref3, ref4, ref5, ref6
children ref1
emotion ref1
perception ref1
private messages ref1
Twitter ref1, ref2, ref3, ref4, ref5, ref6, ref7
Wetiko factor ref1
YouTube ref1, ref2, ref3, ref4, ref5
Soros, George ref1, ref2, ref3, ref4, ref5, ref6
Spain ref1
SPI-B (Scientific Pandemic Insights Group on Behaviours) ref1, ref2, ref3, ref4
spider and the web ref1, ref2, ref3, ref4
Starmer, Keir ref1
Statute Law ref1
Steiner, Rudolf ref1, ref2, ref3
Stockholm syndrome ref1
streptomycin ref1
suicide and self-harm ref1, ref2, ref3, ref4, ref5
Sumer ref1, ref2
Sunstein, Cass ref1, ref2, ref3
swine flu (H1N1) ref1, ref2, ref3
synchronicity ref1
synthetic biology ref1, ref2, ref3, ref4
synthetic meat ref1, ref2

T

technology *see also* **artificial intelligence (AI); Internet;**
social media addiction ref1, ref2, ref3, ref4
Archons ref1, ref2
the cloud ref1, ref2, ref3, ref4, ref5, ref6, ref7
cyber-operations ref1
cyberwarfare ref1
radiation ref1, ref2
social distancing and isolation ref1
technocracy ref1
Tedros Adhanom Ghebreyesus ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10, ref11, ref12, ref13
telepathy ref1
Tenpenny, Sherri ref1
Tesla, Nikola ref1
testosterone levels, decrease in ref1
testing for Covid-19 ref1, ref2
anal swab tests ref1
cancer ref1
China ref1, ref2, ref3
Corman-Drosten test ref1, ref2, ref3, ref4
death certificates ref1, ref2
fraudulent testing ref1
genetic material, amplification of ref1
Lateral Flow Device (LFD) ref1
PCR tests ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8
vaccines ref1, ref2, ref3
Thunberg, Greta ref1, ref2, ref3
Totalitarian Tiptoe ref1, ref2, ref3, ref4
transgender persons
activism ref1
artificial wombs ref1

censorship ref1
child abuse ref1, ref2
Human 2.0 ref1, ref2, ref3
Wokeness ref1, ref2, ref3, ref4, ref5
women, deletion of rights and status of ref1, ref2
young persons ref1
travel restrictions ref1
Trudeau, Justin ref1, ref2, ref3
Trump, Donald ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10, ref11
Twitter **ref1, ref2, ref3, ref4, ref5, ref6, ref7**

U

UKColumn ref1, ref2
United Nations (UN) ref1, ref2, ref3, ref4, ref5 *see also* **Agenda 21/Agenda 2030 (UN)**
United States ref1, ref2
American Revolution ref1
borders ref1, ref2
Capitol Hill riot ref1, ref2
children ref1
China ref1, ref2
CIA ref1, ref2
Daily Pass tracking system ref1
demographics by immigration, changes in ref1
Democrats ref1, ref2, ref3, ref4, ref5, ref6, ref7
election fraud ref1
far-right domestic terrorists, pushbackers as ref1
Federal Reserve ref1
flu/respiratory diseases statistics ref1
Global Cult ref1, ref2
hand sanitisers, FDA warnings on ref1

immigration, effects of illegal ref1
impeachment ref1
Israel ref1, ref2
Judaism ref1, ref2, ref3
lockdown ref1
masks ref1
mass media ref1, ref2
nursing homes ref1
Pentagon ref1, ref2, ref3, ref4
police ref1, ref2, ref3, ref4
pushbackers ref1
Republicans ref1, ref2
borders ref1, ref2
Democrats ref1, ref2, ref3, ref4, ref5
Russia, inquiry into collusion with ref1
Sabbatians ref1, ref2, ref3
September 11, 2001, terrorist attacks ref1, ref2, ref3, ref4
UFO sightings, release of information on ref1
vaccines ref1
white supremacy ref1, ref2, ref3, ref4
Woke Democrats ref1, ref2

V

vaccines ref1, ref2, ref3
adverse reactions ref1, ref2, ref3, ref4, ref5
Africa ref1
anaphylactic shock ref1, ref2, ref3, ref4
animals ref1, ref2
anti-vax movement ref1, ref2, ref3, ref4, ref5
AstraZeneca/Oxford ref1, ref2, ref3, ref4
autoimmune diseases, rise in ref1, ref2
Big Pharma ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8

bioweapon, as real ref1, ref2
black and ethnic minority communities ref1
blood clots ref1, ref2
Brain Computer Interface (BCI) ref1
care homes, deaths in ref1
censorship ref1, ref2, ref3
chief medical officers and scientific advisers, financial interests of ref1, ref2
children ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
China ref1, ref2
clinical trials ref1, ref2, ref3, ref4, ref5, ref6
compensation ref1
compulsory vaccinations ref1, ref2, ref3
computer programs ref1
consciousness ref1
cover-ups ref1
creation before Covid ref1
cytokine storm ref1
deaths and illnesses caused by vaccines ref1, ref2, ref3, ref4, ref5
definition ref1
developing countries ref1
digital tattoos ref1
DNA-manipulation ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
emergency approval ref1, ref2, ref3, ref4, ref5
female infertility ref1
funding ref1
genetic suicide ref1
Global Cult ref1
heart chakras ref1
hesitancy ref1
Human 2.0 ref1, ref2, ref3, ref4
immunity from prosecution ref1, ref2, ref3

implantable technology ref1
Israel ref1
Johnson & Johnson ref1, ref2, ref3, ref4
lockdowns ref1
long-term effects ref1
mainstream media ref1, ref2, ref3, ref4, ref5
masks ref1, ref2, ref3, ref4, ref5
Medicines and Healthcare products Regulatory Agency (MHRA) ref1, ref2
messaging ref1
Moderna ref1, ref2, ref3, ref4, ref5, ref6
mRNA vaccines ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9
nanotechnology ref1, ref2
NHS ref1, ref2, ref3, ref4, ref5
older people ref1, ref2
operating system ref1
passports ref1, ref2, ref3, ref4
Pfizer/BioNTech ref1, ref2, ref3, ref4, ref5, ref6, ref7
polyethylene glycol ref1
pregnant women ref1
psychopathic personality ref1
races, targeting different ref1
reverse transcription ref1
Smart Grid ref1
social distancing ref1
social media ref1
sterility ref1
synthetic material, introduction of ref1
tests ref1, ref2, ref3
travel restrictions ref1
variants ref1, ref2
viruses, existence of ref1
whistle-blowing ref1

WHO ref1, ref2, ref3, ref4
Wokeness ref1
working, vaccine as ref1
young people ref1
Vallance, Patrick ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9
variants ref1, ref2, ref3
vegans ref1
ventilators ref1, ref2
virology ref1, ref2
virtual reality ref1, ref2, ref3
viruses, existence of ref1
visual reality ref1, ref2
vitamin D ref1, ref2
von Braun, Wernher ref1, ref2

W

war-zone hospital myths ref1
waveforms ref1, ref2
wealth ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9 ref10, ref11
wet market conspiracy ref1
Wetiko factor ref1
alcoholism and drug addiction ref1
anti-human, why Global Cult is ref1
Archons ref1, ref2, ref3, ref4
artificial intelligence (AI) ref1
Big Pharma ref1, ref2
children ref1
China ref1
consciousness ref1, ref2
education ref1
Facebook ref1

fear ref1, ref2
frequency ref1, ref2
Gates ref1, ref2
Global Cult ref1, ref2
heart ref1, ref2
lockdowns ref1
masks ref1
Native American concept ref1
psychopathic personality ref1, ref2
reframing/retraining programmes ref1
religion ref1
Silicon Valley ref1
Smart Grid ref1
smartphone addiction ref1, ref2
social media ref1
war ref1, ref2
WHO ref1
Wokeness ref1, ref2, ref3
Yaldabaoth ref1, ref2, ref3, ref4
whistle-blowing ref1, ref2, ref3, ref4, ref5, ref6, ref7
white privilege ref1, ref2
white supremacy ref1, ref2, ref3, ref4, ref5
Whitty, Christopher ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10
'who benefits' ref1
Wi-Fi ref1, ref2, ref3, ref4
Wikipedia ref1, ref2
Wojcicki, Susan ref1, ref2, ref3, ref4, ref5, ref6, ref7
Wokeness
Antifa ref1, ref2, ref3, ref4
anti-Semitism ref1
billionaire social justice warriors ref1, ref2, ref3

Capitol Hill riot ref1, ref2
censorship ref1
Christianity ref1
climate change hoax ref1, ref2
culture ref1
education, control of ref1
emotion ref1
facts ref1
fascism ref1, ref2, ref3
Global Cult ref1, ref2, ref3, ref4
group-think ref1
immigration ref1
indigenous people, solidarity with ref1
inversion ref1, ref2, ref3
left, hijacking the ref1, ref2
Marxism ref1, ref2, ref3
mind control ref1
New Woke ref1
Old Woke ref1
Oneness ref1
perceptual programming ref1
Phantom Self ref1
police ref1
defunding the ref1
reframing ref1
public institutions ref1
Pushbackers ref1, ref2, ref3
racism ref1, ref2, ref3
reframing ref1, ref2
religion, as ref1
Sabbatians ref1, ref2, ref3
Silicon Valley ref1
social justice ref1, ref2, ref3, ref4

transgender ref1, ref2, ref3, ref4, ref5
United States ref1, ref2
 vaccines ref1
 Wetiko factor ref1, ref2, ref3
young people ref1, ref2, ref3
women, deletion of rights and status of ref1, ref2
World Economic Forum (WEF) ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9
World Health Organization (WHO) ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9
 AIDs/HIV ref1
 amplification cycles ref1
 Big Pharma ref1, ref2, ref3
 cooperation in health emergencies ref1
 creation ref1, ref2
 fatality rate ref1
 funding ref1, ref2, ref3
 Gates ref1
 Internet ref1
 lockdown ref1
 vaccines ref1, ref2, ref3, ref4
 Wetiko factor ref1
world number 1 (masses) ref1, ref2
world number 2 ref1
Wuhan ref1, ref2, ref3, ref4, ref5, ref6, ref7 ref8

Y

Yaldabaoth ref1, ref2, ref3, ref4, ref5, ref6
Yeadon, Michael ref1, ref2, ref3, ref4
young people *see also* **children** addiction to technology ref1
 Human 2.0 ref1
 vaccines ref1, ref2

Wokeness ref1, ref2, ref3
YouTube ref1, ref2, ref3, ref4, ref5
WHO 548

Z

Zaks, Tal ref1
Zionism ref1, ref2, ref3
Zuckerberg, Mark ref1, ref2, ref3, ref4, ref5, ref6, ref7, ref8, ref9, ref10, ref11, ref12
Zulus ref1

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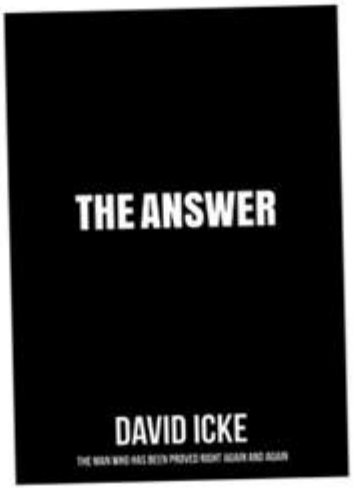
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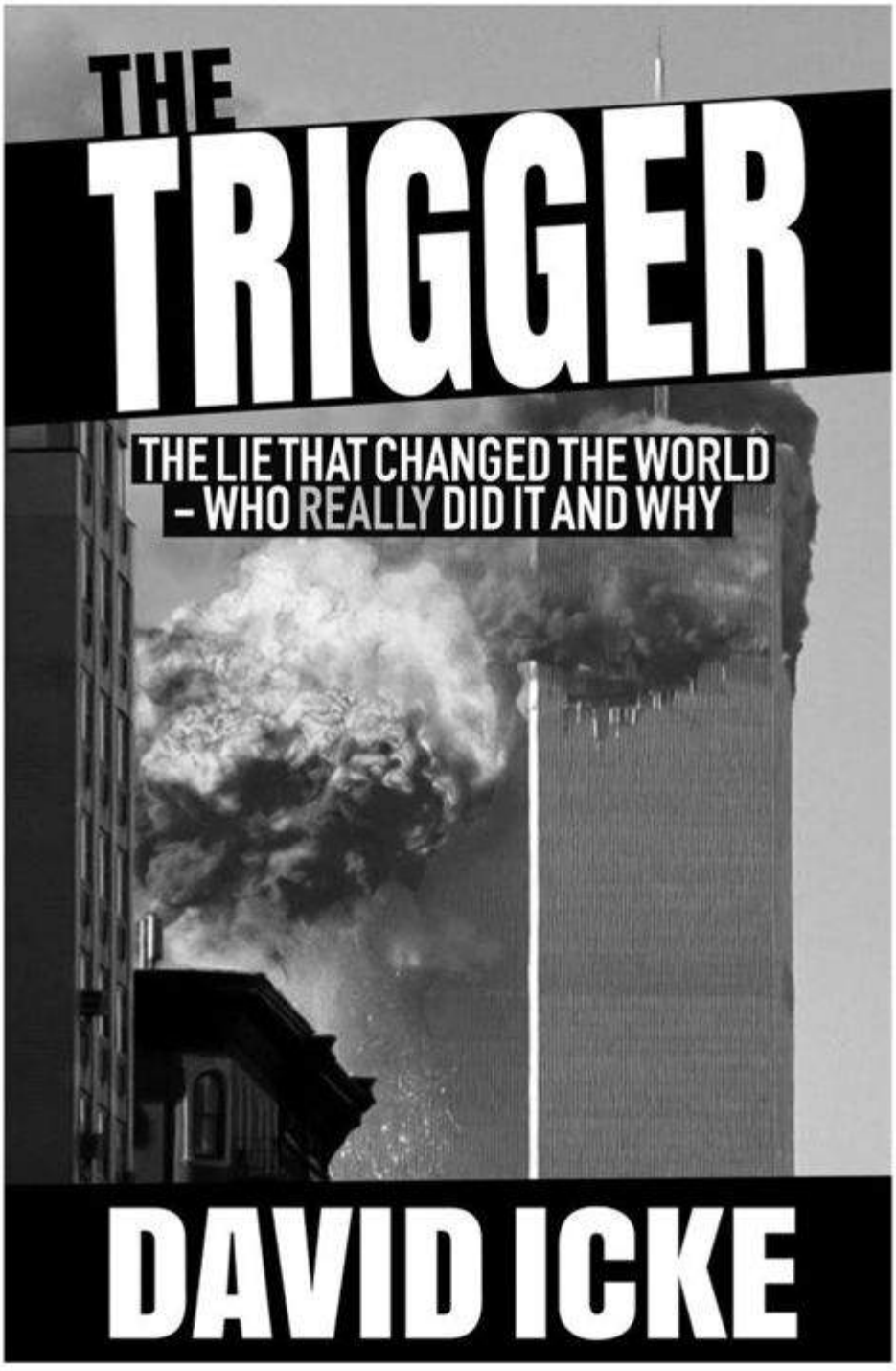
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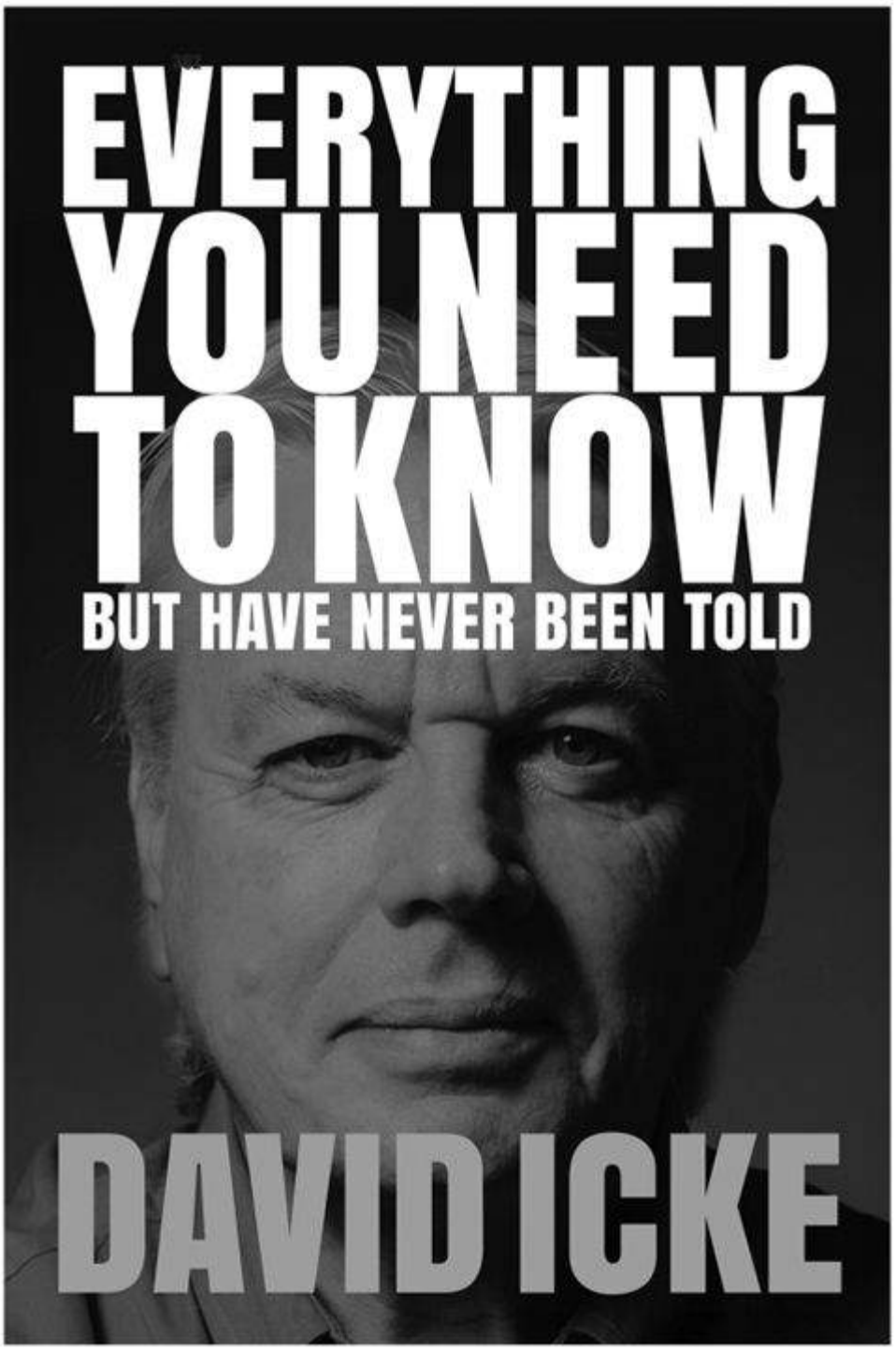


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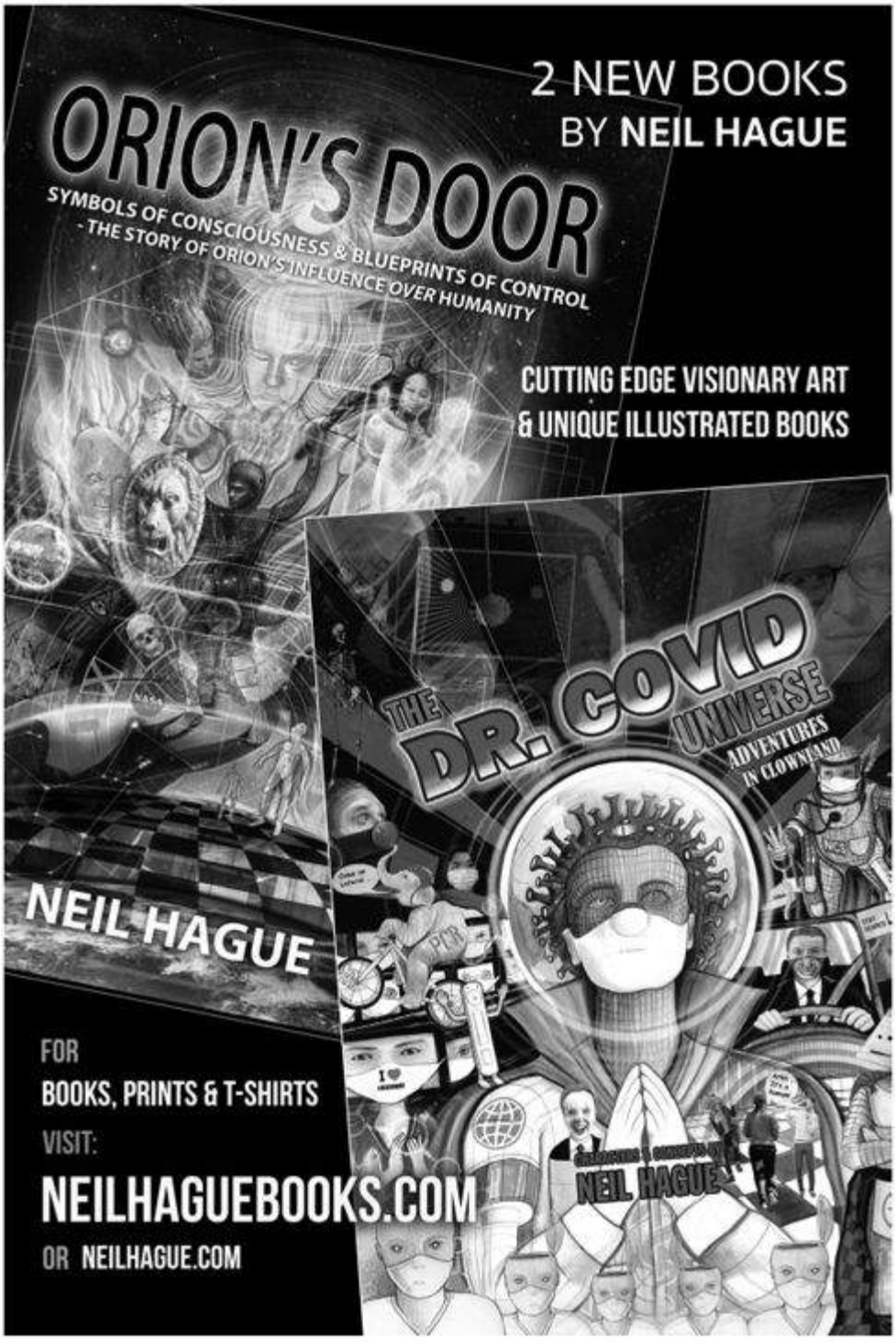
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noun

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